AVERAGES OVER COMPACT LIE GROUPS, TWISTED BY WEYL CHARACTERS AND APPLICATION TO MOMENTS OF DERIVATIVES OF CHARACTERISTIC POLYNOMIALS

A DISSERTATION
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FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Paul-Olivier Dehaye

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Daniel Bump) Principal Adviser

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Persi Diaconis)

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Brian Conrey)

Approved for the University Committee on Graduate Studies.

Abstract

In this thesis, we are concerned with integrals over classical compact Lie groups such as $G(N) = \mathrm{U}(N), \mathrm{SO}(N)$ or $\mathrm{Sp}(N)$, with respect to their Haar measure. Those integrals are of significant interest in many areas of physics and mathematics, but we will exclusively focus on applications to Number Theory.

We address a few different problems:

- We discuss how integrals over $G(N)$ are affected when inserting a Weyl character of $G(N)$ into the integrand, and give a very concise result on asymptotics in N.

- We explain an identity of Bump, Diaconis, Tracy and Widom on integrals over $\mathrm{U}(N)$ twisted by two Weyl characters, and uncover the existence of a differential operator linking those integrals to single-twisted integrals.

- Based on lessons gathered from the previous results, we compute an average of direct interest in Number Theory:

  $$\int_{\mathrm{U}(N)} |\Lambda_g(1)|^{2k-2} |\Lambda_g'(1)|^2 \mathrm{d}g,$$

  where $\Lambda_g(t)$ is the characteristic polynomial. We actually obtain a result which is exact at fixed N.

The techniques we develop along the way are rather complementary to the standard ones for computing averages over compact Lie groups (Selberg integral, after expanding explicitly the Haar measure).

Throughout this thesis, we rely mostly on methods taken from classical representation theory as well as symmetric function theory, and highlight opportunities for future research.

Acknowledgements

I arrived at Stanford five years ago. Maybe it is the weather, maybe it is the process required to get here, but Stanford is a very welcoming environment to work and live in. During those five years, I have met many new people that have made this whole adventure worthwhile.

A few of them stand out, and it is a tremendous pleasure to acknowledge them here, at the risk of forgetting some.

I could not have wished for a better adviser than Prof. Bump. Dan has always listened to me carefully and answered my questions to the perfect level. He glossed over the stupid ones, carefully considered the hard ones and gave me hints on the others. As a result, our weekly meetings always seemed like an animated discussion from equal to equal, which is no small feat on his part.

I have found the professors in the math department to be universally very friendly and approachable. In particular, I would like to thank Profs. Brubaker, Conrey, Diaconis, Papikian and Thiem for answering numerous questions and suggesting many new and exciting avenues of research. Prof. Vakil has also always been very encouraging, particularly with all my mathematical side interests.

Much of the friendly and social atmosphere in the department is also due to my colleagues, the grad students. I have never felt competition between us, only a very strong emulation. My officemates of 381D, Jack, Muzo and Olivier deserve special mention for successfully sharing the second-smallest office in Building 380 with me. I should thank the G-Z group (BD, Beth, Chang, Kâzim, Yannis, Justin, Ryan) for "that excellent Number Theory adventure". Fernando showed me that it is possible to dance salsa in your office every night and still find a job at the same time. Many others kept a social ambiance going through the years: Andre, Andy, Antonio, Brett, Dan (×2), Daniel, David, Dave, Geir, Jessica, Josh, Julie, Kyoung, Lan Hsuan, Leo, Mariel, Olivier, Pierre, Rob, Robin (×2), Véronique, Zach, and one special person I am sure I forgot.

Finally, the support of the department (and NSF grant 0354662) have made quite a few quarters of research possible, when most of the progress in this thesis was made. For instance, I have been able to attend the school at the Newton Institute (organized by Profs. Snaith and Mezzadri) that initially sparked my interest in Random Matrix Theory and its connections with Number Theory. It has also allowed me to have extensive discussions with world experts in the topics I was thinking about at the time. I also wish to thank those experts (Profs. Farmer, Gonek, Hughes, Kohnen and Stein) for their time.

There is life beyond work. I found (vegetarian) food, entertainment, political discourse, a new wardrobe and many friends at Columbae House. Eric (Katz') acknowledgments in 2004 read

> I would like to acknowledge Columbae. It is no exaggeration to say that had Erick Matsen not brought me to Columbae for dinner in late November (or was it early December) of 1999, I would not have continued at Stanford beyond my first year.

The same is true for me. Albert introduced me to Columbae back in 2002, and I can safely say that I am (at least) the second graduate of the math department for whom this applies. I have many memories of wonderful times there, but the best ones would be Midnight Bread, Friday Fries, and watching the State of the Union in the lounge.

Later on, I would follow Albert to another amazing house, Hammarskjöld. I found there pretty much the same good things, minus the Speed-Scrabble plus meat and Monday "movies". I am happy I was still able to meet so many amazing people during my last year at Stanford.

Throughout those years, my main joy was hiking beautiful Northern California. I had never hiked before getting here, and I owe all that I learned to the Redwood

Outdoors Club[1] and the hiking partners it let me encounter: Albert, Amanda, Christina (or was it Columbae?), Jenny, Josh, Karen, Kasia, Diederik (or was it Belgica?), Neel, Steph, etc.

The weekends I wasn't hiking, I always had a nice home to come to. Thank you to all my housemates for keeping me happy through their conversation, dancing, pranks, or general joy of life : Carrie, Diederik (or was it ΣX?), Gabe, Jessica, Eric, Luke, Zak and of course Henry. Little did we know that one IM conversation would lead to five years of cohabitation and a paper or two!

Thanks to a few others too: Bert for the Jamba Juices, Christine for showing me her Montréal, Cédric and Maureen for the Vegas experience and many other crazy adventures (you guys are always welcome for a Hamburger Helper), Pieter for talking cyclism in Flemish with Diederik so I didn't have to and Rolando for somehow keeping me (and most of my friends) legally in this country.

Chapter 5 owes a lot to Michele's patience with me. Thank you Michele for your kindness through the end of this thesis. Thank you also for sharing your world with me.

Merci aussi à tous mes amis de Belgique. Même si mes visites à Bruxelles se faisaient rares, je sais que j'avais toujours leur soutien.

Je voudrais aussi remercier ma famille, pour les encouragements réguliers et soutenus.

Merci à Pierre-Gilles d'être fier de son grand frère. Lui aussi, en cinq ans, est devenu un homme et je ne peux qu'être fier de lui.

Enfin, je suis *aware* de tout ce que je dois à mes parents. A des milliers de kilomètres, j'ai compris la valeur de l'éducation qu'ils m'ont donnée. Je leur dois tout et j'en suis heureux.

Paul-Olivier Dehaye
San Francisco–Stanford
September 2006

[1]Rentals are 8:30pm to 9:00pm, Wednesday's, at the temporary(?) gear shed. Bring your checkbook. See `http://redwood.stanford.edu`.

Contents

Chapter 1

Introduction

Historically, the study of integrals of symmetric functions of eigenvalues over compact classical Lie groups (with respect to Haar measure) has been important for many areas of mathematics and physics. We will not even attempt to describe the relevance of this problem to physics, but refer the reader to the introduction of Mehta's book [Meh91]. On the mathematics side, we would like to mention at least the following works:

- The Heine-Szegö identity and its relations to the strong Szegö limit theorem. This identity expresses averages over unitary groups as determinants of Toeplitz matrices (see Bump and Diaconis [BD02]), while the strong Szegö limit theorem gives asymptotics for such determinants (see the book by Böttcher and Silbermann [BS99]).

- The study of averages of characteristic polynomials over compact classical Lie groups. Keating and Snaith conjectured that their calculations of those averages would serve as good predictors for moments of the Riemann ζ function [KS00b, unitary case] and other data extracted from L-functions [KS00a, other classical groups]. Most of the integrals that appear here are computed using Selberg's integral. However, Selberg's result is not sufficiently general to compute all of the required integrals (see Chapter 5). Our personal interest in Random Matrix theory sparks from this connection with Number

Theory. We will get back repeatedly to this.

- Diaconis and Shahshahani's work [DS94] on averages of products of traces, and further refinements by Johansson [Joh97]. Those papers have a very probabilistic flavor, and rely on separate work for their most important result. Indeed, the answer to their computations turns out to be expressible as values of characters of the Brauer algebra, which were evaluated by Ram [Ram95, Ram97]. This allows Diaconis and Shahshahani [DS94, Theorem 4] to express those moments in terms of independent normals.

Our main goal in this thesis is to ease the study of such integrals, and to try to offer more systematic methods to compute them. We would like our tools to be based in representation theory, and more particularly on Schur-Weyl duality. We see this duality as a powerful tool to bring computations over compact Lie groups onto the domain of representations of the symmetric groups (or at least the combinatorics of Young tableaux).

We will favor using classical objects of the theory of symmetric functions such as Schur polynomials, power polynomials, ... This approach is historically justified (each of those sets of polynomials forms a basis for the symmetric functions, and has been used extensively to investigate symmetric functions), while the recent paper of Bump and Gamburd [BG06] shows how elegant, concise and powerful this method can be.

In this thesis, we will apply those general principles to the study of three (originally) separate problems.

1.1 Twists by Weyl characters

The first problem is to study what happens when we modify an integrand and introduce a Weyl character.

Let G be $\mathrm{U}(n)$, $\mathrm{SO}(2n)$, $\mathrm{SO}(2n+1)$ or $\mathrm{Sp}(2n)$ and let $\mathbf{\Phi}_{n,f}$ be a class function on G, essentially defined by $\mathbf{\Phi}_{n,f}(g) = \prod_i e^{f(t_i)}$, where $\{t_i\}$ is the set[1] of eigenvalues

[1] or half of that set...

of g . There are extra technical conditions on $\mathbf{\Phi}_{n,f}$, but these will be introduced just before the statement of Theorem 2.7, Section 2.2.

The strong Szegö limit theorem gives the asymptotics and the rate of convergence of $\lim_{n\to\infty}(\mathbb{E}_{\mathrm{U}(n)}\mathbf{\Phi}_{n,f})$. Johansson [Joh97] was the first to generalize this theorem to the other classical groups.

The goal of Chapter 2 will be to study how those averages and asymptotics are affected when we introduce irreducible characters of G into the integrand. The characters $\boldsymbol{\chi}_\lambda^G$ were constructed by Weyl for the compact classical Lie groups using his Character Formula. By the Peter-Weyl theorem, these characters form a basis of the Hilbert space of class functions on G and are thus very natural to consider. For the unitary group, they are given by Schur polynomials evaluated at the eigenvalues.

Theorem 2.7 will show that the ratio

$$\frac{\mathbb{E}_G\boldsymbol{\chi}_\lambda^G\mathbf{\Phi}_{n,f}}{\mathbb{E}_G\mathbf{\Phi}_{n,f}}$$

approaches a limit when $n >> 0$. This extends the corresponding results for the unitary groups due to Bump and Diaconis [BD02] to other classical groups. Remarkably, our ratio is independent of the Cartan type of the group G and equal to the ratio they obtained for the unitary groups. It only varies with f and λ. The results of Johansson provide the asymptotics of the denominator, so this Theorem actually determines the asymptotics of the numerator as well.

Along the way, we offer with Theorem 2.1 a self-contained proof of the following result of Diaconis and Shahshahani.

Theorem. *Let λ be a partition, $\lambda \vdash k$ and $n \geq k$. Let $\epsilon = 1$ when $G = \mathrm{Sp}(2n)$ and $\epsilon = 0$ when $G = \mathrm{SO}(2n)$ or $\mathrm{SO}(2n+1)$. If*

$$\mathbf{p}_\lambda(g) := \prod_{i\in\mathbb{N}} \mathrm{tr}(g^{\lambda_i})$$

then

$$\mathbb{E}_G\mathbf{p}_\lambda = \mathrm{sgn}(\lambda)^\epsilon g(\lambda),$$

where $g(\lambda)$ is defined to be the number of matchings[2] *of k points preserved under the action of a given element of $\mathcal{S}_k$ of cycle type λ.*

Diaconis and Shahshahani had already obtained this result, but through a different interpretation for the function g as moments of independent normal distributions. The interpretation in terms of matchings (see for instance [HL72]) is more natural in our proof.

Chapter 2 is organized as follows:

Section 2.1 will first go over notation, then introduce the reader to the representation theory of the compact classical Lie groups (Weyl characters and Branching Rules). Section 2.2 will contain all of the proofs. It will also present the statement of Theorem 2.7, and then shortly discuss its significance in relation to the rest of the literature.

Almost all the results in Chapter 2 appeared first in [Deh05a].

1.2 An identity due to ((Bump and Diaconis) and (Tracy and Widom))

As the title of Chapter 3 tries to suggest, this addresses an identity due to two separate pairs of authors.

Consider a function $\sigma(t) : \mathbb{T} \to \mathbb{C} =: \sum_{k\in\mathbb{Z}} d_k t^k$, where $\mathbb{T}$ is the unit circle. We will later assume symmetry properties on the d_i's such as $d_i = \overline{d_{-i}}$, but this is not immediately needed. Take the Toeplitz matrix with symbol σ:

$$M_n(\sigma) = M_n = \begin{pmatrix} d_0 & d_1 & \cdots & \cdots & d_{n-1} \\ d_{-1} & d_0 & d_1 & \cdots & d_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & d_1 \\ d_{1-n} & \cdots & \cdots & d_{-1} & d_0 \end{pmatrix}_{n\times n} = (d_{i-j})_{n\times n}.$$

[2]We remind the reader that a *matching* of a set S is a perfect partition of S into pairs.

A classical question for Toeplitz matrices is then to consider the asymptotics of the determinant $\det(M_n)$ as n goes to infinity. Our identity will stem from the same question for a slightly altered version of M_n. For λ and μ partitions of length less or equal to n, look at

$$M_n^{\lambda\,\mu}(\sigma) := \left(d_{\lambda_i-\mu_j-i+j}\right)_{n\times n}.$$

Those new matrices are not Toeplitz, but at least they are minors of the Toeplitz matrix $M_m(\sigma)$, for some m larger than n. This is clear once illustrated: set $n := 3$, $m := 5$, $\lambda = (2,1)$, $\mu = (1)$ and for the sake of simplicity take the symbol $d_i = i$. We have the matrices

$$M_3^{\lambda\,\mu}(\sigma) = \begin{pmatrix} 1 & 3 & 4 \\ -1 & 1 & 2 \\ -3 & -1 & 0 \end{pmatrix} \quad \text{and} \quad M_5(\sigma) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 & 3 \\ -2 & -1 & 0 & 1 & 2 \\ -3 & -2 & -1 & 0 & 1 \\ -4 & -3 & -2 & -1 & 0 \end{pmatrix}.$$

Observe that $M_3^{\lambda\,\mu}(\sigma)$ is the minor of $M_5(\sigma)$ obtained by striking its first and third column and its second and fourth row. If m had been bigger, we would only have needed to strike more rows and columns.

Tracy and Widom [TW02] looked at asymptotics of ratios of minors and determinants[3]

$$\mathrm{TW}^{\lambda\,\mu} := \lim_{n\to\infty} \frac{\det M_n^{\lambda\,\mu}}{\det M_n},$$

which they obtained as determinants involving the Fourier coefficients of the Wiener-Hopf factorization of $\sigma(t)$[4].

Independently, Bump and Diaconis [BD02] looked at the same ratios under the restriction that

$$\sigma(t) = \exp\left(\sum_{k>0} \frac{p_k}{k} t^k + \frac{\tilde{p}_k}{k} t^{-k}\right)$$

[3]The asymptotics of the determinants are well known, so this effectively provides the asymptotics for the minors.

[4]See Section 3.4 for more details.

for the sets of constants $\{p_k \in \mathbb{C}\}$ and $\{\tilde{p}_k \in \mathbb{C}\}$.

Using the Heine identity, they quickly translated this problem about ratios of determinants of Toeplitz matrices into a question on twists of integrals over the unitary group (hence our interest).

Their formula for the ratios

$$\mathrm{BD}^{\lambda\,\mu} := \lim_{n\to\infty} \frac{\det M_n^{\lambda\,\mu}}{\det M_n}$$

involved Laguerre polynomials and the coefficients p_k, $\tilde{p}_k$.

Both pairs of authors computed the same quantities, so together they obtained a vast array of equalities

$$\mathrm{BD}^{\lambda\,\mu} = \mathrm{TW}^{\lambda\,\mu},$$

as λ and μ vary.

In Chapter 3, one of our goals is to directly prove[5] these identities (culminating with Theorem 3.1 in Section 3.5). Our main tool will be an operator Δ taking many different forms and acting on the expressions for $\mathrm{BD}^{\lambda\,\mu}$ or $\mathrm{TW}^{\lambda\,\mu}$. The operator Δ is very interesting when considered in the context of integrals over unitary groups or even back to minors of Toeplitz matrices.

Let $\varnothing$ be the trivial partition. The following results are proved:

1. $\Delta\left(\mathrm{BD}^{\lambda\,\varnothing}\cdot\mathrm{BD}^{\varnothing\,\mu}\right) = \mathrm{BD}^{\lambda\,\mu}$ (Theorem 3.2),
2. $\Delta\left(\mathrm{TW}^{\lambda\,\varnothing}\cdot\mathrm{TW}^{\varnothing\,\mu}\right) = \mathrm{TW}^{\lambda\,\mu}$ (Theorem 3.3),
3. $\mathrm{BD}^{\lambda\,\varnothing} = \mathrm{TW}^{\lambda\,\varnothing}$ (Jacobi-Trudi identity, a classical identity in symmetric function theory).

In other words, we will demonstrate how their identity is a differentiated version of the Jacobi-Trudi identity. Beyond this explanation, our results uncover unexpected relations between single-twisted and doubly-twisted integrals.

Many of the results from Chapter 3 are already in [Deh05b].

[5] i.e. not involving Toeplitz determinants

1.3 Intermission

In this very short Chapter, we will discuss the lessons learned from Chapters 2 and 3 and prepare for Chapter 5.

We will emphasize that we should in general roughly expect

$$\lim_{N\to\infty} \frac{\int_{G(N)} f(g)s_\lambda(g)\mathrm{d}g}{\int_{G(N)} f(g)\mathrm{d}g}$$

to relate in a direct fashion to the Schur function $s_\lambda(x_1,\cdots)$ after proper specialization in the ring of symmetric functions in $x_1,\cdots$. The specialization will involve the power polynomials and the (normalized) Fourier coefficients $c_1,\cdots$ in $f(t) := \exp\sum_i \frac{c_i}{i}t^i$ (Theorem 2.7).

Our lesson from Chapter 3 is that we should expect (Theorem 3.1)

$$\int_{\mathrm{U}(N)} f(g)s_\lambda(g)\overline{s_\mu(g)}\mathrm{d}g$$

to relate to

$$\int_{\mathrm{U}(N)} f(g)s_\lambda(g)\mathrm{d}g, \int_{\mathrm{U}(N)} f(g)\overline{s_\mu(g)}\mathrm{d}g \quad \text{and} \quad \int_{\mathrm{U}(N)} f(g)\mathrm{d}g.$$

We will briefly speculate that a similar statement should hold true for more groups than just $\mathrm{U}(N)$.

These lessons should prove useful for many applications.

One application is discussed in Chapter 5. However, the presentation there does not emphasize this dependency. Indeed, while those lessons were initially very helpful in treating the problem of Chapter 5, the results in that Chapter were then significantly improved (from asymptotic in N to exact at finite N), which made the reliance on those old lessons less obvious. We thus decided to include a short informal review in Chapter 4 that we could draw on in Chapter 5.

1.4 Mixed moments of derivatives

When giving a talk at the Number Theory and Random Matrix Theory Conference at the University of Rochester in June 2006, I mentioned how those old lessons could possibly be applied to the problem of computing asymptotics for

$$\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^{2k'} \mathrm{d}g, \tag{1.1}$$

where $\Lambda_g'(t)$ is the derivative of the characteristic polynomial of g[6], and how those lessons predict that asymptotically

$$\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^{2k'} \mathrm{d}g \sim_N f_{k'}(N) \int_{\mathrm{U}(N)} |\Lambda_g(1)|^{2k'} \mathrm{d}g,$$

where $f_{k'}$ is relatively much simpler and has slower growth in N. In other words, the main factor in $\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^{2k'} \mathrm{d}g$ should be $\int_{\mathrm{U}(N)} |\Lambda_g(1)|^{2k'} \mathrm{d}g$.

Chris Hughes, one of the conference organizers, immediately told me he had already empirically observed this factorization for a larger class of integrals, which encouraged me to work more on this.

Along with many other Analytic Number Theorists, his interest in this problem stems from the work of Keating and Snaith [KS00b]: Keating and Snaith originally conjectured that characteristic polynomials over $\mathrm{U}(N)$ serve as good predictors for the Riemann ζ function. For instance, they have conjectured that moments of characteristic polynomials and ζ behave in the same way, and have demonstrated significant numerical agreement. Similarly, one can hope that derivatives of characteristic polynomials would serve as good predictors for derivatives of ζ on the critical line. This would have potential (conjectural) applications to the many theorems in Analytic Number Theory relying on moments of ζ'.

The Random Matrix Theory results presented here concern

$$\int_{\mathrm{U}(N)} |\Lambda_g(1)|^{2k-2} |\Lambda_g'(1)|^2 \mathrm{d}g, \tag{1.2}$$

[6] k' denotes an integer. This notation will make sense in light of the final expression evaluated in Section 5

i.e. some average of mixed moments of a characteristic polynomial and a derivative. Assuming the Keating-Snaith conjectures, those results would predict the Number Theory moments

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2k-2}|\zeta'(1/2+it)|^2\mathrm{d}t.$$

Remark on Chapter 5 The initial results and proofs relied directly on ideas from Chapter 2. The discussion used a technique of Bump and Gamburd and some symmetric function theory to first translate (1.2) into a big sum over combinatorial structures involving Young tableaux (see Section 5.3). The next step required splitting that sum into subsums, and evaluating asymptotics on each using lessons from Chapter 2.

However, those initial results can actually be much improved through a careful analysis of the combinatorics involved (Section 5.7). We end up with a beautiful *exact result* for (1.2) in Theorem 5.11, *at fixed* N.

This result is very recent, so the combinatorial expression we obtain for (1.2), a hypergeometric series, could still be simplified. We actually offer a candidate in Theorem 5.20. This Theorem would extend the range of validity of our results to $k \in \mathbb{C}, \mathrm{Re}\, k > 1/2$, which is significant for analytic Number Theory.

Other extensions of this result that have not been approached yet are discussed in Section 5.11.

Chapter 2

Twists by Weyl characters

The main goal in this Chapter will be to study what happens when we modify an integrand and introduce a Weyl character. Weyl characters are important in the study of integrals over compact Lie groups, because the Peter-Weyl theorem guarantees that they will form a basis of the Hilbert space of class functions on that group.

Before we achieve that goal, we stumble upon a new proof of the following Theorem, as a corollary of some of the work needed.

Theorem 2.1 (Diaconis and Shahshahani [DS94]). *Let λ be a partition, $\lambda \vdash k$ and $n \geq k$. Let $\epsilon = 1$ when $G = \mathrm{Sp}(2n)$ and $\epsilon = 0$ when $G = \mathrm{SO}(2n)$ or $\mathrm{SO}(2n+1)$. If*

$$\mathbf{p}_\lambda(g) := \prod_{i \in \mathbb{N}} \mathrm{tr}(g^{\lambda_i})$$

then

$$\mathbb{E}_G \mathbf{p}_\lambda = \mathrm{sgn}(\lambda)^\epsilon g(\lambda),$$

where $g(\lambda)$ is defined to be the number of matchings[1] *of k points preserved under the action of a given element of $\mathcal{S}_k$ of cycle type λ.*

This result had already been obtained by Diaconis and Shahshahani. Their proof uses the interpretation for g in terms of moments of independent normals.

[1] We remind the reader that a *matching* of a set S is a perfect partition of S into pairs.

If the reader only wants to understand the proof of Theorem 2.1, it might be helpful to observe that Propositions 2.3 and 2.5 include a γ that will only be useful for Theorem 2.7. The reader could thus safely assume that $\gamma = (0, 0, \cdots)$ and still see a full proof of the statement.

If we are willing to restrict the integrand to have $\lambda_i = 1$ for all i, Rains [Rai98, Theorem 3.4] has proved this result in the full range for n. We present only the symplectic case of his result. In our notation, he proved that $\mathbb{E}_{\mathrm{Sp}(2n)}\mathbf{p}_\lambda(g)$ with $\lambda = (1, 1, \cdots, 1) \vdash k$ is equal to the number of fixed-point-free involutions of length k with no decreasing subsequence of length greater than $2n$.

In the *stable range*[2], he effectively counts the number of fixed-point-free involutions of length k, i.e. the number of matchings on k points preserved by the identity permutation on those k points.

The problem of Theorem 2.1 was also solved in full generality by Pastur and Vasilchuk [PV04], although their method of proof is arguably more complicated. We will sketch it in the orthogonal case. Let $F : \mathrm{SO}(m) \to \mathbb{R}$ be a continuously differentiable function and X be any $n \times n$ real antisymmetric matrix. By left-invariance of Haar measure, $\mathbb{E}_{g\in\mathrm{SO}(m)}F(e^{tX}g)$ is independent of the real parameter t and so $\mathbb{E}_{g\in\mathrm{SO}(m)}(F'(g)Xg) = 0$, where F' is the derivative of F. This expression can then be expanded and used to reduce the main expression to simpler ones.

We would like to point out that our proof of Theorem 2.1 involves the hyper-octahedral group $\mathcal{B}_k$. Both Stolz [Sto05] and Rains [Rai95] have already used the same group for this computation.

We now turn to a more complicated problem, our main goal for this Chapter.

Let G be $\mathrm{U}(n)$, $\mathrm{SO}(2n)$, $\mathrm{SO}(2n{+}1)$ or $\mathrm{Sp}(2n)$ and let $\mathbf{\Phi}_{n,f}$ be a class function on G, essentially defined by $\mathbf{\Phi}_{n,f}(g) = \prod_i e^{f(t_i)}$, where $\{t_i\}$ is a subset of eigenvalues of g . There are extra technical conditions on $\mathbf{\Phi}_{n,f}$, but these will be introduced just before the statement of Theorem 2.7, Section 2.2.

The strong Szegö limit theorem gives the asymptotics and the rate of convergence of $\lim_{n\to\infty}(\mathbb{E}_{\mathrm{U}(n)}\mathbf{\Phi}_{n,f})$. Johansson [Joh97] was the first to generalize this theorem to the other classical groups.

[2]See page 17.

The main goal of this Chapter will be to study how those averages and asymptotics are affected when we introduce irreducible characters of G into the integrand. The characters $\boldsymbol{\chi}_\lambda^G$ were constructed by Weyl for the compact classical Lie groups using his Character Formula. By the Peter-Weyl theorem, these characters form a basis of the Hilbert space of class functions on G and are thus very natural to consider.

Theorem 2.7 will show that the ratio

$$\frac{\mathbb{E}_G \boldsymbol{\chi}_\lambda^G \boldsymbol{\Phi}_{n,f}}{\mathbb{E}_G \boldsymbol{\Phi}_{n,f}}$$

approaches a limit when $n >> 0$. This extends the corresponding results for the unitary groups due to Bump and Diaconis [BD02] to other classical groups. Remarkably, our ratio is independent of the Cartan type of the group G and equal to the ratio they obtained for the unitary groups. It only varies with f and λ.

A different point of view is offered in Bump, Diaconis and Keller [BDK02]: we can modify the Haar measure $\mathrm{d}g$ into $\boldsymbol{\chi}_\lambda^G \overline{\boldsymbol{\chi}_\lambda^G} \mathrm{d}g$. We know that $\boldsymbol{\chi}_\lambda^G \overline{\boldsymbol{\chi}_\lambda^G}$ is always positive and of mass 1 by orthogonality of Weyl characters hence $\boldsymbol{\chi}_\lambda^G \overline{\boldsymbol{\chi}_\lambda^G} \mathrm{d}g$ is a measure. With this point of view, Theorem 2.7 would thus partially explain how the average of $\boldsymbol{\Phi}_{n,f}$ with respect to Haar measure $\mathrm{d}g$ is modified when *twisting* the Haar measure by a character (see the last two remarks on page 29).

Thirdly, we would like to mention the recent preprint of Bump and Gamburd [BG06]. They showed how many of the integrals useful for Number Theory can be computed in a unified way. An example of such an integral would be

$$\int_{\mathrm{U}(n)} \prod_i \Lambda_g(e^{\alpha_i})\, \mathrm{d}g,$$

where $\Lambda_g(\cdot)$ is the characteristic polynomial of g, and the α_i's are points on the unit circle. The importance of integrals of this type originates from the work of Keating and Snaith [KS00a, KS00b], where the integrals have been shown to predict the moments of $\zeta(\cdot)$ and of L-functions.

The method of Bump and Gamburd is based on symmetric function theory and classical results (Weyl Character Formula, Littlewood Branching Rules of Theorem 2.2, page 18, and Cauchy Identity). The reader is referred to their introduction for a much more comprehensive survey of all the results their method is known to produce, and how (if) they were proved before.

We aim to use similar principles, relying on classical symmetric function theory to offer more systematic techniques to compute such integrals.

Chapter 2 is organized as follows:

Section 2.1 will first go over notation, then introduce the reader to the representation theory of the compact classical Lie groups (Weyl characters and Branching Rules). Section 2.2 will contain all of the proofs. It will also present the statement of Theorem 2.7, and then shortly discuss its significance in relation to the rest of the literature. Note also Section 4.3 for some speculation concerning the results presented here.

2.1 Representation theory of the classical groups

We now introduce Weyl characters and the branching rules between different classical compact Lie groups. We follow the expositions of [BG06] and [KT87], but our notation is closer to [BG06] (which adds to Macdonald's [Mac95]).

2.1.1 Notation

We now introduce the notation we will need for this Chapter.

Partitions.

A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is a finite decreasing sequence of non-negative integers. We define the weight $|\lambda|$ of λ to be the sum $\sum \lambda_i$. If this weight is k, we also use the notation $\lambda \vdash k$. The length $l(\lambda)$ of λ is the maximal i such that $\lambda_i \neq 0$. The conjugate of λ is denoted λ'. We say that a partition is even if all of its parts λ_i are even. We define the union $\lambda \cup \mu$ to be the partition of $|\lambda| + |\mu|$ whose parts are the union of the parts of λ and μ. There is a partial ordering on partitions: $\lambda \subseteq \mu$ iff $\lambda_i \leq \mu_i$ for all i. Finally, we define the $\lambda(i)$'s so that $(i^{\lambda(i)}) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, i.e. $\lambda(i)$ counts the number of λ_j's equal to i.

Symmetric group.

The symmetric group on k points will be $\mathcal{S}_k$. If $\lambda \vdash k$, elements of type λ are the elements whose cycle types correspond to the partition λ. We use $\mathcal{C}_\lambda$ for the conjugacy class of those elements. We denote a centralizer in the group G by $C_G(\cdot)$, and by z_λ the order of the centralizer of an element of $\mathcal{C}_\lambda$. As usual, the irreducible characters χ_λ of $\mathcal{S}_k$ are indexed by partitions $\lambda \vdash k$. We sometimes abuse notation and take $\chi_\lambda(\mu)$ to mean the value of χ_λ on $\mathcal{C}_\mu$. If χ_λ and χ_μ are characters of $\mathcal{S}_{|\lambda|}$ and $\mathcal{S}_{|\mu|}$, their product $\chi_\lambda \odot \chi_\mu$ in the character ring of symmetric groups will be the character $\mathrm{Ind}_{\mathcal{S}_{|\lambda|} \times \mathcal{S}_{|\mu|}}^{\mathcal{S}_{|\lambda|+|\mu|}}(\chi_\lambda \times \chi_\mu)$ (see Sagan's book [Sag91] for all aspects of the representation theory of symmetric groups, and page 164 for the product of characters $\chi_\lambda \odot \chi_\mu$).

Classical groups.

Let J be the $2n \times 2n$ matrix given by

$$J = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}.$$

We would like to introduce a few classical groups:

$$\begin{aligned}
\mathrm{U}(n) &= \{g \in M_n(\mathbb{C}) \quad | \quad gg^* = I\}, \\
\mathrm{O}(n) &= \{g \in \mathrm{U}(n) \quad | \quad gg^t = I\}, \\
\mathrm{SO}(n) &= \{g \in \mathrm{O}(n) \quad | \quad \det(g) = 1\}, \\
\mathrm{Sp}(2n) &= \{g \in \mathrm{U}(2n) \quad | \quad gJg^t = J\}.
\end{aligned}$$

If G is one of those groups, it is compact for the topology induced by $M_n(\mathbb{C})$ or $M_{2n}(\mathbb{C})$. We can thus consider its Haar measure $\mathrm{d}g$ and normalize it so the total volume of G is 1. We write $\mathbb{E}_G f$ for $\int_G f(g)\,\mathrm{d}g$.

Symmetric functions and power characters.

Let $\mathbb{C}[x_1, \cdots, x_m]^{\mathcal{S}_m}$ be the ring of symmetric polynomials in m variables. We define the power sum symmetric functions $p_i(x_1, \cdots, x_m) = x_1^i + \cdots + x_m^i$ and $p_\lambda(x_1, \cdots, x_m) = \prod_i p_{\lambda_i}(x_1, \cdots, x_m)$. Let $\lambda \vdash k$ and consider the map that takes the Schur function s_λ to the irreducible character χ_λ of $\mathcal{S}_k$. This map extends by linearity to take any symmetric function to a class function. We will abuse notation and denote by p_λ both the power sum symmetric function and its image under that map (see Bump's book [Bum04, Theorem 39.1]). Finally, we define the characters $\mathbf{p}_\lambda$ of $G = \mathrm{U}(m), \mathrm{O}(m), \mathrm{SO}(m)$ or $\mathrm{Sp}(m = 2n)$ by $\mathbf{p}_\lambda(g) := p_\lambda(t_1, t_2, \cdots, t_m)$ where the t_i's are *all* the eigenvalues of g. There is an obvious interpretation of those generalized characters in terms of the trace. For instance, we have $\mathbf{p}_{(3,1,1)}(g) = \mathrm{tr}(g^3) \cdot (\mathrm{tr}\, g)^2$.

2.1.2 Weyl characters

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be a partition. Let i and j be indices running between 1 and n. Guided by the Weyl Character Formula, we define the following symmetric functions of $\{x_1, \cdots, x_n\}$, actually polynomials in $\mathbb{Z}[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]$:

$$\begin{aligned}
s_\lambda(x_1, \cdots, x_n) &= \frac{\left|x_i^{\lambda_j+n-j}\right|}{\left|x_i^{n-j}\right|}, \\
\chi_\lambda^{\mathrm{SO}(2n+1)}(x_1, \cdots, x_n) &= \frac{\left|x_i^{\lambda_j+n-j+1/2} - x_i^{-(\lambda_j+n-j+1/2)}\right|}{\left|x_i^{n-j+1/2} - x_i^{-(n-j+1/2)}\right|}, \\
\chi_\lambda^{\mathrm{Sp}(2n)}(x_1, \cdots, x_n) &= \frac{\left|x_i^{\lambda_j+n-j+1} - x_i^{-(\lambda_j+n-j+1)}\right|}{\left|x_i^{n-j+1} - x_i^{-(n-j+1)}\right|}.
\end{aligned}$$

The $s_\lambda(\cdot)$ are the regular Schur polynomials that appear in the representation theory of the symmetric group. Take $g \in \mathrm{U}(n)$ (resp. $\mathrm{SO}(2n{+}1)$ or $\mathrm{Sp}(2n)$). Label the eigenvalues of g by $\{t_1, \cdots, t_n\}$ (resp. $\{t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}, 1\}$ or $\{t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}\}$). This allows us to define the functions $\mathbf{s}_\lambda(g)$, $\boldsymbol{\chi}_\lambda^{\mathrm{Sp}(2n)}(g)$ or $\boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n+1)}(g)$ through the values of the respective function on the subset $\{t_1, \cdots, t_n\}$.

When $G = \mathrm{SO}(2n+1)$ (resp. $\mathrm{Sp}(2n)$), Weyl showed that the character $\boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n+1)}$ (resp. $\boldsymbol{\chi}_\lambda^{\mathrm{Sp}(2n)}$) is irreducible when $l(\lambda) \leq n$. This is called the *stable range* for λ[3].

Due to the involution in the Dynkin diagram of type D_n, the case of $\boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n)}$ is actually special. We will again define $\boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n)}(g)$ as the value of a function $\chi_\lambda^{\mathrm{SO}(2n)}$ on an appropriate subset of n eigenvalues of g. The difference in this case is that we only have $\lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_n|$ for the index set. If λ is a regular partition, we define $\lambda_+ := \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ and $\lambda_- := (\lambda_1, \lambda_2, \cdots, -\lambda_n)$. The characters $\boldsymbol{\chi}_{\lambda^+}^{\mathrm{SO}(2n)}$ and $\boldsymbol{\chi}_{\lambda^-}^{\mathrm{SO}(2n)}$ are exchanged by the involution on the Dynkin diagram, i.e by conjugation by an element of $\mathrm{O}(2n)$ of negative determinant[4].

The Weyl character formula defines the functions

$$\chi_\lambda^{\mathrm{SO}(2n)}(x_1, \cdots, x_n) = \frac{\left|x_i^{\lambda_j+n-j} + x_i^{-(\lambda_j+n-j)}\right| + \left|x_i^{\lambda_j+n-j} - x_i^{-(\lambda_j+n-j)}\right|}{\left|x_i^{n-j} + x_i^{-(n-j)}\right|}.$$

If we set $\chi_\lambda^{\mathrm{O}(2n)} := \chi_{\lambda^+}^{\mathrm{SO}(2n)} + \chi_{\lambda^-}^{\mathrm{SO}(2n)}$ when $\lambda_n \neq 0$ and $\chi_\lambda^{\mathrm{O}(2n)} := \chi_{\lambda^+}^{\mathrm{SO}(2n)}$ otherwise, then

$$\chi_\lambda^{\mathrm{O}(2n)}(x_1, \cdots, x_n) = \frac{\left|x_i^{\lambda_j+n-j} + x_i^{-(\lambda_j+n-j)}\right|}{\left|x_i^{n-j} + x_i^{-(n-j)}\right|}.$$

The character $\boldsymbol{\chi}_\lambda^{\mathrm{O}(2n)}(g)$ is defined similarly by evaluating $\chi_\lambda^{\mathrm{O}(2n)}$ on eigenvalues.

It is still a consequence of Weyl's work that $\boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n)}$ is an irreducible character of $\mathrm{SO}(2n)$ when $l(\lambda) \leq n$. However, $\boldsymbol{\chi}_\lambda^{\mathrm{O}(2n)}$ will merely be the character of the representation of $\mathrm{SO}(2n)$ which is obtained by restricting an irreducible representation

[3]The book of Goodman and Wallach [GW98, Chapter 10] is the standard reference for this. See also the paper of Koike and Terada [KT87].

[4]It might be helpful for the reader to observe that in the odd orthogonal case, $\mathrm{O}(2n+1) \cong \mathrm{SO}(2n+1) \times \mathbb{Z}/2$ so the involution acts trivially.

of O($2n$) to SO($2n$), *not* the character of a representation of O($2n$).

For the sake of uniformity in the orthogonal case, we will sometimes want to use $\boldsymbol{\chi}_\lambda^{\mathrm{O}(2n+1)} := \boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n+1)}$.

We also use the notational shortcut $\boldsymbol{\chi}_\lambda^G$ where G is one of the Lie groups defined above. It might be good at this point to remind the reader that χ_λ denotes a character of a symmetric group.

The irreducibility of the various characters considered guarantees certain orthogonality properties, which we will only describe as needed in the proofs.

2.1.3 Branching rules

Let $G = \mathrm{SO}(m)$ or $\mathrm{Sp}(m)$. Since $G \subset \mathrm{U}(m)$, the restriction of $\mathbf{s}_\lambda$ to G is a class function for G and can be expressed as a sum of $\boldsymbol{\chi}_\mu^G$'s. The branching rules describe more precisely how to do that (see the paper of Koike and Terada [KT87, page 492] for a modern and complete proof).

Theorem 2.2 (Littlewood)**.** *Let λ be a partition of length less than or equal to n. Then*

$$\begin{aligned}
\boldsymbol{s}_\lambda\Big\downarrow_{\mathrm{Sp}(2n)}^{\mathrm{U}(2n)} &= \sum_{\mu\subseteq\lambda}\left(\sum_{\nu \text{ even}} c_{\nu'\mu}^{\lambda}\right)\boldsymbol{\chi}_\mu^{\mathrm{Sp}(2n)},\\
\boldsymbol{s}_\lambda\Big\downarrow_{\mathrm{SO}(2n+1)}^{\mathrm{U}(2n+1)} &= \sum_{\mu\subseteq\lambda}\left(\sum_{\nu \text{ even}} c_{\nu\mu}^{\lambda}\right)\boldsymbol{\chi}_\mu^{\mathrm{O}(2n+1)}\\
\boldsymbol{s}_\lambda\Big\downarrow_{\mathrm{SO}(2n)}^{\mathrm{U}(2n)} &= \sum_{\mu\subseteq\lambda}\left(\sum_{\nu \text{ even}} c_{\nu\mu}^{\lambda}\right)\boldsymbol{\chi}_\mu^{\mathrm{O}(2n)},
\end{aligned}$$

where $\boldsymbol{s}_\lambda\Big\downarrow_G^{\mathrm{U}(n)}$ indicates the restriction to G of the character $\boldsymbol{s}_\lambda$ of $\mathrm{U}(n)$ and $c_{\nu\mu}^\lambda$ are the Littlewood-Richardson coefficients.

Remark.

This is where the eigenvalue 1 "disappears" in the $\mathrm{SO}(2n+1)$ case. Let $g \in \mathrm{SO}(2n+1) \subset \mathrm{U}(2n+1)$, with eigenvalues $\{1, t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{1}\}$. The left-hand side is

$$\mathbf{s}_\lambda(g) = s_\lambda(1, t_1, \cdots, t_n, t_1^{-1}, \cdots, t_n^{-1}),$$

while the right-hand side only involves terms of the form

$$\boldsymbol{\chi}_\mu^{\mathrm{O}(2n+1)}(g) = \chi_\mu^{\mathrm{O}(2n+1)}(t_1, \cdots, t_n).$$

2.2 Proofs

We will now present the main derivation. This is vaguely similar to a few steps of the proof of [DE01, Theorem 2.1] in the unitary case.

Proposition 2.3. *Let $\lambda \vdash k$ and $n \geq k$. Then*

$$\mathbb{E}_{\mathrm{Sp}(2n)} \boldsymbol{\chi}_\gamma^{\mathrm{Sp}(2n)} \mathbf{p}_\lambda = \sum_{\substack{\beta' \text{ even} \\ \gamma \cup \beta \vdash k}} \langle \chi_\gamma \odot \chi_\beta, p_\lambda \rangle_{\mathfrak{S}_k}.$$

Similarly (but with β instead of β'), we have

$$\mathbb{E}_{\mathrm{SO}(2n+1)} \boldsymbol{\chi}_\gamma^{\mathrm{SO}(2n+1)} \mathbf{p}_\lambda = \sum_{\substack{\beta \text{ even} \\ \gamma \cup \beta \vdash k}} \langle \chi_\gamma \odot \chi_\beta, p_\lambda \rangle_{\mathfrak{S}_k} = \mathbb{E}_{\mathrm{SO}(2n)} \boldsymbol{\chi}_\gamma^{\mathrm{SO}(2n)} \mathbf{p}_\lambda$$

Note: when $|\gamma| > |\lambda| = k$ or when $k - |\gamma|$ is odd, those sums are indeed trivial and give a value of 0.

Proof. The general method of proof is to use the branching rules from Section 2.1.3 to eventually transfer the problem to a symmetric group.

For definiteness, we will only prove this for $\mathrm{Sp}(2n)$ and discuss at the end the minor changes needed in the orthogonal cases. Let $g \in \mathrm{Sp}(2n)$ have eigenvalues

$\{t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}\}$. Then

$$\begin{aligned}\mathbf{p}_\lambda(g) &= \sum_{\mu \vdash k} \chi_\mu(\lambda)\mathbf{s}_\mu(g) \\ &= \sum_{\mu \vdash k} \chi_\mu(\lambda) \sum_{\nu \subseteq \mu} \left(\sum_{\beta' \text{ even}} c^{\mu}_{\nu\beta} \right) \boldsymbol{\chi}^{\mathrm{Sp}(2n)}_\nu(g),\end{aligned}$$

where the first line follows from the usual decomposition of power polynomials into Schur polynomials given by the character table of a symmetric group (see Sagan [Sag91, Equation (4.23)]). The second line follows by applying the branching rule for each $\mu \vdash k$. The branching rule is only valid when $l(\mu) \leq n$. This explains our final restriction of $n \geq k$.

We know that $\mathbb{E}_{\mathrm{Sp}(2n)} \boldsymbol{\chi}^{\mathrm{Sp}(2n)}_\gamma \boldsymbol{\chi}^{\mathrm{Sp}(2n)}_\nu = 1$ when $\gamma = \nu$ and 0 otherwise (this is a consequence of the theory of the Weyl Character formula). Hence

$$\mathbb{E}_{\mathrm{Sp}(2n)} \boldsymbol{\chi}^{\mathrm{Sp}(2n)}_\gamma \mathbf{p}_\lambda = \sum_{\mu \vdash k} \left(\chi_\mu(\lambda) \sum_{\beta' \text{ even}} c^{\mu}_{\gamma\beta} \right),$$

where the condition that $\nu = \gamma \subseteq \mu$ is still present implicitly in the Littlewood-Richardson coefficient ($c^{\mu}_{\gamma\beta} = 0$ if $\gamma \not\subseteq \mu$). For the same reason, we see that this sum is trivial when $|\gamma| > |\mu| = k$.

The final statement follows from observing that $\sum_{\mu \vdash k} c^{\mu}_{\gamma\beta} \chi_\mu = \chi_\gamma \odot \chi_\beta$ and $\chi(\lambda) = \langle \chi, p_\lambda \rangle_{\mathfrak{S}_k}$.

For the orthogonal groups, the only difference is that two characters will pop up when $\lambda_n \neq 0$. Let $m = 2n$ or $2n + 1$. The branching rules will involve $\boldsymbol{\chi}^{\mathrm{O}(m)}_\lambda$ while the twist that we introduce comes from a character of type $\boldsymbol{\chi}^{\mathrm{SO}(m)}_\lambda$. Fortunately, all we need for the same proof to work is $\mathbb{E}_G \boldsymbol{\chi}_\lambda^{\mathrm{O}(m)} \boldsymbol{\chi}_\lambda^{\mathrm{SO}(m)} = 1$:

$$\begin{aligned}
\mathbb{E}_G \boldsymbol{\chi}_\lambda^{\mathrm{O}(2n)} \boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n)} &= \mathbb{E}_G \boldsymbol{\chi}_{\lambda^+}^{\mathrm{SO}(2n)} \boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n)} + \mathbb{E}_G \boldsymbol{\chi}_{\lambda^-}^{\mathrm{SO}(2n)} \boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n)} \\
&= 1 + 0 \text{ by orthonormality for } \mathrm{SO}(2n). \\
\mathbb{E}_G \boldsymbol{\chi}_\lambda^{\mathrm{O}(2n+1)} \boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n+1)} &= \mathbb{E}_G \boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n+1)} \boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n+1)} \\
&= 1 \text{ by orthonormality for } \mathrm{SO}(2n+1).
\end{aligned}$$

□

We would like to remind the reader at this point of a few facts from the representation theory of the symmetric group.

Lemma 2.4. *Let* sgn *be the sign character in* $\mathcal{S}_k$*.*

1. *If* $\beta \vdash k$*, then* $\chi_{\beta'} = \mathrm{sgn} \otimes \chi_\beta$*,*

2. *If* $\beta \vdash k$*, then*

$$p_\beta \otimes \mathrm{sgn} = \mathrm{sgn}(\beta) p_\beta$$

3. *Restrict* k *to be even. Then*

$$\sum_{\substack{\beta \text{ even} \\ \beta \vdash k}} \chi_\beta = \mathrm{Ind}_{\mathcal{B}_k}^{\mathcal{S}_k} 1,$$

where $\mathcal{B}_k$ *is the centralizer of the chosen permutation* $(1,2)\ (3,4) \cdots (k-1,k)$ *in* $\mathcal{S}_k$*.*

4. *Restrict* k *to be even. Then*

$$\mathrm{sgn} \otimes \mathrm{Ind}_{\mathcal{B}_k}^{\mathcal{S}_k} 1 = \mathrm{Ind}_{\mathcal{B}_k}^{\mathcal{S}_k} (\mathrm{Res}_{\mathcal{B}_k}^{\mathcal{S}_k} \mathrm{sgn}).$$

Proof. 1. This is in Bump's book [Bum04, Theorem 39.3].

2. This is immediate:

$$p_\lambda \otimes \mathrm{sgn} = \sum_{\mu \vdash k} \chi_\mu(\lambda)\chi_\mu \otimes \mathrm{sgn} = \sum_{\mu \vdash k} \chi_\mu(\lambda)\chi_{\mu'}$$

   and, by part 1,

$$= \sum_{\mu \vdash k} \chi_{\mu'}(\lambda)\chi_\mu = \sum_{\mu \vdash k} \chi_\mu(\lambda)\,\mathrm{sgn}(\lambda)\chi_\mu$$

3. See [Bum04, Theorem 45.4].

4. This is a consequence of Frobenius Reciprocity.

□

This lemma leads immediately to a second version of Proposition 2.3.

Proposition 2.5. *Let $\lambda \vdash k$ and $n \geq k$. Let $\epsilon = 1$ when $G = \mathrm{Sp}(2n)$ and $\epsilon = 0$ when $G = \mathrm{SO}(2n)$ or $\mathrm{SO}(2n+1)$. Then*

$$\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda = \left\langle \mathrm{Ind}_{\mathcal{S}_{|\gamma|} \times \mathcal{B}_{k-|\gamma|}}^{\mathcal{S}_k} \left(\chi_\gamma \otimes \mathrm{sgn}^\epsilon\right), p_\lambda \right\rangle_{\mathcal{S}_k},$$

where by a slight abuse of notation, we confuse sgn *and* $\mathrm{Res}_{\mathcal{B}_k}^{\mathcal{S}_k} \mathrm{sgn}$.

Proof. All the steps required are applications of Lemma 2.4 to the statement of Proposition 2.3.

$$\begin{aligned} \mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda &= \sum_{\substack{\beta \text{ even} \\ \gamma \cup \beta \vdash k}} \langle \chi_\gamma \odot (\mathrm{sgn}^\epsilon) \otimes \chi_\beta, p_\lambda \rangle_{\mathcal{S}_k} \\ &= \left\langle \chi_\gamma \odot \left(\mathrm{sgn}^\epsilon \otimes \mathrm{Ind}_{\mathcal{B}_{k-|\gamma|}}^{\mathcal{S}_{k-|\gamma|}} 1 \right), p_\lambda \right\rangle_{\mathcal{S}_k} \end{aligned}$$

We now apply Lemma 2.4.4 to get the result stated.

□

2.2.1 Discussion of Theorem 2.1

As a special case to Proposition 2.5, we are now ready to compute integrals of traces directly, without involving the Brauer algebra as in Ram [Ram97].

Proof of Theorem 2.1. We want here to compute $\mathbb{E}_G \mathbf{p}_\lambda$, so we are now in the simplest case of Proposition 2.5, when $|\gamma| = 0$. When k is odd, there is simply no matching on k points. On the other hand, it was a consequence of Proposition 2.3 that $\mathbb{E}_G \mathbf{p}_\lambda = 0$ as $k - |\gamma| = k$ is odd. We can thus restrict our attention to the k even case. We have thanks to Lemma 2.4 that

$$\begin{aligned}
\mathbb{E}_G \mathbf{p}_\lambda &= \left\langle \operatorname{Ind}_{\mathcal{B}_k}^{\mathcal{S}_k} 1, p_\lambda \otimes \operatorname{sgn}^\epsilon \right\rangle_{\mathcal{S}_k} \\
&= \operatorname{sgn}(\lambda)^\epsilon \left\langle 1, \operatorname{Res}_{\mathcal{B}_k}^{\mathcal{S}_k} p_\lambda \right\rangle_{\mathcal{B}_k} \\
&= \frac{z_\lambda \operatorname{sgn}(\lambda)^\epsilon}{|\mathcal{B}_k|} \# \left\{ \sigma \in C_{\mathcal{S}_k}((1,2)\cdots(k-1,k)) \mid \operatorname{type}(\sigma) = \lambda \right\},
\end{aligned}$$

since p_λ is an indicator function for the conjugacy class of permutations of type λ in $\mathcal{S}_k$.

If $\sigma \in C_{\mathcal{S}_k}((1,2)\cdots(k-1,k))$ then σ preserves the matching $\{\{1,2\}, \cdots \{k-1,k\}\}$, i.e. it sends a pair to a pair. We use this to switch to the language of matchings.

$$\begin{aligned}
\mathbb{E}_G \mathbf{p}_\lambda &= \frac{\operatorname{sgn}(\lambda)^\epsilon}{|\mathcal{C}_\lambda|} \frac{|\mathcal{S}_k|}{|\mathcal{B}_k|} \# \left\{ \sigma \in C_{\mathcal{S}_k}((1,2)(3,4)\cdots(k-1,k)) \cap \mathcal{C}_\lambda \right\} \\
&= \frac{\operatorname{sgn}(\lambda)^\epsilon}{|\mathcal{C}_\lambda|} \sum_{\text{matching } M \text{ of } k \text{ points}} \# \left\{ \sigma \in \mathcal{C}_\lambda \mid \sigma(M) = M \right\} \\
&= \frac{\operatorname{sgn}(\lambda)^\epsilon}{|\mathcal{C}_\lambda|} \#\{(M,\sigma) \mid M \text{ a matching of } k \text{ points}, \\
&\qquad\qquad \sigma \in \mathcal{C}_\lambda, \sigma(M) = M\} \\
&= \frac{\operatorname{sgn}(\lambda)^\epsilon}{|\mathcal{C}_\lambda|} \sum_{\sigma \in \mathcal{C}_\lambda} \# \left\{ \text{matchings preserved by } \sigma \right\}.
\end{aligned}$$

The last steps make use of a double-counting argument. All the summands in the last line are equal, and there are $|\mathcal{C}_\lambda|$ of them so we have

$$\mathbb{E}_G \mathbf{p}_\lambda = \mathrm{sgn}(\lambda)^\epsilon g(\lambda),$$

where $g(\lambda)$ is the number of matchings preserved by a permutation of cycle type λ. □

As mentioned earlier, this offers a combinatorial interpretation for a result first proved by Diaconis-Shahshahani [DS94]. Naturally, we have to check that our definition of g agrees with the definition they gave. This is a purely combinatorics problem.

Proposition 2.6. *Let $\lambda \vdash k$. Then $g(\lambda) = \prod_j g_j(\lambda(j))$, where $g_j(\cdot)$ is given by*

$$\begin{aligned} \text{if } j \text{ is odd} \quad g_j(a) &= \begin{cases} 0 & \text{if } a \text{ is odd} \\ j^{a/2}(a-1)(a-3)\cdots 1 & \text{if } a \text{ is even,} \end{cases} \\ \text{if } j \text{ is even} \quad g_j(a) &= \sum_t \binom{a}{2t} j^t (2t-1)(2t-3)\cdots 1 \end{aligned}$$

Proof. From our combinatorial definition of g, it is immediate that

$$g(\lambda) = \prod_j g((j^{\lambda(j)}))$$

. All we have left to prove is $g((j^a)) = g_j(a)$.

if j is odd: Take $\sigma \in \mathcal{C}_{(j^a)}$. Since each cycle of σ is of odd length, any matching of points preserved by σ must match cycles as well. If a is odd there is no such matching. If a is even, any matching of points will also match cycles. There are $(a-1)(a-3)\cdots 1$ possible matchings of cycles. Once a matching of cycles is chosen, we still have to decide on how to match points in each individual pair of cycles. There are j choices for each of the $a/2$ pairs of cycles.

if j is even: This is more subtle, as matchings of points inside the same cycle are allowed. Say there are $2t$ cycles whose points are matched with points in another cycle (the *external* cycles) and thus $a-2t$ cycles whose points are matched with a point within the same cycle (the *internal* cycles). There are $\binom{a}{2t}$ ways of choosing which cycles will be external, and then $(2t-1)(2t-3)\cdots 1$ ways of matching external cycles. Once we have a pair of external cycles, there are j ways of matching points between the two cycles. On the other hand, there is a unique way of matching points within an internal cycle: a point has to be paired with the point most distant for the ordering given by the cycle.

□

2.2.2 Discussion of Theorem 2.7

Let $\mathbb{T} = \{t \in \mathbb{C} \mid |t| = 1\}$, and let $\sigma(t) = \sum_{i\in\mathbb{Z}} d_i t^i = \exp\left(c_0 + \sum_{i\in\mathbb{Z}} \frac{c_i}{i} t^i\right) = e^{f(t)}$ be a function on $\mathbb{T}$.

We will always assume $f(t^{-1}) = f(t)$ (i.e. $c_i = c_{-i}$).

We also *define*[5] two extra conditions:

Condition (A)

$$\sum_{i>0} \frac{|c_i|}{i} < \infty$$

Condition (B)

$$\sum_{i>0} \frac{|c_i|^2}{i} < \infty$$

Those conditions were already relevant to the work of Bump and Diaconis [BD02], and the whole field of Toeplitz matrices[6].

We would like now to define a class function on G. Define $\{t_1, \cdots, t_n\}$ such that the eigenvalues of $g \in G$ are

1. $t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}$ when $G = \mathrm{Sp}(2n)$ or $G = \mathrm{SO}(2n)$,

[5] We will not assume those conditions by default Condition (B) will only appear much later.

[6] The book by Böttcher and Silbermann [BS99] gives a very clear introduction to the analytic theory of Toeplitz matrices. Theorem 5.2 in [BS99] uses those conditions. Sets of functions satisfying Conditions (A) and (B) are denoted $W(\mathbb{T})$ and $B_2^{1/2}(\mathbb{T})$ respectively.

2. $t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}, 1$ when $G = \mathrm{SO}(2n+1)$.

Then $\mathbf{\Phi}_{n,f}(g) = \prod_{k=1}^{n} \sigma(t_k)$ is a class function on G (the symmetry condition $f(t^{-1}) = f(t)$ guarantees that $\mathbf{\Phi}_{n,f}$ is independent of our choice of a subset $\{t_1, \cdots, t_n\}$ of the eigenvalues). We even have

$$\mathbf{\Phi}_{n,f}(g) = e^{nc_0} \exp\left(\sum_{i>0} \frac{c_i}{i} \mathbf{p}_{(i)}(g)\right).$$

Theorem 2.7. *Assume that f satisfies Condition (A). For simplicity of notation, take $\boldsymbol{\chi}_\gamma^G = \boldsymbol{\chi}_\gamma^{\mathrm{SO}(2n+1)}$ (resp. $\boldsymbol{\chi}_\gamma^{\mathrm{Sp}(2n)}$, $\boldsymbol{\chi}_\gamma^{\mathrm{SO}(2n)}$) if $G = \mathrm{SO}(2n+1)$ (resp. $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n)$). Then*

$$\lim_{n\to\infty} \frac{\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{\Phi}_{n,f}}{\mathbb{E}_G \mathbf{\Phi}_{n,f}} = R(\gamma, (c_i)),$$

where

$$R(\gamma, (c_i)) = \sum_{\lambda \vdash |\gamma|} \chi_\gamma(\lambda) \left(\prod_{i=1}^{\infty} \frac{c_i^{\lambda(i)}}{i^{\lambda(i)} \lambda(i)!}\right).$$

We delay comments on this Theorem to page 29 and start with the proof.

Proof. As a first approximation to $\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{\Phi}_{n,f}$, we will actually study $\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda$ for $\lambda \vdash k \le n$. It will be useful to split up λ into subpartitions. To avoid confusion with notation previously used for partition parts $(\lambda_1, \lambda_2, \cdots, \lambda_n)$, we will use $\lambda_a \cup \lambda_b = \lambda$ in this proof only.

We start from the final equation in Proposition 2.5 and apply Frobenius Reciprocity to get

$$\begin{aligned}
\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda &= \left\langle \chi_\gamma \otimes \mathrm{Res}_{\mathcal{B}_{k-|\gamma|}}^{\mathcal{S}_{k-|\gamma|}} \mathrm{sgn}^\epsilon, \mathrm{Res}_{\mathcal{S}_{|\gamma|}\times\mathcal{B}_{k-|\gamma|}}^{\mathcal{S}_k} p_\lambda \right\rangle_{\mathcal{S}_{|\gamma|}\times\mathcal{B}_{k-|\gamma|}} \\
&= \frac{z_\lambda}{|\mathcal{S}_{|\gamma|}||\mathcal{B}_{k-|\gamma|}|} \sum_{\substack{(\rho_a,\rho_b)\in\mathcal{S}_{|\gamma|}\times\mathcal{B}_{k-|\gamma|} \\ \mathrm{type}(\rho_a)=\lambda_a\vdash|\gamma| \\ \mathrm{type}(\rho_b)=\lambda_b\vdash k-|\gamma| \\ \lambda_a\cup\lambda_b=\lambda}} \chi_\gamma(\rho_a)\,\mathrm{sgn}^\epsilon(\rho_b),
\end{aligned}$$

where $\epsilon = 1$ when $G = \mathrm{Sp}(2n)$ and 0 otherwise. We now sum over conjugacy classes (i.e. cycle types) instead. The correction factor for the ρ_a's of type λ_a will be $\frac{|\mathcal{S}_{|\gamma|}|}{z_{\lambda_a}} = |\mathcal{C}_{\lambda_a}|$, so

$$\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda \;=\; \frac{z_\lambda}{|\mathcal{B}_{k-|\gamma|}|} \sum_{\substack{\lambda_a \vdash |\gamma| \\ \lambda_a \cup \lambda_b = \lambda}} \frac{\chi_\gamma(\lambda_a)\operatorname{sgn}(\lambda_b)^\epsilon}{z_{\lambda_a}} |\mathcal{B}_{k-|\gamma|} \cap \mathcal{C}_{\lambda_b}|.$$

Observe from the proof of Theorem 2.1, with λ replaced by λ_b, that

$$\mathbb{E}_G \mathbf{p}_{\lambda_b} = \frac{z_{\lambda_b}\operatorname{sgn}(\lambda_b)^\epsilon}{|\mathcal{B}_{k-|\gamma|}|} |\mathcal{B}_{k-|\gamma|} \cap \mathcal{C}_{\lambda_b}|.$$

The hypothesis $n \geq |\lambda_b|$ of Theorem 2.1 is automatically satisfied since we already assume $n \geq |\lambda|$ and $\lambda = \lambda_a \cup \lambda_b$.

We now have the much simpler

$$\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda = \sum_{\substack{\lambda_a \vdash |\gamma| \\ \lambda_a \cup \lambda_b = \lambda}} \frac{z_\lambda}{z_{\lambda_a} z_{\lambda_b}} \chi_\gamma(\lambda_a) \mathbb{E}_G \mathbf{p}_{\lambda_b}$$

or even

$$\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda = \sum_{\substack{\lambda_a \vdash |\gamma| \\ \lambda_a \cup \lambda_b = \lambda}} \frac{\lambda!}{\lambda_a! \lambda_b!} \chi_\gamma(\lambda_a) \mathbb{E}_G \mathbf{p}_{\lambda_b} \tag{2.1}$$

where $\lambda! = \prod_{i \geq 1} (\lambda(i)!)$.

We can now deal with $\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f}$. As in *Toeplitz minors* [BD02], absolute convergence is guaranteed by Condition (A), the bound $|\operatorname{tr}(g^i)| \leq m$ when $g \in \mathrm{U}(m), \mathrm{SO}(m)$ or $\mathrm{Sp}(m)$ and compactness of those groups:

$$\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f} \;\leq\; e^{nc_0} \int_G \max_{g \in G} (|\boldsymbol{\chi}_\gamma^G|) \exp\left(\sum_{i>0} \frac{|c_i|}{i} |\operatorname{tr}(g^i)| \right).$$

We are thus allowed to permute sums and products in the full expansion of $\mathbf{\Phi}_{n,f}$:

$$\begin{aligned}
\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{\Phi}_{n,f} &= e^{nc_0} \mathbb{E}_G \boldsymbol{\chi}_\gamma^G \exp\left(\sum_{i>0} \frac{c_i}{i} \mathbf{p}_{(i)} \right) \\
&= e^{nc_0} \mathbb{E}_G \boldsymbol{\chi}_\gamma^G \prod_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{(c_i \mathbf{p}_{(i)})^j}{i^j j!} \\
&= e^{nc_0} \mathbb{E}_G \boldsymbol{\chi}_\gamma^G \sum_{(\alpha_i)} \prod_{i=1}^{\infty} \frac{(c_i \mathbf{p}_{(i)})^{\alpha_i}}{i^{\alpha_i} \alpha_i!} \\
&= e^{nc_0} \mathbb{E}_G \boldsymbol{\chi}_\gamma^G \sum_{(\alpha_i)} \prod_{i=1}^{\infty} \frac{c_i^{\alpha_i}}{i^{\alpha_i} \alpha_i!} \mathbf{p}_{(i^{\alpha_i})} \\
&= e^{nc_0} \sum_{\substack{(\alpha_i) \\ \lambda := (i^{\alpha_i})}} \left(\prod_{i=1}^{\infty} \frac{c_i^{\alpha_i}}{i^{\alpha_i} \alpha_i!} \right) \mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda,
\end{aligned}$$

From this definition of λ, we observe that $\lambda(j) = \alpha_j$, which explains the notation: $\alpha_j <> \lambda_j$ in general.

Once $n \geq |\lambda|$, we are allowed to substitute for every term $\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda$ the r.h.s. of Equation (2.1). For a given n, this only applies for the terms at the head of the series, but any term in the series will eventually be substituted, when $n \geq |\lambda|$. Combined with absolute convergence, this guarantees the asymptotics

$$\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{\Phi}_{n,f} \overset{n\to\infty}{\sim} e^{nc_0} \sum_{(\alpha_i)} \left(\left(\prod_{i=1}^{\infty} \frac{c_i^{\alpha_i}}{i^{\alpha_i} \alpha_i!} \right) \sum_{\substack{\lambda_a \vdash |\gamma| \\ \lambda_a \cup \lambda_b = (i^{\alpha_i}) =: \lambda}} \frac{\lambda!}{\lambda_a! \lambda_b!} \chi_\gamma(\lambda_a) \mathbb{E}_G \mathbf{p}_{\lambda_b} \right).$$

We now switch the sums, and change the index of one sum from (α_i) with $(i^{\alpha_i}) = \lambda$ to (β_i) with $(i^{\beta_i}) = \lambda_b$. This implies $\lambda_a(j) + \beta_j = \lambda(j) = \alpha_j$. We get

$$\begin{aligned}
\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{\Phi}_{n,f} &\overset{n\to\infty}{\sim} e^{nc_0} \sum_{\lambda_a \vdash |\gamma|} \left(\left(\frac{\chi_\gamma(\lambda_a)}{\lambda_a!} \prod_{i=1}^{\infty} \frac{c_i^{\lambda_a(i)}}{i^{\lambda_a(i)}} \right) \sum_{(\beta_i)} \left(\prod_{i=1}^{\infty} \frac{c_i^{\beta_i}}{\beta_i!} \right) \mathbb{E}_G \mathbf{p}_{(i^{\beta_i})} \right) \\
&= R(\gamma, (c_i)) \mathbb{E}_G \mathbf{\Phi}_{n,f},
\end{aligned}$$

and finally

$$\lim_{n\to\infty} \frac{\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f}}{\mathbb{E}_G \boldsymbol{\Phi}_{n,f}} = R(\gamma, (c_i)) = \sum_{\lambda \vdash |\gamma|} \chi_\gamma(\lambda) \left(\prod_{i=1}^{\infty} \frac{c_i^{\lambda(i)}}{i^{\lambda(i)} \lambda(i)!} \right).$$

□

Remarks.

- As mentioned earlier, this ratio $R(\gamma, (c_i))$ already appears in Theorem 6 of Bump and Diaconis [BD02], when $G = \mathrm{U}(n)$. It is striking that this ratio is independent of the Cartan type of G.

- Even more striking is that

$$s_\gamma(x_1, \cdots) = \sum_{\lambda \vdash |\gamma|} \chi_\gamma(\lambda) \left(\prod_{i=1}^{\infty} \frac{p_i(x_1, \cdots)^{\lambda(i)}}{i^{\lambda(i)} \lambda(i)!} \right) = R(\gamma, p_i(x_1, \cdots))$$

  after the trivial extension of the definition of R to $\mathbb{C}[x_1, \cdots, x_m]^{\mathfrak{S}_m}$.

  In other words, the value of the RHS in Theorem 2.7 is merely the Schur polynomial s_γ with the power polynomials specialized to the c_i's.

  This will be explained in more details in Chapter 4, and more particularly Section 4.1.1.

- The authors went a bit further in [BD02] and modified the integrand using two characters (one of them appeared conjugated). There is no real need to do this here, as the characters $\boldsymbol{\chi}_\lambda^G$ are real in the non-unitary cases, and we would just end up with a product of two characters. Koike and Terada [KT87, Corollary 2.5.3] have shown that the multiplication rules are also essentially[7] independent of the Cartan type of G, i.e. that

$$\boldsymbol{\chi}_\mu^G \cdot \boldsymbol{\chi}_\nu^G = \sum_\lambda c_{\mu\nu}^\lambda \boldsymbol{\chi}_\lambda^G.$$

[7] This is only valid for $n \geq l(\mu) + l(\nu)$, and the case $G = \mathrm{SO}(2n)$ is slightly different.

This can be combined with Theorem 2.7 to show that there will also be an asymptotic ratio for $\frac{\mathbb{E}_G \boldsymbol{\chi}_\mu^G \boldsymbol{\chi}_\nu^G \boldsymbol{\Phi}_{n,f}}{\mathbb{E}_G \boldsymbol{\Phi}_{n,f}}$, independent of the Cartan type of G.

- Johansson [Joh97, Theorem 3.8.i with $\eta = i$] was the first to generalize the strong Szegö limit theorem to all the classical groups. He found asymptotics for $\mathbb{E}_G \boldsymbol{\Phi}_{n,f}$ as $n \to \infty$. Bump and Diaconis [BD] later found a new proof of Johansson's result that actually inspired our own work and an extension of this result. We state here a weaker version of Johansson's result in a style closer to our own. Note that this is the first time we need Condition (B).

Theorem 2.8 (Johansson [Joh97], Bump and Diaconis [BD])**.** *Let $f(t) = \sum_i \frac{c_i}{i} t^i$ satisfy Conditions (A) and (B) in addition to the usual symmetry condition $f(t) = f(t^{-1})$. Then*

$$\begin{aligned}
\mathbb{E}_{\mathrm{SO}(2n+1)} \boldsymbol{\Phi}_{n,f} &= \exp\left(\sum_{i=1}^{\infty} \frac{c_i^2}{2i} - \sum_{i=1}^{\infty} \frac{c_{2i-1}}{2i-1} + o(1) \right) \\
\mathbb{E}_{\mathrm{Sp}(2n)} \boldsymbol{\Phi}_{n,f} &= \exp\left(\sum_{i=1}^{\infty} \frac{c_i^2}{2i} - \sum_{i=1}^{\infty} \frac{c_{2i}}{2i} + o(1) \right) \\
\mathbb{E}_{\mathrm{SO}(2n)} \boldsymbol{\Phi}_{n,f} &= \exp\left(\sum_{i=1}^{\infty} \frac{c_i^2}{2i} + \sum_{i=1}^{\infty} \frac{c_{2i}}{2i} + o(1) \right)
\end{aligned}$$

We can thus combine Theorems 2.7 and 2.8 to get the asymptotics for $\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f}$, i.e. for the Haar measure twisted by a character of type $\boldsymbol{\chi}_\lambda^G$.

Chapter 3

On an identity due to (Bump and Diaconis) and (Tracy and Widom)

3.1 Introduction

3.1.1 Origin: Toeplitz determinants

Fix $\sigma(t)$ to be a function of the unit circle $\mathbb{T}$ in $\mathbb{C}$ that can be written in the form

$$\sigma(t) = \exp\left(\sum_{k>0} \frac{p_k}{k} t^k + \frac{\tilde{p}_k}{k} t^{-k}\right)$$

for the sets of constants $\{p_k \in \mathbb{C}\}$ and $\{\tilde{p}_k \in \mathbb{C}\}$[1]. This requires σ to have winding number 0 around the origin (since $\log \sigma(t)$ is defined, see [BS99, pp. 15–17] for more details). This also defines a set of constants $\{d_k\}$ so that $\sum_{k\in\mathbb{Z}} d_k t^k := \sigma(t)$ (i.e. the d_k's are the Fourier coefficients of $\sigma(t)$). We will further assume that the $|p_k|$'s and $|\tilde{p}_k|$'s decrease fast enough, i.e. that all of the sums $\sum_k \frac{|p_k|}{k}$, $\sum_k \frac{|\tilde{p}_k|}{k}$ and $\sum_k \frac{|p_k \tilde{p}_k|}{k}$ are bounded.

We now construct a matrix M_n having constant entries on diagonals parallel

[1] This implies that $\sigma(0) = 1$, a condition that is merely there for exposition.

to the main diagonal (Toeplitz property with *symbol* σ):

$$M_n(\sigma) = M_n = \begin{pmatrix} d_0 & d_1 & \cdots & \cdots & d_{n-1} \\ d_{-1} & d_0 & d_1 & \cdots & d_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & d_1 \\ d_{1-n} & \cdots & \cdots & d_{-1} & d_0 \end{pmatrix}_{n\times n} = (d_{i-j})_{n\times n}$$

A classical question for Toeplitz matrices is then to consider the asymptotics of the determinant $\det(M_n)$ as n goes to infinity. Bump and Diaconis [BD02] looked at asymptotics of minors of M_n, i.e. (due to the Toeplitz property) matrices of the form

$$M_n^{\lambda\mu} = \left(d_{\lambda_i-\mu_j-i+j}\right)_{n\times n},$$

where λ and μ are partitions. They proved that

$$\lim_{n\to\infty} \frac{\det(M_n^{\lambda\mu})}{\det(M_n)} = \mathrm{R}^{\lambda\mu} \in \mathbb{C}$$

when all of the sums $\sum_k \frac{|p_k|}{k}$, $\sum_k \frac{|\tilde{p}_k|}{k}$ and $\sum_k \frac{|p_k\tilde{p}_k|}{k}$ are bounded. We will assume those conditions for the rest of this Chapter.

Bump and Diaconis not only proved that this limit exists, they also found an expression for this constant $\mathrm{R}^{\lambda\mu}$ as a sum over symmetric groups involving Laguerre polynomials. We will denote this expression $\mathrm{BD}^{\lambda\mu}$ (see Section 3.3).

Independently, Tracy and Widom [TW02] obtained other expressions $\mathrm{TW}^{\lambda\mu}$ (see Section 3.4) for those ratios as determinants involving the Wiener-Hopf factorization of $\sigma(t)$, i.e.

$$\begin{aligned} \sigma(t) &= \exp\left(\sum_{k>0} \frac{p_k}{k} t^k\right) \cdot \exp\left(\sum_{k>0} \frac{\tilde{p}_k}{k} t^{-k}\right) \\ &=: \sum_{k\geq 0} h_k t^k \cdot \sum_{k\geq 0} \tilde{h}_k t^{-k}, \end{aligned} \tag{3.1}$$

where the last equality defines the h_k's and $\tilde{h}_k$'s.

Tracy and Widom's theorems are valid under slightly more general conditions than Bump and Diaconis'. Lyons [Lyo03] discusses this point in detail.

Joining results on $\mathrm{BD}^{\lambda\mu}$ and $\mathrm{TW}^{\lambda\mu}$ together, we have the main identity

Theorem 3.1. *[BD02, TW02] Let λ, μ be partitions. Then, if $\sum_k |p_k|, \sum_k |\tilde{p}_k|$ and $\sum_k k|p_k\tilde{p}_k|$ are bounded, we have*

$$\mathrm{BD}^{\lambda\mu} \left(= \mathrm{R}^{\lambda\mu}\right) = \mathrm{TW}^{\lambda\mu}.$$

3.1.2 Results

If one forgets its origins, Theorem 3.1 is a mysterious combinatorial identity $\mathrm{BD}^{\lambda\mu} = \mathrm{TW}^{\lambda\mu}$. Our main goal for this Chapter will be to prove this identity more directly, without relying on Toeplitz determinants. We will show how this identity is just a differentiated version of the Jacobi-Trudi identity. Along the way, we will prove various other results on the combinatorial representations $\mathrm{TW}^{\lambda\mu}$ and $\mathrm{BD}^{\lambda\mu}$ (and thus also on those ratios $\mathrm{R}^{\lambda\mu}$).

In Section 3.2, we will review the notions of symmetric function theory which we need. We first define classical symmetric functions, and then introduce a linear operator $\boldsymbol{\Delta}$ acting on symmetric functions. Three equivalent formulas will be given for $\boldsymbol{\Delta}$.

We will also define what we mean by "specialization" from an algebra of symmetric functions to $\mathbb{C}$.

This will implicitly define a differential operator

$$\Delta := \exp\left(\sum_{k>0} k\partial_{p_k}\partial_{\tilde{p}_k}\right)$$

acting on functions of the variables p_k and $\tilde{p}_k$ as the image of $\boldsymbol{\Delta}$ under the specialization.

Our notation will be similarly consistent throughout this Chapter:

"object" is "symmetric" and specializes to "object", i.e. a boldface **"object"** (is a)/(acts on) symmetric function(s), and specializes to "object" which is of similar nature but without the symmetry component.

In Section 3.3, we will first define the expressions $\mathrm{BD}^{\lambda\,\mu}$ and $\mathbf{BD}^{\lambda\,\mu}$ and then prove

Theorem 3.2.

$$\Delta\left(\mathrm{BD}^{\lambda\,\varnothing}\cdot\mathrm{BD}^{\varnothing\,\mu}\right)=\mathrm{BD}^{\lambda\,\mu}\quad\textit{and}\quad\boldsymbol{\Delta}\left(\mathbf{BD}^{\lambda\,\varnothing}\cdot\mathbf{BD}^{\varnothing\,\mu}\right)=\mathbf{BD}^{\lambda\,\mu}.$$

Section 3.4 will proceed similarly for the expressions $\mathrm{TW}^{\lambda\,\mu}$ and $\mathbf{TW}^{\lambda\,\mu}$ and prove

Theorem 3.3.

$$\Delta\left(\mathrm{TW}^{\lambda\,\varnothing}\cdot\mathrm{TW}^{\varnothing\,\mu}\right)=\mathrm{TW}^{\lambda\,\mu}\quad\textit{and}\quad\boldsymbol{\Delta}\left(\mathbf{TW}^{\lambda\,\varnothing}\cdot\mathbf{TW}^{\varnothing\,\mu}\right)=\mathbf{TW}^{\lambda\,\mu}.$$

Looking back to ratios of Toeplitz determinants, Theorem 3.1 along with Theorem 3.2 or 3.3 immediately prove the following corollary:

Corollary 3.4.

$$\Delta\left(\mathrm{R}^{\lambda\,\varnothing}\cdot\mathrm{R}^{\varnothing\,\mu}\right)=\mathrm{R}^{\lambda\,\mu}.$$

Section 3.5 will be devoted to the proof of Theorem 3.1.

Finally, we give in Section 3.6 a couple of noteworthy relations on the $\mathrm{R}^{\lambda\,\mu}$'s.

3.2 General definitions and notations

We summarize the definitions and notations employed in this Chapter.

3.2.1 Partitions and symmetric groups

A partition $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)$ is a finite decreasing sequence of non-negative integers. We define the weight $|\lambda|$ of λ to be the sum $\sum\lambda_i$. If this weight is k, we also use the notation $\lambda\vdash k$. If $k=0$, we denote the trivial partition $(0,0,0,0,\cdots)$ by $\varnothing$. The length $l(\lambda)$ of λ is the maximal i such that $\lambda_i\neq 0$.

There is a partial ordering on partitions: $\lambda \subseteq \mu$ iff $\lambda_i \leq \mu_i$ for all i. In a probable break of standard notation, $\lambda(i)$ counts the number of λ_j's equal to i, so that $(i^{\lambda(i)}) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$. In an even greater offense, if π is a permutation, we will use $\pi(i)$ for the number of elements of i's in the cycle type of π, **not** for the image of point i under π (this usage is limited to Section 3.3, with no risk of notational confusion in the whole Chapter).

As usual, partitions of fixed weight k index conjugacy classes in the symmetric group on k points $\mathcal{S}_k$. We set $z_\lambda := \prod_i i^{\lambda(i)} i!$. This is the order of the centralizer of a permutation in $\mathcal{S}_{|\lambda|}$ of cycle-type λ.

In order to present the formula of Bump and Diaconis, we will also need the irreducible characters of the symmetric groups. For a fixed k, all irreducible representations of $\mathcal{S}_k$ are indexed by partitions of weight k (see the book by Sagan [Sag91] for a friendly introduction). If $\lambda \vdash k$, we will use χ^λ for the character of the representation corresponding to λ.

3.2.2 Symmetric functions

We now introduce a few functions in the graded rings $\mathbf{\Lambda}(\overline{X})$ and $\mathbf{\Lambda}(\overline{Y})$ of symmetric functions in countably many independent variables $\overline{X} := \{x_1, x_2, x_3, \cdots\}$ and $\overline{Y} := \{y_1, y_2, y_3, \cdots\}$ over $\mathbb{Q}$. The former can be most directly thought of as the ring of formal sums $S(x_1, \cdots)$ of monomials in the variables x_i that have the symmetry property $S(x_{\rho(1)}, x_{\rho(2)}, \cdots) = S(x_1, x_2, \cdots)$ for all $\rho \in \mathcal{S}_\infty$. The most classic reference on the topic is Macdonald's book [Mac95, Sections 1.2-1.5].

We will use the notation $\mathbf{p}_\lambda$, $\mathbf{h}_\lambda$, $\mathbf{s}_\lambda$ and $\mathbf{s}_{\lambda/\mu}$ for the various interesting functions living in $\mathbf{\Lambda}(\overline{X})$. They will be respectively the power sum, complete, Schur and skew Schur functions in the variables $\{x_i\}$ associated to the partition λ (to the skew partition λ/μ for the latter). Similarly, we use $\tilde{\mathbf{p}}_\lambda$, $\tilde{\mathbf{h}}_\lambda$, $\tilde{\mathbf{s}}_\lambda$ and $\tilde{\mathbf{s}}_{\lambda/\mu}$ for the same functions in $\mathbf{\Lambda}(\overline{Y})$. In general, the boldface font will be used for functions in $\mathbf{\Lambda}(\cdot)$. A tilde indicates the variable set $\overline{Y}$, while the default (when there is no tilde) is to assume that the variable set is $\overline{X}$.

One can define an inner product on $\mathbf{\Lambda}(\cdot)$ by setting the Schur polynomials to be orthonormal: $\langle \mathbf{s}_\lambda, \mathbf{s}_\mu \rangle_{\mathbf{\Lambda}(\overline{X})} = \delta_{\lambda\mu}$. The $\langle \cdot, \cdot \rangle_{\mathbf{\Lambda}(\overline{X})}$ indicates that this inner product is for $\mathbf{\Lambda}(\overline{X})$. We will need the fact that the $\mathbf{p}_\lambda$'s form an orthogonal base: $\langle \mathbf{p}_\lambda, \mathbf{p}_\mu \rangle_{\mathbf{\Lambda}(\overline{X})} = z_\lambda \delta_{\lambda\mu}$.

We will also need to consider the ring of symmetric functions in two sets of variables

$$\mathbf{\Lambda}(\overline{X}, \overline{Y}) = \mathbf{\Lambda}(\overline{X}) \otimes \mathbf{\Lambda}(\overline{Y}).$$

This comes equipped with an induced inner product defined by extending linearly

$$\left\langle \mathbf{a} \cdot \tilde{\mathbf{a}}, \mathbf{b} \cdot \tilde{\mathbf{b}} \right\rangle_{\mathbf{\Lambda}(\overline{X}, \overline{Y})} = \left\langle \mathbf{a}, \mathbf{b} \right\rangle_{\mathbf{\Lambda}(\overline{X})} \cdot \left\langle \tilde{\mathbf{a}}, \tilde{\mathbf{b}} \right\rangle_{\mathbf{\Lambda}(\overline{Y})}.$$

3.2.3 The derivations $\mathbf{p}_n^\perp$ and $\tilde{\mathbf{p}}_n^\perp$

Let us first consider just the set of variables $\overline{X}$.

Following Macdonald [Mac95, Example 3, Section 1.5, page 75], we define the ring homomorphism $^\perp : \mathbf{\Lambda}(\overline{X}) \longrightarrow \mathrm{End}(\mathbf{\Lambda}(\overline{X}))$ in such a way that

$$\left\langle \mathbf{f}^\perp \mathbf{u}, \mathbf{v} \right\rangle_{\mathbf{\Lambda}(\overline{X})} = \langle \mathbf{u}, \mathbf{f}\mathbf{v} \rangle_{\mathbf{\Lambda}(\overline{X})}$$

for all $u, v \in \mathbf{\Lambda}(\overline{X})$. This is the adjoint of multiplication in the ring $\mathbf{\Lambda}(\overline{X})$.

Macdonald (following Foulkes) shows that $\mathbf{p}_n^\perp = n\partial_{\mathbf{p}_n}$ and so that $\mathbf{p}_n^\perp$ is a derivation. Indeed, we have

$$\begin{aligned}
\left\langle \mathbf{p}_n^\perp(\mathbf{p}_\lambda), \mathbf{p}_\mu \right\rangle_{\mathbf{\Lambda}(\overline{X})} &= \langle \mathbf{p}_\lambda, \mathbf{p}_\mu \mathbf{p}_n \rangle_{\mathbf{\Lambda}(\overline{X})} \\
&= \begin{cases} 0 & \text{if } \lambda \neq (n) \cup \mu \\ z_\lambda & \text{if } \lambda = (n) \cup \mu \end{cases} \\
&= \begin{cases} 0 & \text{if } \mu \neq \lambda \setminus (n) \\ z_\lambda & \text{if } \mu \neq \lambda \setminus (n) \end{cases} \\
&= \left\langle z_\lambda z_\mu^{-1} \mathbf{p}_{\lambda \setminus (n)}, \mathbf{p}_\mu \right\rangle_{\mathbf{\Lambda}(\overline{X})}.
\end{aligned}$$

But $z_\lambda z_{\lambda\setminus(n)}^{-1} = n\lambda(n)$, so $\mathbf{p}_n^\perp(\mathbf{p}_\lambda) = n\partial_{\mathbf{p}_n}(\mathbf{p}_\lambda)$ and we get our claim that $\mathbf{p}_n^\perp = n\partial_{\mathbf{p}_n}$.

A similar result is of course true for $\mathbf{\Lambda}(\overline{Y})$ (for the adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathbf{\Lambda}(\overline{Y})}$).

Observe that

$$(\mathbf{a} \cdot \tilde{\mathbf{a}})^{\perp} = \mathbf{a}^{\perp} \otimes \tilde{\mathbf{a}}^{\perp},$$

and so $^{\perp}$ is a homomorphism $\mathbf{\Lambda}(\overline{X}, \overline{Y}) \longrightarrow \mathrm{End}(\mathbf{\Lambda}(\overline{X}, \overline{Y}))$.

3.2.4 Specializing symmetric objects

Again, we first consider only the variables $\overline{X}$.

For a given $\sigma(t) = \exp\left(\sum_{k>0} \frac{p_k}{k} t^k + \frac{\tilde{p}_k}{k} t^{-k}\right)$ We define an algebra homomorphism

$$\begin{array}{rcl} H_{\overline{X}} : \mathbf{\Lambda}(\overline{X}) & \longrightarrow & \mathbb{C} \\ \mathbf{p}_k & \longmapsto & p_k. \end{array}$$

Following on Section 3.2.3, $H_{\overline{X}}$ actually also preserves derivations, so $H_{\overline{X}}(n\partial_{\mathbf{p}_k}) = H_{\overline{X}}(\mathbf{p}_k^{\perp}) = k\partial_{p_k}$.

We also observe that the generating function for the h_k is the same as the generating function for the $\mathbf{h}_n$, i.e. compare Equation 3.1 with the generating function identity

$$\exp\left(\sum_{k>0} \frac{\mathbf{p}_k}{k} t^k\right) = \sum_{k \geq 0} \mathbf{h}_k t^k.$$

This very classical identity (Newton's identity describing the roots of a polynomial) was already discussed in the context of Pólya's enumeration theory in the paper by Bump and Diaconis [BD02]. In any case, this guarantees that

$$H_{\overline{X}}(\mathbf{h}_k) = h_k.$$

Of course, a similar specialization $H_{\overline{Y}} : \mathbf{\Lambda}(\overline{Y}) \longrightarrow \mathbb{C}$ exists, and induces a specialization $H : \mathbf{\Lambda}(\overline{X}, \overline{Y}) \longrightarrow \mathbb{C}$.

The concept of "specialization" as applied here will be discussed in more details in Chapter 4.

3.2.5 Differential operators Δ and $\boldsymbol{\Delta}$

Consider the vector space V of all the polynomials in the variables $p_k, \tilde{p}_k$. We can define a (generalized) differential operator Δ as

$$\Delta = \exp\left(\sum_k k\partial_{p_k}\partial_{\tilde{p}_k}\right) = \prod_{k>0}\sum_{i\geq 0}\frac{k^i}{i!}(\partial_{p_k}\partial_{\tilde{p}_k})^i,$$

where $(\partial_{p_k}\partial_{\tilde{p}_k})^i$ is composition. Note that sums and products will be finite for any element of V but that the order of Δ is not uniformly bounded on V.

We define the operator $\boldsymbol{\Delta}$ on $\boldsymbol{\Lambda}(\overline{X}, \overline{Y})$ in the same way, which implies that they commute:

$$H \circ \boldsymbol{\Delta} = \Delta \circ H.$$

It follows from the previous sections that

$$\boldsymbol{\Delta} = \exp\left(\sum_k \frac{\mathbf{p}_k^{\perp}\tilde{\mathbf{p}}_k^{\perp}}{k}\right).$$

3.3 The Bump-Diaconis side

Assume $\lambda \vdash m$ and $\mu \vdash p$. Then Bump and Diaconis define for $\sigma(t) = \exp\left(\sum_{k>0}\frac{p_k}{k}t^k\right)\cdot \exp\left(\sum_{k>0}\frac{\tilde{p}_k}{k}t^{-k}\right)$ the expression

$$\mathrm{BD}^{\lambda\mu} = \frac{1}{m!}\sum_{\pi\in S_m}\frac{1}{p!}\sum_{\rho\in S_p}\chi^{\lambda}(\pi)\chi^{\mu}(\rho)\prod_{k>0}\mathrm{F}_k(\pi,\rho),$$

with

$$\mathrm{F}_k(\pi,\rho) = \begin{cases} k^{\rho(k)}\rho(k)!L_{\rho(k)}^{(\pi(k)-\rho(k))}\left(-\frac{p_k\tilde{p}_k}{k}\right)p_k^{\pi(k)-\rho(k)} & \text{if } \pi(k)\geq\rho(k)^2 \\ k^{\pi(k)}\pi(k)!L_{\pi(k)}^{(\rho(k)-\pi(k))}\left(-\frac{p_k\tilde{p}_k}{k}\right)\tilde{p}_k^{\rho(k)-\pi(k)} & \text{if } \rho(k)\geq\pi(k) \end{cases}$$

[2]We remind the reader of our unconventional usage of $\pi(k)$ for the number of $k-$cycles in π.

and where

$$L_n^{(\alpha)}(t) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-t)^k}{k!} = \sum_{k=0}^{n} \binom{n+\alpha}{k} \frac{(-t)^{n-k}}{(n-k)!}$$

is the usual Laguerre polynomial (the former expression is the standard definition, while the latter formula is only a reindexing of it that will be more useful here).

We define similarly $\mathbf{BD}^{\lambda\,\mu}(\sigma)$ and $\mathbf{F}_k(\pi,\rho)$.

Lemma 3.5. *Let* $\max_k = \max(\pi(k),\rho(k))$ *and* $\min_k = \min(\pi(k),\rho(k))$*. Then,*

$$\mathbf{F}_k(\pi,\rho) = \sum_{i=0}^{\min_k} k^i i! \binom{\max_k}{i} \binom{\min_k}{i} \mathbf{p}_k^{\pi(k)-i} \tilde{\mathbf{p}}_k^{\rho(k)-i}.$$

Proof. We just need to expand the Laguerre polynomial in the definition of $\mathbf{F}_k$ while keeping track of the degrees in $\mathbf{p}_k$ and $\tilde{\mathbf{p}}_k$. The key is to observe that all the monomials will have the correct degrees, i.e. will be $\mathbf{p}_k^{\pi(k)-i}\tilde{\mathbf{p}}_k^{\rho(k)-i}$ for $0 \le i \le \min(\rho(k),\pi(k))$. □

Proof of Theorem 3.2. When one of the partitions is trivial, the $\mathbf{BD}^{\lambda\,\mu}$ reduce[3] to

$$\mathbf{BD}^{\lambda\,\varnothing} = \frac{1}{m!}\sum_{\pi\in\mathfrak{S}_m} \chi^\lambda(\pi)\mathbf{p}_\pi \quad \text{and} \quad \mathbf{BD}^{\varnothing\,\mu} = \frac{1}{p!}\sum_{\rho\in\mathfrak{S}_p} \chi^\mu(\rho)\tilde{\mathbf{p}}_\rho$$

We thus need to evaluate

$$\begin{aligned} \boldsymbol{\Delta}\left(\mathbf{BD}^{\lambda\,\varnothing}\cdot\mathbf{BD}^{\varnothing\,\mu}\right) &= \boldsymbol{\Delta}\left(\frac{1}{m!}\sum_{\pi\in\mathfrak{S}_m}\chi^\lambda(\pi)\mathbf{p}_\pi \cdot \frac{1}{p!}\sum_{\rho\in\mathfrak{S}_p}\chi^\mu(\rho)\tilde{\mathbf{p}}_\rho\right) \\ &= \frac{1}{m!}\sum_{\pi\in\mathfrak{S}_m}\frac{1}{p!}\sum_{\rho\in\mathfrak{S}_p}\chi^\lambda(\pi)\chi^\mu(\rho)\Delta\left(\mathbf{p}_\pi\tilde{\mathbf{p}}_\rho\right). \end{aligned}$$

[3]We use here permutations as index for the power sums functions. We mean by $\mathbf{p}_\pi$ the function $\mathbf{p}_\lambda$, where λ is the cycle-type of π.

Each term is of the form

$$\mathbf{\Delta}\left(\mathbf{p}_\pi \tilde{\mathbf{p}}_\rho\right) = \left[\prod_{k>0} e^{k\partial_{\mathbf{p}_k}\partial_{\tilde{\mathbf{p}}_k}}\right]\left(\prod_{k>0} \mathbf{p}_k^{\pi(k)} \tilde{\mathbf{p}}_k^{\rho(k)}\right) = \prod_{k>0}\left[e^{k\partial_{\mathbf{p}_k}\partial_{\tilde{\mathbf{p}}_k}}\left(\mathbf{p}_k^{\pi(k)} \tilde{\mathbf{p}}_k^{\rho(k)}\right)\right],$$

where

$$\begin{aligned}
\left[e^{k\partial_{\mathbf{p}_k}\partial_{\tilde{\mathbf{p}}_k}}\right]\left(\mathbf{p}_k^{\pi(k)} \tilde{\mathbf{p}}_k^{\rho(k)}\right) &= \sum_{i\geq 0} \frac{(k\partial_{\mathbf{p}_k}\partial_{\tilde{\mathbf{p}}_k})^i}{i!}\left(\mathbf{p}_k^{\pi(k)} \tilde{\mathbf{p}}_k^{\rho(k)}\right) \\
&= \sum_{i\geq 0} k^i i! \binom{\pi(k)}{i} \mathbf{p}_k^{\pi(k)-i} \binom{\rho(k)}{i} \tilde{\mathbf{p}}_k^{\rho(k)-i} \\
&= \mathbf{F}_k(\pi,\rho)
\end{aligned}$$

by Lemma 3.5.

Summing over all terms, we have

$$\begin{aligned}
\mathbf{\Delta}\left(\mathbf{BD}^{\lambda\,\varnothing} \cdot \mathbf{BD}^{\varnothing\,\mu}\right) &= \frac{1}{m!}\sum_{\pi\in\mathfrak{S}_m}\frac{1}{p!}\sum_{\rho\in\mathfrak{S}_p}\chi^\lambda(\pi)\chi^\mu(\rho)\mathbf{\Delta}\left(\mathbf{p}_\pi\tilde{\mathbf{p}}_\rho\right) \\
&= \frac{1}{m!}\sum_{\pi\in\mathfrak{S}_m}\frac{1}{p!}\sum_{\rho\in\mathfrak{S}_p}\chi^\lambda(\pi)\chi^\mu(\rho)\prod_{k>0}\mathbf{F}_k(\pi,\rho) \\
&= \mathbf{BD}^{\lambda\,\mu}.
\end{aligned}$$

The identity

$$\Delta\left(\mathrm{BD}^{\lambda\,\varnothing}\cdot\mathrm{BD}^{\varnothing\,\mu}\right) = \mathrm{BD}^{\lambda\,\mu}$$

follows from

$$\begin{aligned}
\mathrm{BD}^{\lambda\,\mu} &= H(\mathbf{BD}^{\lambda\,\mu}) \\
&= H(\mathbf{\Delta}\left(\mathbf{BD}^{\lambda\,\varnothing}\cdot\mathbf{BD}^{\varnothing\,\mu}\right)) \\
&= \mathbf{\Delta}\left(H(\mathbf{BD}^{\lambda\,\varnothing}\cdot\mathbf{BD}^{\varnothing\,\mu})\right) \\
&= \Delta\left(\mathrm{BD}^{\lambda\,\varnothing}\cdot\mathrm{BD}^{\varnothing\,\mu}\right),
\end{aligned}$$

completing our proof of Theorem 3.2. □

We will need an additional lemma later.

Lemma 3.6.

$$\mathbf{BD}^{\lambda\,\varnothing} = \mathbf{s}_\lambda \quad \textit{and} \quad \mathbf{BD}^{\varnothing\,\mu} = \tilde{\mathbf{s}}_\mu.$$

Proof. This is immediate from the definitions of $\mathbf{BD}^{\lambda\,\varnothing}$ and $\mathbf{BD}^{\varnothing\,\mu}$: we get the expansions[4]

$$\frac{1}{|\lambda|!}\sum_{\pi\in\mathfrak{S}_{|\lambda|}} \chi^\lambda(\pi)\mathbf{p}_\pi = \mathbf{s}_\lambda \quad \text{and} \quad \frac{1}{|\lambda|!}\sum_{\pi\in\mathfrak{S}_{|\lambda|}} \chi^\lambda(\pi)\tilde{\mathbf{p}}_\pi = \tilde{\mathbf{s}}_\lambda$$

for Schur polynomials in terms of power sums, a fact that was already presented by Bump and Diaconis in their paper. □

3.4 The Tracy-Widom side

Since $\sigma(t) = \exp\left(\sum_{k>0} \frac{p_k}{k} t^k + \frac{\tilde{p}_k}{k} t^{-k}\right)$, it is reasonable to consider the functions

$$\begin{aligned} \sigma^+(t) :=& \quad \textstyle\sum_{k\geq 0} h_k t^k := & \exp\left(\sum_{k>0} \frac{p_k}{k} t^k\right) \\ \text{and } \sigma^-(t) :=& \quad \textstyle\sum_{k\geq 0} \tilde{h}_k t^{-k} := & \exp\left(\sum_{k>0} \frac{\tilde{p}_k}{k} t^{-k}\right). \end{aligned}$$

It is a classical theorem from operator theory for Toeplitz matrices (see Böttcher and Silbermann's book [BS99, page 15]) that we then have

$$\lim_{n\to\infty} (M_n(\sigma^+) \cdot M_n(\sigma^-))_{ij} = \lim_{n\to\infty} M_n(\sigma)_{ij}.$$

This is called the Wiener-Hopf factorization of the symbol σ.

[4] We use here permutations as index for the power sums functions. For instance, we mean by $\mathbf{p}_\pi$ the function $\mathbf{p}_\lambda$, where λ is the cycle-type of π.

Tracy and Widom use the Fourier coefficients h_k's and $\tilde{h}_k$'s of σ^+ and σ^- to formulate their result.

We are now ready to define $\mathbf{TW}^{\lambda\,\mu}$ for the partitions $\lambda \vdash m$ and $\mu \vdash p$. Let d be an integer large enough that $\lambda_{d+1} = \mu_{d+1} = 0$. Obviously, $d = \max(l(\lambda), l(\mu))$ would do, but d could be taken larger without affecting the result. Then we set

$$
\begin{aligned}
\mathbf{TW}^{\lambda\,\mu} &= \det\left(\left(\tilde{\mathbf{h}}_{i-j+\mu_{d-i+1}}\right)_{d\times\infty} \cdot \left(\mathbf{h}_{j-i+\lambda_{d-j+1}}\right)_{\infty\times d}\right) \\
&= \det\left(\begin{pmatrix}
\tilde{\mathbf{h}}_{\mu_d} & & \succ & \tilde{\mathbf{h}}_{1-d+\mu_d} & \tilde{\mathbf{h}}_{-d+\mu_d} & \succ & \succ \tilde{\mathbf{h}}_0 & 0 & \cdots & & 0\cdots \\
& \nwarrow & \succ & & & \succ & \succ \tilde{\mathbf{h}}_0 & 0 & \cdots & & 0\cdots \\
& \tilde{\mathbf{h}}_{\mu_{d-i+1}} & & & & \succ & & \succ \tilde{\mathbf{h}}_0 & 0 & \cdots & 0\cdots \\
& & \nwarrow & & & \succ & & \succ \tilde{\mathbf{h}}_0 & 0 & \cdots & 0\cdots \\
\tilde{\mathbf{h}}_{d-1+\mu_1} & & \succ & \tilde{\mathbf{h}}_{\mu_1} & \tilde{\mathbf{h}}_{-1+\mu_1} & \succ & & & \succ \tilde{\mathbf{h}}_0 & 0\cdots & 0\cdots
\end{pmatrix}_{d\times\infty}\right. \\
&\qquad \cdot \left.\begin{pmatrix}
\mathbf{h}_{\lambda_d} & & & & \mathbf{h}_{d-1+\lambda_1} \\
\curlyvee & \nwarrow & & & \curlyvee \\
& & \mathbf{h}_{\lambda_{d-i+1}} & & \\
\curlyvee & & & \nwarrow & \curlyvee \\
\mathbf{h}_{1-d+\lambda_d} & & & & \mathbf{h}_{\lambda_1} \\
\mathbf{h}_{-d+\lambda_d} & & & & \mathbf{h}_{-1+\lambda_1} \\
\curlyvee & \curlyvee & \curlyvee & \curlyvee & \curlyvee \\
\curlyvee & \curlyvee & & & \\
\mathbf{h}_0 & \mathbf{h}_0 & \curlyvee & \curlyvee & \\
0 & 0 & \mathbf{h}_0 & \mathbf{h}_0 & \\
\vdots & \vdots & 0 & 0 & \curlyvee \\
& & \vdots & \vdots & \mathbf{h}_0 \\
& & & & \vdots \\
0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}_{\infty\times d}\right).
\end{aligned}
$$

The structure of those matrices is important. We now attempt to describe it in words.

We have here the determinant of a product of two "half-strip" matrices of sizes $d \times \infty$ and $\infty \times d$. The entries along the main diagonal (marked by the arrows $\nwarrow$) are all of the form $\mathbf{h}_{\lambda_i}$ or $\tilde{\mathbf{h}}_{\mu_i}$. The first matrix (resp. second) has a privileged direction, $\succ$ (resp. $\curlyvee$), in which the indices of $\mathbf{h}_\star$ (resp. $\tilde{\mathbf{h}}_\star$) are decreasing. This guarantees that the product is well-defined: each line on the first column has only finitely many non-zero entries[5].

[5]This is not important, bu there is also a "cascading effect" among non-zero entries: in the

We define $\mathrm{TW}^{\lambda\,\mu} := H(\mathbf{TW}^{\lambda\,\mu})$.

The matrices involved in the definitions of $\mathbf{TW}^{\lambda\,\mu}$ and $\mathrm{TW}^{\lambda\,\mu}$ are obviously very similar to the Jacobi-Trudi matrix. We remind the reader that the Jacobi-Trudi matrix of dimension $d \times d$ for the partition λ ($d \geq l(\lambda)$) is the matrix

$$\mathbf{JT}^d_\lambda = \begin{pmatrix} \mathbf{h}_{\lambda_1} & \prec & & \prec & \mathbf{h}_{d-1+\lambda_1} \\ & \searrow & & \prec & \\ & & \mathbf{h}_{\lambda_i} & & \\ & \prec & & \searrow & \\ \mathbf{h}_{1-d+\lambda_d} & \prec & & \prec & \mathbf{h}_{\lambda_d} \end{pmatrix}_{d\times d},$$

where we respected the same conventions with arrows. We define $\tilde{\mathbf{JT}}^d_\lambda$ in a totally analogous way (i.e. using $\tilde{\mathbf{h}}$'s). It is a central theorem of the theory of symmetric functions that $\det(\mathbf{JT}^d_\lambda) = \mathbf{s}_\lambda$ (see [Bum04, Theorem 37.1]) and is thus independent of d (as long as $d \geq l(\lambda)$). Similarly, $\det(\tilde{\mathbf{JT}}^d_\lambda) = \tilde{\mathbf{s}}_\lambda$.

We are now ready to comment on the result of Tracy and Widom a bit further.

Lemma 3.7.

$$\mathbf{TW}^{\lambda\,\varnothing} = \mathbf{s}_\lambda \quad \textit{and} \quad \mathbf{TW}^{\varnothing\,\mu} = \tilde{\mathbf{s}}_\mu.$$

Proof. We will only do the case $\mu = \varnothing$. Pick $d \geq l(\lambda)$. The left-hand side matrix in the definition of $\mathbf{TW}^{\lambda\,\varnothing}$ is then lower triangular, with 1's on the main diagonal. Without affecting the final determinant, we can row-reduce this matrix to $(\delta_{ij})_{d\times\infty}$, with δ_{ij} the Kronecker delta.

---

first matrix for instance, the last non-zero entry on each row (i.e. $\tilde{\mathbf{h}}_0$) has to be (weakly) to the right of any non-zero entry on the rows above.

Hence we easily compute

$$\begin{aligned}\mathbf{TW}^{\lambda\varnothing} &= \begin{pmatrix} \mathbf{h}_{\lambda_d} & & & \mathbf{h}_{d-1+\lambda_1} \\ \curlyvee & \nwarrow & & \curlyvee \\ & & \mathbf{h}_{\lambda_{d-i+1}} & \\ \curlyvee & & \nwarrow & \curlyvee \\ \mathbf{h}_{1-d+\lambda_d} & & & \mathbf{h}_{\lambda_1} \end{pmatrix}_{d\times d} \\ &= \det\left(\left(\mathbf{JT}^d_\lambda\right)_{d+1-j,d+1-i}\right) = \det \mathbf{JT}^d_\lambda = \mathbf{s}_\lambda .\end{aligned}$$

The key observation is thus that the $d \times d$ truncation of the right-hand side matrix in the Tracy-Widom determinant is the anti-transpose[6] of the Jacobi-Trudi matrix, and that a determinant is not affected under anti-transposition. □

We can now get started on the proof of Theorem 3.3.

Proof of Theorem 3.3. We need to compute $\mathbf{\Delta}\left(\mathbf{TW}^{\lambda\varnothing} \cdot \mathbf{TW}^{\varnothing\mu}\right)$. We have

$$\mathbf{\Delta} = \exp\left(\sum_k k\partial_{\mathbf{p}_k}\partial_{\tilde{\mathbf{p}}_k}\right) = \exp\left(\sum_k \frac{\mathbf{p}_k\tilde{\mathbf{p}}_k}{k}\right)^{\perp}.$$

The exponential can easily be expanded to obtain

$$\mathbf{\Delta} = \left(\sum_\nu \frac{1}{z_\nu}\mathbf{p}_\nu\tilde{\mathbf{p}}_\nu\right)^{\perp},$$

where the sum is over all partitions ν. We now make use of the Cauchy identity

$$\sum_\nu \frac{1}{z_\nu}\mathbf{p}_\nu\tilde{\mathbf{p}}_\nu = \prod_{\substack{x_i\in\overline{X}\\ y_j\in\overline{Y}}} \frac{1}{1-x_iy_j} = \sum_\nu \mathbf{s}_\nu\tilde{\mathbf{s}}_\nu$$

[6]The anti-transpose of a matrix is its transposed along the main anti-diagonal.

and obtain our final expression:

$$\mathbf{\Delta} = \left(\sum_\nu \frac{1}{z_\nu}\mathbf{p}_\nu\tilde{\mathbf{p}}_\nu\right)^\perp = \left(\sum_\nu \mathbf{s}_\nu\tilde{\mathbf{s}}_\nu\right)^\perp .$$

Coming back to our original computation, we just obtained

$$\mathbf{\Delta}\left(\mathbf{TW}^{\lambda\,\varnothing}\cdot\mathbf{TW}^{\varnothing\,\mu}\right) = \sum_\nu \mathbf{s}_\nu^\perp(\mathbf{s}_\lambda)\tilde{\mathbf{s}}_\nu^\perp(\tilde{\mathbf{s}}_\mu). \tag{3.2}$$

Observe that

$$\begin{aligned}\mathbf{s}_\nu^\perp(\mathbf{s}_\lambda) &= \sum_\mu \left\langle \mathbf{s}_\nu^\perp(\mathbf{s}_\lambda), \mathbf{s}_\mu\right\rangle \mathbf{s}_\mu \\ &= \sum_\mu \left\langle \mathbf{s}_\lambda, \mathbf{s}_\mu\cdot\mathbf{s}_\nu\right\rangle \mathbf{s}_\mu \\ &= \sum_\mu c^\lambda_{\mu\nu}\mathbf{s}_\mu = \mathbf{s}_{\lambda/\nu}.\end{aligned}$$

The last sum, which involves the Littlewood-Richardson coefficients, is precisely the definition of $\mathbf{s}_{\lambda/\nu}$.

Armed with this observation, we can thus rework Equation (3.2) into

$$\mathbf{\Delta}\left(\mathbf{TW}^{\lambda\,\varnothing}\cdot\mathbf{TW}^{\varnothing\,\mu}\right) = \sum_\nu \mathbf{s}_{\lambda/\nu}\tilde{\mathbf{s}}_{\mu/\nu}.$$

When ν runs through all partitions, the skew function $\mathbf{s}_{\lambda/\nu}$ runs through all $d\times d$ minors $\left(\mathbf{h}_{j-i-\nu_i+\lambda_{d-j+1}}\right)_{d\times d}$ of the matrix $\left(\mathbf{h}_{j-i+\lambda_{d-j+1}}\right)_{\infty\times d}$. Similarly, $\tilde{\mathbf{s}}_{\mu/\nu}$ will run through the minors $\left(\tilde{\mathbf{h}}_{i-j-\nu_j+\mu_{d-i+1}}\right)_{d\times d}$ of $\left(\tilde{\mathbf{h}}_{i-j+\mu_{d-i+1}}\right)_{d\times\infty}$. Moreover, the minors obtained with $\mathbf{s}_\nu^\perp$ and $\tilde{\mathbf{s}}_\nu^\perp$ are paired up just as in the Cauchy-Binet identity. Therefore, we obtain

$$\begin{aligned}\mathbf{\Delta}\left(\mathbf{TW}^{\lambda\,\varnothing}\cdot\mathbf{TW}^{\varnothing\,\mu}\right) &= \det\left(\left(\tilde{\mathbf{h}}_{i-j+\mu_{d-i+1}}\right)_{d\times\infty}\cdot\left(\mathbf{h}_{j-i+\lambda_{d-j+1}}\right)_{\infty\times d}\right) \\ &= \mathbf{TW}^{\lambda\,\mu}\end{aligned}$$

and we are done. The proof for $\mathrm{TW}^{\lambda\,\mu}$ follows from applying the homomorphism H.

□

3.5 The proof of Theorem 3.1

Proof of Theorem 3.1. We have from Lemmas 3.6 and 3.7 that

$$\mathbf{BD}^{\lambda\,\varnothing} = \mathbf{s}_\lambda = \mathbf{TW}^{\lambda\,\varnothing} \quad \text{and} \quad \mathbf{BD}^{\varnothing\,\mu} = \tilde{\mathbf{s}}_\mu = \mathbf{TW}^{\varnothing\,\mu}.$$

Tracing back to those lemmas, this is a direct consequence of the Jacobi-Trudi identity.

The Theorem now follows. We have

$$\begin{aligned} \mathbf{BD}^{\lambda\,\mu} &= \mathbf{\Delta}\left(\mathbf{BD}^{\lambda\,\varnothing}\cdot\mathbf{BD}^{\varnothing\,\mu}\right) && \text{(Theorem 3.2)} \\ &= \mathbf{\Delta}\left(\mathbf{TW}^{\lambda\,\varnothing}\cdot\mathbf{TW}^{\varnothing\,\mu}\right) \\ &= \mathbf{TW}^{\lambda\,\mu} && \text{(Theorem 3.3).} \end{aligned}$$

□

3.6 Some relations among $\mathrm{R}^{\lambda\,\mu}$'s

Two very natural properties of $\mathrm{R}^{\lambda\mu}$ pop out of the presentation due to Tracy and Widom. The proofs rely only on basic properties of determinants and the Tracy-Widom expression $\mathbf{TW}^{\lambda\,\mu}$.

Proposition 3.8. *Let (r) and (s) denote partitions with just one part each, of size $r \geq 1$ and $s \geq 1$ and let λ, μ be partitions, with $\max(l(\lambda), l(\mu)) \leq d$. Then,*

$$\mathrm{R}^{(r)\,(s)} = \mathrm{R}^{(r)\,\varnothing}\cdot\mathrm{R}^{\varnothing\,(s)} + \mathrm{R}^{(r-1)\,(s-1)} \tag{3.3}$$

and

$$\mathrm{R}^{\lambda\,\mu} = \det\left(\mathrm{R}^{(\lambda_i+d-i)\,(\mu_j+d-j)}\right)_{1\leq i,j\leq d}. \tag{3.4}$$

Proof. Both results follow from the same fact:

$$\begin{aligned}
\mathbf{TW}^{(r)\,(s)} &= \det\left(\left(\tilde{\mathbf{h}}_{1-j+s}\right)_{1\times\infty}\cdot\left(\mathbf{h}_{1-i+r}\right)_{\infty\times 1}\right)\\
&= \tilde{\mathbf{h}}_s\mathbf{h}_r+\tilde{\mathbf{h}}_{s-1}\mathbf{h}_{r-1}+\cdots\\
&= \tilde{\mathbf{h}}_s\mathbf{h}_r+\mathbf{TW}^{(r-1)\,(s-1)}\\
&= \mathbf{TW}^{(r)\,\varnothing}\mathbf{TW}^{\varnothing\,(s)}+\mathbf{TW}^{(r-1)\,(s-1)},
\end{aligned}$$

which proves Equation 3.3.

For Equation 3.4, we just need to observe that $\mathbf{TW}^{\lambda\,\mu}$ is defined as the determinant of a matrix $\mathbf{M}$ which itself is a product of two matrices. The coefficient on the i^{th} row and the j^{th} column of $\mathbf{M}$ is given by

$$\mathbf{M}_{ij}=\sum_{k=0}^{\infty}\tilde{\mathbf{h}}_{i-1-k+\mu_{d+1-i}}\mathbf{h}_{j-1-k+\lambda_{d+1-j}},$$

where this sum is actually finite (because the terms eventually vanish).

By the reasonning for Equation 3.3, we actually know that

$$\mathbf{M}_{ij}=\mathbf{TW}^{(j-1+\lambda_{d+1-j})\,(i-1+\mu_{d+1-i})}.$$

Equation 3.4 then follows from the invariance of determinants under transposition and anti-transposition. □

Chapter 4

Intermission

We would like to highlight the main results from the previous chapters, present them informally, and put them in context for the next chapter.

In order to make the ideas clear, I will gloss over many issues of convergence in Section 4.1.1. However, similar results will be rigorously reproved in Proposition 4.1. It will be shown in Section 4.1.3 that those results are actually exactly the same, thus offering one rigorous proof and one intuitive guess for the same result.

4.1 Summary of results of Chapters 2 and 3

4.1.1 Chapter 2

The main result of Chapter 2 (Theorem 2.7) tells us that under certain conditions on $f(t) = \sum_{i\in\mathbb{Z}} \frac{c_i}{|i|} t^i$,

$$\lim_{n\to\infty} \frac{\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f}}{\mathbb{E}_G \boldsymbol{\Phi}_{n,f}} = R(\gamma, (c_i)),$$

where

$$R(\gamma, (c_i)) = \sum_{\lambda \vdash |\gamma|} \chi_\gamma(\lambda) \left(\prod_{i=1}^{\infty} \frac{c_i^{\lambda(i)}}{i^{\lambda(i)} \lambda(i)!} \right),$$

and we mentioned in the remarks on page 29 that the RHS could be seen as a Schur polynomial, once we have specialized in the ring of symmetric functions the power polynomials p_i to the Fourier coefficients[1] c_i of f.

4.1.2 A Theorem of Bump and Gamburd

In [BG06, Corollary 1], Bump and Gamburd prove that

$$\int_{\mathrm{U}(N)} |\det(\mathrm{Id}+g)|^{2k} \mathrm{d}g = s_{\langle N^k \rangle}\left([1]^{2k}\right).$$

Their result can actually be slightly improved to

Proposition 4.1.

$$\int_{\mathrm{U}(N)} |\det(\mathrm{Id}+g)|^{2k} s_\gamma(g) \mathrm{d}g = s_{\langle N^k \rangle}\left([1]^{2k}\right) \cdot s_{\gamma'}\left([1]^{2k}\right).$$

Proof. This proof is entirely rigorous. We have

$$\begin{aligned}
\int_{\mathrm{U}(N)} |\det(\mathrm{Id}+g)|^{2k} s_\gamma(g) \mathrm{d}g &= \int_{\mathrm{U}(N)} |\det(\mathrm{Id}+g)|^{2k} \overline{s_\gamma(g)} \mathrm{d}g = \\
&\int_{\mathrm{U}(N)} \det(\mathrm{Id}+g)^{2k} \overline{s_\gamma(g) \det(g)}^{k} \mathrm{d}g = \\
&\int_{\mathrm{U}(N)} \sum_\lambda s_\lambda\left([1]^{2k}\right) s_{\lambda'}(g) \overline{s_\gamma(g) s_{\langle k^N \rangle}(g)} \mathrm{d}g = \\
&\qquad s_{\langle N^k \rangle}\left([1]^{2k}\right) s_{\gamma'}\left([1]^{2k}\right),
\end{aligned}$$

where the third equality follows from identifying the determinant as a Schur function and the dual Cauchy identity

$$\prod_{i=1}^{K} \det(\mathrm{Id}+\alpha_i g) = \sum_\lambda s_\lambda(\alpha_1, \cdots, \alpha_K) s_{\lambda'}(t_1, \cdots, t_N).$$

[1] rescaled by $1/i$

□

4.1.3 Agreement so far

We have now offered in Section 4.1.1 an intuitive guess at a statement, and rigorously proved Proposition 4.1 in Section 4.1.2. We will now show that results from both Sections are identical, thus offering an intuitive derivation and a formal proof of the same result.

In the language of Section 4.1.1,

$$\mathbf{\Phi}_{n,f}(g) = \det(\mathrm{Id} + g)^{2k}$$

and hence

$$\begin{aligned} f(t) = \log\left((1+t)^{2k}(1+t^{-1})^{2k}\right) \\ = 2k\left(\log(1+t) + \log(1+t^{-1})\right) \\ = 2k\left(\cdots - \frac{t^{-4}}{4} + \frac{t^{-3}}{3} - \frac{t^{-2}}{2} + t^{-1} + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots\right), \end{aligned}$$

or in other words all the $c_i = (-1)^{i+1}2k$.

Notice that we do not care whether those values for $p_i = c_i$ can be achieved for a certain set of $x_1, x_2, \cdots$. Indeed, Theorem 2.7 merely says that

$$\lim_{N\to\infty} \frac{\int_{\mathrm{U}(N)} |\det(\mathrm{Id} + g)|^{2k} s_\gamma(g)\mathrm{d}g}{\int_{\mathrm{U}(N)} |\det(\mathrm{Id} + g)|^{2k}}$$

equals the image of s_γ under the homomorphism from $\Lambda(x_1, x_2, \cdots)$ to $\mathbb{C}$ that extends the map $p_i \longrightarrow c_i = (-1)^{i+1}2k$.

We would like to show agreement between the two techniques from Sections 4.1.1 and 4.1.2 to compute

$$\lim_{N\to\infty} \frac{\int_{\mathrm{U}(N)} |\det(\mathrm{Id} + g)|^{2k} s_\gamma(g)\mathrm{d}g}{\int_{\mathrm{U}(N)} |\det(\mathrm{Id} + g)|^{2k}}.$$

i.e. show

$$s_{\gamma'}\left([1]^{2k}\right) = s_\gamma\Big|_{p_i:=(-1)^{i+1}2k}.$$

The notation stands for "specialization", as explained in Section 4.1.1.

We make use of the involution $\iota \in \mathrm{End}(\Lambda)$ that exchanges

$$s_\lambda \text{ and } s_{\lambda'}$$

and

$$E(t) = \prod_i (1 + x_i t) = \sum_i e_i t^i, \text{ and } H(t) = \prod_i (1 - x_i t)^{-1} = \sum_i h_i t^i,$$

where e_i are the elementary polynomials and h_i the complete polynomials.

This involution thus also exchanges

$$\frac{E'(t)}{E(t)} = \sum_i (-1)^i p_{i+1} t^i \text{ and } \frac{H'(t)}{H(t)} = \sum_i p_{i+1} t^i.$$

Hence,

$$s_\gamma\Big|_{p_i:=(-1)^{i+1}2k} = s_{\gamma'}\Big|_{p_i:=2k}.$$

This is significant because $x_1, x_2, \cdots, x_{2k} = 1$ and $x_{2k+1}, x_{2k+2}, \cdots = 0$ implies $p_i = 2k$ for all i. We thus get

$$s_\gamma\Big|_{p_i:=(-1)^{i+1}2k} = s_{\gamma'}\Big|_{p_i:=2k} = s_{\gamma'}\left([1]^{2k}\right),$$

showing agreement between Sections 4.1.1 an 4.1.2.

4.1.4 Chapter 3

We do not see the proof of the (Bump-Diaconis)-(Tracy-Widom) identity as the main result of Chapter 3, but rather Corollary 3.4:

$$\Delta\left(\mathrm{R}^{\lambda\,\varnothing} \cdot \mathrm{R}^{\varnothing\,\mu}\right) = \mathrm{R}^{\lambda\,\mu}.$$

This Corollary establishes that single-twist integrals are related to double-twists integrals through a differential operator Δ.

This differential operator is acting on a function of Fourier coefficients.

4.2 Interpretation

In all of these results, a common theme is to replace power sums with the Fourier coefficients of the logarithm of the function applied to eigenvalues in the integrand.

This theme is present in the statements of Theorem 2.7, Proposition 4.1, Section 4.1.3 and the proofs of Chapter 3.

The recipe is always the same. Take $f(t) = \exp\left(\sum_i \frac{c_i}{|i|} t^i\right)$ and assume the original integrand is given by $\prod_{\text{eigenvalues } \theta_i} f(\theta_i)$. Twist the integrand by a symmetric function. The limit of ratios of the twisted integral to the original integral is then given by the image of that symmetric function under the specialization homomorphism on Λ sending the power sum functions to the Fourier coefficients c_i.

This does not constitute a proof, but it is a quick way to guess what the integral should be.

Actually, one can even see the Strong Szegö Theorem as more evidence of something similar. That theorem states that

$$\lim_{N\to\infty} \int_{\mathrm{U}(N)} \mathbf{\Phi}_{N,f} \mathrm{d}g = \exp\left(\sum_{i\geq 1} \frac{c_i c_{-i}}{i}\right),$$

where the right-hand side, in light of the discussion page 44 equals

$$\exp\left(\sum_{i\geq 1} \frac{p_i \tilde{p}_i}{i}\right)\Bigg|_{p_i := c_i, \tilde{p}_i := c_{-i}} = \sum_{\mu} s_\mu \tilde{s}_{\mu'}\Bigg|_{p_i := c_i, \tilde{p}_i := c_{-i}},$$

i.e. another specialization of a symmetric function using Fourier coefficients instead of power sum functions. Note that $p_i = 2k$ for all i *can* be achieved by setting the first $2k$ variables $x_1, \cdots, x_{2k}$ to be 1 and all the others to be 0.

4.3 Speculation

It is of course natural to try to explain Corollary 3.4 in the language of Toeplitz determinants and give an interpretation for this operator Δ. So far we have been unsuccessful at this task.

We can only observe that this operator is very reminiscent of vertex operators, and appears as a twisted product or "Cliffordization" in [FJ04].

Note also that other papers [FJKW06, FJK05] by a similar set of authors present the branching rules that would be needed to extend the results of Chapter 2 to a more general class of groups. Those groups would be compact subgroups of $GL(N)$ preserving a Young tensor ($\mathrm{O}(m)$ preserving a symmetric bilinear form, and $\mathrm{Sp}(2n)$ preserving a skew-symmetric bilinear form).

Chapter 5

An application: mixed moments of derivatives of characteristic polynomials

5.1 Introduction

As explained in Chapter 1, a natural object of study if one is interested in moments of derivatives of the Riemann ζ-function is

$$M_N(2k, 2k') = \int_{\mathrm{U}(N)} |\Lambda_g'(1)|^{2k'} |\Lambda_g(1)|^{2k-2k'} \mathrm{d}g,$$

where $\Lambda_g(t) = \det(t\,\mathrm{Id} - g)$ is the characteristic polynomial of g. Note that this integral does not appear directly computable through Selberg's integral formula, which explains that a complete result for $M_N(2k, 2k')$ is still unavailable. Partial results ($k' = 1, 2$ and 3 were obtained by Chris Hughes in his Ph.D. thesis [Hug03, Hug01] and in unpublished notes).

After a reminder on the Murnaghan-Nakayama rule in Section 5.2, we will express $M_N(2k, 2k')$) in terms of symmetric functions in Section 5.3 (for k and k' integers).

This expression looks informally like

$$M_N(2k, 2k') = \int_{\mathrm{U}(N)} |\Lambda_g(1)|^{2k} \cdot \phi_{k'}(g) \mathrm{d}g,$$

for a certain symmetric function $\phi_{k'}$. This is very encouraging, as the results from Chapter 2, as presented in Section 4.1, tell us that

$$\lim_{N\to\infty} \frac{M_N(2k, 2k')}{M_N(2k)} = \text{specialization of } \phi_{k'}.$$

The specialization of $\phi_{k'}$ is decided by the coefficients c_i in $f(t) = \exp(\sum_i \frac{c_i}{i} t^i)$, with

$$\det(\mathrm{Id} - g)^{2k} = \prod_{\text{eigenvalues } \theta_i \text{ of } g} f(\theta_i).$$

Obviously, $f(t) = (1-t)^{2k}$, so the Taylor expansion $\log(1-t) = \sum_i \frac{t^i}{i}$ implies $c_i = 2k$. Specializing the power polynomials p_i to $2k$ for all i can actually be achieved: set the $2k$ first variables to 1, and all the others to 0.

Of course, nothing of this is valid, as $|\theta_i| = 1$, and $\sum_i \frac{c_i}{i}$ diverge. If we are willing to go along, however, this would predict that

$$M_N(2k, 2k') \sim_N M_N(2k) \cdot \phi_{k'}(\underbrace{1, 1, 1, \cdots, 1}_{2k}, 0, \cdots).$$

We will actually obtain an identity similar to this (when $k' = 1$), but valid at fixed N (see Theorem 5.11).

After the work we need to do finding $\phi_{k'}$, we focus exclusively on the simpler case $k' = 1$. We present a general outline of our method of computation in Section 5.4, which will steer the discussion towards combinatorics of Young tableaux.

At that point, we will have an expression for $M_N(2k, 2k')$ as a giant sum over operations on Young tableaux of specific values of Schur polynomials. We describe this indexing set in a few different ways in Section 5.5. That section will include a roadmap to that set, given in Figure 5.1. We give the basics on how to evaluate Schur polynomials in Section 5.6.

The main part of the argument will then consist of reorganizing the sum appropriately, following the roadmap given earlier. This is Section 5.7, which will be split into many subsections dealing with each case separately.

Summing up all of those cases, we get to our main proved results on $M_N(2k, 2)$, presented in Section 5.8 (fixed N) and Section 5.9 (asymptotic in N). Extracting asymptotic results from results at fixed N will be a difficult process, but will improve our understanding.

Section 5.10 will present an unexpected conjectural identity. Very little in this identity is conjectural: there is extensive numerical evidence to support it, and it is even clear that the theory of hypergeometric summation developed by Wilf and Zeilberger should prove it. It is however very surprising, and powerful: it would provide us with a very simple final statement and would extend the range of validity of our result from $k \in \mathbb{N}$ to $k \in \mathbb{C}, \operatorname{Re} k > 1/2$ (which is important for Number Theory applications)[1].

Finally, in Section 5.11, we look back at the proof and discuss various side considerations: what if we do not evaluate $\Lambda_g(t)$ at $t = 1$, but at another point instead? What about $k' > 1$?

As an encouragement to the reader, we now state the final result presented in this Chapter (some notation will be explained later, and this Theorem will actually be proved in installments). We use the standard notation for the rising factorial

$$\begin{aligned} X^{(k)} &:= X \cdot (X+1) \cdot \dots \cdot (X+k-1), \\ &= \frac{\Gamma(X+k)}{\Gamma(X)} \end{aligned}$$

and $[1]^{2k}$ to represent $(\underbrace{1, 1, \cdots, 1}_{2k})$.

[1] This could probably be done independently of Conjecture 5.17, as the purported analytic continuation of the bracketed expression in Theorem 5.1 is clear.

Theorem 5.1. *For $k \in \mathbb{N}$, with the notation described above,*

$$M_N(2k,2) = \int_{\mathrm{U}(N)} |\Lambda_g'(1)|^2 |\Lambda_g(1)|^{2k-2} \mathrm{d}g =$$
$$s_{\langle N^k \rangle}\left([1]^{2k}\right) \cdot \left[\frac{N^2 - N}{2} + \sum_{0 \le U \le N-1} \frac{(N-U)^{(k)}(U+1)^{(k)}}{(k-1)!} \cdot \right.$$
$$\left. \sum_{0 \le Y \le k-1} (-1)^{Y+1} \binom{k-1}{Y} \frac{U-Y-1}{(N+Y+1)^{(k)}(U+Y+1)} \right] \sim$$
$$\frac{k^2}{(2k-1)(2k+1)} \cdot \frac{\prod_{i=1}^{k} \Gamma(i)!^2}{\prod_{i=1}^{2k} \Gamma(i)} N^{k^2+2}. \quad (5.1)$$

Furthermore, there is compelling numerical evidence that the bracketed expression in (5.1) equals

$$P(N,k) = \frac{2N^2k^2 + Nk}{2(2k-1)(2k+1)},$$

which would prove that for $k \in \mathbb{C}$, with $\operatorname{Re} k > 1/2$,

$$\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^2 |\Lambda_g(1)|^{2k-2} \mathrm{d}g = \frac{2N^2k^2 + Nk}{2(2k-1)(2k+1)} \cdot \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2}.$$

As explained earlier,

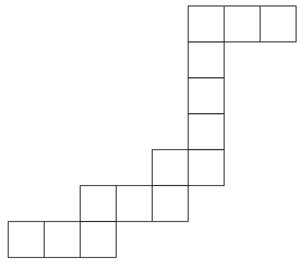
$$\int_{\mathrm{U}(N)} |\Lambda_g(1)|^{2k} \mathrm{d}g = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2}$$

should be present as a factor in the asymptotic result, and part of the surprise is that it also present at the fixed N level.

5.2 Reminder: The Murnaghan-Nakayama rule

We define a ribbon to be a connected skew Young tableaux not containing any 2×2 block. If the ribbon contains i blocks, we talk of an i-ribbon. The height of the ribbon μ/λ, denoted by $\mathrm{ht}(\mu/\lambda)$ is the number of vertical steps it includes.

For instance, the 14-ribbon



is of height 6.

The Murnaghan-Nakayama rule expresses products of Schur polynomials and power polynomials in terms of Schur polynomials.

The formula reads:

$$s_\lambda p_i = \sum_\mu (-1)^{\mathrm{ht}(\mu/\lambda)} s_\mu,$$

where μ is obtained from λ by adding an i-ribbon of length i.

5.3 Mixed moments in terms of symmetric functions

Let $\Lambda_g(t) = \det(t\,\mathrm{Id} - g)$ be the characteristic polynomial of the matrix g. By the dominated convergence theorem, we have

$$\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^{2k'} |\Lambda_g(1)|^{2k-2k'} \mathrm{d}g = \lim_{t\to 1} \int_{\mathrm{U}(N)} |\Lambda_g'(t)|^{2k'} |\Lambda_g(t)|^{2k-2k'} \mathrm{d}g.$$

We have that

$$\sum_{\text{eigenvalues } \theta_i} \frac{1}{t-\theta_i} = \frac{1}{t}\left(N + \frac{p_1(g)}{t} + \frac{p_2(g)}{t^2} + \cdots\right).$$

Hence, by logarithmic differentiation,

$$\int_{\mathrm{U}(N)} |\Lambda_g'(t)|^{2k'} |\Lambda_g(t)|^{2k-2k'} \mathrm{d}g \quad = \quad \int_{\mathrm{U}(N)} |\Lambda_g(t)|^{2k} \frac{1}{t^{2k'}} \left| \sum_{i=0} \frac{p_i(g)}{t^i} \right|^{2k'} \mathrm{d}g,$$

where by convention $p_0(g) = N$.

The dual Cauchy identity

$$\sum_\mu s_\mu(\overline{x}) s_{\mu'}(\overline{y}) = \prod_{i,j}(1 + x_i y_j)$$

gives (exactly as in Bump-Gamburd [BG06] and Section 4.1.2) a new expansion

$$= \frac{1}{t^{2k'}} \int_{\mathrm{U}(N)} \sum_\mu s_\mu([-t]^k, [-1/t]^k) s_{\mu'}(g) s_{\langle k^N \rangle}(\overline{g}) \left| \sum_{i=0} \frac{p_i(g)}{t^i} \right|^{2k'} \mathrm{d}g,$$

where $([-t]^k, [-1/t]^k)$ is shorthand for $(\underbrace{-t, \cdots, -t}_{k}, \underbrace{-1/t, \cdots, -1/t}_{k})$. Notice that the $p_i(g)$ are bounded by N, so as long as $t > 1$, the series in i and μ converge absolutely.

We now focus exclusively on the simpler case $k' = 1$, which reduces everything so far to

$$\int_{\mathrm{U}(N)} |\Lambda_g'(t)|^2 |\Lambda_g(t)|^{2k-2} \mathrm{d}g =$$
$$\frac{1}{t^2} \int_{\mathrm{U}(N)} \sum_\mu s_\mu([-t]^k, [-1/t]^k) s_{\mu'}(g) s_{\langle k^N \rangle}(\overline{g}) \left(\sum_{i=0} \frac{p_i(g)}{t^i} \right) \left(\sum_{j=0} \frac{p_j(\overline{g})}{t^j} \right) \mathrm{d}g =$$
$$\frac{1}{t^2} \int_{\mathrm{U}(N)} (-)^{|\mu'|} \sum_\mu s_\mu([t]^k, [1/t]^k) s_{\mu'}(g) s_{\langle k^N \rangle}(\overline{g}) \left(\sum_{i=0} \frac{p_i(g)}{t^i} \right) \left(\sum_{j=0} \frac{p_j(\overline{g})}{t^j} \right) \mathrm{d}g, \quad (5.2)$$

where the last line uses the general fact $s_\lambda(-t_1, \cdots, -t_r) = (-1)^{|\lambda|} s_\lambda(t_1, \cdots, t_r)$.

5.4 General outline

For fixed N, we want to evaluate the limit for Equation (5.2) as $t \to 1$.

Thanks to the Murnaghan-Nakayama rule, we have a very combinatorial interpretation[2] for the RHS of Equation (5.2): $s_{\langle k^N \rangle}(\overline{g}) \sum_j \frac{p_j(\overline{g})}{t^j}$ gives a (signed) sum

[2]In a first reading, one should probably ignore the t's and set $t := 1$. They are needed for

of Schur functions for the partitions obtained by adding a ribbon to the rectangle $\langle k^N \rangle$. We thus have

$$s_{\langle k^N \rangle}(\overline{g}) \sum_{j>0} \frac{p_j(\overline{g})}{t^j} = \sum_{\delta \in D} (-1)^{\mathrm{ht}(\delta / \langle k^N \rangle)} \frac{1}{t^{|\delta| - Nk}} s_\delta(\overline{g}) \tag{5.3}$$

for δ ranging over the appropriate set D. Similarly, for each μ', we get a relatively simple second signed sum of Schur functions, i.e.

$$s_{\mu'}(g) \sum_{i>0} \frac{p_i(g)}{t^i} = \sum_{\epsilon \in E_{\mu'}} (-1)^{\mathrm{ht}(\epsilon/\mu')} \frac{1}{t^{|\epsilon| - |\mu'|}} s_\epsilon(g). \tag{5.4}$$

We know that some version of orthogonality is true for those Schur functions: when integrating $\int_{\mathrm{U}(N)} s_\epsilon(g) s_\delta(\overline{g}) \mathrm{d}g$ we get 1 when $\epsilon = \delta$ and $l(\delta) \leq N$, and 0 otherwise. Thus one only needs to find the similar terms in both signed sums, and track up their signs (one term is for g, the other one for $\overline{g}$). For each such pair $s_\epsilon(g) s_\delta(\overline{g})$, we will get a contribution of

$$s_\mu \left([t]^k, [1/t]^k \right) \frac{(-)^{\mathrm{ht}(\delta/\langle k^N \rangle) + \mathrm{ht}(\epsilon/\mu') + |\mu'|}}{t^{|\delta| + |\epsilon| - |\mu'| - kN}} \in \mathbb{Q}[[t, 1/t]]$$

to the final sum we are estimating.

Remarks

- This is 0 unless $l(\mu) \leq 2k$, so we get a condition on μ'.
- There is a second condition on μ': since $l(\epsilon) \leq N$, we also have $l(\mu') \leq N$. Together with the first remark, this implies that μ sits within the rectangle $\langle N^{2k} \rangle$ and that only finitely many μ will be involved in this sum.
- This implies that $s_\mu([t]^k, [1/t]^k)$ is $O(t^{2k^2 N})$ in $\mathbb{Q}[[t, 1/t]]$ and hence that

$$s_\mu \left([t]^k, [1/t]^k \right) \frac{(-)^{\mathrm{ht}(\delta/\langle k^N \rangle) + \mathrm{ht}(\epsilon/\mu') + |\mu'|}}{t^{|\delta| + |\epsilon| - |\mu'| - kN}} \in t^{2k^2 N} \mathbb{Q}[[1/t]],$$

---

convergence purposes, but the beauty of the argument in this chapter is that those convergence issues are very isolated. Hence the t's will only be needed in Section 5.7.4.

which will help to prove convergence once $t > 1$.

- Finally, there will only be finitely many contributions for each exponent of t, so the final sum over μ, i and j in Equation (5.2) will be well-defined in $\mathbb{Q}[[t, 1/t]]$.
- Equations (5.3) and (5.4) are only true with the sums starting at $i = 1$ or $j = 1$. Indeed, if say $i = 0$, then $p_i(g) = N$ by convention (see Section 5.3), so a coefficient would be present for $\epsilon = \mu'$.

For clarity, we summarize what we know so far in the following proposition.

Proposition 5.2. *The following identity is true:*

$$\int_{\mathrm{U}(N)} |\Lambda_g'(t)|^2 |\Lambda_g(t)|^{2k-2} \mathrm{d}g = \lim_{t\to 1} \sum_{\mu,\epsilon} (-)^{\mathrm{ht}(\epsilon/\langle k^N\rangle)+\mathrm{ht}(\epsilon/\mu')+|\mu'|} \frac{s_\mu\left([t]^k, [1/t]^k\right)}{t^{|\epsilon|+|\epsilon|-|\mu'|-kN}}, \quad (5.5)$$

where μ, ϵ range through all partitions such that

- *ϵ is obtained from μ' by adding a ribbon, or alternatively by adding a ribbon to $\langle k^N\rangle$;*
- *$l(\mu') \leq 2k$;*
- *$l(\epsilon) \leq N$;*
- *If no ribbon is added to μ' to obtain ϵ, then the associated term in the sum in Equation (5.5) should have an extra coefficient of N (which is omitted for clarity but is important);*
- *Similarly, if no ribbon is added to $\langle k^N\rangle$ to get to ϵ, one needs to add an extra coefficient of N (one will need to add a coefficient of N^2 if $\mu' = \epsilon = \langle k^N\rangle$).*

The sum in the limit is well-defined, and the limit exists when approaching from the right.

We will call the sum in Proposition 5.2 the **main sum**.

We now give an alternative take on μ, ϵ, and the process of adding ribbons.

5.5 Ways to add/remove ribbons

We are now trying to evaluate the main sum from Equation (5.5). In Section 5.5.1, we will show how to reparametrize the index set for the main sum, and in Section 5.5.2, we will split this set to obtain subsums of the main sum.

Both of those Sections will use very visual combinatorics of the partitions.

5.5.1 Alternative approach

For each partition μ, we have the following set-up for the partitions $\langle i \rangle, \langle j \rangle, \delta, \epsilon$ and $\langle k^N \rangle$:

$$\begin{array}{rcccl} & \epsilon & = & \delta & \\ \text{Add an } i\text{-ribbon} & \uparrow & & \uparrow & \text{Add a } j\text{-ribbon} \\ & \mu' & & \langle k^N \rangle & \end{array}$$

with the additional conditions that $l(\epsilon) \leq N$ and $l(\mu) \leq 2k$.

In our current view, this diagram is obtained by starting on the lower row, *then* requiring the equality on the first row between ϵ and δ. A second approach to this diagram is to start with the rectangle $\langle k^N \rangle$, use equality between δ and ϵ, and then finish at μ':

$$\begin{array}{rcccl} & \epsilon & = & \delta & \\ \text{Remove an } i\text{-ribbon} & \downarrow & & \uparrow & \text{Add a } j\text{-ribbon} \\ & \mu' & & \langle k^N \rangle, & \end{array}$$

still with the additional conditions that $l(\epsilon) \leq N$ and $l(\mu) \leq 2k$.

Finally, we will prefer to think of the diagram of transposed partitions:

$$\begin{array}{ccccc} & \epsilon' & = & \delta' & \\ \text{Remove an } i\text{-ribbon} & \downarrow & & \uparrow & \text{Add a } j\text{-ribbon} \\ & \mu & & \langle N^k \rangle, & \end{array} \tag{5.6}$$

still with the additional conditions that $l(\epsilon) \leq N$ and $l(\mu) \leq 2k$.

We will now study all the different methods for obtaining μ following the recipe diagram (5.6). This will provide a very natural and useful way to split the sum in Equation (5.5).

5.5.2 "Topologically" different ways to modify a rectangle with ribbons

There are many different ways to add/remove ribbons. We wish to group them "topologically", i.e. up to the length of the arms of the ribbons involved. We will call each equivalence class a *method*.

Since $\langle N^k \rangle'$ is already of length N, there is only one method of adding a ribbon to this rectangle and obtain a δ' such that $l(\delta) \leq N$: the ribbon has to be added to the bottom of the rectangle $\langle N^k \rangle$, not to the right, and has at most one joint, not two or three.

On the other hand, there are several different methods of removing an i-ribbon. A list is compiled on the next pages, in the second column.

There are essentially nine cases, indexed in the first column $\mathcal{A}$ to $\mathcal{I}$. In cases $\mathcal{A}$ and $\mathcal{B}$, the length of the ribbon that is added is 0. In the other cases, the length is $U + W + Y + Z + 1$ (replacing varibles by 0 if they do not appear in a given case). The black squares indicate the ribbon that is removed. If nothing is black, as in cases $\mathcal{A}$ and $\mathcal{C}$, then we are not removing a ribbon ($j = 0$). The naming might appear random, but there is some reason behind it: X, Y and Z are used for parts of vertical stretches in a ribbon, while U, V and W account for parts of horizontal stretches. Furthermore, we used the convention that Y is for a part that is only added, X is for a part that is only removed, and Z is for a part that is added then removed.

The partition μ is represented by what is left white and the union of the white and black boxes represents $\delta' = \epsilon'$. Observe that the same μ can be the outcome of different methods (for instance, the methods $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{G}$ could lead to the same final μ, and similarly for $\mathcal{B}$ and $\mathcal{F}$, or $\mathcal{H}$ and $\mathcal{I}$).

We will often abuse notation and denote by $s_{\mathcal{C}(U,Y)}$ *the Schur function of the partition* μ *resulting from applying method* $\mathcal{C}(U,Y)$ (for instance). This is harmless.

The **bounds** column indicates bounds on the variables U, V, W, X, Y, Z that are used to parametrize each specific partition within a general method for adding-removing ribbons.

The **parity** column indicates the sign[3] for each term. Some of it comes from the Murnaghan-Nakayama rule (and thus the height of the added and removed ribbons), and some comes from the $(-)^{|\mu'|}$ in Equation (5.2). Since we view each μ as obtained from $\langle N^k \rangle$ after modifications, we omit $(-)^{kN}$ from each term.

In this section, we have learned the following:

Proposition 5.3. *The sum in Equation (5.5) can be seen as a sum over all diagrams similar to diagram (5.6). The sum can then naturally be split into nine subsums, one for each* method $\mathcal{A}, \mathcal{B}, \cdots, \mathcal{I}$. *Three of these subsums (associated to methods* $\mathcal{D}, \mathcal{E}$ *and* $\mathcal{F}$*) are infinite, but all the others are finite.*

The last statement follows from carefully considering Figure 5.1.

[3]or rather the exponent to put to -1 to get the sign...

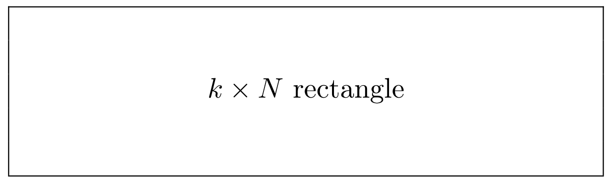
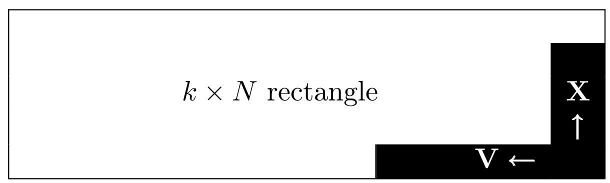
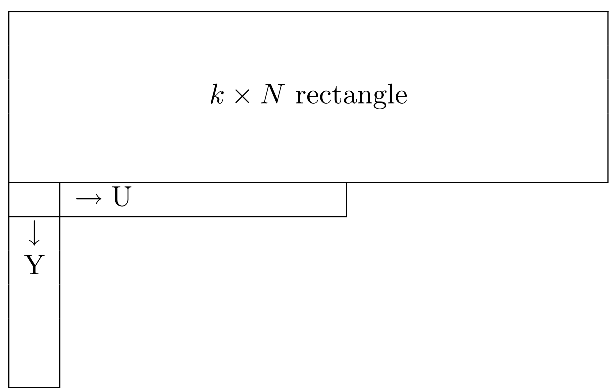
name	method diagram	bounds	parity
$\mathcal{A}$			$+$
$\mathcal{B}(V, X)$		$0 \le V \le N-1$ $0 \le X \le k-1$	$X+1$
$\mathcal{C}(U, Y)$		$0 \le U \le N-1$ $0 \le Y \le k-1$	$Y+1$

Figure 5.1: The nine methods to add and remove a ribbon as in diagram (5.6)

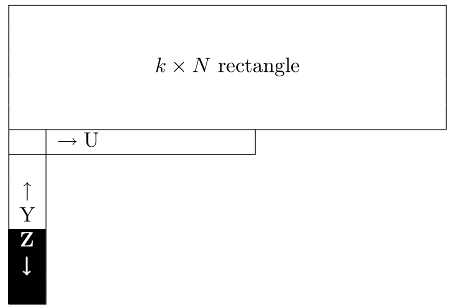
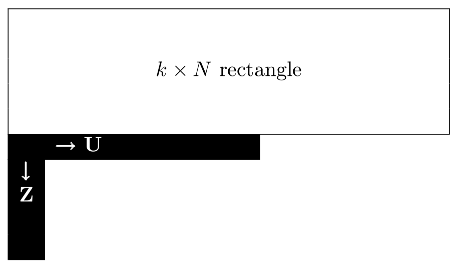
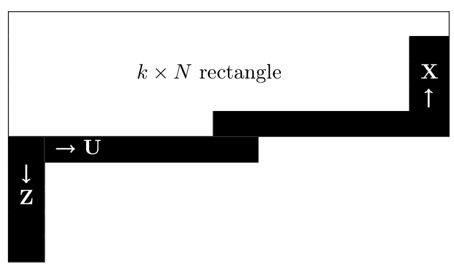
name	method diagram	bounds	parity
$\mathcal{D}(U, Y, Z)$		$0 \leq U \leq N-1$ $0 \leq Y \leq k-1$ $1 \leq Z$	$Y+1$
$\mathcal{E}(U, Z)$		$0 \leq U \leq N-1$ $0 \leq Z$	$+$
$\mathcal{F}(U, X, Z)$		$0 \leq U \leq N-1$ $0 \leq X \leq k-1$ $0 \leq Z$	$X+1$

Figure 5.1: The nine methods to add and remove a ribbon as in diagram (5.6) (continued)

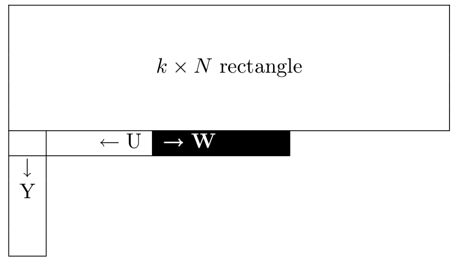
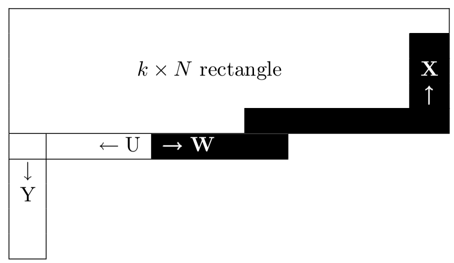
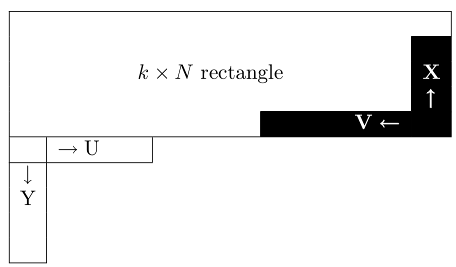
name	method diagram	bounds	parity
$\mathcal{G}(U, W, Y)$		$0 \leq U \leq N-2$ $1 \leq W \leq N-U-1$	Y
$\mathcal{H}(U, W, X, Y)$		$0 \leq U \leq N-2$ $1 \leq W \leq N-U-1$ $0 \leq X \leq k-1$	$X+Y-1$
$\mathcal{I}(U, V, X, Y)$		$0 \leq U \leq N-2$ $0 \leq V \leq N-U-2$ $0 \leq X \leq k-1$ $0 \leq Y \leq k-1$	$X+Y$

Figure 5.1: The nine methods to add and remove a ribbon as in diagram (5.6) (continued)

5.6 Evaluation of Schur polynomials

Most methods will turn out to produce a contribution expressed as a Schur polynomial evaluated at $\left([1]^{2k}\right)$. Fortunately, this can be evaluated using l'Hopital rule on the Weyl character formula, to obtain the Weyl dimension formula. Assume $l(\lambda) \leq 2k$. Then

$$s_\lambda\left([1]^{2k}\right) = \frac{\prod_{1\leq i<j\leq 2k}(\lambda_i - \lambda_j + j - i)}{\prod_{1\leq i<j\leq 2k}(j-i)}, \tag{5.7}$$

where $\lambda = (\lambda_1, \cdots, \lambda_{l(\lambda)}, 0, \cdots, 0)$ might be fitted with trailing 0's in order to define $\lambda_{l(\lambda)+1}, \cdots, \lambda_{2k}$.

The denominator in this formula is really a constant depending only on k, and will be of very little importance. Its actual value is $\prod_{i=1}^{2k-1} i!$.

The numerator has a geometric interpretation: look at the Young tableaux of the partition λ, and consider this as a tableau of length $2k$, regardless of the actual length of λ. In this view, some rows at the end possibly contain 0 blocks. The numerator can then be seen as a product over pairs of distinct rows of the sum of the differences in lengths and indices of the pair. In other words, we have a product over pairs of rows of the taxicab distance between the trailing boxes in that pair.

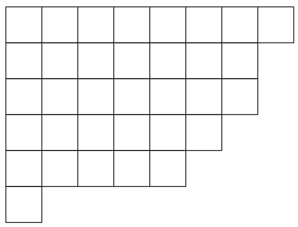
The following proposition follows from the geometric interpretation.

Proposition 5.4 (Symmetry property)**.** *Let k, N be fixed integers. Let μ and ν be partitions that fit together to form a partition of a $2k \times N$-rectangle when ν is rotated* $180°$*. Then,*

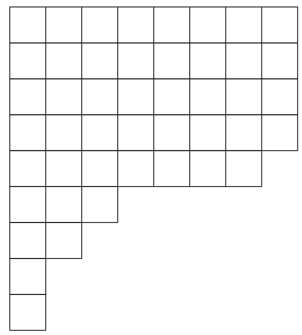
$$s_\mu\left([1]^{2k}\right) = s_\nu\left([1]^{2k}\right),$$

where $[1]^{2k}$ means that the Schur polynomials are evaluated at $(\underbrace{1, 1, \cdots, 1}_{2k})$.

For instance, when $k = 5$ and $N = 8$, the partitions



and



partition a 10×8 rectangle.

Proof. When we use the geometric interpretation of the Weyl dimension formula, only the relative distance of trailing boxes in each row is important.

Color those trailing boxes in both μ and ν. Rotate ν 180°. Assemble them into the rectangle. The two sets of colored boxes are then horizontal translates of each other, which shows that the relative distances within each set are the same. $\square$

This proposition will be useful to identify contributions to the main sum coming from different methods.

5.7 Reorganizing the main sum

We now split the terms in the main sum (which appeared in Proposition 5.2) according to which method it comes from.

5.7.1 Contributions due to method $\mathcal{A}$

There is just one term, so we have no problem taking the limit

$$\lim_{t \to 1} s_{\langle N^k \rangle} \left([t]^k, [1/t]^k \right) = s_{\langle N^k \rangle} \left([1]^{2k} \right).$$

It is important to observe that there is an extra coefficient of N^2 needed this method (see Proposition 5.2).

We thus get:

Proposition 5.5. *The subsum associated to method $\mathcal{A}$ evaluates to*

$$N^2 \cdot s_{\langle N^k \rangle}\left([1]^{2k}\right).$$

5.7.2 Contributions due to methods $\mathcal{H}$ and $\mathcal{I}$

These two methods turn out to play beautifully together:

Proposition 5.6. *The subsums associated to methods $\mathcal{H}$ and $\mathcal{I}$ cancel.*

Proof. If we look at Figure 5.1, we see that each partition that appears in method $\mathcal{H}$ also appears in method $\mathcal{I}$, but with opposite sign (take X, Y and U to be equal, and W in $\mathcal{H}$ equal to $N-1-V-U$ in $\mathcal{I}$). □

5.7.3 Contributions due to methods $\mathcal{B}, \mathcal{C}$ and $\mathcal{G}$

There are only finitely many terms in this case. Therefore, we can take the limit $t \to 1$ without worries.

As a first step, we observe that methods $\mathcal{B}(U,Y)$ and $\mathcal{C}(U,Y)$ end up producing complementary partitions μ, just as in the hypothesis of Proposition 5.4. Hence we have

$$s_{\mathcal{B}(U,Y)}\left([1]^{2k}\right) = s_{\mathcal{C}(U,Y)}\left([1]^{2k}\right).$$

Actually, those terms are not only equal but they come up with the same sign. We can thus limit ourselves to evaluating one of the two methods $\mathcal{B}$ or $\mathcal{C}$ (say $\mathcal{C}$), and double the result.

Furthermore, method $\mathcal{G}(U,W,Y)$ produces the same μ's as method $\mathcal{C}(U,Y)$, except that W can range anywhere between 1 and $N-1-U$ (if $W=0$ was allowed, we would fall back on method $\mathcal{C}$). However, in method $\mathcal{G}$, the sign is the exact opposite as the sign obtained through method $\mathcal{C}$.

Therefore, if we group this information, we get that

$$\sum_{\substack{0\le V\le N-1\\ 0\le X\le k-1}} N(-1)^{X+1} s_{\mathcal{B}(V,X)}\left([1]^{2k}\right) +$$
$$\sum_{\substack{0\le U\le N-1\\ 0\le Y\le k-1}} N(-1)^{Y+1} s_{\mathcal{C}(U,Y)}\left([1]^{2k}\right) + \sum_{\substack{0\le U\le N-2\\ 0\le Y\le k-1\\ 1\le W\le N-U-1}} (-1)^{Y} s_{\mathcal{G}(U,W,Y)}\left([1]^{2k}\right) =$$
$$\sum_{\substack{0\le U\le N-1\\ 0\le Y\le k-1}} (\mathbf{2N}-(\mathbf{N}-\mathbf{U}-\mathbf{1}))(-1)^{Y+1} s_{\mathcal{C}(U,Y)}\left([1]^{2k}\right) \quad (5.8)$$

(the extra factors of N are explained just as in Proposition 5.2).

We simplify this and get

Proposition 5.7. *Methods $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{G}$ together contribute*

$$\sum_{\substack{0\le U\le N-1\\ 0\le Y\le k-1}} (N+U+1)(-1)^{Y+1} s_{\mathcal{C}(U,Y)}\left([1]^{2k}\right).$$

5.7.4 Contributions due to methods $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$

This is the only case that has infinitely many terms. Essentially, we had to drag all the t's so far just to handle convergence issues in this case.

However, we are helped in that these terms will be infinitely many copies of the same finite few. This will be crucial in proving the convergence of the series as t approaches 1.

A first key fact is that $\mathcal{D}(U,Y,Z+R)$ and $\mathcal{D}(U,Y,Z)$ produce the same partition μ. Therefore, in our big sum, their terms would be

$$\frac{s_{\mathcal{D}(U,Y,Z)}\left([t]^k,[1/t]^k\right)}{t^{U+Y+1+2Z}} \text{ and } \frac{s_{\mathcal{D}(U,Y,Z+R)}\left([t]^k,[1/t]^k\right)}{t^{U+Y+1+2Z+2R}}.$$

Notice how the left and right ones only differ by a factor of $\frac{1}{t^{2R}}$.

The same will be true for methods $\mathcal{E}$ and $\mathcal{F}$[4].

As a consequence, the sum of all the terms associated with these methods will factor out very neatly into

$$\underbrace{\left[\sum_{\mu\in S}\frac{(-1)^{x(\mu)}s_\mu\left([t]^k,[1/t]^k\right)}{t^{w(\mu)}}\right]}_{F(t)}\cdot\left(1+\frac{1}{t^2}+\frac{1}{t^4}+\frac{1}{t^6}+\cdots\right),$$

for an appropriate finite set S and appropriate functions w and x. The function x is the parity function, which is described in the last column of Figure 5.1. We will not explicitly describe w or S, except to say that S is the set of all the partitions that can be obtained through methods $\mathcal{D}(U,Y,1)$, $\mathcal{E}(U,0)$ or $\mathcal{F}(U,X,0)$ (repeats are allowed: the same μ can be obtained through more than one of those ways, and $w(\cdot)$ will then take different values on the same μ, as explained later).

The cardinality of S is exactly $(N-1)((k-1)+1+(k-1))$. Therefore, $F(t)$ is well-defined, and the sum of all those terms will evaluate to

$$\lim_{t\to 1}F(t)\cdot\frac{t^2}{t^2-1}. \tag{5.9}$$

We are guaranteed by convergence of the integral $\int_{\mathrm{U}(N)}|\Lambda_g'(1)|^2|\Lambda_g(1)|^{2k-2}\mathrm{d}g$ that the limit in (5.9) exists. Hence $F(1)=0$, and this limit evaluates (by L'Hopital) to $F'(1)/2$.

Lemma 5.8. *If* $F(t)=\sum_{\mu\in S}\frac{(-1)^{x(\mu)}s_\mu\left([t]^k,[1/t]^k\right)}{t^{w(\mu)}}$ *as before, then*

$$F'(1)=\sum_{\mu\in S}-(-1)^{x(\mu)}w(\mu)s_\mu\left([1]^{2k}\right).$$

Proof. This is a simple consequence of the chain rule and the symmetry of Schur

[4]There is a notable difference between method $\mathcal{D}$ and methods $\mathcal{E}$ and $\mathcal{F}$, however: the bounds on Z are different ($Z\geq 1$ for the former, $Z\geq 0$ for the latter). Indeed, if $Z=0$ in the latter cases, other blocks are removed are still removed, while this is not true for method $\mathcal{D}$, where $Z=0$ makes us fall back on method $\mathcal{C}$.

polynomials in their variables:

$$\frac{\mathrm{d}}{\mathrm{d}t} s_\mu\left([t]^k,[1/t]^k\right)\Big|_{t=1} = k\frac{\partial}{\partial x_1} s_\mu\left([t]^k,[1/t]^k\right)\Big|_{t=1} - \frac{k}{t^2}\frac{\partial}{\partial x_1} s_\mu\left([1/t]^k,[t]^k\right)\Big|_{t=1} = 0$$

In this equation, $\frac{\partial}{\partial x_1} s_\mu$ means the derivative of that Schur polynomial with respect to its first variable.

Hence, in the sum

$$F'(1) = \sum_{\mu\in S}(-1)^{x(\mu)}\left[\frac{\mathrm{d}}{\mathrm{d}t} s_\mu\left([t]^k,[1/t]^k\right)\Big|_{t=1} - w(\mu) s_\mu\left([1]^{2k}\right)\right],$$

only every other term survives and we thus get the statement. □

We have now essentially proved this statement:

Proposition 5.9. *The contributions due to methods $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ sum up to*

$$F'(1)/2.$$

This can alternatively be expressed as a signed (by x), weighted (by w) sum of the terms coming from method $\mathcal{C}$ and the term coming from method $\mathcal{A}$.

Proof. We only have to remark that when $\mu \in S$, $s_\mu\left([1]^{2k}\right)$ is equal to a Schur function showing up in either Proposition 5.5 or Proposition 5.7. □

This can be made more precise, in particular by computing $w(\cdot)$ explicitly.

Proposition 5.10. *The contributions due to methods $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ sum up to*

$$\sum_{\substack{0\le U\le N-1\\0\le Y\le k-1}} \frac{(-1)^Y(U+Y+3)}{2} s_{\mathcal{C}(U,Y)}\left([1^{2k}]\right) + \sum_{0\le U\le N-1}\frac{-(2U+2)}{2} s_{\mathcal{A}}\left([1]^{2k}\right) + \sum_{\substack{0\le U\le N-1\\0\le Y\le k-1}} \frac{(-1)^Y(2N+Y+1-U)}{2} s_{\mathcal{C}(U,Y)}\left([1^{2k}]\right)$$

Proof. For $\mathcal{D}$, we need to first use Proposition 5.4. Then, $w(\cdot)$ for $\mathcal{D}(U,Y,1)$ (resp. $\mathcal{E}(U,0)$, resp. $\mathcal{F}(N-1-U,Y,0)$) is $U+Y+3$ (resp. $2U+2$, resp. $2N+Y+1-U$). The value of w for $\mathcal{D}(U,Y,1)$ might be surprising, but one needs to remember that $Z \geq 1$, unlike $\mathcal{E}(U,0)$ or $\mathcal{F}(N-1-U,Y,0)$, where $Y \geq 0$ (see footnote 4, page 73). □

5.8 An exact formula for fixed N

Putting together Propositions 5.5, 5.6, 5.7 and 5.10, we get the following identity

Theorem 5.11.

$$\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^2 |\Lambda_g(1)|^{2k-2} \mathrm{d}g = \sum_{\substack{0 \leq U \leq N-1 \\ 0 \leq Y \leq k-1}} (-1)^{Y+1}(U-Y-1) s_{\mathcal{C}(U,Y)}\left([1^{2k}]\right) + \frac{N^2-N}{2} s_{\mathcal{A}}\left([1]^{2k}\right) \quad (5.10)$$

Observe that this is the first version of the formula predicted in Section 5.1.

Proof. We just add up the coefficients for each Schur function. Relevant identities are

$$(N+U+1) - \frac{U+Y+3}{2} - \frac{2N+Y+1-U}{2} = U-Y-1$$

and

$$N^2 - \sum_{U=0}^{N-1}(U+1) = \frac{N^2-N}{2}.$$

□

This provides an exact formula, using Equation (5.7). It is best expressed with the standard notation for the rising factorial

$$X^{(k)} := X \cdot (X+1) \cdot \dots \cdot (X+k-1).$$

Theorem 5.12.

$$\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^2 |\Lambda_g(1)|^{2k-2} \mathrm{d}g = s_{\langle N^k \rangle}\left([1]^{2k}\right) \cdot \left[\frac{N^2-N}{2} + \sum_{0 \le U \le N-1} \frac{(N-U)^{(k)}(U+1)^{(k)}}{(k-1)!} \cdot \sum_{0 \le Y \le k-1} (-1)^{Y+1} \binom{k-1}{Y} \frac{U-Y-1}{(N+Y+1)^{(k)}(U+Y+1)}\right] \quad (5.11)$$

Proof. Starting with Theorem 5.11, we only need to compute $s_{\mathcal{C}(U,Y)}$ and $s_{\mathcal{A}}$. This can be done easily using the Weyl Dimension Formula (Formula (5.7)).

Evaluation of $s_{\mathcal{A}}$: We split the rows of the partition into two sets and compute the relative distance of trailing boxes. The following table compiles the products of factors due to pairs (i, j) of each type:

	$1 \le i \le k$	$k+1 \le i \le 2k$
$1 \le j \le k$	$\prod_{i=0}^{k-1} i!$	nothing
$k+1 \le j \le 2k$	$\prod_{i=1}^{k} (N+i)^{(k)}$	$\prod_{i=0}^{k-1} i!$

The empty slot occurs because we always have the requirement $i < j$.

Hence, taking the product of all those terms and dividing by $\prod_{1 \le i < j \le 2k} (j-i)$

(as described in Equation (5.7)), we get

$$s_{\mathcal{A}} = \frac{\prod_{i=0}^{k-1} i!^2 \prod_{i=1}^{k} (N+i)^{(k)}}{\prod_{i=1}^{2k-1} i!}.$$

Evaluation of $s_{\mathfrak{C}(U,Y)}\left([1]^{2k}\right)$**:** The computation for $s_{\mathfrak{C}(U,Y)}\left([1]^{2k}\right)$ is similar but slightly harder. The following table compiles the products of factors due to pairs (i,j) of each type:

	$1 \le i \le k$	$i = k+1$	$k+1 < i \le 2k$
$1 \le j \le k$	$\prod_{i=0}^{k-1} i!$	nothing	nothing
$j = k+1$	$(N-U)^{(k)}$	nothing	nothing
$k+1 < j \le 2k$	$\frac{\prod_{i=1}^{k}(N+i)^{(k)}}{(N+Y+1)^{(k)}}$	$\frac{(U+1)^{(k)}}{U+Y+1}$	$\frac{\prod_{i=0}^{k-1} i!}{Y!(k-Y-1)!}$

Again, the empty slots occur because of the permanent condition $i < j$, and we get

$$s_{\mathfrak{C}(U,Y)}\left([1]^{2k}\right) = \frac{\prod_{i=1}^{k-1} i!^2}{\prod_{i=1}^{2k-1} i!} \cdot \frac{\binom{k-1}{Y}}{(k-1)!} \cdot \frac{\prod_{i=1}^{k}(N+i)^{(k)}}{(N+Y+1)^{(k)}} \cdot \frac{(N-U)^{(k)}(U+1)^{(k)}}{U+Y+1}.$$

Hence, based on Theorem 5.11, and some clever factoring out of $s_{\mathcal{A}}$, we get the statement. □

Remark. All the proofs in this Chapter are relatively intricate. For peace of mind, the statement in Theorem 5.12 has been numerically tested. The LHS in Equation (5.11) can be numerically evaluated using Pari or Mathematica[5] and the

[5]I wish to thank Henry Segerman for his help with Mathematica.

explicit formula for the Haar measure on $U(N)$:

$$\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^2 |\Lambda_g(1)|^{2k-2} \mathrm{d}g = \int \cdots \int_{[0,2\pi]^N} f(\theta_1, \cdots, \theta_N) \prod_{1 \le j < k \le N} \left| e^{i\theta_k} - e^{i\theta_j} \right|^2 \frac{\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_N}{N!(2\pi)^N},$$

where

$$f(\theta_1, \cdots, \theta_N) = \left| \left(\prod_{1 \le j \le N} (1 - e^{i\theta_j}) \right)^k \cdot \left(\sum_{1 \le j \le N} \frac{1}{1 - e^{i\theta_j}} \right) \right|^2.$$

The following table summarizes the data for $\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^2 |\Lambda_g(1)|^{2k-2} \mathrm{d}g$, with boldface numbers indicating results checked through this method while the lighter numbers are the results obtained only through the RHS of Equation (5.11).

$k \backslash N$	1	2	3	4
1	**1**	**5**	**14**	30
2	**2**	**24**	**130**	476
3	**6**	**195**	**2394**	17640
4	**20**	**1904**	**58800**	975744
5	**70**	**20580**	**1718640**	69652440
6	**252**	**237600**	**56571372**	5951885940
7	**924**	**2873871**	2031199170	581823618376
8	**3432**	**35990240**	77947661792	63152935276288
9	**12870**	**463047156**	3152975688720	7454881820996064
10	**48620**	**6087542240**	133107303325360	942715942296972480
	up to 100	**up to 20**		

As displayed in the table, Mathematica gives up already on the quadruple integral.

It is striking that those values are only integers. This was not originally expected, so the presentation so far does not really explain it. However it follows from expressing $\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^2 |\Lambda_g(1)|^{2k-2} \mathrm{d}g$ in a way similar to Equation 5.2 (but as a finite sum). Each $s_\mu([1^{2k}])$ being an integer, that integrality property for the sum will follow.

We will get back to results at fixed N in Section 5.10, but we now temporarily focus on asymptotic results.

5.9 Asymptotic result

While Theorem 5.12 gives an exact expression, it can also be misleading. If we naively estimate the size of the factor appearing in that Proposition by ignoring the sign $(-1)^{Y+1}$, we get

$$\frac{N^2-N}{2}+\sum_{0\le U\le N-1}\frac{(N-U)^{(k)}(U+1)^{(k)}}{(k-1)!}\cdot \sum_{0\le Y\le k-1}\binom{k-1}{Y}\frac{U-Y-1}{(N+Y+1)^{(k)}(U+Y+1)}. \quad (5.12)$$

If we use that $N^{(k)}\sim N^k$ and $U\sim N$ in the large N limit, we see that Expression (5.12) is on the order of

$$N^2+\sum_{0\le U\le N-1}N^kU^k\frac{U}{N^kU}\sim N^{k+1}.$$

In other words, this mean that the second term in Expression (5.12), the sum over U, *seems* to entirely dominates the first term. We will show that this is actually inaccurate: a lot of cancellation occurs[6] in the sum over U, and both terms will turn out to be of order N^2.

In order to prove this, we start with a combinatorial lemma.

[6]Not quite as dramatic as in the estimate for methods $\mathcal{H}$ and $\mathcal{I}$, which entirely cancelled out. By the way, a similar naive estimate for those methods would lead to a sum of even higher order: N^{k^2+2k}. See Section 5.11.2 for more on this.

Lemma 5.13. *The following identities are true*

$$\sum_{0\le Y\le k-1}(-1)^{Y+1}\binom{k-1}{Y}Y^j=\begin{cases}0 & \text{for } 0\le j<k-1\\ (-1)^k(k-1)! & \text{for } j=k-1\end{cases}$$

Proof. Set $f(X)=(X-1)^{k-1}$ and expand using the binomial theorem. Evaluate $f(1), f'(1), f''(1), \cdots f^{(k-2)}(1)$ by taking successive derivatives of that expansion. □

We directly put this lemma to good use:

Lemma 5.14. *For large N and $U>k$,*

$$\sum_{0\le Y\le k-1}(-1)^{Y+1}\binom{k-1}{Y}\frac{U-Y-1}{(N+Y+1)^{(k)}(U+Y+1)}\sim_N$$
$$\sum_{0\le Y\le k-1}(-1)^{Y+1}\binom{k-1}{Y}\frac{U-Y}{(N+Y)^k(U+Y+1)}.$$

Proof. We will expand

$$\frac{1}{(N+Y+1)^{(k)}(U+Y+1)}=\frac{1}{N^k(U+1)}\cdot\frac{1}{\left[\prod_{i=1}^k\left(1+\frac{Y+i}{N}\right)\right]\left(1+\frac{Y}{U+1}\right)}=$$
$$\frac{1}{N^k(U+1)}\left(1-\frac{Y}{U+1}+\left(\frac{Y}{U+1}\right)^2+\cdots\right)\prod_{i=1}^k\left(1-\frac{Y+i}{N}+\left(\frac{Y+i}{N}\right)^2+\cdots\right)$$

By Lemma 5.13, we need to focus on terms with exponent at least $k-1$ on Y. Furthermore, we want to have the lowest possible power of N on the denominator. Therefore, when expanding $\left(\frac{Y+i}{N}\right)^j$, we will only be concerned with the term $\frac{Y^j}{N^j}$.

Hence,

$$\sum_{0\le Y\le k-1} (-1)^{Y+1}\binom{k-1}{Y}\frac{U-Y-1}{(N+Y+1)^{(k)}(U+Y+1)} \sim_N$$
$$\sum_{0\le Y\le k-1}\left[\frac{(-1)^{Y+1}\binom{k-1}{Y}(U-Y)}{N^k(U+1)}\left(1-\frac{Y}{U+1}+\left(\frac{Y}{U+1}\right)^2+\cdots\right)\cdot\right.$$
$$\left.\prod_{i=1}^{k}\left(1-\frac{Y}{N}+\frac{Y^2}{N^2}+\cdots\right)\right] = \sum_{0\le Y\le k-1}(-1)^{Y+1}\binom{k-1}{Y}\frac{U-Y}{(U+Y+1)(N+Y)^k}.$$

□

We need one last combinatorial lemma:

Lemma 5.15. *In $\mathbb{Q}[[r,s,X]]$, the sum of the terms of degree $k-1$ in X in*

$$(1-rX+r^2X^2-\cdots)^k(1-sX+s^2X^2-\cdots),$$

is

$$(-1)^{k+1}\sum_{i=0}^{k-1}\binom{k-1+i}{i}r^i s^{k-1-i}X^{k-1},$$

and the sum of the terms of degree $k-2$ in X is

$$(-1)^{k}\sum_{i=0}^{k-2}\binom{k-1+i}{i}r^i s^{k-2-i}X^{k-2},$$

Proof. We first observe that

$$(1+aX+a^2X^2-\cdots)^k = \sum_{i=0}\binom{i+k-1}{k-1}a^iX^i$$

has a very simple combinatorial proof: we have to split i (ordered) powers of X into k subsets, and hence use $k-1$ dividers. We use this equality with $a := -r$.

The rest of the proof is immediate by expanding the product with $(1-sX+s^2X^2-\cdots)$. □

This is all we need for the following theorem.

Theorem 5.16. *In the large N limit,*

$$\int_{\mathrm{U}(N)} |\Lambda'_g(t)|^2 |\Lambda_g(t)|^{2k-2} \mathrm{d}g \sim \frac{k^2 N^2}{(2k-1)(2k+1)} s_{\langle N^k \rangle}\left([1]^{2k}\right) \sim \frac{k^2}{(2k-1)(2k+1)} \cdot \frac{\prod_{i=0}^{k-1} i!^2}{\prod_{i=0}^{2k-1} i!} N^{k^2+2}.$$

Proof. We start with Lemma 5.14, and then combine this (twice) with Lemma 5.15 for $X := Y$, $r := \frac{1}{N}$ and $s := \frac{1}{U+1}$ to extract the terms of degree $k-1$ in Y. We thus get that the expansion of

$$\frac{U - Y}{(N + Y + 1)^{(k)}(U + Y + 1)}$$

as a power series in Y has coefficient asymptotic to

$$\frac{(-1)^{k+1}}{N^k(U+1)} \left(\binom{2k-2}{k-1} \frac{U}{N^{k-1}} + \sum_{i=0}^{k-2} \binom{k-1+i}{i} \frac{2U+1}{N^i (U+1)^{k-i-1}} \right)$$

for Y^{k-1}, in the large N limit.

We now apply Lemma 5.13, and get that the sum over U in the RHS of Equation (5.11) is asymptotic to

$$\sum_{0 \le U \le N-1} \frac{-(N-U)^k}{N^k} \left(\binom{2k-2}{k-1} \frac{U^k}{N^{k-1}} + \sum_{i=0}^{k-2} \binom{k-1+i}{i} \frac{2U^k}{N^i U^{k-i-1}} \right) \quad (5.13)$$

as $N \to \infty$ (the sum is now positive, allowing us to throw away any term of lower combined order in U and N).

We now perform the change of variable $u := U/N$, try to isolate as much of N's as possible and find that Expression (5.13) is equal to

$$-N \cdot \sum_{u \in \{0, \frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N}\}} (1-u)^k \left(\binom{2k-2}{k-1} u^k + \sum_{i=0}^{k-2} 2 \binom{k-1+i}{i} u^{1+i} \right).$$

This can be seen as a Riemann sum that divides the interval $[0,1]$ into N subintervals. In the large N limit, this expression approaches the integral

$$-N^2 \int_{[0,1]} (1-u)^k \left(\binom{2k-2}{k-1} u^k + \sum_{i=0}^{k-2} 2 \binom{k-1+i}{i} u^{1+i} \right) \mathrm{d}u.$$

The extra factor of N appears because we divide by the length of each subinterval (which is $1/N$).

This can be expressed in terms of beta integrals as

$$-N^2 \left(\binom{2k-2}{k-1} B(k+1,k+1) + 2 \sum_{i=0}^{k-2} \binom{k-1+i}{i} B(k+1,i+2) \right),$$

where $B(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1}\mathrm{d}t$ is the usual beta function[7].

We now use the identity

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)},$$

so

$$\binom{2k-2}{k-1} B(k+1,k+1) + 2 \sum_{i=0}^{k-2} \binom{k-1+i}{i} B(k+1,i+2) = \frac{k}{2(2k+1)(2k-1)} + 2 \sum_{i=0}^{k-2} \frac{k(i+1)}{(k+i)(k+i+1)(k+i+2)}.$$

This last sum can be estimated using partial fraction decomposition and telescoping the sum over i, or can be evaluated as an hypergeometric series. In any case,

$$\sum_{i=0}^{k-2} \frac{k(i+1)}{(k+i)(k+i+1)(k+i+2)} = \frac{k-1}{4(2k-1)},$$

[7]The very attentive reader might want to figure out what integrals would appear if multiple ribbons were added and removed. This will be discussed in Section 5.11.3.

so not forgetting the term $\frac{N^2-N}{2}$, the leading term in Expression (5.11) will be

$$\left(1/2 - \frac{k}{2(2k+1)(2k-1)} - \frac{k-1}{2(2k-1)}\right) N^2 = \frac{k^2}{(2k+1)(2k-1)} N^2,$$

which proves the first statement.

The second statement follows easily from the equality

$$s_{\langle N^k \rangle}\left([1]^{2k}\right) = \frac{\prod_{i=0}^{k-1} i!^2}{\prod_{i=0}^{2k-1} i!} \prod_{i=1}^{k} (N+i)^{(k)}.$$

□

All the discussion in this Section can be bypassed if one proves Conjecture 5.17 in the next Section.

5.10 An unexpected identity

We observe empirically the following Conjecture.

Conjecture 5.17. *For $N \geq 1$ and $k \geq 1$,*

$$\frac{N^2-N}{2} + \sum_{0 \leq U \leq N-1} \frac{(N-U)^{(k)}(U+1)^{(k)}}{(k-1)!} \cdot \sum_{0 \leq Y \leq k-1} (-1)^{Y+1} \binom{k-1}{Y} \frac{U-Y-1}{(N+Y+1)^{(k)}(U+Y+1)} = \frac{2N^2k^2+Nk}{2(2k-1)(2k+1)}. \quad (5.14)$$

This is verified on computer for $0 < N \leq 100$ and $0 < k \leq 100$ (or $0 < N < 10000$, $0 < k < 10$). The case $k = 1$ is the pyramidal square numbers identity. The case $N = 1$ is the hypergeometric identity

$$\frac{k}{k+1} \; {}_2F_1\left({2 \;\; 1-k \atop k+2}; 1\right) = \frac{k}{2(2k-1)}.$$

Conjecture 5.17 along with the formula in Theorem 5.12 immediately leads to the following Theorem (this conveniently skips Section 5.9).

Theorem 5.18. *We remind the reader that*

$$s_{\langle N^k \rangle}\left([1]^{2k}\right) = \frac{\prod_{i=0}^{k-1} i!^2}{\prod_{i=0}^{2k-1} i!} \prod_{i=1}^{k} (N+i)^{(k)} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2}.$$

Then, if Conjecture 5.17 is true, for $N \geq 1$ and $k \geq 1$,

$$\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^2 |\Lambda_g(1)|^{2k-2} \mathrm{d}g = \frac{2N^2k^2 + Nk}{2(2k-1)(2k+1)} \cdot s_{\langle N^k \rangle}\left([1]^{2k}\right).$$

This Theorem not only (conjecturally) simplifies the expression in Theorem 5.11, but it would also extend its validity, thanks to Carlson's Theorem.

Theorem 5.19 (Carlson, see [AAR99, Theorem 2.8.1, p. 110])**.** *Let $f(z)$ be an analytic function for $\operatorname{Re} z \geq 0$ such that*

- *$f(n) = 0$ for $n \in \mathbb{N}$,*
- *f is of exponential type (i.e. $|f(z)| \leq Ce^{\tau|z|}$ for some τ),*
- *the type of f along a fixed vertical axis is $< \pi$ (i.e. there exists C', c and ϵ such that $|f(c+iy)| \leq C'e^{(\pi-\epsilon)|y|}$).*

Then, f is identically zero.

This allows us to prove the following:

Theorem 5.20. *Provided Conjecture 5.17 is true,*

$$\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^2 |\Lambda_g(1)|^{2k-2} \mathrm{d}g = \frac{2N^2k^2 + Nk}{2(2k-1)(2k+1)} \cdot \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2} \tag{5.15}$$

is also valid for complex k, with $\operatorname{Re} k > 1/2$. Moreover, the meromorphic continuation of the LHS to the whole complex plane is given by the RHS.

Proof. The first step is to show that

$$\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^{2k'} |\Lambda_g(1)|^{2k-2k'} \mathrm{d}g, \quad k', k \in \mathbb{Z}$$

admits an analytic continuation (keeping k' unchanged) to $k \in \mathbb{C}, \operatorname{Re} k > k' - 1/2$.

The integral over $\mathrm{U}(N)$ equals an integral over $[0, 2\pi]^N$, making the Haar measure more explicit, or even $[0,1]^N$ after a change of variables.

In other words,

$$\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^{2k'} |\Lambda_g(1)|^{2k-2k'} \mathrm{d}g = \int \cdots \int_{[0,1]^N} f(\theta_1, \cdots, \theta_N) \prod_{1 \le j < k \le N} \left| e^{2\pi i \theta_k} - e^{2\pi i \theta_j} \right|^2 \frac{\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_N}{N!}, \quad (5.16)$$

with

$$f(\theta_1, \cdots, \theta_N) = \left(\prod_j \left| 1 - e^{2\pi i \theta_j} \right|^{2k} \right) \cdot \left| \sum_j \frac{1}{1 - e^{2\pi i \theta_j}} \right|^{2k'}.$$

Instead of computing (5.16) directly, we first compute the complex integral

$$\int \cdots \int_{\gamma^N} f(\theta_1, \cdots, \theta_N) \prod_{1 \le j < k \le N} \left| e^{2\pi i \theta_k} - e^{2\pi i \theta_j} \right|^2 \frac{\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_N}{N!}, \quad (5.17)$$

where γ is the Pochhammer contour (see Figure 5.2).

Here the contour starts on $[0,1]$, goes to the point 1 and encircles it in the clockwise direction (at distance R), goes towards the point 0, encircles it in the clockwise direction (at distance R), returns to 1 and repeats everything in the opposite orientation, so that the winding number of γ around each point is zero.

The limit for $R \to 0^+$ of (5.17) will exist as long as the integrals over the small circles converge, which happens when $2k - 2k' > -1$ (the same condition implies convergence around the points 0 and 1). Since those slight deformations of γ do not change the value of (5.17), we have established that (for each variable) the integral over γ equals the integral over 4 segments $[0,1]$, but on different branches. Dividing by the appropriate factor (up to a sign, $(1 - e^{2\pi i(2k-2k'+1)})^{2N}$), we obtain

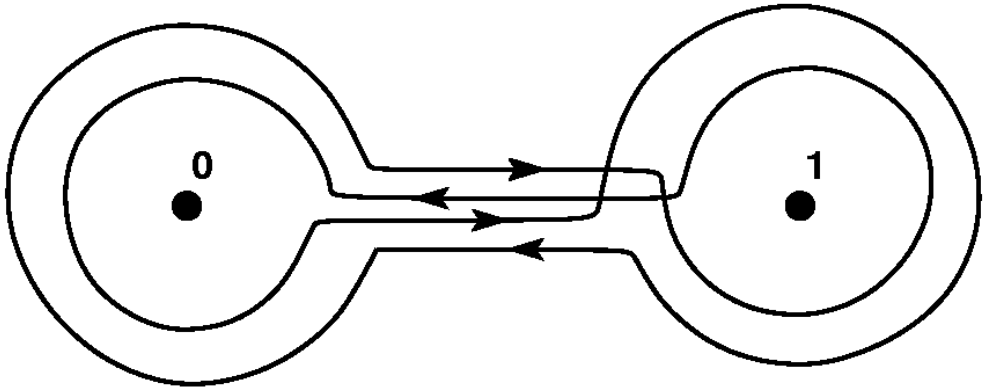


Figure 5.2: The Pochhammer contour.

an integral expression that equals when $k - k' > 1/2$ the original integral (5.16) but has the advantage of making the meromorphic continuation of (5.16) to $k \in \mathbb{C}$ clear. Note that the singularities that we introduced at half-integers when dividing by this factor $(1 - e^{2\pi i(2k-2k'+1)})^{2N}$ are all removable.

At this point, we have proved that the LHS of Equation (5.15) admits a meromorphic continuation to the whole complex plane and is actually holomorphic when $\operatorname{Re} k > 1/2$ (take $k' = 1$ in what we have done so far).

We would like now to prove that this meromorphic continuation is the RHS. The natural way to proceed is now to use Carlson's Theorem on Theorem 5.18: under the assumption of Conjecture 5.17, we know that the difference of the RHS and the LHS of Equation 5.15 is zero for $k \in \mathbb{N} \setminus \{0\}$. We only need to prove the bounds presented in the statement of Theorem 5.19.

This is rather quick:

- We have to prove the type conditions for the analytic continuation in k of $\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^2 |\Lambda_g(1)|^{2k-2} \mathrm{d}g$. For fixed k' and for γ the Pochhammer contour, this analytic continuation is given by

$$I(k) = \int \cdots \int_{\gamma^N} f(\theta_1, \cdots, \theta_N) \prod_{1 \le j < k \le N} \left| e^{2\pi i \theta_k} - e^{2\pi i \theta_j} \right|^2 \frac{\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_N}{N!}, \quad (5.18)$$

with

$$f(\theta_1, \cdots, \theta_N) = \left(\prod_j \left|1 - e^{2\pi i \theta_j}\right|^{2k} \right) \cdot \left| \sum_j \frac{1}{1 - e^{2\pi i \theta_j}} \right|^{2k'}.$$

We have a bound along γ: $\left|1 - e^{2\pi i \theta_j}\right| \leq 2 + \epsilon < 3$. We intend to take $c = 1$ in Theorem 5.19. If $a \in \mathbb{R}$, $|a^b| = a^{\operatorname{Re} b}$ so we easily find

$$|I(1 + iy)| \leq I(1) \quad \text{and} \quad |I(z)| \leq 3^{2 \operatorname{Re} z - 2} \cdot I(1).$$

Hence $I(k)$ is of exponential type at most $\tau = 2 \ln 3$, and of type 0 along the vertical axis $1 + i\mathbf{R}$.

- As a rational function in k, we have no problem to expect from $\frac{2N^2k^2 + Nk}{2(2k-1)(2k+1)}$. Its type is 0.

- We now look at the type of

$$\chi(k) = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j + 2k)}{\Gamma(j + k)^2}.$$

Fortunately, Stirling's formula gives very precise bounds on the Γ-function. One of them (see [AAR99, Corollary 1.4.4]) says that if $x = a + ib$, $a_0 \leq a \leq a_1$ and $|b| \to \infty$, then

$$|\Gamma(a + ib)| = \sqrt{2\pi} |b|^{a - 1/2} e^{-\pi |b|/2} [1 + O(1/|b|)],$$

where the constant implied by O depends only on a_0 and a_1. Take $a_0 = 0$ and $a_1 = 1$. This bounds predicts that on the strip $0 \leq \operatorname{Re} z \leq 1$, χ has type 0 (all the terms $e^{-\pi|b|/2}$ end up canceling). Thus χ has type 0 over all $\operatorname{Re} z \geq 0$, because

$$|\chi(k + 1)| = \left| \prod_{j=1}^{N} \frac{j(j + 2k)}{(j + k)^2} \chi(k) \right| \leq \frac{N^N |2k + N|^N}{|k|^{2N}} |\chi(k)|.$$

As a consequence, Carlson's Theorem applies. □

5.11 Odds and ends

We now discuss possible extensions of the results of this Chapter.

5.11.1 For $k' > 1$

The next obvious step is to consider what would happen when $k' > 1$. The discussion of Section 5.3 would still be valid: we would add k' consecutive ribbons and then remove k' of them. The complexity of the argument would thus be greatly increased, but a first look shows that cancellations akin to Proposition 5.6 occur when adding ribbons. For instance, for any method that will add ribbons "in horizontal layers", there will be an associated method that also adds ribbons "in layers" and that will cancel it out. As a consequence, all the ribbons added have to actually partition a single ribbon in order for the method to be of any interest.

Hopefully, our discussion will carry similarly for other k''s, and will allow for an extension to real k' just as in Theorem 5.20.

5.11.2 At $t \neq 1$

This is a direct analogue of work of Hughes [Hug03, Hug01] for characteristic polynomials.

We could consider instead

$$M_N(2k, 2k'; s, t) = \int_{\mathrm{U}(N)} |\Lambda_g'(t)|^{2k'} |\Lambda_g(s)|^{2k-2k'} \mathrm{d}g.$$

Our discussion of Section 5.3 would still be valid, and the general argument from the following sections would still be the same. Notice that the delicate cancellation of methods $\mathcal{H}$ and $\mathcal{I}$ would not necessarily occur anymore. This is important, as the individual terms associated to those methods have highest intrinsic order, and are expected to increase the order of N^{k^2+2} to a higher one.

5.11.3 From beta integral to Selberg integral

This is a clarification on the footnote of page 83.

The beta integral appears in Section 5.9 as a limit of a Riemann sum. The index set for this Riemann sum is the possible step sizes in the partition $\mathcal{C}(U,Y)$, i.e. the variable U. The crucial fact is that there is only one step in those partitions, so we end up needing to evaluate beta integrals

$$B(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1}\mathrm{d}x.$$

In either of the two possible extensions presented above, i.e. when $t \neq 1$ or $k' > 1$, it is most likely that partitions with more than one step (say n) will enter the spotlight (for instance coming out of methods $\mathcal{H}$ and $\mathcal{I}$ when $t \neq 1$). In taking a similar limit as before, a Riemann sum over those partitions will then become a multivariable analog of a beta integral, i.e. a Selberg integral

$$\int_{[0,1]^n} \prod_{i=1}^{n} x_i^a (1-x_i)^b \prod_{1\leq i<j\leq n} |x_i - x_j|^{2c}\mathrm{d}\underline{x}.$$

The analogy with the geometric interpretation of the Weyl Dimension Formula presented in Equation (5.7) is striking:

- Each x_i will represent the length of one of the steps (think U before).
- Each term x_i^a will represent the distance between trailing boxes in a row with a step and one of the later rows (of length $\leq k << N$, since we are adding at most k' ribbons) .
- Each term $(1-x_i)^b$ will represent the distance between trailing boxes in a row with a step and one of the earlier rows (of length $\geq N - k'$, since we are removing at most k' ribbons; think $(N-U)^b$ before).
- Each term $(x_i - x_j)$ will represent the distance between trailing boxes in two rows with a step.

We find this connection between Selberg integral, the original integrals computed, and the combinatorial fixed N analogues very inspiring.

5.11.4 Other classical Lie groups

As explained earlier, there is also significant interest in Number Theory for similar statistics over other classical Lie groups $(\mathrm{SO}(m), \mathrm{Sp}(2n))$. We expect branching rules (see Section 2.1.3) to be useful once again in bridging the new difficulties that arise over those groups.

Chapter 6

Conclusion

In this thesis, we have presented theorems that we hope will be useful for a more systematic study of integrals over compact Lie groups of symmetric functions in eigenvalues.

We have first shown how those integrals are affected when introducing Weyl characters in the integrand.

We have then discussed the case of two simultaneous such twists in relations to two successive ones.

Finally, we applied those ideas and a few others to the computation of

$$\int_{\mathrm{U}(N)} |\Lambda_g'(1)|^2 |\Lambda_g(1)|^{2k-2} \mathrm{d}g.$$

Our result is valid for $k \in \mathbb{Z}, k > 1$ and conjecturally for $k \in \mathbb{C}, \operatorname{Re} k > 1/2$ as well. This conjecture depends on a hypergeometric identity.

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