

Joint moments of derivatives of characteristic polynomials

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Motivation

Theorem ($k = 1$: Hardy-Littlewood, $k = 2$: Ingham):

$$\lim_{T \rightarrow \infty} \frac{1}{T(\log T)^{k^2}} \int_0^T |\zeta(1/2 + it)|^{2k} dt = f(k)a(k),$$

where $a(k)$ is a product over primes, $f(1) = 1$, $f(2) = 1/12$.

Conjectures

Conrey and Ghosh: $f(3) = 42/9!$

Conrey and Gonek: $f(4) = 24024/16!$

Keating and Snaith: $f(k) = \lim_{N \rightarrow \infty} \frac{1}{N^{k^2}} \langle |Z_U(0)|^{2k} \rangle_{U(N)}$,

where

$$Z_U(\theta) = \prod_{j=1}^N \left(1 - e^{i(\theta_j - \theta)} \right).$$

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Motivation

If you want to conjecture

$$\lim_{T \rightarrow \infty} \frac{1}{T(\log T)^{k^2+2h}} \int_0^T |\zeta(1/2+it)|^{2k-2h} |\zeta'(1/2+it)|^{2h} dt,$$

you need the associated RMT averages

$$(*) = \lim_{N \rightarrow \infty} \frac{1}{N^{k^2+2h}} \left\langle |Z_U(0)|^{2k-2h} |Z'_U(0)|^{2h} \right\rangle_{U(N)}.$$

Problem: Formulas have been obtained for $(*)$ and many other similar averages, but we certainly do not know the analytic continuation $k, h \in \mathbb{C}$.

Chris Hughes proved $(*)$ had the **right normalization** and experimentally observed some structure for fixed $h \in \mathbb{N}$.

Motivation (cont.)

Something fun is happening...

Theorem (Conrey-Ghosh):

$$\lim_{T \rightarrow \infty} \frac{1}{T(\log T)^2} \int_0^T |\mathcal{Z}(t)\mathcal{Z}'(t)| dt = \frac{e^2 - 5}{4\pi},$$

with $\mathcal{Z}(t)$ is Hardy's function, i.e. a real function such that $|\mathcal{Z}(t)| = |\zeta(1/2 + it)|$.

Number Theory tells us here something about random matrices... but how does this strange fraction appear out of averages of characteristic polynomials?

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Main result

Theorem. Fix $r \in \mathbb{N}$. Then,

$$\lim_{N \rightarrow \infty} \frac{\langle |Z_U(0)|^{2k} \left(\frac{Z'_U(0)}{Z_U(0)} \right)^r \rangle_{U(N)}}{N^r \langle |Z_U(0)|^{2k} \rangle_{U(N)}} = \left(-\frac{i}{2} \right)^r \frac{X_r(2k)}{Y_r(2k)},$$

where $X_r(u)$ and $Y_r(u)$ are even polynomials, with $\deg X_r = \deg Y_r$. Moreover,

$$Y_r(u) = \prod_{\substack{1 \leq a \leq r \\ a \text{ odd}}} (u^2 - a^2)^{\alpha_a(r)},$$

$$\text{with } \alpha_a(r) = \left\lfloor \frac{-a + \sqrt{a^2 + 4r}}{2} \right\rfloor.$$

I obtain more results, for different polynomials, and possibly generalizations for higher derivatives or other locations.

Obstruction

We want to compute averages over compact Lie groups of symmetric functions of the eigenvalues.

When the integrand is multiplicative in the eigenvalues, for instance for $|Z_U(0)|^s = \prod_{j=1}^N |1 - e^{i2\pi\theta_j}|^s$, we can apply the Selberg integral and evaluate the RMT problem easily.

For more general integrands, it is also possible to use the Heine identity to transfer the problem to computing a determinant of a growing Toeplitz matrix (with N), but the computations get very complicated very quickly.

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Proof due to Bump and Gamburd (I)

Bump and Gamburd have a very original way to compute

$$\left\langle |Z_U(0)|^{2k} \right\rangle_{U(N)}$$

which uses the dual Cauchy identity

$$\sum_{\lambda \text{ partitions}} s_{\lambda}(x_1, x_2, \dots, x_M) s_{\lambda^t}(y_1, y_2, \dots, y_N) = \prod_{m,n}^{M,N} (1 + x_m y_n).$$

$$\begin{aligned} \sum_{\lambda} s_{\lambda}([1^{2k}]) \overline{s_{\lambda^t}(U)} &= \det(\text{Id} + \overline{U})^{2k} \\ &= \overline{\det(U)^k} |\det(\text{Id} + U)|^{2k} \\ &= \overline{s_{\langle kN \rangle}(U)} |\det(\text{Id} + U)|^{2k} \end{aligned}$$

or (replacing U by $-U$)

$$|Z_U(0)|^{2k} = (-1)^{kN} \overline{s_{\langle kN \rangle}(U)} \sum_{\lambda} (-1)^{|\lambda|} s_{\lambda}([1^{2k}]) \overline{s_{\lambda^t}(U)}.$$

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Proof due to Bump and Gamburd (II)

$$|Z_U(0)|^{2k} = (-1)^{kN} s_{\langle kN \rangle}(U) \sum_{\lambda} (-1)^{|\lambda|} s_{\lambda}([1^{2k}]) \overline{s_{\lambda t}(U)}$$

So

$$\begin{aligned} \langle |Z_U(0)|^{2k} \rangle_{U(N)} &= s_{\langle Nk \rangle}([1^{2k}]) \\ &\sim_N \frac{G(k+1)^2}{G(2k+1)} N^{k^2}. \end{aligned}$$

with $G(z)$ the Barnes G -function with property

$$G(k+1) = \prod_{j=1}^{k-1} j!$$

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Why does this work?

- ▶ Very natural to expand a symmetric function of eigenvalues in one of the classical bases for those functions.
- ▶ Very easy to integrate when we choose the Schur functions, due to their orthogonality.

It works in 2 clear steps, the first one leading to an expression as symmetric functions, the second actually evaluating this expression.

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Extension for derivatives

Observe that

$$\left(\frac{Z'_U(0)}{Z_U(0)} \right)^r = \left(\sum_{j=1}^N \frac{ie^{i\theta_j}}{1 - e^{i\theta_j}} \right)^r = \left(i \sum_{m \geq 1} p_m(U) \right)^r.$$

This is of course not allowed, but trust me.

We will use this and results on symmetric functions to reexpress the integrand as a sum of Schur functions.

This fits in a more general setting of modifying an integrand multiplicatively (see my thesis).

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Set up

We get

$$\left\langle |Z_U(0)|^{2k} \left(\frac{Z'_U(0)}{Z_U(0)} \right)^r \right\rangle_{U(N)} =$$

$$i^r \left\langle \sum_{\mu} (-1)^{|\mu|} s_{\mu}([1]^{2k}) \overline{s_{\mu^t}(U)} s_{\langle k^N \rangle}(U) \cdot \left(\sum_{m \geq 1} p_{(m)}(U) \right)^r \right\rangle_{U(N)}$$

Problem: How do we multiply

$$s_{\langle k^N \rangle}(U) p_{(m_1)}(U) \cdots p_{(m_j)}(U)$$

and express the result as a Schur function?

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Murnaghan-Nakayama rule

The formula reads

$$s_{\lambda} p(m) = \sum_{\mu} (-1)^{\text{ht}(\mu/\lambda)} s_{\mu},$$

where μ is obtained from λ by adding a(n?) m -ribbon.

For instance, when $\lambda = \square\square$ and $m = 5$,

$$s_{\square\square} \cdot p(5) = s_{\begin{array}{|c|} \hline \square \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \end{array}} - s_{\begin{array}{|c|} \hline \square \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \end{array}} + s_{\begin{array}{|c|} \hline \square \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \end{array}} + s_{\begin{array}{|c|} \hline \square \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \end{array}} + s_{\begin{array}{|c|} \hline \square \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \end{array}}.$$

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Restrictions on partitions occurring in the sum

We need to evaluate

$$\left\langle \sum_{\mu} (-1)^{|\mu|} s_{\mu}([1]^{2k}) \overline{s_{\mu^t}(U)} \cdot s_{\langle kN \rangle}(U) \left(\sum_{m \geq 1} p_{(m)}(U) \right)^r \right\rangle_{U(N)}.$$

We get $l(\mu) \leq 2k$, $l(\mu^t) \leq N$, i.e. μ fits inside a $2k \times N$ rectangle.

Hence, we know the sum involves **finitely** many μ s and thus **m is bounded**. This justifies our expansion of the geometric sum earlier, **provided** $r - 2k < 1$.

In any case, the combinatorics is much simpler now, because it all happens inside a $k \times N$ box.

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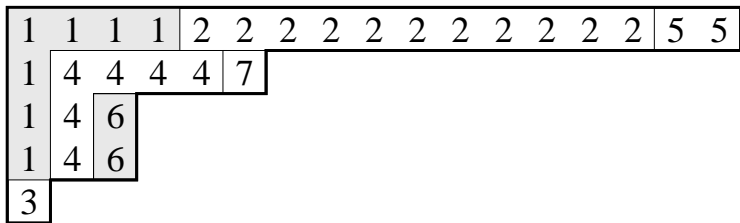
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Iterated Murnaghan-Nakayama rule



$$\chi_{(17,6,3,3,1)}((7, 11, 1, 6, 2, 2, 1)) = \sum_T (-1)^{\text{ht}(T)} =$$

$$\dots + (-1)^3 (-1)^0 (-1)^0 (-1)^2 (-1)^0 (-1)^1 (-1)^0 + \dots$$

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Proposition

$$\left\langle |Z_U(0)|^{2k} \left(\frac{Z'_U(0)}{Z_U(0)} \right)^r \right\rangle_{U(N)} = (-i)^r \sum_{\bar{\mu} \in \mathbb{N}_0^r} \sum_{\lambda \text{ within } k \times N} \chi_{\bar{\mu}}^\lambda s_{\langle N^k \rangle_{U\lambda}} \left([1^{2k}] \right)$$

Evaluation

We want to evaluate

$$\sum_{\substack{\lambda \text{ within} \\ k \times N}} \sum_{\bar{\mu} \in \mathbb{N}_0^r} \chi_{\bar{\mu}}^{\lambda} s_{\langle N^k \rangle_{U\lambda}} \left([1^{2k}] \right),$$

which a priori is hard because of the values of characters of symmetric groups.

We do this using 2 results.

1. A formula of El-Samra and King giving $s_{\lambda}([1^{2k}])$ as a determinant in terms of the Frobenius coordinates of λ . The formula is antisymmetric in those coordinates.
2. A “straighter” alternative to the M-N rule due to Borodin.

Both combine extremely well (signs cancel), and taking a sum over $\bar{\mu}$ makes things tractable. We only have to count combinatorial structures, **which might be harder for some generalizations.**

Finite N

$$\sum_{r>0} \left\langle |Z_U(0)|^{2k} \left(\frac{Z'_U(0)}{Z_U(0)} \right)^r \right\rangle_{U(N)} \frac{(iz)^r}{r!} = s_{\langle N^k \rangle} \left([1^{2k}] \right)$$

$$\sum_{d=1}^{\infty} \sum_{\vec{s}, \vec{t} \in \mathbb{N}^d} \left| \frac{z^{1+s_i+t_j}}{s_i! t_j! (1+s_i+t_j)} \right|_{d \times d}$$

$$\sum_{\substack{\vec{p} \in [0, N-1]^d \\ \vec{q} \in [0, k-1]^d}} \left| k \frac{\binom{p_i}{s_i} \binom{q_j}{t_j} \binom{k+p_i}{p_i} \binom{k-1}{q_j} (N-p_i)^{(k)} (-1)^{q_j}}{(N+q_j+1)^{(k)} (1+p_i+q_j)} \right|_{d \times d}$$

The variables \vec{p}, \vec{q} refer to the partition λ with the conditions $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d$ and $\lambda_1^t \geq \lambda_2^t \geq \dots \geq \lambda_d^t$ removed.

The variables \vec{s}, \vec{t} correspond to the number of blocks in the structure of λ , and somehow also parametrize a partition.

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$$\sum_{r>0} \left\langle |Z_U(0)|^{2k} \left(\frac{Z'_U(0)}{Z_U(0)} \right)^r \right\rangle_{U(N)} \frac{(iz)^r}{r!} \sim_N \frac{G(k+1)^2}{G(2k+1)} N^{k^2}$$

$$\sum_{d=1}^{\infty} \sum_{\vec{s}, \vec{t} \in \mathbb{N}^d} \left| \frac{1}{s_i! t_j! (1 + s_i + t_j)} \right|_{d \times d}^2$$

$$\prod_{i,j=1}^d \left(\prod_{a=-t_j}^{s_i} \frac{k+a}{2k+a} \right) (Nz)^{d+\sum s_i+t_j},$$

and the result that this expression is a rational function of k with poles as prescribed earlier follows quickly (poles at $k = a$ odd and between $1 - r$ and $r - 1$, with each reaching highest order d when say $\vec{s} = \vec{0}$ and t_i s are close, but different). Also, what we obtain is even in k by symmetry between \vec{s} and \vec{t} (transposition).

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$$\left\langle |Z_U(0)|^{2k} \sum_{r \geq 1} \frac{(iz)^r}{r!} \left(\frac{Z'_U(0)}{Z_U(0)} \right)^r \right\rangle_{U(N)} \sim_N \frac{G(k+1)^2}{G(2k+1)} N^{k^2}$$

$$\sum_{\substack{\lambda \\ \{\vec{s} \\ \vec{t}\} := \lambda \\ d := \text{Frob.rk}(\lambda)}} \left| \frac{1}{s_i! t_j! (1 + s_i + t_j)} \right|_{d \times d}^2$$

$$\prod_{i,j=1}^d \left(\prod_{a=-t_j}^{s_i} \frac{k+a}{2k+a} \right) (Nz)^{|\lambda|}$$

which only completely makes sense when $r - 2k < 1$.

More expressions

1. As hypergeometric multisums
2. Back in terms of Schur functions, using the El-Samra+King formula again (leads to Plancherel measure)
3. In terms of the “Macdonald’s ninth variation of the Schur function” (see a paper by Nakagawa, Noumi, Shirakawa, and Yamada)
4. As asymptotics for the determinant of a growing matrix, imitating a proof of the Cauchy identity
5. Future work: Using orthogonal polynomials, we should also get recurrence formulas for those rational functions (joint with Alexei Borodin, Caltech)

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Relations for real characteristic polynomials

Define

$$V_U(\theta) := e^{iN(\theta+\pi)/2} e^{-i\sum_{j=1}^N \theta_j/2} Z_U(\theta).$$

It is easily checked that for real θ , $V_U(\theta)$ is real and $|V_U(\theta)| = |Z_U(\theta)|$.

Also,

$$\left| \frac{V'_U(\theta)}{V_U(\theta)} \right|^2 = \left(\frac{Z'_U(\theta)}{Z_U(\theta)} \right)^2 + iN \left(\frac{Z'_U(\theta)}{Z_U(\theta)} \right) - \frac{N^2}{4},$$

so we can recover

$$\left\langle |V_U(0)|^{2k} \left| \frac{V'_U(0)}{V_U(0)} \right|^{2h} \right\rangle_{U(N)} \quad h \in \mathbb{N}$$

from what we have computed so far, and we again get rational functions of k .

Situation for Conrey-Ghosh

We only know

$$\left\langle |V_U(0)|^{2k} \left| \frac{V'_U(0)}{V_U(0)} \right|^{2h} \right\rangle_{U(N)} \quad h \in \mathbb{N}.$$

This means that we cannot recover (because there $h = 1/2$)

$$\left\langle |V_U(0)|^2 \left| \frac{V'_U(0)}{V_U(0)} \right| \right\rangle_{U(N)}.$$

Remember this would “explain” the $\frac{e^2-5}{4\pi}$ that Conrey and Ghosh found when looking at

$$\lim_{T \rightarrow \infty} \frac{1}{T(\log T)^2} \int_0^T |\mathcal{Z}(t)\mathcal{Z}'(t)| dt.$$

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Theorem. Fix $h \in \mathbb{N}$. Then,

$$\lim_{N \rightarrow \infty} \frac{\left\langle |V_U(0)|^{2k} \left| \frac{V'_U(0)}{V_U(0)} \right|^{2h} \right\rangle_{U(N)}}{N^{2h} \langle |V_U(0)|^{2k} \rangle_{U(N)}} = C(2h) \frac{\tilde{X}_{2h}(2k)}{Y_{2h}(2k)},$$

where $\tilde{X}_{2h}(u)$ and $Y_{2h}(u)$ are even polynomials, with $\deg \tilde{X}_{2h} \leq \deg Y_{2h}$. Moreover, as before,

$$Y_r(u) = \prod_{\substack{1 \leq a \leq r \\ a \text{ odd}}} (u^2 - a^2)^{\alpha_a(r)},$$

with $\alpha_a(r) = \left\lfloor \frac{-a + \sqrt{a^2 + 4r}}{2} \right\rfloor$.

We do not understand $\tilde{X}_{2h}(u)$.

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$2h$	$\tilde{X}_{2h}(u)$
2	1
4	1
6	$u^2 - 9$
8	$u^2 - 33$
10	$u^4 - 90u^2 + 1497$
12	$u^6 - 171u^4 + 6867u^2 - 27177$
14	$u^8 - 316u^6 + 30702u^4 - 982572u^2 + 6973305$
16	$u^8 - 484u^6 + 76902Ku^4 - 4461348u^2 + 67692705$
18	$u^{12} - 766u^{10} + 215847u^8 - 27766980u^6 + 1653656895u^4 - 41530140126u^2 + 337968054585$
20	$u^{14} - 1055u^{12} + 421093u^{10} - 79486155u^8 + 7242179715u^6 - 290444510205u^4 + 4099101803991u^2 - 8381907513945$

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Chris Hughes observed that those polynomials tended to have real roots. Indeed, the first numerator with some complex roots is \tilde{X}_{42} , i.e. this first only occurs for

$$\left\langle |V_U(0)|^{2k} \left| \frac{V'_U(0)}{V_U(0)} \right|^{42} \right\rangle_{U(N)} .$$

The numerator is of degree 44 (in k) and has only 4 non-real roots.

Numerators for moments of real char. polys

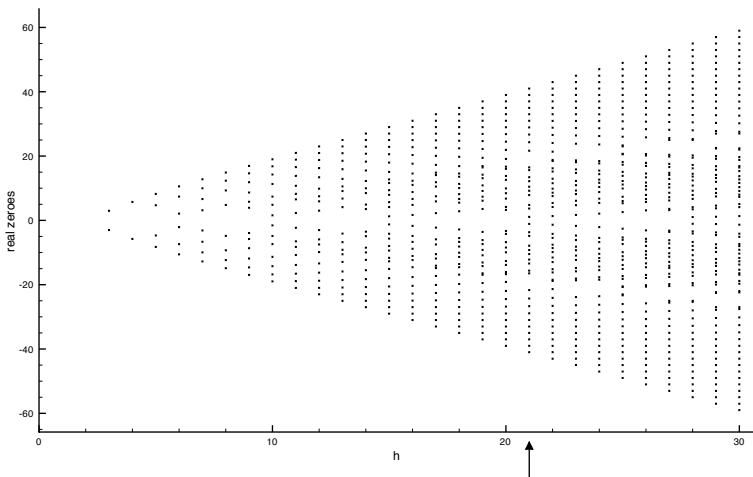


Figure: Location of the roots (in k) on the real line of the numerators for the real moments $\left\langle |V_U(0)|^{2k} \left| \frac{V'_U(0)}{V_U(0)} \right|^{2h} \right\rangle_{U(N)}$.

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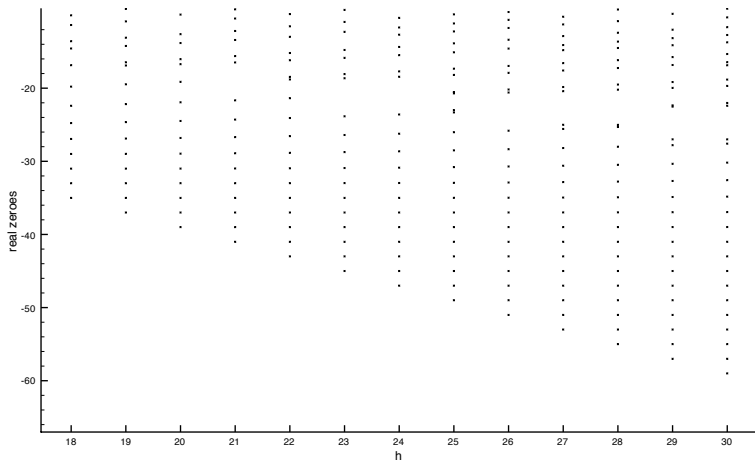


Figure: Location of the roots (in k) on the real line of the numerators for the real moments $\left\langle |V_U(0)|^{2k} \left| \frac{V'_U(0)}{V_U(0)} \right|^{2h} \right\rangle_{U(N)}$.

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One form for the final formula

$$\left\langle |Z_U(0)|^{2k} \sum_{r \geq 1} \frac{(iz)^r}{r!} \left(\frac{Z'_U(0)}{Z_U(0)} \right)^r \right\rangle_{U(N)} \sim_N \frac{G(k+1)^2}{G(2k+1)} N^{k^2}$$

$$\sum_{\substack{\lambda \\ \left\{ \begin{smallmatrix} \vec{s} \\ \vec{t} \end{smallmatrix} \right\} := \lambda \\ d := \text{Frob.rk}(\lambda)}} \left| \frac{1}{s_i! t_j! (1 + s_i + t_j)} \right|_{d \times d}^2$$

$$\prod_{i,j=1}^d \left(\prod_{a=-t_j}^{s_i} \frac{k+a}{2k+a} \right) (Nz)^{|\lambda|}$$

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References

[math.NT/0703440](#) *Joint moments of derivatives of characteristic polynomials or of the Riemann zeta function.*

[math.RT/0504399](#) *Averages over classical compact Lie groups and Weyl characters.*
Philosophy on why this approach should work more generally.

Talk available (soon) at:

<http://www.maths.ox.ac.uk/~pdehaye/>

Moments of
derivatives of
characteristic
polynomials

transpose

S_k^N

$s(U) = 0$ when length
rectangle $< N^k >$

Frobenius

vect

sort

ones

character symmetric group