# Moments of derivatives of characteristic polynomials

Paul-Olivier Dehaye
Merton College, University of Oxford
pdehaye@maths.ox.ac.uk

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http://www.maths.ox.ac.uk/~pdehaye

# Outline

- Motivation: Number Theory
- Original problem: Random Matrix Theory
- Techniques: Representation Theory, Combinatorics (+ Hypergeometric summation)
- Results: Many different presentations

# Motivation

Let

$$\zeta(s) \quad := \quad \sum_{n\in\mathbb{N}} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

when $\Re s > 1$, and defined by meromorphic continuation otherwise.

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One can also ask

$$\frac{1}{T}\int_0^T |\zeta(1/2+\mathrm{i}t)|^{2k}\,\mathrm{d}t \sim_T ?,$$

which gives information about large values of $\zeta$ or zero-density estimates.

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**Theorem** ($k=1$: Hardy-Littlewood, $k=2$: Ingham):

$$\lim_{T\to\infty} \frac{1}{T(\log T)^{k^2}} \int_0^T |\zeta(1/2+\mathrm{i}t)|^{2k}\,\mathrm{d}t = f(k)a(k),$$

where $a(k)$ is a product over primes, and $f(1)=1$, $f(2)=1/12$.

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Conrey and Ghosh: $f(3) = 42/9!$
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$$Z_U(\theta) = \prod_{j=1}^{N} \left(1 - e^{\mathrm{i}(\theta_j - \theta)}\right),$$

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New understanding of the conjecture of Keating and Snaith, due to Gonek, Hughes and Keating.

$$\zeta(1/2+\mathrm{i}t) \approx P_X(t) \times Z_X(t) \approx$$

$$\prod_{p\leq X}\left(1-p^{-\frac{1}{2}-\mathrm{i}t}\right)^{-1} \times \prod_{\substack{\gamma_n \\ |t-\gamma_n|<1/\log X}} \left(\mathrm{i}(t-\gamma_n)e^{\gamma}\log X\right)$$

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Splitting Conjecture: Let $X$ and $T \to \infty$ with $X = O((\log T)^{2-\epsilon})$. Then for $k > -1/2$ we have

$$\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2k} \mathrm{d}t \sim_T \left(\frac{1}{T}\int_0^T \left|P_X\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2k} \mathrm{d}t\right) \times \left(\frac{1}{T}\int_0^T \left|Z_X\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2k} \mathrm{d}t\right)$$

# Motivation

If you want to conjecture

$$\lim_{T\to\infty}\frac{1}{T(\log T)^{k^2+2h}}\int_0^T |\zeta(1/2+\mathrm{i}t)|^{2k-2h}|\zeta'(1/2+\mathrm{i}t)|^{2h}\,\mathrm{d}t$$

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you need the associated RMT averages

$$(*) = \lim_{N\to\infty} \frac{1}{N^{k^2+2h}} \left\langle |Z_U(0)|^{2k-2h} |Z_U'(0)|^{2h} \right\rangle_{\mathrm{U}(N)}.$$

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Problem: Formulas have been obtained for $(*)$ and many other similar averages, but we certainly do not know the analytic continuation $k, h \in \mathbb{C}$.

Chris Hughes proved that $(*)$ had the correct normalization and experimentally observed some structure for fixed $h \in \mathbb{N}$.

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Something fun is happening...

Theorem (Conrey-Ghosh):

$$\lim_{T\to\infty} \frac{1}{T(\log T)^2} \int_0^T |\mathcal{Z}(t)\mathcal{Z}'(t)| \,\mathrm{d}t = \frac{e^2-5}{4\pi},$$

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Number Theory tells us here something about random matrices... but how does this strange fraction appear out of averages of characteristic polynomials?

$$\lim_{N\to\infty} \frac{1}{N^2} \left\langle \left| V_U(0) V_U'(0) \right| \right\rangle_{\mathrm{U}(N)} \overset{?}{=} \frac{e^2-5}{4\pi}$$

with

$$V_U(\theta) = e^{\mathrm{i}N(\theta+\pi)/2} e^{-\mathrm{i}\sum_{j=1}^N \theta_j/2} Z_U(\theta).$$

# Main result

Theorem: Fix $r \in \mathbb{N}$. Then,

$$\lim_{N\to\infty} \frac{\left\langle |Z_U(0)|^{2k} \left(\frac{Z_U'(0)}{Z_U(0)}\right)^r \right\rangle_{\mathrm{U}(N)}}{N^r \left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)}} = \left(-\frac{\mathrm{i}}{2}\right)^r \frac{X_r(2k)}{Y_r(2k)},$$

where $X_r(u)$ and $Y_r(u)$ are even polynomials, with $\deg X_r = \deg Y_r$. Moreover,

$$Y_r(u) = \prod_{\substack{1\le a\le r\\ a \text{ odd}}} (u^2 - a^2)^{\alpha_a(r)},$$

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Results for other polynomials can be deduced from this particular problem.

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When the integrand is multiplicative in the eigenvalues, for instance for $|Z_U(0)|^s = \prod_{j=1}^{N} |1 - e^{\mathrm{i}2\pi\theta_j}|^s$, we can apply the Selberg integral and evaluate the RMT problem fairly easily.

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For more general integrands, it is also possible to use the Heine identity to transfer the problem to computing a determinant of a growing Toeplitz matrix (with $N$), but the computations get very complicated very quickly.

# Proof due to Bump and Gamburd (I)

Bump and Gamburd have a very original way to compute

$$\left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)}$$

which uses the dual Cauchy identity

$$\sum_{\lambda \text{ partitions}} s_\lambda(x_1, x_2, \cdots, x_M) s_{\lambda^t}(y_1, y_2, \cdots, y_N) = \prod_{m,n}^{M,N} (1 + x_m y_n).$$

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or (replacing $U$ by $-U$)

$$|Z_U(0)|^{2k} = (-1)^{kN} s_{\langle k^N \rangle}(U) \sum_{\lambda} (-1)^{|\lambda|} s_\lambda\left([1^{2k}]\right) \overline{s_{\lambda^t}(U)}.$$

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$$\sim_N \frac{G(k+1)^2}{G(2k+1)} N^{k^2}.$$

with $G(z)$ the Barnes $G$-function with property

$$G(k+1) = \prod_{j=1}^{k-1} j!$$

# Why does this work?

Consider the advantages over computing directly

$$\left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)} = \frac{1}{N!(2\pi)^N} \times$$

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{m=1}^{N} \left| 1 - e^{i\theta_m} \right|^{2k} \prod_{1 \le m < n \le N} \left| e^{i\theta_m} - e^{i\theta_n} \right|^2 \prod_m \mathrm{d}\theta_m.$$

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- Very natural to expand a symmetric function of eigenvalues in one of the classical bases for those functions.
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- Very easy to integrate when we choose the Schur functions, due to their orthogonality.

It works in two clear steps, the first one leading to an expression as symmetric functions, the second actually evaluating this expression.

# Extension for derivatives

Observe that

$$\begin{aligned}\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r &= \left(\sum_{j=1}^{N} \frac{\mathrm{i}e^{\mathrm{i}\theta_j}}{1-e^{\mathrm{i}\theta_j}}\right)^r \\ &= \left(\mathrm{i}\sum_{m\geq 1} p_m(U)\right)^r.\end{aligned}$$

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$$\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r = \lim_{Y\to 1^-} \left(\sum_{j=1}^{N} \frac{\mathrm{i}e^{\mathrm{i}\theta_j}Y}{1-e^{\mathrm{i}\theta_j}Y}\right)^r$$

$$= \lim_{Y\to 1^-} \left(\mathrm{i}\sum_{m\geq 1} p_m(U)Y^m\right)^r.$$

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We will use this and results on symmetric functions to reexpress the integrand as a sum of Schur functions.

# Extension for derivatives

Observe that

$$\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r = \left(\sum_{j=1}^{N} \frac{\mathrm{i}e^{\mathrm{i}\theta_j}}{1-e^{\mathrm{i}\theta_j}}\right)^r = \left(\mathrm{i}\sum_{m\geq 1} p_m(U)\right)^r.$$

This is of course not allowed, but trust me.

We will use this and results on symmetric functions to reexpress the integrand as a sum of Schur functions.

This fits in a more general setting of modifying an integrand multiplicatively (see my thesis).

# Set up

We get

$$\left\langle |Z_U(0)|^{2k}\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r\right\rangle_{\mathrm{U}(N)} =$$

$$\mathrm{i}^r\left\langle \sum_{\mu}(-)^{|\mu|}s_\mu([1]^{2k})\overline{s_{\mu^{\mathrm{t}}}(U)}\,s_{\langle k^N\rangle}(U)\cdot\left(\sum_{m\geq 1}p_{(m)}(U)\right)^r\right\rangle_{\mathrm{U}(N)}$$

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Problem: How do we multiply

$$s_{\langle k^N \rangle}(U) p_{(m_1)}(U) \cdots p_{(m_j)}(U)$$

and express the result as a Schur function?

# Murnaghan-Nakayama rule

The formula reads

$$s_\lambda p_{(m)} = \sum_{\mu} (-1)^{\mathrm{ht}(\mu/\lambda)} s_\mu,$$

where $\mu$ is obtained from $\lambda$ by adding a(n?) $m$-ribbon.

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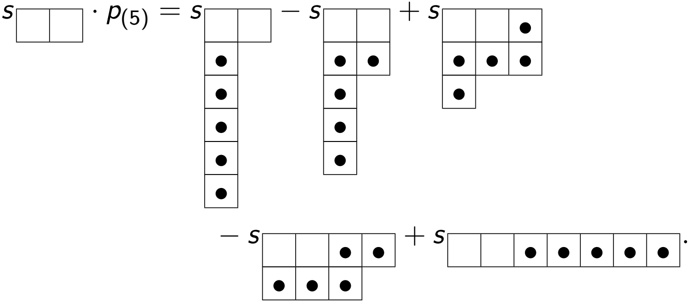
For instance, when $\lambda = \ydiagram{2}$ and $m = 5$,



$$s_{\ydiagram{2}} \cdot p_{(5)} = s_{\begin{ytableau} {} & {} \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{ytableau}} - s_{\begin{ytableau} {} & {} \\ \bullet & \bullet \\ \bullet \\ \bullet \\ \bullet \end{ytableau}} + s_{\begin{ytableau} {} & {} & \bullet \\ \bullet & \bullet & \bullet \\ \bullet \end{ytableau}} - s_{\begin{ytableau} {} & {} & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{ytableau}} + s_{\begin{ytableau} {} & {} & \bullet & \bullet & \bullet & \bullet & \bullet \end{ytableau}}.$$

# Restrictions on partitions occuring in the sum

We need to evaluate

$$\left\langle \sum_{\mu} (-)^{|\mu|} s_{\mu^{\mathrm{t}}}([1]^{2k}) \overline{s_{\mu}(U)} \cdot s_{\langle k^N \rangle}(U) \left( \sum_{m \geq 1} p_{(m)}(U) \right)^{r} \right\rangle_{\mathrm{U}(N)} .$$

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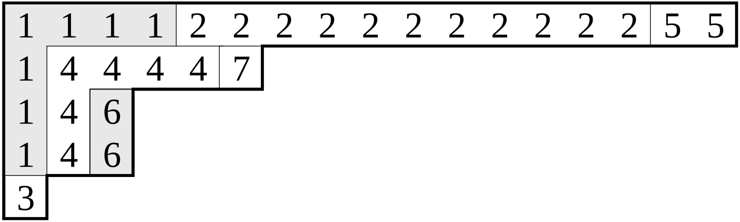
In any case, the combinatorics is much simpler now, because it all happens inside a $k \times N$ box.

# Iterated Murnaghan-Nakayama rule

The iterated M-N rule deals with the process of adding successive ribbons to form a partition $\lambda$, or alternatively the process of cutting up a partition $\lambda$ into ribbons.



| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 5 | 5 |
| 1 | 4 | 4 | 4 | 4 | 7 | | | | | | | | | | | |
| 1 | 4 | 6 | | | | | | | | | | | | | | |
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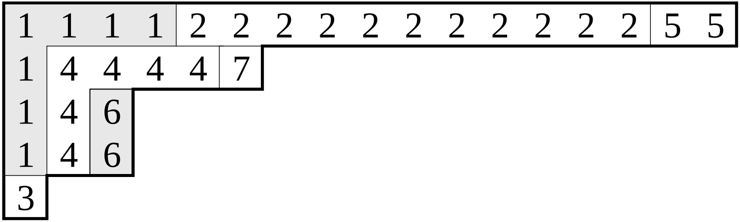

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
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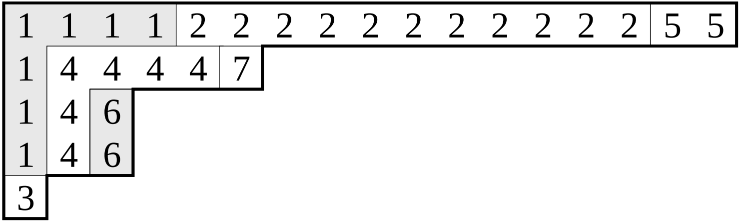

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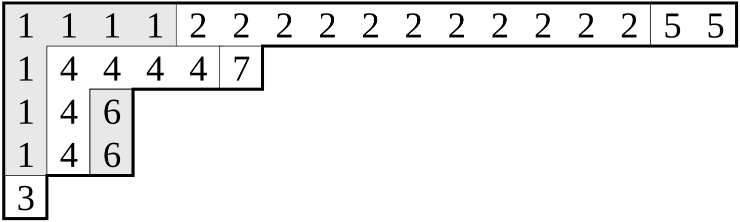

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 5 | 5 |
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$$\cdots + (-1)^3(-1)^0(-1)^0(-1)^2(-1)^0(-1)^1(-1)^0 + \cdots$$

# Everything together

## Proposition

$$\left\langle |Z_U(0)|^{2k} \left( \frac{Z_U'(0)}{Z_U(0)} \right)^r \right\rangle_{\mathrm{U}(N)} = \\ (-\mathrm{i})^r \sum_{\overline{\boldsymbol{\mu}} \in \mathbb{N}_0^r} \sum_{\substack{\lambda \text{ within} \\ k \times N}} \chi_\lambda^{\mathcal{S}_{|\lambda|}}(\overline{\boldsymbol{\mu}})\, s_{\langle N^k \rangle \cup \lambda}\left( \left[ 1^{2k} \right] \right)$$

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Both combine extremely well (signs cancel), and taking a sum over $\overline{\mu}$ makes things tractable. We end up only having to count some combinatorial structures.

# Finite $N$

$$\sum_{r\geq 0}\left\langle |Z_U(0)|^{2k}\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r\right\rangle_{U(N)}\frac{(iz)^r}{r!} = s_{\langle N^k\rangle}\left(\left[1^{2k}\right]\right)$$

$$\sum_{d=0}^{\infty}\sum_{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d}\left[\prod_i \frac{(N+t_i)!(2k-t_i-1)!(k+s_i)!}{(N-s_i-1)!(2k+s_i)!(k-t_i-1)!}\right]$$

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The variables $\vec{s}$, $\vec{t}$ surprisingly correspond to the number of blocks in the structure of $\lambda$.

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The variables $\vec{\mathbf{s}}$, $\vec{\mathbf{t}}$ surprisingly correspond to the number of blocks in the structure of $\lambda$.
There is massive cancellation happening between the different ways of forming a partition $\lambda$.

# Finite $N$

$$\sum_{r\geq 0}\left\langle |Z_U(0)|^{2k}\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r\right\rangle_{\mathrm{U}(N)}\frac{(\mathrm{i}z)^r}{r!}=s_{\langle N^k\rangle}\left(\left[1^{2k}\right]\right)$$

$$\sum_{d=0}^{\infty}\sum_{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d}\left[\prod_i\frac{(N+t_i)!(2k-t_i-1)!(k+s_i)!}{(N-s_i-1)!(2k+s_i)!(k-t_i-1)!}\right]$$

$$\left|\frac{1}{s_i!t_j!(1+s_i+t_j)}\right|_{d\times d}^2 z^{d+\sum s_i+t_i}$$

The variables $\vec{\mathbf{s}}$, $\vec{\mathbf{t}}$ surprisingly correspond to the number of blocks in the structure of $\lambda$.
There is massive cancellation happening between the different ways of forming a partition $\lambda$.
This also uses a new and very effective hypergeometric summation package, SIGMA, by Carsten Schneider (Linz).

# Asymptotics

$$\sum_{r\geq 0}\left\langle |Z_U(0)|^{2k}\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r\right\rangle_{\mathrm{U}(N)}\frac{(\imath z)^r}{r!}\sim_N \frac{G(k+1)^2}{G(2k+1)}N^{k^2}$$

$$\sum_{d=0}^{\infty}\sum_{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d}\left|\frac{1}{s_i!t_j!(1+s_i+t_j)}\right|^2_{d\times d}$$

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The coefficient of $z^r$ or $z^{|\lambda|}$ is only correct when $k$ is large enough ($r<2k+1$). Fortunately, there are infinitely such $k$ so this provides the meromorphic continuations in $k$.

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This expression is a rational function of $k$ with poles as prescribed earlier follows quickly (poles at $k = a$ odd and between $1 - r$ and $r - 1$, with each reaching highest order $d$ when say $\overrightarrow{\mathbf{s}} = \overrightarrow{\mathbf{0}}$ and $t_i$s are close, but different).

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This expression is a rational function of $k$ with poles as prescribed earlier follows quickly (poles at $k=a$ odd and between $1-r$ and $r-1$, with each reaching highest order $d$ when say $\overrightarrow{\mathbf{s}}=\overrightarrow{\mathbf{0}}$ and $t_i$s are close, but different).
Also, what we obtain is even in $k$ by symmetry between $\overrightarrow{\mathbf{s}}$ and $\overrightarrow{\mathbf{t}}$ (transposition).

# Variation of Schur functions

We use the "ninth variation of Schur functions" due to Macdonald (see a paper by Nakagawa, Noumi, Shirakawa, and Yamada).

$$\sum_{r\geq 0}\left\langle |Z_U(0)|^{2k}\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r\right\rangle_{\mathrm{U}(N)}\frac{(\mathrm{i}z)^r}{r!}$$

$$\sim_N \frac{G(k+1)^2}{G(2k+1)}N^{k^2}\sum_{\lambda}\tilde{s}_{\lambda}^{(2k)}s_{\lambda}\left(\left[1^k\right]\right)(Nz)^{|\lambda|}$$

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# Plancherel measure (with A. Borodin, Caltech)

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$$\dim \lambda = \dim \chi_\lambda = \#\text{paths to } \lambda \text{ in the Young graph.}$$

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The formula obtained before then takes the form

$$\sum_{\lambda} \frac{m(\lambda)}{|\lambda|!} f(\lambda) z^{|\lambda|},$$

for $f(\lambda)$ given by a product over the boxes of the partition $\lambda$.

# Orthogonal polynomials (with Borodin, Caltech)

Reëxpress

$$\sum_{\substack{\lambda \\ \left\{ \vec{\mathbf{s}} \atop \vec{\mathbf{t}} \right\} := \lambda \\ d := \mathrm{Frob.rk}(\lambda)}} \left| \frac{1}{s_i!\,t_j!\,(1+s_i+t_j)} \right|^2_{d \times d} \prod_{i=1}^{d} \left( \prod_{a=-t_i}^{s_i} \frac{k+a}{2k+a} \right) z^{|\lambda|}$$

$$= C \sum_{x_1 > \cdots > x_k \geq 0} \prod_{i<j} (x_i - x_j)^2 \prod_i \omega_k(x_i),$$

where $\omega_k(s) = \frac{z^s}{s!(k+s)!}$.

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where $\omega_k(s) = \frac{z^s}{s!(k+s)!}$.
Then the RHS can be obtained as the product of norms of orthogonal polynomials for the weight $\omega_k$.
This leads to multiple Riemann-Hilbert problems linking the coefficients of $z^r$ and $z^{r+1}$, at $k$ and $k+1$. Their solutions are given by discrete Painlevé equations which give recurrence relations $k \to k+1$, $r \to r+1$.
This should also work at finite $N$ and give recurrence relations $N \to N+1$.

# Relations for real characteristic polynomials

Define

$$V_U(\theta) \quad := \quad e^{iN(\theta+\pi)/2} e^{-i\sum_{j=1}^N \theta_j/2} Z_U(\theta).$$

It is easily checked that for real $\theta$, $V_U(\theta)$ is real and $|V_U(\theta)| = |Z_U(\theta)|$.

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Also,

$$\left|\frac{V_U'(\theta)}{V_U(\theta)}\right|^2 = \left(\frac{Z_U'(\theta)}{Z_U(\theta)}\right)^2 + \mathrm{i}N\left(\frac{Z_U'(\theta)}{Z_U(\theta)}\right) - \frac{N^2}{4}$$

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so we can recover

$$\left\langle |V_U(0)|^{2k} \left|\frac{V_U'(0)}{V_U(0)}\right|^{2h} \right\rangle_{\mathrm{U}(N)} \qquad h \in \mathbb{N}$$

from what we have computed so far, and we again get rational functions of $k$.

# Situation for Conrey-Ghosh

We only know

$$\left\langle |V_U(0)|^{2k} \left| \frac{V_U'(0)}{V_U(0)} \right|^{2h} \right\rangle_{\mathrm{U}(N)} \qquad h \in \mathbb{N}.$$

This means that we cannot recover (because there $h = 1/2$)

$$\left\langle |V_U(0)|^{2} \left| \frac{V_U'(0)}{V_U(0)} \right| \right\rangle_{\mathrm{U}(N)} .$$

Remember this would "explain" the $\frac{e^2-5}{4\pi}$ that Conrey and Ghosh found when looking at

$$\lim_{T\to\infty} \frac{1}{T(\log T)^2} \int_0^T |\mathcal{Z}(t)\mathcal{Z}'(t)| \, \mathrm{d}t.$$

# Numerators for moments of real char. polys

Theorem. Fix $h \in \mathbb{N}$. Then,

$$\lim_{N\to\infty} \frac{\left\langle |V_U(0)|^{2k} \left| \frac{V_U'(0)}{V_U(0)} \right|^{2h} \right\rangle_{\mathrm{U}(N)}}{N^{2h} \left\langle |V_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)}} = C(2h) \frac{\tilde{X}_{2h}(2k)}{Y_{2h}(2k)},$$

where $\tilde{X}_{2h}(u)$ and $Y_{2h}(u)$ are even polynomials, with $\deg \tilde{X}_{2h} \leq \deg Y_{2h}$. Moreover, as before,

$$Y_r(u) = \prod_{\substack{1\leq a\leq r \\ a \text{ odd}}} (u^2 - a^2)^{\alpha_a(r)},$$

with $\alpha_a(r) = \left\lfloor \frac{-a+\sqrt{a^2+4r}}{2} \right\rfloor$.

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We do not understand $\tilde{X}_{2h}(u)$.

# Numerators for moments of real char. polys

| $2h$ | $\tilde{X}_{2h}(u)$ |
|---|---|
| 2 | $1$ |
| 4 | $1$ |
| 6 | $u^2 - 9$ |
| 8 | $u^2 - 33$ |
| 10 | $u^4 - 90u^2 + 1497$ |
| 12 | $u^6 - 171u^4 + 6867u^2 - 27177$ |
| 14 | $u^8 - 316u^6 + 30702u^4 - 982572u^2 + 6973305$ |
| 16 | $u^8 - 484u^6 + 76902u^4 - 4461348u^2 + 67692705$ |
| 18 | $u^{12} - 766u^{10} + 215847u^8 - 27766980u^6 + 1653656895u^4 - 41530140126u^2 + 337968054585$ |
| 20 | $u^{14} - 1055u^{12} + 421093u^{10} - 79486155u^8 + 7242179715u^6 - 290444510205u^4 + 4099101803991u^2 - 8381907513945$ |

# Numerators for moments of real char. polys

Chris Hughes observed that those polynomials tended to have real roots. Indeed, the first numerator with some complex roots is $\tilde{X}_{42}$, i.e. this first only occurs for

$$\left\langle |V_U(0)|^{2k} \left| \frac{V_U'(0)}{V_U(0)} \right|^{42} \right\rangle_{\mathrm{U}(N)} .$$

The numerator is of degree 44 (in $k$) and has only 4 non-real roots.

# Numerators for moments of real char. polys

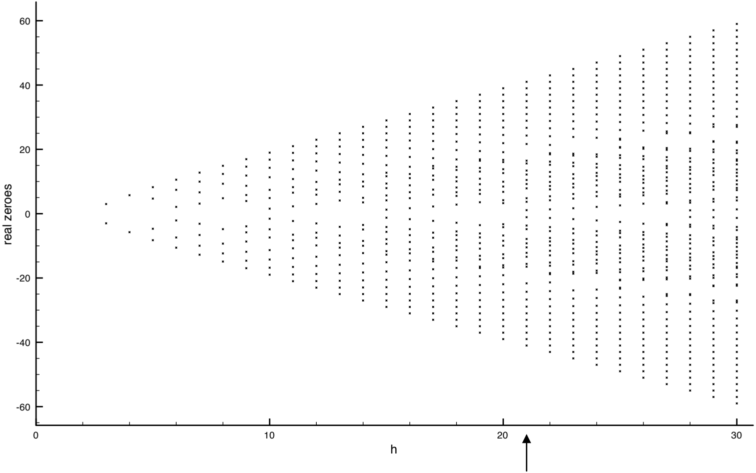

Figure: Location of the roots (in $k$) on the real line of the numerators for the real moments $\left\langle |V_U(0)|^{2k} \left| \frac{V_U'(0)}{V_U(0)} \right|^{2h} \right\rangle_{\mathrm{U}(N)}$.

# Numerators for moments of real char. polys

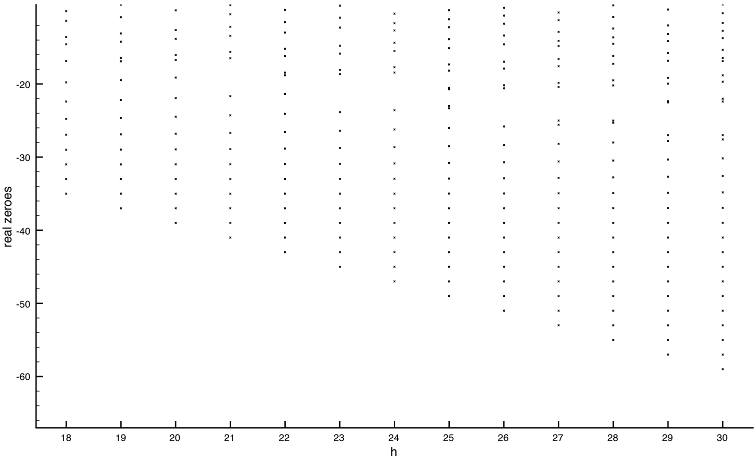

Figure: Location of the roots (in $k$) on the real line of the numerators for the real moments $\left\langle |V_U(0)|^{2k} \left| \frac{V_U'(0)}{V_U(0)} \right|^{2h} \right\rangle_{\mathrm{U}(N)}$.

# One form for the final formula

$$\sum_{r\geq 0}\left\langle |Z_U(0)|^{2k}\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r\right\rangle_{\mathrm{U}(N)}\frac{(\mathrm{i}z)^r}{r!}\sim_N \frac{G(k+1)^2}{G(2k+1)}N^{k^2}$$

$$\sum_{\substack{\lambda\\ \left\{\vec{\mathbf{s}} \atop \vec{\mathbf{t}}\right\}:=\lambda\\ d:=\mathrm{Frob.rk}(\lambda)}} \left|\frac{1}{s_i!\,t_j!\,(1+s_i+t_j)}\right|^2_{d\times d}$$

$$\prod_{i=1}^{d}\left(\prod_{a=-t_i}^{s_i}\frac{k+a}{2k+a}\right)(Nz)^{|\lambda|}$$

# References

math.NT/0703440 *Joint moments of derivatives of characteristic polynomials.* Accepted in *Algebra and Number Theory Journal*.

math.RT/0504399 *Averages over classical Lie groups, twisted by characters.*
Philosophy on why this approach should work more generally.

Acknowledgments: Alexei Borodin, Dan Bump, Brian Conrey, Persi Diaconis, Chris Hughes, Masatoshi Noumi, Hidetaka Sakai, Carsten Schneider.

Talk available (soon) at:

`http://www.maths.ox.ac.uk/~pdehaye/`