Integrals over compact Lie groups twisted by Weyl characters.

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August 23rd 2005

Main problem:

* $G(n) := U(n), SO(2n), Sp(2n), SO(2n+1)$

  $U(n) = \{ g \in Mat_n(\mathbb{C}) \mid gg^* = Id. \}$

  $O(n) = \{ g \in U(n) \mid gg^t = Id. \}$

  $SO(n) = \{ g \in O(n) \mid \det(g) = 1 \}$

  $Sp(2n) = \{ g \in U(2n) \mid g J g^t = J \}$ $J = \begin{pmatrix} & -Id_n \\ Id_n & \end{pmatrix}$

  $|t_i| = 1$; in $O(n), Sp(2n)$, the e.v. come in <u>pairs</u> $\{t_i, t_i^{-1}\}$

* $\sigma : \mathbb{T} \to \mathbb{C}$ (maybe $\sigma(t) = \sigma(t^{-1})$)

  Construct $\Phi_\sigma : G(n) \to \mathbb{C}$

  $$g \longmapsto \prod_{\text{half of } t_i\text{'s}} \sigma(t_i)$$

  well-defined!

* $\{\chi^{(n)}\}$ sequence of chars of $G(n)$

$$\lim_{n \to \infty} \frac{\int_{G(n)} \Phi_\sigma(g) \chi^{(n)}(g)\, dg}{\int_{G(n)} \Phi_\sigma(g)\, dg} = ?$$

Weyl characters.

Let T be the torus of $G(n)$, C^+ be the pos. Weyl chamber

Let $\lambda \in X^*(T) \cap C^+$, written multiplicatively.

$\rho = \frac{1}{2}$ sum of positive roots.

Define
$$\chi_\lambda^{(n)} := \frac{\sum_{w \in W} (-1)^w e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^w e^{w(\rho)}} \quad \text{(W.C.F.)}$$

<u>Properties:</u> * λ (almost-) partition $n \geq \ell(\lambda)$ $\longleftrightarrow$ $\lambda \in X^*(T) \cap C^+$

* $n < \ell(\lambda) \Rightarrow \chi_\lambda^{(n)} \equiv 0$

* One λ defines a family $\{\chi_\lambda^{(n)}\}$

* Fixed n, $n \geq \ell(\lambda), \ell(\mu)$

$$\mathbb{E}_{G(n)} \chi_\lambda^{(n)} \chi_\mu^{(n)} = \langle \chi_\lambda^{(n)}, \chi_\mu^{(n)} \rangle = \delta_\mu^\lambda$$

$\Rightarrow$ Irreducible and orthogonal.

Example: $Sp(4)$ $(n=2)$

$$T = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_1^{-1} & \\ & & & t_2^{-1} \end{pmatrix} ; \ t_1, t_2 \in \mathbb{C} \right\}$$

$X^*(T) \cong \mathbb{Z}^2$.

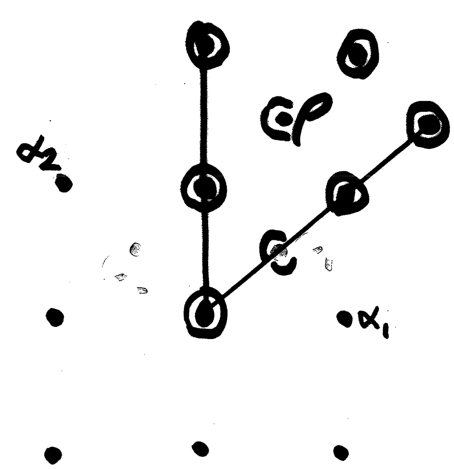
Basis of $\mathbb{Z}^2 = \{e_1, e_2\}$

Written multiplicatively: $e^{\mu}(t) = t_1^{\mu_1} \cdot t_2^{\mu_2}$ if $\mu = \mu_1 e_1 + \mu_2 e_2$

$Sp(4)$ has root system C_2, 4 long roots $\pm 2e_i$

4 short roots $\pm e_1 \pm e_2$. Simple pos. roots

are $\alpha_1 = e_1 - e_2$, $\alpha_2 = 2e_2$



α_i

$\rho = 2e_1 + e_2$

$\odot \in X^*(T) \cap C^+$

Rem: $\lambda \in X^*(T) \cap C^+ \Rightarrow \lambda_1 \geq \lambda_2 \geq 0$

$\Rightarrow \lambda$ is a partition

Examples (II)

$$\chi_\lambda^{Sp(2n)}(t_1,\dots,t_{\underline{\underline{n}}}) = \frac{\left| t_i^{\lambda_j+n-j+1} - t_i^{-(\lambda_j+n-j+1)} \right|_{n\times n}}{\left| t_i^{n-j+1} - t_i^{-(n-j+1)} \right|_{n\times n}}$$

only half of e.v!

$$\chi_\lambda^{U(n)}(t_1,\dots,t_n) = \frac{\left| x_i^{\lambda_j+n-j} \right|_{n\times n}}{\left| x_i^{n-j} \right|_{n\times n}} = \text{Schur polynomial}$$

Branching Rules

$$Sp(2n) \subseteq U(2n)$$

$$\chi_\lambda^{U(2n)}\Big\downarrow_{Sp(2n)} = \sum_{\mu \subseteq \lambda} \Big(\sum_{\nu \text{ even}} c^{\lambda}_{\nu',\mu} \Big) \chi_\mu^{Sp(2n)}$$

(when $n \geq \ell(\lambda)$)

Koike-Terada (J. Algebra 1987)

Odds + Ends on S_k

* $\chi_\mu^{S_k}$, $\mu \vdash k$
* $\chi_\mu^{S_k}(\lambda)$ = value on C_λ, conjugacy class of type λ
* $z_\lambda = \prod_i i^{\lambda(i)} \cdot \lambda(i)!$ = order centralizer of elt in C_λ
* Character table = transition matrix:

Let $p_{(i)}(x_1, \ldots) = x_1^i + x_2^i + \ldots$

$$p_\lambda(x_1, \ldots) = \prod_i p_{(\lambda_i)}(x_1, \ldots)$$

Then $$p_\lambda(x_1, \ldots) = \sum_{\mu \vdash k} \chi_\mu^{S_k}(\lambda)\, \underset{\substack{\shortparallel \\ \text{Schur polynomial} \\ \shortparallel \\ \chi_\mu^{U(\cdot)}}}{s_\mu(x_1, \ldots)}$$

Observe that $p_\lambda(g) := p_\lambda(\text{eigenvalues})$

$$\Rightarrow p_\lambda(g) = \prod_i tr(g^{\lambda_i})$$

* Littlewood - Richardson coefficients

$$c_{\nu\mu}^\lambda$$

Lemma: Let B_{2k} be the centralizer of $(1,2)(3,4)\cdots(2k-1,2k)$ in S_{2k}. Assume $\lambda \vdash k$, $\gamma \vdash \ell$, $n \geq k$. Then,

$$\mathbb{E}_{Sp(2n)} \chi_\gamma^{Sp(2n)} P_\lambda = \langle \mathrm{Ind}_{S_\ell \times B_{k-\ell}}^{S_k} (\chi_\gamma \otimes \mathrm{sgn}), z_\lambda 1_{C_\lambda} \rangle_{S_k}$$

* In $SO(n)$ case, no sign present
* $k-\ell$ odd or negative $\Rightarrow \langle \, , \rangle_{S_k} = 0$
* transfers computation to S_k.

Proof:

$$P_\lambda(g) = P_\lambda(t_1, t_1^{-1}, t_2, t_2^{-1}, \dots, t_n, t_n^{-1})$$

$$= \sum_{\mu \vdash k} \chi_\mu^{S_k}(\lambda)\, s_\mu(t_1, t_1^{-1}, \dots, t_n, t_n^{-1})$$

need $\ell(\mu) \leq n$ $\forall \mu \vdash k$ $\Rightarrow n \geq k$

$$= \sum_{\mu \vdash k} \chi_\mu^{S_k}(\lambda)\, \chi_\mu^{U(2n)}(g)$$

Branching rules

$$= \sum_{\mu \vdash k} \chi_\mu^{S_k}(\lambda) \Big(\sum_{\beta' \text{ even}} c^\mu_{\nu\beta} \Big) \chi_\nu^{Sp(2n)}(g)$$

Proof (cont.)

$$\mathbb{E}_{Sp(2n)} \chi_\gamma^{Sp(2n)} p_\lambda = \sum_{\mu \vdash k} \left(\chi_\mu^{S_k}(\lambda) \sum_{\beta' \text{ even}} c^\mu_{\gamma\beta} \right)$$

$$= \sum_{\beta' \text{ even}} \sum_{\mu \vdash k} \chi_\mu^{S_k}(\lambda) c^\mu_{\gamma\beta}$$

$$= \sum_{\beta' \text{ even}} \mathrm{Ind}_{S_{|\gamma|} \times S_{|\beta|}}^{S_k} \left(\chi_\gamma^{S_{|\gamma|}} \otimes \chi_\beta^{S_{|\beta|}} \right)(\lambda)$$

$\chi_{\beta'} = \chi_\beta \otimes \mathrm{sgn}$

$$= \sum_{\beta \text{ even}} \langle \chi_\gamma^{S_{|\gamma|}} \odot (\mathrm{sgn} \otimes \chi_\beta^{S_{|\beta|}}), z_\lambda 1\!\!1_{C_\lambda} \rangle_{S_k}$$

$\sum_{\beta \text{ even}} \chi_\beta = \mathrm{Ind}_{B_k}^{S_k} 1\!\!1$ when $\beta \vdash k$

$$= \langle \mathrm{Ind}_{S_{|\gamma|} \times B_{k-|\gamma|}}^{S_k} (\chi_\gamma^{S_{|\gamma|}} \otimes \mathrm{sgn}), z_\lambda 1\!\!1_{C_\lambda} \rangle_{S_k}$$

$\square$

Special case:

A matching of k points is a 2-partition of those k pts.

Let $\lambda \vdash k$, $n \geq k$.

Let $g(\lambda) :=$ # matchings of k pts preserved by one elt of C_λ

Thm: $\mathbb{E}_{Sp(2n)} P_\lambda = \int_{Sp(2n)} \prod_i tr(g^{\lambda_i})\, dg = sgn(\lambda)\, g(\lambda)$

γ trivial in lemma $\quad = \langle \mathrm{Ind}_{B_k}^{S_k} sgn, z_\lambda 1_\lambda \rangle_{S_k}$

$= z_\lambda \, sgn(\lambda) \langle 1, \mathrm{Res}_{B_k}^{S_k} 1_\lambda \rangle_{B_k}$

$= z_\lambda \, sgn(\lambda) \cdot \dfrac{\# B_k \cap C_\lambda}{\# B_k}$

} double counting

$= sgn(\lambda)\, g(\lambda)$. $\square$

Diaconis-Shashahani: $g(\lambda) = \prod_j g_j(\lambda(j))$

j odd: $g_j(a) = \begin{cases} 0 & a \text{ odd} \\ j^{a/2}(a-1)!! & a \text{ even} \end{cases}$

j even: $g_j(a) = \sum_t \binom{a}{2t} j^t (2t-1)!!$

* $G(n) = Sp(2n)$
* $\sigma(t) = \sum_{i\in\mathbb{Z}} d_i t^i = \exp\left(\sum_{\mathbb{Z}} \frac{c_i}{|i|} t^i\right) = \exp(f(t))$
* $f(t) = f(t^{-1})$ (i.e. $c_i = c_{-i}$)
* $\sum \left|\frac{c_i}{i}\right| < \infty$
* $\Phi_\sigma(g) = \prod_{\text{half}} \sigma(t_k) = \exp \sum_{i>0} \frac{c_i}{i} p_{(i)}(g) \cdot e^{nc_0}$

__Thm:__

$$\lim_{n\to\infty} \frac{\mathbb{E}_{Sp(2n)} \chi_\gamma^{Sp(2n)} \Phi_\sigma(g)}{\mathbb{E}_{Sp(2n)} \Phi_\sigma(g)} = R(\gamma, (c_i))$$

$$R(\gamma, (c_i)) = \sum_{\lambda \vdash |\gamma|} \frac{\chi_\gamma^{S_{|\gamma|}}(\lambda)}{z_\lambda} \left(\prod_{i=1}^{\ell(\lambda)} c_{\lambda_i}\right)$$

__Rmk:__
* Same ratio/prod for $SO(n)$; ratio for $U(n)$
* Twice the same char. gives new measure $|\chi_\gamma|^2 dg$
* Asymptotics of denominators known (Johansson)
* Bump + Gamburd: $\int_{G(n)} |\det(I-g)|^{2k} dg$

Applies for $\int_{G(n)} \prod |1-t_i|^{2k} \chi_\gamma^{G(n)}(g)\, dg$ as well.

$\sigma(t) =$ $f(t) = (1-t)^k (1-t^{-1})^k \Rightarrow c_i = k \ \forall i$

Idea of proof.

$$1^\circ)\ \left|\mathbb{E}_{Sp(2m)}\chi_\gamma^{Sp(2m)}\Phi_\delta\right| \leq \int_{Sp(2m)} e^{mc_0}\max|\chi_\gamma^{Sp(2m)}|\cdot\exp\Big(\sum_{i>0}\frac{|c_i|}{i}|\mathrm{tr}(g^i)|\Big)\,dg$$

$$2^\circ)\ \mathbb{E}_{Sp(2m)}\chi_\gamma^{Sp(2m)}P_\lambda = \Big\langle \chi_\gamma^{Sp(2m)}\otimes \mathrm{Res}^{S_{k-|\gamma|}}_{B_{k-|\gamma|}}\mathrm{sgn},$$

$$\mathrm{Res}^{S_k}_{S_{|\gamma|}\times B_{k-|\gamma|}} z_\lambda \mathbb{1}_{C_\lambda}\Big\rangle_{S_{|\gamma|}\times B_{k-|\gamma|}}$$

$$= \mathrm{cst.}\sum_{\substack{\lambda_a\vdash|\gamma|\\ \lambda_a\cup\lambda_b=\lambda}} \chi_\gamma^{Sp(2m)}(\lambda_a)\cdot\mathrm{sgn}(\lambda_b)$$

$$= \sum_{\substack{\lambda_a\vdash|\gamma|\\ \lambda_a\cup\lambda_b=\lambda}} \frac{\lambda!}{\lambda_a!\,\lambda_b!}\chi_\gamma^{Sp(2m)}(\lambda_a)\ \mathbb{E}_{Sp(2m)}P_{\lambda_b}$$

U(n) only!

$\sigma(t) = e^{g(t)} = \exp\left(\sum_{i>0} \frac{p_i}{i} t^i + \frac{\tilde{p}_i}{i} t^{-i}\right) = \sigma^+(t) \cdot \sigma^-(t) = \left(\sum d_i^+ t^i\right) \cdot \left(\sum d_i^- t^{-i}\right)$

$= \sum d_i t^i$

(Wiener Hopf)

Heine identity $\quad M_n := \int_{U(n)} \Phi_\sigma(g)\, dg = \det(d_{j-i})_{n\times n} = \det\left[(d^+_{j-i})_{n\times\infty} \cdot (d^-_{i-j})_{\infty\times n}\right]$

(Toeplitz)

Twisted Heine $\quad M_n^{\lambda\mu} := \int_{U(n)} \Phi_\sigma(g)\, s_\lambda(g)\, \overline{s_\mu(g)}\, dg = \det(d_{\lambda_i-\mu_j-i+j})_{n\times n} = \det\left[(d^+_{j-i+\lambda_i}) \cdot (d^-_{i-j+\mu_j})\right]$

$R^{\lambda\phi} := \lim_n \frac{M_n^{\lambda\phi}}{M_n} \quad BD^{\lambda\phi} := \sum_{\gamma\vdash|\lambda|} \frac{\chi^\lambda(\gamma)}{z_\gamma} \prod_i p_{\gamma_i} \overset{JT}{=} Tw^{\lambda\phi} := \det\left(d^+_{i-j+\lambda_{\ell-i+1}}\right)_{\ell\times\ell}$

$BD^{\lambda\phi} \cdot BD^{\phi\mu} = Tw^{\lambda\phi} \cdot Tw^{\phi\mu}$

$\Delta = \Delta$

$R^{\lambda\mu} := \lim_n \frac{M_n^{\lambda\mu}}{M_n} \quad BD^{\lambda\mu} :=$ huge expression $\quad Tw^{\lambda\mu} := \det\left[(d^+_{i-j}\ \lambda\text{-shifted} \to)_{\ell\times\infty}\right.$

(Laguerre polynomials)

$\left.(d^-_{j-i}\ \mu\text{-shifted} \downarrow)_{\infty\times\ell}\right]$

$$\sigma(t) = \exp(f(t)) = \exp\left(\sum_{i\geq 1} \frac{p_i}{i} t^i + \frac{\tilde{p}_i}{i} t^{-i}\right)$$

$$= \sigma^+ \cdot \sigma^- = \left(\sum d_i^+ t^i\right) \Big/ \left(\sum d_i^- t^{-i}\right)$$

Define $p_k(\bar{X}) = p_k(x_1, \dots) = \sum_{i\geq 0} x_i^k$

$$h_k(\bar{X}) = \sum_{i_1 \leq i_2 \leq \dots \leq i_k} x_{i_1} \cdot x_{i_2} \cdots x_{i_k}$$

(all monomials of deg. k)

Then:
$$e^{\sum \frac{p_k(\bar{x})\, t^k}{k}} = \sum h_k(\bar{x}) t^k$$

Specialization : $d_i^+ \to h_i(\bar{X})$, $p_i \to p_i(\bar{X})$

$\left(d_i^- \to h_i(\bar{Y}),\ \tilde{p}_i \to p_i(\bar{Y}) \right)$

$$D^{\lambda \phi} = \sum_{\gamma \vdash |\lambda|} \frac{\chi^\lambda(\gamma)}{z_\gamma} \prod p_{\gamma_i} \longrightarrow s_\lambda(\bar{x})$$

$\parallel$ Jacobi-Trudi

$$Tw^{\lambda \phi} = \det\left(d^+_{i-j-\lambda_i}\right) \longrightarrow \det\left(h_{i-j-\lambda_i}(\bar{x})\right)$$

$\Delta := \exp\left(\sum_k k\,\partial_{p_k} \tilde{p}_k\right)$

In $\Lambda(X)$ $k\,\partial_{p_k} = p_k^{\perp}$

In $\Lambda(Y)$ $k\,\partial_{\tilde{p}_k} = \tilde{p}_k^{\perp}$

$$\Rightarrow \Delta = \sum_\nu \frac{1}{z_\nu} p_\nu^{\perp} \tilde{p}_\nu^{\perp}$$

Cauchy identity:

Cauchy Kernel

$$\sum_\nu \frac{1}{z_\nu} p_\nu \tilde{p}_\nu = \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\nu s_\nu \tilde{s}_\nu$$

$\Rightarrow \Delta =$ adj. Cauchy Kernel $= \sum_\nu s_\nu^{\perp} \tilde{s}_\nu^{\perp}$

but $s_\nu^{\perp}(s_\lambda) = s_{\lambda/\nu}$

$$\Rightarrow \Delta(TW^{\lambda\phi} \cdot TW^{\phi\mu}) = \sum_\nu s_{\lambda/\nu} \tilde{s}_{\mu/\nu}$$

$\underset{=}{\text{Cauchy}\cdot\text{Binet}}$ $TW^{\lambda\mu}$