Averages over compact Lie groups (and applications for Number Theory predictions)

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In memoriam: Isolde Field

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The first part of this work is my thesis, done at Stanford under the supervision of Daniel Bump.
It benefited greatly of discussions with Persi Diaconis and Brian Conrey.

Motivation

Theorem ($k = 1$: Hardy-Littlewood, $k = 2$: Ingham):

$$\lim_{T\to\infty} \frac{1}{T(\log T)^{k^2}} \int_0^T |\zeta(1/2 + \mathrm{i}t)|^{2k} \,\mathrm{d}t = f(k)a(k),$$

where $a(k)$ is a product over primes, $f(1) = 1$, $f(2) = 1/12$.

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"The ζ-function is modeled in leading order by averages over characteristic polynomials".

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Theorem (Conrey-Gosh):

$$\lim_{T\to\infty} \frac{1}{T(\log T)^2} \int_0^T |\mathcal{Z}(t)\mathcal{Z}'(t)|\,\mathrm{d}t = \frac{e^2-5}{4\pi},$$

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This, along with many other results, suggests that we look at equivalent problems for (Hardy) characteristic polynomials.

In particular, how does this strange fraction appear out of averages of characteristic polynomials?

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With more generality, we consider

$$\lim_{T\to\infty}\frac{1}{T(\log T)^{k^2+2h}}\int_0^T |\zeta(1/2+\mathrm{i}t)|^{2k-2h}|\zeta'(1/2+\mathrm{i}t)|^{2h}\,\mathrm{d}t$$

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with the associated RMT averages

$$(*) = \int_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k-2h}|\Lambda_g'(0)|^{2h}\,\mathrm{d}g,$$

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Chris Hughes showed that $(*) \sim_N N^{k^2+2h}$ and guessed some facts about the leading coefficient when $h \in \mathbb{N}$.

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$$\int_{[0,1]^N} \prod_{j=1}^{N} x_j^{a-1}(1-x_j)^{b-1} \prod_{1\leq j<k\leq N} |x_j - x_k|^{2c} \,\mathrm{d}\overline{x} = \prod_{j=0}^{N-1} \frac{\Gamma(a+jc)\Gamma(b+jc)\Gamma((j+1)c)}{\Gamma(a+b+(n+j-1)c)\Gamma(c)}.$$

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For more general integrands, it is also possible to use the Heine identity to transfer the problem to computing a determinant of a growing Toeplitz matrix (with N), but the computations get very complicated very quickly.

Main idea

Observe that

$$\frac{|\Lambda_g(0)|^{2k-2h}|\Lambda'_g(0)|^{2h}}{|\Lambda_g(0)|^{2k}} = \left|\frac{\Lambda'_g(0)}{\Lambda_g(0)}\right|^{2h} = \left|\sum_{j=1}^{N} \frac{\mathrm{i}e^{\mathrm{i}\theta_j}}{1-e^{\mathrm{i}\theta_j}}\right|^{2h}.$$

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We will present general results (Ratios) and illustrate the method on the extended Conrey-Gosh example (Moments).

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- Let $\sigma : \mathbb{T} \to \mathbb{C}$, with $\sigma(t) = \sigma(t^{-1})$.
  Define $\Phi_\sigma : \mathrm{G}(N) \to \mathbb{C}, g \mapsto \prod_{\text{(half of) } t_i\text{'s}} \sigma(t_i)$.

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- $$\lim_{N\to\infty} \frac{\int_{\mathrm{U}(N)} \Phi_\sigma(g)\chi^{G(N)}(g)\,\mathrm{d}g}{\int_{\mathrm{U}(N)} \Phi_\sigma(g)\,\mathrm{d}g} = ?$$
- How is an average over $\mathrm{G}(N)$ affected when introducing a character into a multiplicative integrand?

Definitions

- Characters of $\mathrm{G}(N)$ are given by the Weyl Character Formula, where characters are indexed by (almost)-partitions.
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- $N < l(\lambda) \Rightarrow \chi_\lambda^{\mathrm{G}(N)} \equiv 0$.
- For a fixed N, $N \geq l(\lambda), l(\mu)$,

$$\mathbb{E}_{\mathrm{G}(N)}\, \chi_\lambda^{\mathrm{G}(N)} \chi_\mu^{\mathrm{G}(N)} = \delta_\mu^\lambda$$

$\Rightarrow$ For a fixed N, set of orthogonal irreducible characters of $\mathrm{G}(N)$ indexed by partitions of length less than N.

Examples

- $\chi_\lambda^{\mathrm{U}(N)}(t_1, \cdots, t_N) = \dfrac{\left| t_i^{\lambda_j + N - j} \right|_{N \times N}}{\left| t_i^{n-j} \right|_{N \times N}}$

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- $$\chi_\lambda^{\mathrm{Sp}(2N)}(t_1, \cdots, t_N) = \frac{\left| t_i^{\lambda_j + N - j + 1} - t_i^{-(\lambda_j + N - j + 1)} \right|_{N \times N}}{\left| t_i^{N - j + 1} - t_i^{-(N - j + 1)} \right|_{N \times N}}$$

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- As announced, those characters can be seen as symmetric functions of the eigenvalues.
  For instance, for $\mathrm{U}(N)$,
  $$\begin{aligned} \chi_\lambda^{\mathrm{U}(N)}(t_1, \cdots, t_N) &= s_\lambda(t_1, t_2, t_3, \cdots, t_N, 0, 0, \cdots), \\ &=: s_\lambda(g) \end{aligned}$$
  where the s_λ's are Schur polynomials.

Power polynomials

- Define

$$\begin{aligned} p_{(i)}(x_1, x_2, \cdots) &= x_1^i + x_2^i + \cdots \\ p_\lambda(x_1, \cdots) &= \prod_i p_{(\lambda_i)}(x_1, \cdots) \end{aligned}$$

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- Transition matrices between Schur polynomials and power polynomials are well-known, and given by the character tables of symmetric groups.

$$p_\lambda(x_1, \cdots) = \sum_{\mu \vdash k} \chi_\mu^{\mathcal{S}_k}(\lambda) s_\mu(x_1, \cdots)$$

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then

Theorem (math.RT/0504399):

$$\lim_{N\to\infty} \frac{\mathbb{E}_{\mathrm{Sp}(2N)}\, \chi_\gamma^{\mathrm{Sp}(2n)}\, \Phi_\sigma}{\mathbb{E}_{\mathrm{Sp}(2N)}\, \Phi_\sigma} = \sum_{\lambda\vdash|\gamma|} \chi_\gamma(\lambda) \left(\prod_{i=1}^{\infty} \frac{c_i^{\lambda(i)}}{i^{\lambda(i)}\lambda(i)!}\right)$$

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Start by evaluating

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For instance, the inclusion $\mathrm{Sp}(2N) \leq \mathrm{U}(2N)$ induces

$$\chi_\lambda^{\mathrm{U}(2N)} \Big\downarrow_{\mathrm{Sp}(2N)} = \sum_{\mu \subseteq \lambda} \left(\sum_{\nu \text{ even}} c^{\lambda}_{\nu' \mu} \right) \chi_\mu^{\mathrm{Sp}(2N)},$$

when $N \geq l(\lambda)$.
(The $c^{\lambda}_{\nu' \mu}$ are the Littlewood-Richardson coefficients.)

Koike - Terada (J. Algebra 1987), improving on Littlewood.

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$$\mathbb{E}_{\mathrm{Sp}(2N)}\,\chi_\gamma^{\mathrm{Sp}(2N)} p_\lambda \quad = \quad \mathbb{E}_{\mathrm{Sp}(2N)} \sum_{\mu\vdash k} \chi_\mu^{\mathcal{S}_k}(\lambda)\chi_\gamma^{\mathrm{Sp}(2N)}\chi_\mu^{\mathrm{U}(2N)}$$

$$= \sum_{\mu\vdash k} \chi_\mu^{\mathcal{S}_k}(\lambda) \sum_{\beta\subseteq\mu} \sum_{\nu \text{ even}} c^{\mu}_{\nu'\beta}\, \mathbb{E}_{\mathrm{Sp}(2N)}\,\chi_\beta^{\mathrm{Sp}(2N)}\chi_\gamma^{\mathrm{Sp}(2N)}$$

$$= \sum_{\mu\vdash k} \chi_\mu^{\mathcal{S}_k}(\lambda) \sum_{\nu \text{ even}} c^{\mu}_{\nu'\gamma} = \Big(\sum_{\nu \text{ even}} \mathrm{Ind}_{\mathcal{S}_{|\nu|}\times\mathcal{S}_{|\gamma|}}^{\mathcal{S}_k}\big(\chi_{\nu'}^{\mathcal{S}_{|\nu'|}}\otimes\chi_\gamma^{\mathcal{S}_{|\gamma|}}\big)\Big)(\lambda)$$

Idea of the proof

Start by evaluating

$$\mathbb{E}_{\mathrm{Sp}(2N)}\, \chi_\gamma^{\mathrm{Sp}(2N)} p_\lambda .$$

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$$\mathbb{E}_{\mathrm{Sp}(2N)}\, \chi_\gamma^{\mathrm{Sp}(2N)} p_\lambda \;=\; \mathbb{E}_{\mathrm{Sp}(2N)} \sum_{\mu\vdash k} \chi_\mu^{\mathcal{S}_k}(\lambda)\chi_\gamma^{\mathrm{Sp}(2N)}\chi_\mu^{\mathrm{U}(2N)}$$

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$$\sum_{\nu \text{ even}} \chi_{\nu'}^{\mathcal{S}_{2l}} = \mathrm{Ind}^{\mathcal{S}_{2l}}_{\mathcal{B}_{2l}}\, \mathrm{sgn} .$$

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Idea of the proof (cont.)

Everything is reduced to combinatorics of the symmetric group.

Since

$$\Phi_\sigma(g) = \prod_{\text{half}} \sigma(t_i) = \exp\left(\sum_{i>0} \frac{c_i}{i} p_{(i)}(g)\right),$$

we can evaluate

$$\mathbb{E}_{\mathrm{Sp}(2N)}\, \chi_\gamma^{\mathrm{Sp}(2N)} \Phi_\sigma$$

using our knowledge for

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The limit is needed: this derivation was only valid for large N.

Recap

Theorem (math.RT/0504399, Bump-Diaconis for $\mathrm{U}(N)$):
For $\mathrm{G}(N) = \mathrm{U}(N), \mathrm{SO}(2N+1), \mathrm{SO}(2N)$ or $\mathrm{Sp}(2N)$,

$$\lim_{N\to\infty} \frac{\mathbb{E}_{\mathrm{G}(N)}\, \chi_\gamma^{\mathrm{G}(N)} \Phi_\sigma}{\mathbb{E}_{\mathrm{G}(N)}\, \Phi_\sigma} = \sum_{\lambda \vdash |\gamma|} \chi_\gamma(\lambda) \left(\prod_{i=1}^{\infty} \frac{c_i^{\lambda(i)}}{i^{\lambda(i)} \lambda(i)!} \right)$$

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Remarks:

- The proof is exactly the same in the $\mathrm{SO}(N)$ cases as in the $\mathrm{Sp}(2N)$ case.

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Remarks:

- The proof is exactly the same in the SO(N) cases as in the Sp($2N$) case.
- The result is the same in all cases!

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- The proof is exactly the same in the $SO(N)$ cases as in the $Sp(2N)$ case.
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- The proof is exactly the same in the $\mathrm{SO}(N)$ cases as in the $\mathrm{Sp}(2N)$ case.
- The result is the same in all cases!
- Asymptotics of the denominator are entirely known (K. Johansson).
- Could be used for a new measure $|\chi_\gamma|^2\, \mathrm{d}g$.

Restatement (specialization)

Theorem. For $\mathrm{G}(N) = \mathrm{U}(N), \mathrm{SO}(2N+1), \mathrm{SO}(2N)$ or $\mathrm{Sp}(2N)$,

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- When Φ_σ is taken to be $|\Lambda_g(0)| = |\det(\mathrm{Id} - g)|^{2k}$

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$$s_\gamma \Big|_{p_{(i)} := k} = s_\gamma(\underbrace{1, \cdots, 1}_{k \text{ times}}, 0, 0, \cdots) =: s_\gamma\left([1]^k\right).$$
- Specialization happens in a few more instances: see my thesis and *On an identity due to ((Bump and Diaconis) and (Tracy and Widom))*, math.CO/0601348

Part II

This second part is a simplification and extension of results in my thesis.
I am indebted to Chris Hughes (York & Bristol) for sharing some of his unpublished results and ideas.

Moments of derivatives of char. polynomials

I promised an application for

$$\int_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k-2h} |\Lambda_g'(0)|^{2h} \,\mathrm{d}g,$$

with characteristic polynomial $\Lambda_g(\theta) = \prod_{j=1}^{N} \left(1 - e^{\mathrm{i}(\theta_j - \theta)}\right)$.

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$$\frac{|\Lambda_g(0)|^{2k-2h} |\Lambda_g'(0)|^{2h}}{|\Lambda_g(0)|^{2k}} = \left| \frac{\Lambda_g'(0)}{\Lambda_g(0)} \right|^{2h} =$$

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this last step clearly requiring more caution.

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Moments of derivatives of char. polynomials (cont.)

So

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and the previous results indicate that

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should be evaluated first, then use what is known for $\lim_{N\to\infty} \mathbb{E}_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k}$.

Moments of derivatives of char. polynomials (cont.)

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This works, but we will from now on look at finite N.

Proof due to Bump and Gamburd

Bump and Gamburd have a very original way to compute

$$\int_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k}\,\mathrm{d}g \qquad (*)$$

which uses the dual Cauchy identity

$$\prod_{m,n}(1+x_m y_n) = \sum_{\lambda} s_\lambda(x_1, x_2, \cdots) s_{\lambda'}(y_1, y_2, \cdots).$$

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$$= \mathbb{E}_{\mathrm{U}(N)} \overline{s_{\langle k^N \rangle}} \cdot \sum_{\lambda} s_\lambda\left([1]^{2k}\right) s_{\lambda'}$$

$$\left(s_{\langle k^N \rangle} = (e_N)^k, e_N(\theta_1, \theta_2, \cdots, \theta_N) = \theta_1 \theta_2 \cdots \theta_N\right)$$

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$$= \mathbb{E}_{\mathrm{U}(N)}\, \overline{s_{\langle k^N\rangle}} \cdot \sum_{\lambda} s_\lambda\left([1]^{2k}\right) s_{\lambda'} = s_{\langle N^k\rangle}([1]^{2k})$$

$$\left(s_{\langle k^N\rangle} = (e_N)^k, e_N(\theta_1,\theta_2,\cdots,\theta_N) = \theta_1\theta_2\cdots\theta_N\right)$$

Proof due to Bump and Gamburd (cont.)

But $s_{\langle N^k\rangle}([1]^{2k})$ can be evaluated, for instance using the Weyl Dimension Formula. This leads to

$$\int_{\mathrm{U}(N)} |\det(\mathrm{Id}+g)|^{2k}\,\mathrm{d}g = s_{\langle N^k\rangle}([1]^{2k}) \sim_N \frac{G(k+1)^2}{G(2k+1)} N^{k^2},$$

with $G(z)$ the Barnes G-function with property

$$G(k+1) = \prod_{i=1}^{k} \Gamma(i).$$

Setting up the computation (trick)

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One can easily show that for real θ,

$$V_g(\theta) := \mathrm{i}^N e^{\mathrm{i}\frac{N}{2}\theta} e^{-\frac{i}{2}\operatorname{tr} g} \Lambda_g(\theta) \quad \in \mathbb{R}$$

and

$$\frac{V_g'(\theta)}{V_g(\theta)} = \frac{1}{2}\mathrm{i}N + \frac{\Lambda_g'(\theta)}{\Lambda_g(\theta)}$$

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so

$$\left|\frac{\Lambda_g'(\theta)}{\Lambda_g(\theta)}\right|^2 = \frac{N^2}{4} + \left|\frac{V_g'(\theta)}{V_g(\theta)}\right|^2 = \frac{N^2}{4} + \left(\frac{V_g'(\theta)}{V_g(\theta)}\right)^2$$

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$$\begin{aligned}\left|\frac{\Lambda_g'(\theta)}{\Lambda_g(\theta)}\right|^2 &= \frac{N^2}{4} + \left|\frac{V_g'(\theta)}{V_g(\theta)}\right|^2 \\ &= \frac{N^2}{4} + \left(\frac{V_g'(\theta)}{V_g(\theta)}\right)^2 \\ &= \left(\frac{\Lambda_g'(\theta)}{\Lambda_g(\theta)}\right)^2 + \mathrm{i}N\frac{\Lambda_g'(\theta)}{\Lambda_g(\theta)}.\end{aligned}$$

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Setting up the computation (trick)

We have reduced computing

$$\mathbb{E}_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k-2h} |\Lambda_g'(0)|^{2h}$$

to computing

$$\mathbb{E}_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k} \left(\frac{\Lambda_g'(0)}{\Lambda_g(0)} \right)^j, \text{ for } 1 \leq j \leq 2h.$$

Remark: This trick is not necessary for the method to apply (see my thesis for how to proceed).

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From now on, we assume h and j's are integers.

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I will show how to evaluate

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Problem: How do we multiply

$$s_{\langle k^N \rangle}(g) p_{(m_1)}(g) \cdots p_{(m_j)}(g)?$$

Murnaghan-Nakayama rule

The formula reads

$$s_\lambda p_{(m)} = \sum_{\mu} (-1)^{\mathrm{ht}(\mu/\lambda)} s_\mu,$$

where μ is obtained from λ by adding a(n?) m-ribbon.

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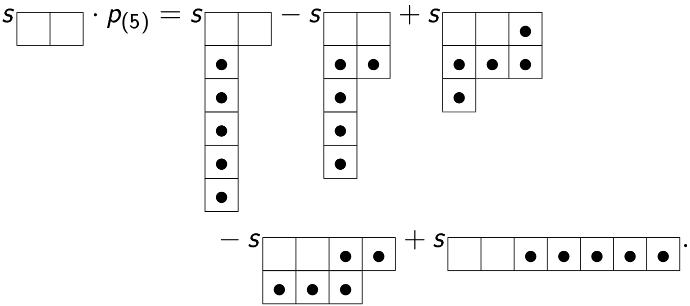
For instance, when $\lambda = \square\square$ and $m = 5$,



General outline

Remember we assume j to be an integer.

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We need to evaluate

$$\sum_{\mu} (-)^{|\mu'|} s_{\mu}([1]^{2k}) \quad \mathbb{E}_{\mathrm{U}(N)} \overline{s_{\mu'}} \cdot s_{\langle k^N \rangle} \Big(\sum_{m=1} p_{(m)} \Big)^{j}.$$

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Again, we will use orthogonality relations, hence we need to match partitions in $s_{\langle k^N \rangle} p_{(m_1)} \cdots p_{(m_j)}$ with $\overline{s_{\mu'}}$, when expanding the former using the Murnaghan-Nakayama rule.

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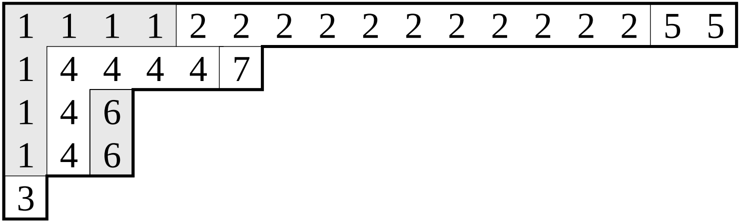
Remarks:

- As long as $l(\mu') \leq N$.
- $s_\mu\left([1]^{2k}\right) = 0$ if $l(\mu) > 2k$.
- The μ's fit in a $2k \times N$ box, so the sum is now finite.

Full Murnaghan-Nakayama rule



1 1 1 1 2 2 2 2 2 2 2 2 2 2 2 5 5
1 4 4 4 4 7
1 4 6
1 4 6
3

$$\chi_{(17,6,3,3,1)}\left((7,11,1,6,2,2,1)\right)$$

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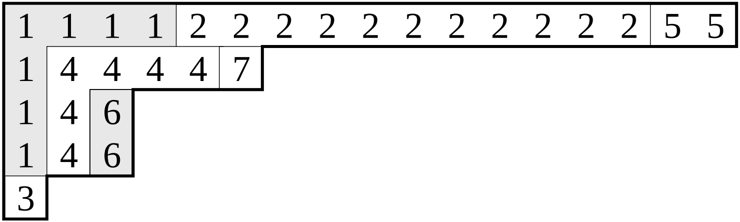

1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	5	5
1	4	4	4	4	7											
1	4	6														
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3																

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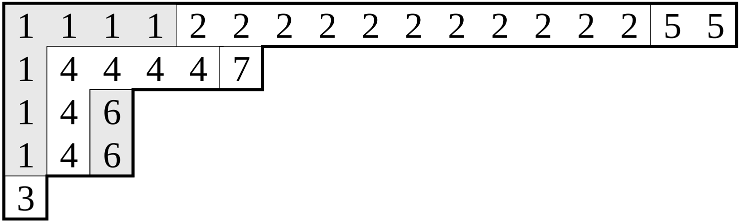

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$$\chi_{(17,6,3,3,1)}((7,11,1,6,2,2,1)) = \sum_{T} (-1)^{\mathrm{ht}(T)} =$$

$$\cdots + (-1)^{3}(-1)^{0}(-1)^{0}(-1)^{2}(-1)^{0}(-1)^{1}(-1)^{0} + \cdots$$

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There is a lot of complexity in each $\chi_\lambda(\bar{m})$, but fortunately we are summing over many $\bar{m}$'s.
Results of El Samra and King on values of Schur functions in Frobenius coordinates, and Borodin on combinatorics of the M-N rule, can be combined to reduce this to a counting problem.

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These exact formulas can be numerically tested (sigh!).

Final exact result

$$\frac{\sum_{j=1} \mathbb{E}_{\mathrm{U}(N)} \left|\Lambda_g(0)\right|^{2k} \left(\frac{\Lambda_g'(0)}{\Lambda_g(0)}\right)^j z^j}{\mathbb{E}_{\mathrm{U}(N)} \left|\Lambda_g(0)\right|^{2k}} =$$

$$\sum_{d=1}^{\infty} \sum_{\vec{s},\vec{t}\in\mathbb{N}^d} \left(d + \sum(s_i + t_i)\right)! \left|\frac{(-\imath z)^{1+s_i+t_j}}{s_i!\,t_j!\,(1 + s_i + t_j)}\right|_{d\times d}$$

$$\sum_{\substack{\bar{a}\in[0,N-1]^d \\ \bar{b}\in[0,k-1]^d}} \left|\frac{\binom{a_i}{s_i}\binom{b_j}{t_j}\binom{k+a_i}{a_i}\binom{k-1}{b_j}(N - a_i)^{(k)}(-1)^{b_j}}{(N + b_j + 1)^{(k)}(1 + a_i + b_j)}\right|_{d\times d}$$

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Asymptotics

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$$\frac{\sum_{j=1} \mathbb{E}_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k} \left(\frac{\Lambda_g'(0)}{\Lambda_g(0)}\right)^j z^j}{\mathbb{E}_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k}} \sim_N$$

$$\sum_{d=1}^{\infty} \sum_{\vec{s},\vec{t}\in\mathbb{N}^d} \left(d+\sum(s_i+t_i)\right)! \left|\frac{1}{s_i!t_j!(1+s_i+t_j)}\right|_{d\times d} \left|\frac{H^{k,s_i,t_j}}{s_i!t_j!(1+s_i+t_j)}\right|_{d\times d} (-\mathrm{i}Nz)^{d+\sum(s_i+t_i)}$$

Asymptotics

$$\frac{\sum_{j=1} \mathbb{E}_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k} \left(\frac{\Lambda'_g(0)}{\Lambda_g(0)}\right)^j z^j}{\mathbb{E}_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k}} \sim_N$$

$$\sum_{d=1}^{\infty} \sum_{\vec{s},\vec{t}\in\mathbb{N}^d} \left(d+\sum(s_i+t_i)\right)! \left|\frac{1}{s_i!t_j!(1+s_i+t_j)}\right|_{d\times d}$$

$$\left|\frac{H^{k,s_i,t_j}}{s_i!t_j!(1+s_i+t_j)}\right|_{d\times d} (-\mathrm{i}Nz)^{d+\sum(s_i+t_i)},$$

with

$$H^{k,s,t} = \prod_{i=-t}^{s} \frac{k+i}{2k+i},$$

a rational function in k with poles when $2k$ is an odd integer between $-s$ and t.

Asymptotics (cont.)

Theorem. Let $h \in \mathbb{N}$. Then,

$$\lim_{N\to\infty} \frac{\mathbb{E}_{\mathrm{U}(N)} \left|\Lambda_g(0)\right|^{2k-2h} \left|\Lambda_g'(0)\right|^{2h}}{N^{2h}\,\mathbb{E}_{\mathrm{U}(N)} \left|\Lambda_g(0)\right|^{2k}} = \frac{P_h(2k)}{2^{2h} Q_h(2k)},$$

where $P_h(u)$ and $Q_h(u)$ are polynomials, P_h is even, with $\deg P_h \leq \deg Q_h$. Moreover,

$$Q_h(u) = \prod_{l=1}^{h} (u^2 - (2l-1)^2)^{\alpha_{2l-1}},$$

with $\alpha_x = \left\lfloor \frac{-x+\sqrt{x^2+8h}}{2} \right\rfloor$.

Polynomials $P_h(u)$ (first obtained by Hughes)

$$
\begin{aligned}
P_1(u) &= u^2 \\
P_2(u) &= u^4 - 8u^2 - 6 \\
P_3(u) &= u^8 - 33u^6 + 198u^4 + 74u^2 - 360 \\
P_4(u) &= u^{10} - 81u^8 + 1740u^6 - 8284u^4 - 7716u^2 + 34020 \\
P_5(u) &= u^{14} - 170u^{12} + 9597u^{10} - 215560u^8 + 1846928u^6 \\
&\quad -4247400u^4 - 12317076u^2 + 42366240 \\
P_6(u) &= u^{18} - 291u^{16} + 30177u^{14} - 1379507u^{12} + 28177518u^{10} - 236602818u^8 \\
&\quad +604630084u^6 + 1570591476u^4 - 10008266040u^2 + 7829929800 \\
P_7(u) &= u^{22} - 484u^{20} + 90384u^{18} - 8378492u^{16} + 415889897u^{14} - 11196067680u^{12} \\
&\quad +157699171570u^{10} - 1023611526808u^8 + 1699483809828u^6 \\
&\quad +11589901952544u^4 - 62361799232760u^2 + 44754182272800 \\
P_8(u) &= u^{24} - 708u^{22} + 198590u^{20} - 28525892u^{18} + 2275085529u^{16} \\
&\quad -102837376096u^{14} + 2598141390568u^{12} - 34807690054560u^{10} \\
&\quad +213458763180152u^8 - 261862022455104u^6 - 3402805264433280u^4 \\
&\quad +19256263380043200u^2 - 11718802173078000 \\
P_9(u) &= u^{30} - 1054u^{28} + 460431u^{26} - 109299828u^{24} + 15577804767u^{22} \\
&\quad -1394331670638u^{20} + 79872695247657u^{18} - 2932723486507728u^{16} \\
&\quad +68022586503825552u^{14} - 962308385613255088u^{12} + 7682283932820069016u^{10} \\
&\quad -26475220331016986304u^8 - 59889950570120914224u^6 \\
&\quad +976582356673028315040u^4 - 3441287004848413282800u^2 \\
&\quad +1366282646437284576000.
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&\quad +13662826464372845760 00.
\end{aligned}
$$

What is $P_{1/2}(u)$?

Back to NT

Conjecture. Let $h \in \mathbb{N}$. Then,

$$\lim_{T\to\infty} \frac{\int_0^T |\zeta(1/2+\mathrm{i}t)|^{2k-2h}|\zeta'(1/2+\mathrm{i}t)|^{2h}\,\mathrm{d}t}{(\log T)^{2h}\int_0^T |\zeta(1/2+\mathrm{i}t)|^{2k}\,\mathrm{d}t} = \frac{P_h(2k)}{2^{2h}Q_h(2k)},$$

where $P_h(u)$ and $Q_h(u)$ are polynomials, P_h is even, with $\deg P_h \leq \deg Q_h$. Moreover,

$$Q_h(u) = \prod_{l=1}^{h}(u^2-(2l-1)^2)^{\alpha_{2l-1}},$$

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Conclusion

- The first series of results on asymptotic ratios can be used to compute averages for non-multiplicative integrands.

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Conclusion

- The first series of results on asymptotic ratios can be used to compute averages for non-multiplicative integrands.
- Armed with the intuition from ratios, techniques similar to Bump' and Gamburd's can be applied to those non-multiplicative integrands to give exact results.
- This is shown to work in the special case of mixed moments and to improve on previous results.

References

math.RT/0504399 Averages over classical compact Lie groups and Weyl characters.

math.CO/0601348 On an identity due to ((Bump and Diaconis) and (Tracy and Widom)).

In preparation All the mixed moments results.

Thesis and talk available at:

`http://www.maths.ox.ac.uk/~pdehaye/`