Averages over compact Lie groups, twisted by Weyl characters

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Motivation (for all)

Theorem ($k = 1$: Hardy-Littlewood, $k = 2$: Ingham)

$$\lim_{T\to\infty} \frac{1}{(\log T)^{k^2}} \frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2k} \, \mathrm{d}t = f(k)a(k),$$

where $a(k)$ is a product over primes, $f(1) = 1$, $f(2) = 1/12$.

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$$\int_{\mathrm{U}(N)} |\Lambda_g|^{s}\,\mathrm{d}g = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+s)}{(\Gamma(j+s/2))^2}.$$

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$$\int_{[0,1]^N} \prod_{j=1}^{N} x_j^{a-1}(1-x_j)^{b-1} \prod_{1\leq j<k\leq N} |x_j - x_k|^{2c} \,\mathrm{d}\overline{x} = \prod_{j=0}^{N-1} \frac{\Gamma(a+jc)\Gamma(b+jc)\Gamma((j+1)c)}{\Gamma(a+b+(n+j-1)c)\Gamma(c)}.$$

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It is also possible to use the Heine identity to transfer the problem to computing a determinant of a growing Toeplitz matrix (with N), but the computations get very complicated very quickly.

Motivation (for RMTists, cont.)

We might want to consider

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with the associated RMT averages

$$(*) = \int_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k-2h} |\Lambda_g'(0)|^{2h} \, \mathrm{d}g,$$

where $\Lambda_g(\theta) = \prod_{j=1}^N \left(1 - e^{i(\theta_j - \theta)}\right)$.

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We do know that $(*) \sim_N N^{k^2+2h}$. The leading coefficient seems to be a rational function in k of degrees bounded by a linear function of h.

Motivation (main idea)

Observe that

$$\frac{|\Lambda_g(0)|^{2k-2h}|\Lambda_g'(0)|^{2h}}{|\Lambda_g(0)|^{2k}} = \left|\frac{\Lambda_g'(0)}{\Lambda_g(0)}\right|^{2h} = \left|\sum_{j=1}^{N}\frac{ie^{i\theta_j}}{1-e^{i\theta_j}}\right|^{2h}.$$

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We will present general results and illustrate the method on this example.

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- $G(N) = U(N), SO(2N), Sp(2N), SO(2N+1)$ $|t_i| = 1$, with the eigenvalues coming in conjugate pairs for the non-unitary case.

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- Let $\sigma : \mathbb{T} \to \mathbb{C}$, with $\sigma(t) = \sigma(t^{-1})$. Define $\Phi_\sigma : \mathrm{G}(N) \to \mathbb{C}, g \mapsto \prod_{(\text{half of}) \ t_i\text{'s}} \sigma(t_i)$.

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- $$\lim_{N\to\infty} \frac{\int_{\mathrm{U}(N)} \Phi_\sigma(g)\chi^{(N)}(g)\,\mathrm{d}g}{\int_{\mathrm{U}(N)} \Phi_\sigma(g)\,\mathrm{d}g} = ?$$
- How is an average over $\mathrm{G}(N)$ affected when introducing a character in the integrand?

Definitions

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- $N < l(\lambda) \Rightarrow \chi_\lambda^{\mathsf{G}(N)} \equiv 0$.
- For a fixed N, $N \geq l(\lambda), l(\mu)$,

$$\mathbb{E}_{\mathsf{G}(N)}\, \chi_\lambda^{\mathsf{G}(N)} \chi_\mu^{\mathsf{G}(N)} = \delta_\mu^\lambda$$

$\Rightarrow$ For a fixed N, set of orthogonal irreducible characters of $\mathsf{G}(N)$ indexed by partitions of length less than N.

Examples

- $\chi_{\lambda}^{\mathrm{U}(N)}(t_1, \cdots, t_N) = \dfrac{\left| t_i^{\lambda_j + N - j} \right|_{N \times N}}{\left| t_i^{n-j} \right|_{N \times N}}$

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- $$\chi_\lambda^{\mathrm{Sp}(2N)}(t_1, \cdots, t_N) = \frac{\left| t_i^{\lambda_j + N - j + 1} - t_i^{-(\lambda_j + N - j + 1)} \right|_{N\times N}}{\left| t_i^{N-j+1} - t_i^{-(N-j+1)} \right|_{N\times N}}$$

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- As announced, those characters can be seen as symmetric functions of the eigenvalues.
  For instance, for $\mathrm{U}(N)$,

$$\begin{aligned} \chi_\lambda^{\mathrm{U}(N)}(t_1,\cdots,t_N) &= s_\lambda(g) \\ &= s_\lambda(t_1,t_2,t_3,\cdots,t_N,0,0,\cdots), \end{aligned}$$

where the s_λ's are Schur polynomials.

Power polynomials

- Define

$$\begin{aligned} p_{(i)}(x_1, x_2, \cdots) &= x_1^i + x_2^i + \cdots \\ p_\lambda(x_1, \cdots) &= \prod_i p_{(\lambda_i)}(x_1, \cdots) \end{aligned}$$

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- Transition matrices between Schur polynomials and power polynomials are well-known, and given by the character tables of symmetric groups.

$$p_\lambda(x_1, \cdots) = \sum_{\mu \vdash k} \chi_\mu^{\mathcal{S}_k}(\lambda) s_\mu(x_1, \cdots)$$

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Theorem (math.RT/0504399):

$$\lim_{N\to\infty} \frac{\mathbb{E}_{\mathrm{Sp}(2N)}\, \chi_\gamma^{\mathrm{Sp}(2n)} \Phi_\sigma}{\mathbb{E}_{\mathrm{Sp}(2N)}\, \Phi_\sigma} = \sum_{\lambda\vdash|\gamma|} \chi_\gamma(\lambda) \left(\prod_{i=1}^{\infty} \frac{c_i^{\lambda(i)}}{i^{\lambda(i)}\lambda(i)!}\right)$$

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Aside: branching rules

Characters of a group reduce to characters of a subgroup.

For instance, the inclusion $\mathrm{Sp}(2N) \leq \mathrm{U}(2N)$ induces

$$\chi_\lambda^{\mathrm{U}(2N)}\Big\downarrow_{\mathrm{Sp}(2N)} = \sum_{\mu\subseteq\lambda}\left(\sum_{\nu\ \text{even}} c^{\lambda}_{\nu'\mu}\right)\chi_\mu^{\mathrm{Sp}(2N)},$$

when $N \geq l(\lambda)$.
(The $c^{\lambda}_{\nu'\mu}$ are the Littlewood-Richardson coefficients.)

Koike - Terada (J. Algebra 1987), improving on Littlewood.

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$$= \sum_{\mu\vdash k}\chi_\mu^{\mathcal{S}_k}(\lambda)\sum_{\nu \text{ even}} c^{\mu}_{\nu'\gamma} = \Big(\sum_{\nu \text{ even}} \mathrm{Ind}_{\mathcal{S}_{|\nu|}\times\mathcal{S}_{|\gamma|}}^{\mathcal{S}_k}\big(\chi_{\nu'}^{\mathcal{S}_{|\nu'|}}\otimes\chi_\gamma^{\mathcal{S}_{|\gamma|}}\big)\Big)(\lambda)$$

Idea of the proof

Start by evaluating

$$\mathbb{E}_{\mathrm{Sp}(2N)}\, \chi_\gamma^{\mathrm{Sp}(2N)} p_\lambda .$$

Idea: Transfer computations to a symmetric group!

$$\mathbb{E}_{\mathrm{Sp}(2N)}\, \chi_\gamma^{\mathrm{Sp}(2N)} p_\lambda = \mathbb{E}_{\mathrm{Sp}(2N)} \sum_{\mu \vdash k} \chi_\mu^{\mathcal{S}_k}(\lambda) \chi_\gamma^{\mathrm{Sp}(2N)} \chi_\mu^{\mathrm{U}(2N)}$$

$$= \sum_{\mu \vdash k} \chi_\mu^{\mathcal{S}_k}(\lambda) \sum_{\beta \subseteq \mu} \sum_{\nu \text{ even}} c^{\mu}_{\nu' \beta}\, \mathbb{E}_{\mathrm{Sp}(2N)}\, \chi_\beta^{\mathrm{Sp}(2N)} \chi_\gamma^{\mathrm{Sp}(2N)}$$

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then use

$$\sum_{\nu \text{ even}} \chi_{\nu'}^{\mathcal{S}_{2l}} = \mathrm{Ind}^{\mathcal{S}_{2l}}_{\mathcal{B}_{2l}} \mathrm{sgn} .$$

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Everything is reduced to combinatorics of the symmetric group.

Since

$$\Phi_\sigma(g) = \prod_{\text{half}} \sigma(t_i) = \exp\left(\sum_{i>0} \frac{c_i}{i} p_{(i)}(g)\right),$$

we can evaluate

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The limit is needed: this derivation was only valid for large N.

Recap

Theorem (math.RT/0504399, Bump-Diaconis for $\mathrm{U}(N)$):
For $\mathrm{G}(N) = \mathrm{U}(N), \mathrm{SO}(2N+1), \mathrm{SO}(2N)$ or $\mathrm{Sp}(2N)$,

$$\lim_{N\to\infty} \frac{\mathbb{E}_{\mathrm{G}(N)}\, \chi_\gamma^{\mathrm{G}(N)} \Phi_\sigma}{\mathbb{E}_{\mathrm{G}(N)}\, \Phi_\sigma} = \sum_{\lambda\vdash|\gamma|} \chi_\gamma(\lambda) \left(\prod_{i=1}^{\infty} \frac{c_i^{\lambda(i)}}{i^{\lambda(i)}\lambda(i)!} \right)$$

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Remarks:

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- The proof is exactly the same in the $\mathrm{SO}(N)$ cases as in the $\mathrm{Sp}(2N)$ case.
- Asymptotics of the denominator are entirely known (K. Johansson).

Recap

Theorem (math.RT/0504399, Bump-Diaconis for U(N)):
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Remarks:

- The proof is exactly the same in the SO(N) cases as in the Sp($2N$) case.
- Asymptotics of the denominator are entirely known (K. Johansson).
- Could be used for a new measure $|\chi_\gamma|^2 \, \mathrm{d}g$.

Restatement (specialization)

Theorem For $\mathrm{G}(N) = \mathrm{U}(N), \mathrm{SO}(2N+1), \mathrm{SO}(2N)$ or $\mathrm{Sp}(2N)$,

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- When Φ_σ is taken to be $|\Lambda_g(0)| = |\det(\mathrm{Id} - g)|^{2k}$

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- Specialization happens in a few more instances: see my thesis and *On an identity due to ((Bump and Diaconis) and (Tracy and Widom))*, math.CO/0601348

Moments of derivatives of char. polynomials

I promised an application for

$$\int_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k-2h} |\Lambda_g'(0)|^{2h} \, \mathrm{d}g,$$

with characteristic polynomial $\Lambda_g(\theta) = \prod_{j=1}^{N} \left(1 - e^{i(\theta_j - \theta)}\right)$.

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with characteristic polynomial $\Lambda_g(\theta) = \prod_{j=1}^{N} \left(1 - e^{i(\theta_j - \theta)}\right)$. Observe as before that

$$\frac{|\Lambda_g(0)|^{2k-2h} |\Lambda_g'(0)|^{2h}}{|\Lambda_g(0)|^{2k}} = \left|\frac{\Lambda_g'(0)}{\Lambda_g(0)}\right|^{2h} =$$

$$\left|\sum_{j=1}^{N} \frac{ie^{i\theta_j}}{1 - e^{i\theta_j}}\right|^{2h} = \left| i \sum_{m=1}^{\infty} p_{(m)}(g)\right|^{2h}.$$

Moments of derivatives of char. polynomials (cont.)

So

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and the previous results indicate that

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should be evaluated first, then use what is known for $\lim_{N\to\infty} \mathbb{E}_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k}$.

Moments of derivatives of char. polynomials (cont.)

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This works, but we will from now on look at finite N.

Proof due to Bump and Gamburd

Bump and Gamburd have a very original way to compute

$$\int_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k}\,\mathrm{d}g \qquad (*)$$

which uses the dual Cauchy identity

$$\prod_{m,n}(1+x_m y_n) = \sum_{\lambda} s_\lambda(x_1, x_2, \cdots) s_{\lambda'}(y_1, y_2, \cdots).$$

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$$\left(s_{\langle k^N \rangle} = (e_N)^k,\ e_k(\theta_1, \theta_2, \cdots, \theta_N) = \theta_1 \theta_2 \cdots \theta_N\right)$$

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$$\left(s_{\langle k^N \rangle} = (e_N)^k, \, e_k(\theta_1, \theta_2, \cdots, \theta_N) = \theta_1 \theta_2 \cdots \theta_N \right)$$

Proof due to Bump and Gamburd (cont.)

Weyl Dimension Formula

$$s_\lambda([1]^r) = \prod_{1 \le i < j \le r} \frac{(\lambda_i - i) - (\lambda_j - j)}{j - i}$$

Proof due to Bump and Gamburd (cont.)

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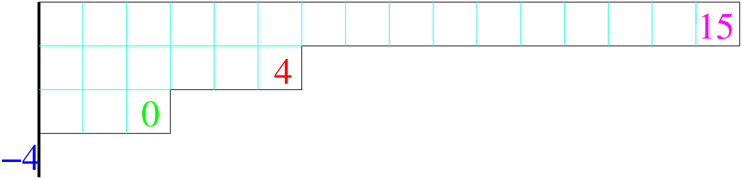
$$s_\lambda([1]^r) = \prod_{1\le i<j\le r} \frac{(\lambda_i - i) - (\lambda_j - j)}{j - i}$$

Example $\chi_{(16,6,3)}(1,1,1,1)$



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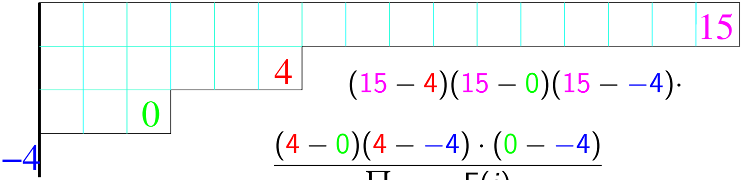

$$\frac{(15-4)(15-0)(15--4)\cdot (4-0)(4--4)\cdot(0--4)}{\prod_{1\le i\le 4}\Gamma(i)}$$

Proof due to Bump and Gamburd (cont.)

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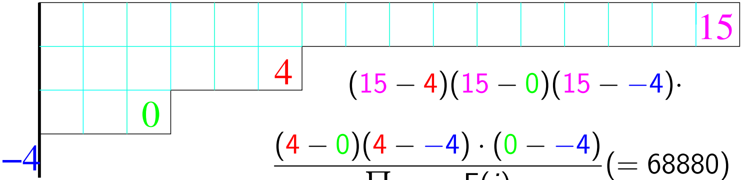

$$(15-4)(15-0)(15--4)\cdot$$

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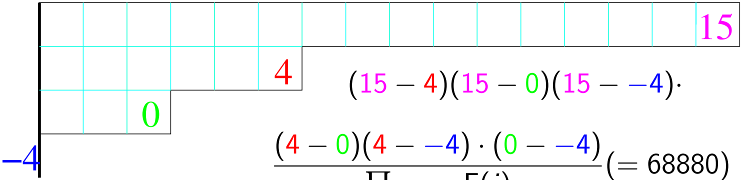

$$\frac{(15-4)(15-0)(15--4)\cdot(4-0)(4--4)\cdot(0--4)}{\prod_{1\le i\le 4}\Gamma(i)} (= 68880)$$

Hence

$$\int_{U(N)} |\Lambda_g(0)|^{2k}\,\mathrm{d}g = s_{\langle N^k\rangle}([1]^{2k}) = \frac{\prod_{i=1}^{k}(N+i)^{(k)}\Gamma(i)^2}{\prod_{i=1}^{2k}\Gamma(i)},$$

with $X^{(k)} = X(X+1)\cdots(X+k-1)$.

Setting up the computation (trick)

One can easily show that for real θ,

$$V_g(\theta) := i^N e^{i\frac{N}{2}\theta} e^{-\frac{i}{2}\operatorname{tr} g} \Lambda_g(\theta) \quad \in \mathbb{R}$$

and

$$\frac{V_g'(\theta)}{V_g(\theta)} = \frac{1}{2} iN + \frac{\Lambda_g'(\theta)}{\Lambda_g(\theta)}$$

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Setting up the computation (trick)

We have reduced computing

$$\mathbb{E}_{\mathrm{U}(N)}\,|\Lambda_g(0)|^{2k-2h}|\Lambda_g'(0)|^{2h}$$

to computing

$$\mathbb{E}_{\mathrm{U}(N)}\,|\Lambda_g(0)|^{2k}\left(\frac{\Lambda_g'(0)}{\Lambda_g(0)}\right)^j, \text{ for } 1 \le j \le 2h.$$

Remark: This trick is not necessary for the method to apply (see my thesis for how to proceed).

Following Bump-Gamburd

I will only show how to evaluate

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Problem: How do we multiply

$$s_{\langle k^N \rangle}(g) p_{(m)}(g)?$$

Murnaghan-Nakayama rule

The formula reads

$$s_\lambda p_{(m)} = \sum_{\mu} (-1)^{\mathrm{ht}(\mu/\lambda)} s_\mu,$$

where μ is obtained from λ by adding a(n?) m-ribbon.

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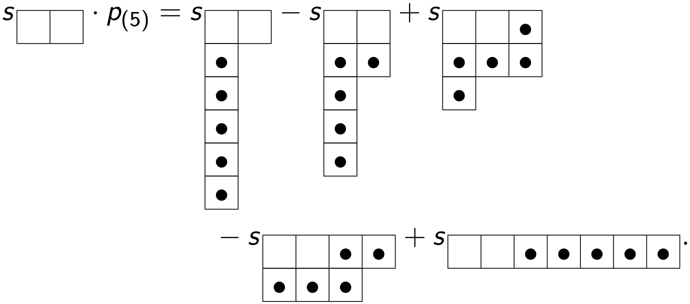
For instance, when $\lambda =$  and $m = 5$,



General outline

We need to evaluate

$$\sum_{\mu} (-)^{|\mu'|} s_{\mu}([1]^{2k}) \quad \mathbb{E}_{\mathrm{U}(N)}\, \overline{s_{\mu'}} \cdot s_{\langle k^N \rangle} \Big(\sum_{m=1} p_{(m)} \Big)^2 .$$

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- The μ's fit in a $2k \times N$ box, so the sum is now finite.

Methods

A "topological" classification of the different ways of adding ribbons.

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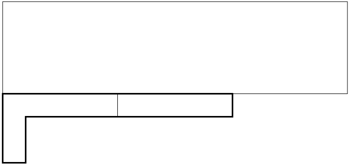

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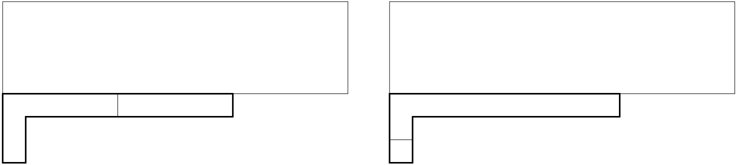

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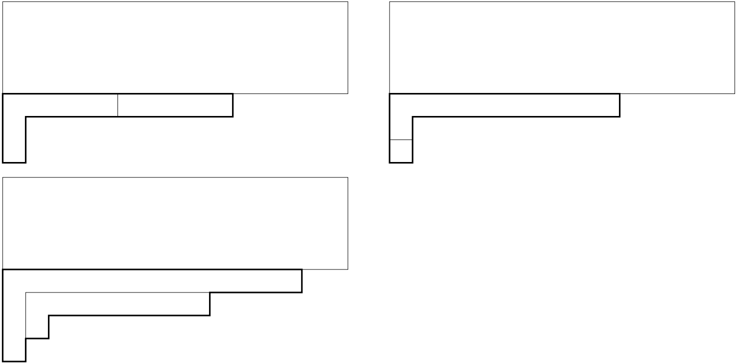

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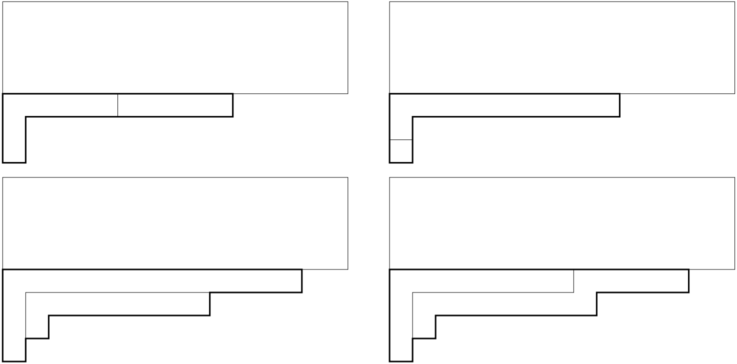

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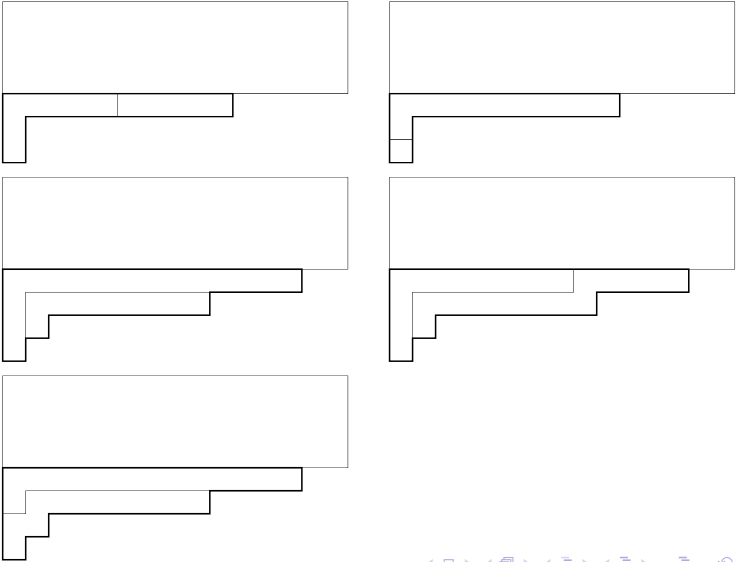

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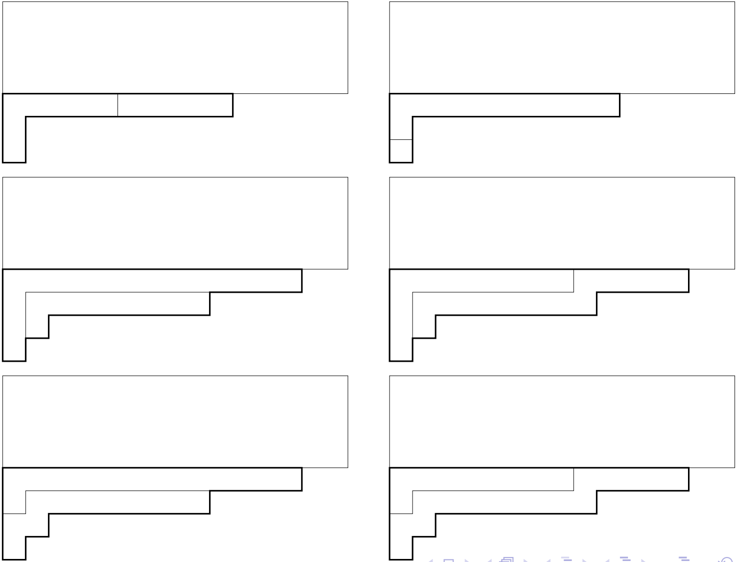
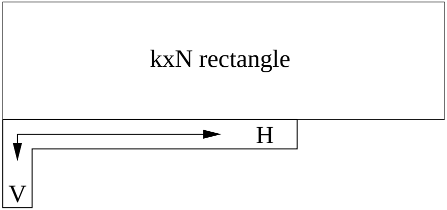


Computation (each term)

Let $\mu = \mu(H, V)$.



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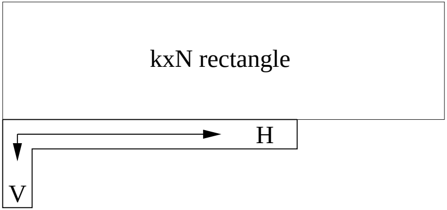

Then the Weyl Dimension Formula gives

$$\frac{s_{\mu(H,V)}\left([1]^{2k}\right)}{s_{\langle N^k\rangle}\left([1]^{2k}\right)} = \frac{(N-H+1)^{(k)}H^{(k)}}{(V-1)!(k-V)!(N+V)^{(k)}(H+V-1)}.$$

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If we go through everything, we get

Theorem

$$\int_{\mathrm{U}(N)} |\Lambda_g'(0)|^2 |\Lambda_g(0)|^{2k-2}\,\mathrm{d}g =$$

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- Numerics agree!

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- Numerics agree!
- Agrees with results of Chris Hughes in his thesis (2001) and unpublished notes.

Asymptotics

Theorem

$$\int_{\mathrm{U}(N)} |\Lambda_g'(0)|^2 |\Lambda_g(0)|^{2k-2} \, \mathrm{d}g \sim_N$$

$$\frac{k^2}{(2k-1)(2k+1)} \frac{\prod_{i=0}^{k-1} i!^2}{\prod_{i=0}^{2k-1} i!} N^{k^2+2},$$

which is caused by massive cancellations.

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Proof Drop every factor not of highest order in N, simplify the sum over V. Then, we need to estimate

$$\sum_{1\le H\le N} \frac{(N-H)^k}{N^k} \cdot \left(\sum_{i=0}^{k-1} \binom{k-1+i}{i} \frac{H^i}{N^i} + \sum_{i=0}^{k-2} \binom{k-1+i}{i} \frac{H^{i+1}}{N^{i+1}} \right)$$

and we use Selberg's integral/Beta integral.

“Conjecture” “messy=nice”

- $$\sum_{\substack{1\le H\le N\\1\le V\le k}} \frac{(-1)^{V-1}(N+V-H)(N-H+1)^{(k)}H^{(k)}}{(V-1)!(k-V)!(N+V)^{(k)}(H+V-1)} =$$

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- Would simplify the computations of the asymptotics.

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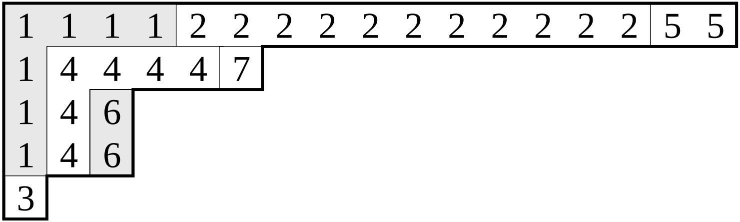
- Allows a continuation to $k\in\mathbb{C}$, $Re\ k > 1/2$.
- Would simplify the computations of the asymptotics.
- $k=1$ is the pyramidal square numbers identity, $N=1$ is
  $${}_2F_1\left({1\quad -k \atop k+1};1\right)=1/2,$$
  which is Gauss’ hypergeometric sum.

Higher mixed moments

In general ($h > 1$), we need to compute

$$(*) = \mathbb{E}_{\mathrm{U}(N)} \left|\Lambda_g(0)\right|^{2k} \left(\frac{\Lambda_g'(0)}{\Lambda_g(0)}\right)^j, \text{ for } 1 \le j \le 2h.$$

Full Murnaghan-Nakayama rule



1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	5	5
1	4	4	4	4	7											
1	4	6														
1	4	6														
3																

$$\chi^{(17,6,3,3,1)}_{(7,11,1,6,2,2,1)}$$

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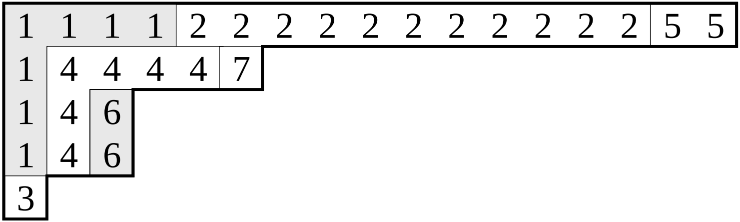

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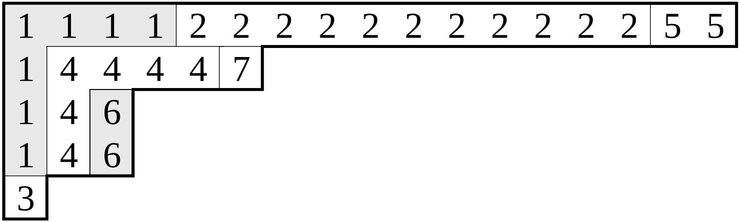

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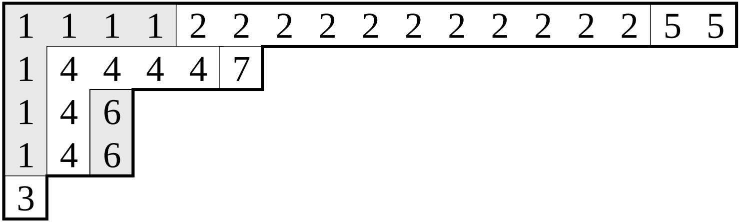

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Note: The simplification that occurs with just two ribbons is a statement about characters of symmetric groups evaluated on conjugacy classes of permutations with just two cycles.

Higher mixed moments (cont.)

In general ($h > 1$), we need to compute

$$(*) = \mathbb{E}_{\mathrm{U}(N)} \left|\Lambda_g(0)\right|^{2k} \left(\frac{\Lambda_g'(0)}{\Lambda_g(0)}\right)^j, \text{ for } 1 \leq j \leq 2h.$$

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$$\frac{s_{\langle N^k\rangle\cup\lambda}\left([1]^{2k}\right)}{s_{\langle N^k\rangle}\left([1]^{2k}\right)} \sim_N s_\lambda\left([1]^k\right).$$

Higher mixed moments (cont.)

In general ($h > 1$), we need to compute

$$(*) = \mathbb{E}_{\mathrm{U}(N)} |\Lambda_g(0)|^{2k} \left(\frac{\Lambda'_g(0)}{\Lambda_g(0)} \right)^j, \text{ for } 1 \leq j \leq 2h.$$

The simplification that occurs for just two ribbons does not happen then, but the full Murnaghan-Nakayama rule gives

$$(*) = \sum_{\substack{(\mu_1,\cdots,\mu_j) \\ \in \mathbb{N}_0^j}} \sum_{\substack{\lambda \text{ within} \\ k \times N}} \chi^{\lambda}_{\mathrm{sort}(\mu)} s_{\langle N^k \rangle \cup \lambda} \left([1]^{2k} \right)$$

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$$\frac{s_{\langle N^k \rangle \cup \lambda}\left([1]^{2k}\right)}{s_{\langle N^k \rangle}\left([1]^{2k}\right)} \sim_N s_\lambda\left([1]^k\right).$$

Observe that again this is a specialization of a symmetric function.

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- In the case of mixed moments, this is very close to giving a complete answer.

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The end.

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Thesis and talk available at:

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