

Summary

We use **asymptotic Plancherel partitions** for **random matrix theory** problems, with applications in **analytic number theory**.

Question

We are interested in

$$g(2k, 2h) := \lim_{N \rightarrow \infty} \frac{1}{N^{k^2+2h}} \left\langle |Z_U(0)|^{2k-2h} |Z'_U(0)|^{2h} \right\rangle_{U(N)},$$

with U a random $N \times N$ unitary matrix distributed according to Haar measure and with characteristic polynomial

$$Z_U(\theta) = \prod_{j=1}^N (1 - e^{i(\theta_j - \theta)}).$$

We could also study the large k and h behaviour at finite N .

Motivation

Hughes has conjectured that for $0 \leq h \leq k$, with $k, h \in \mathbb{N}$,

$$\lim_{T \rightarrow \infty} \frac{1}{T(\log T)^{k^2+2h}} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k-2h} \left| \zeta'\left(\frac{1}{2} + it\right) \right|^{2h} dt = a_k g(2k, 2h),$$

with a_k a (known) Euler product.

The characteristic polynomials of random matrices serve as model for ζ , but **this conjectural model itself needs to be better understood!**

Those moments are tied to the existence of much-larger-than-average gaps between zeroes of ζ , as shown by Hall.

Definitions

Hooklengths and contents

1210	8	7	6	5	3	2	1	0	1	2	3	4	5	6	7	8	
8	6	4	3	2	1	-1	0	1	2	3	4						
3	1	-2	-1														
1	-3																

The hooklengths $h(\square)$ and contents $c(\square)$ for $\square \in (9, 6, 2, 1)$.

We set

$$h(\lambda) = \prod_{\square \in \lambda} h(\square) \quad \text{and} \quad k \uparrow \lambda = \prod_{\square \in \lambda} (k + c(\square))$$

Classical symmetric functions

We use $s_\lambda(X)$ for the Schur symmetric functions in the set of variables X . If U is an $N \times N$ matrix with eigenvalues $\{e^{i\theta_j}\}$, we set

$$s_\lambda(U) = s_\lambda(e^{i\theta_1}, \dots, e^{i\theta_N}).$$

Properties

Orthogonal characters

The Schur functions are orthonormal characters of $U(N)$, for large N :

$$\left\langle s_\lambda(U) \overline{s_\mu(U)} \right\rangle_{U(N)} = \delta_{\lambda\mu} \delta_{\text{length}(\lambda) \leq N}$$

Hook-content formula

$$s_\lambda(\{1\}^K) = \frac{K \uparrow \lambda}{h(\lambda)}$$

Result 1: Main formula

Proposition (main formula): For $0 \leq r \leq 2k$,

$$\lim_{N \rightarrow \infty} \frac{(2i)^r \left\langle |Z_U(0)|^{2k} \left(\frac{Z'_U(0)}{Z_U(0)} \right)^r \right\rangle_{U(N)}}{\left\langle |Z_U(0)|^{2k} \right\rangle_{U(N)}} = \sum_{\lambda \vdash r} \frac{r!}{h(\lambda)^2} \frac{2^r (k \uparrow \lambda)}{(2k) \uparrow \lambda} = \sum_{\lambda \vdash r} m(\lambda) \frac{P_\lambda(k)}{Q_\lambda(k)},$$

a sum over partitions of r of rational functions, which is even in k .

Kerov's Central Limit Theorem for Plancherel measure

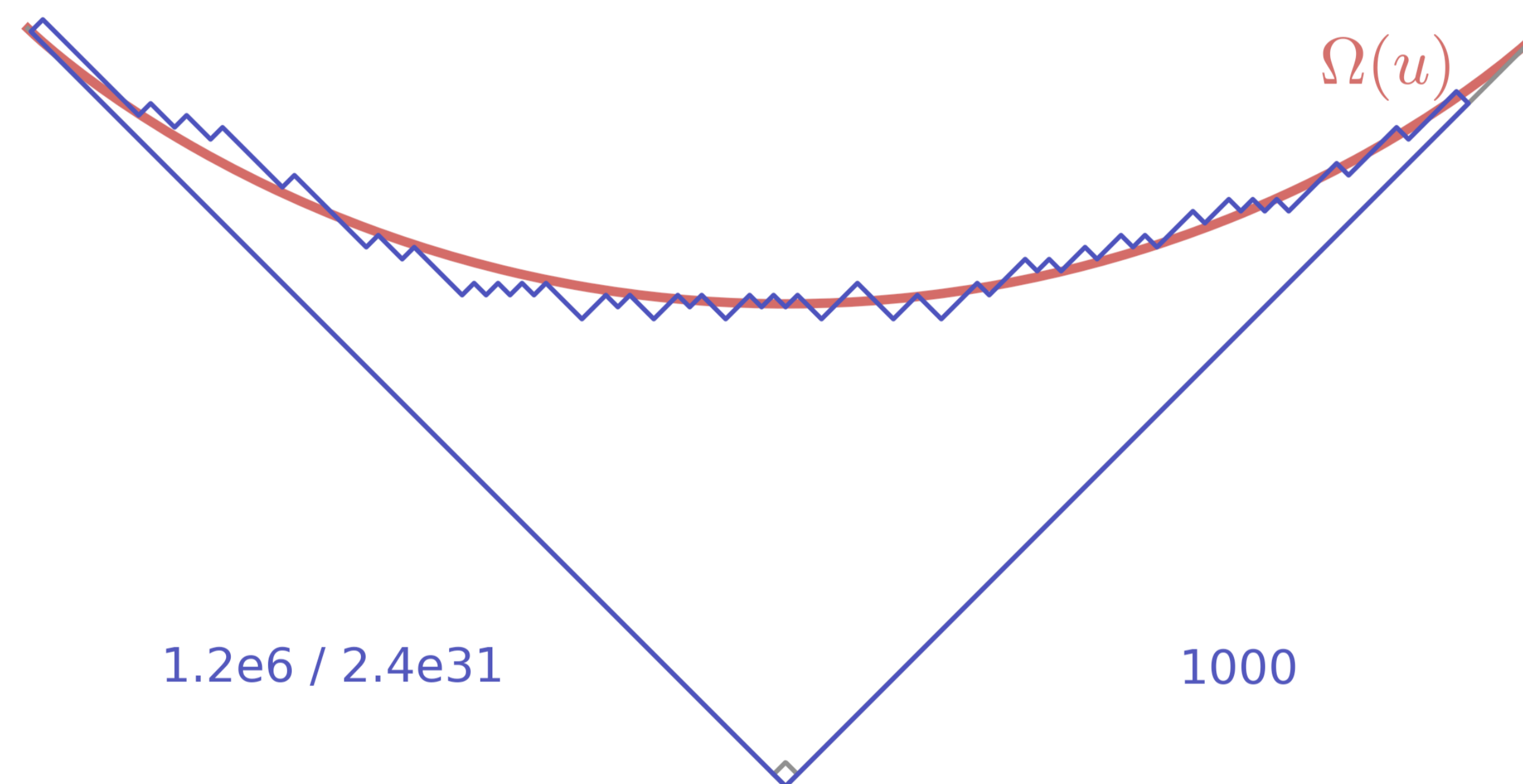
The Plancherel measure on partitions of r is

$$m(\lambda) := \frac{|\lambda|!}{h(\lambda)^2}$$

and tends for large r to the **δ -measure on a limit shape**

$$\Omega(u) = \frac{2}{\pi} \left(u \arcsin \frac{u}{2} + \sqrt{4 - u^2} \right) \quad (\text{for } |u| < 2, |u| \text{ otherwise})$$

after renormalization $\tilde{\lambda}(u) := \frac{1}{\sqrt{r}} \lambda(\sqrt{r}u)$ of the Russian diagrams,



in the sense that

$$\lim_{n \rightarrow \infty} \sum_{\lambda \vdash n} m(\lambda) \int_{-\infty}^{\infty} (\tilde{\lambda}(u) - \Omega(u)) g(u) du = 0.$$

The **fluctuations** $\tilde{\lambda}(u) - \Omega(u)$ have been studied by Kerov, Ivanov and Olshanski: their coefficients in the Chebyshev basis tend to Gaussians.

Result 2: Nekrasov-Okounkov style asymptotics

By a refinement of a method of Nekrasov and Okounkov, after expressing $\frac{2^r (k \uparrow \lambda)}{(2k) \uparrow \lambda}$ as an integral against $\lambda(u)$, we use these fluctuations to get

Theorem: For each $0 \leq r \leq 2k$, set

$$\sum_{\lambda \vdash r} m(\lambda) \frac{2^r (k \uparrow \lambda)}{(2k) \uparrow \lambda} = \frac{1}{k^0} + f_1(r) \frac{1}{k^2} + f_2(r) \frac{1}{k^4} + \dots$$

Then,

$$f_j(r) \sim \frac{(-1)^j r^{2j}}{8^j j!}$$

Result 3: Stanley's polynomiality of Plancherel averages

Let $F \in \Lambda(X)$ be a symmetric polynomial. Then, Stanley has shown that

$$\sum_{\lambda \vdash r} m(\lambda) F(\{c(\square) : \square \in \lambda\}) \in \mathbb{Q}[r],$$

with this polynomial explicitly given when $F = e_\mu$ (elementary symmetric).

Theorem: We have $f_j(r) \in \mathbb{Q}[r]$.

Proof of main formula, part 1: Bump-Gamburd derivation

Bump and Gamburd use the **dual Cauchy identity**

$$\sum_{\lambda} s_\lambda(\bar{X}) s_{\lambda^t}(\bar{Y}) = \prod_{m,n} (1 + x_m y_n)$$

and then **orthonormality** to compute

$$\begin{aligned} \left\langle |Z_U(0)|^{2k} \right\rangle_{U(N)} &= (-1)^{kN} \left\langle \det(U)^k \det(I - U)^{2k} \right\rangle_{U(N)} \\ &= (-1)^{kN} \left\langle \frac{1}{s_{\langle kN \rangle}(U)} \sum_{\lambda} s_\lambda(\{-1\}^{2k}) s_{\lambda^t}(U) \right\rangle_{U(N)} = s_{\langle N^k \rangle}(\{1\}^{2k}) \end{aligned}$$

Proof of main formula, part 2: Okounkov-Olshanski formula

Let s_μ^* be the "shifted Schur functions". Then,

$$\frac{s_\lambda(1 + a_1, \dots, 1 + a_n)}{s_\lambda(\{1\}^n)} = \sum_{\mu \vdash n} \frac{s_\mu^*(\lambda_1, \dots, \lambda_n)}{n \uparrow \mu} s_\mu(a_1, \dots, a_n).$$

Some of the s_μ^* admit nice expressions. For instance,

$$s_{\mu^*}(\{N\}^k) = (-1)^{|\mu|} \frac{(-N \uparrow \mu)(k \uparrow \mu)}{h(\mu)},$$

and so $s_{\langle N^k \rangle}(1 + a_1, \dots, 1 + a_n)$ can be expressed as a sum over partitions.

"Taylor series of a Schur function at the identity"

Open questions

This also gives the **analytic continuation** in the parameter k , but not h . Alternative interpretations using determinantal point processes or hypergeometric functions of matrix arguments are also interesting.

Additional references

- **Bump-Gamburd** *On the averages of characteristic polynomials from classical groups*
- **Hughes** *On the Characteristic Polynomial of a Random Unitary Matrix and the Riemann Zeta Function*
- **Ivanov-Olshanski** *Kerov's central limit theorem for the Plancherel measure on Young diagrams*
- **Nekrasov-Okounkov** *Seiberg-Witten Theory and Random Partitions*
- **Stanley** *Some combinatorial properties of hook lengths, contents, and parts of partitions*