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My research involves two branches of mathematics: analytic number theory (my personal motivation) and algebraic combinatorics (my tools). I also need to know about random matrix theory (the model). This personal motivation is in some sense interchangeable, as random matrix theory has been used as a model in much of mathematical physics for instance, and a different approach to these models would therefore appeal to many.

Main project

Our starting point is a conjecture of Conrey, Farmer, Keating, Rubinstein and Snaith. Let $k \in \mathbb{N}$. They give a sequence P_k of polynomials such that, for all $\epsilon > 0$, conjecturally,

$$\int_0^T \left|\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2k} \mathrm{d}t \overset{?}{=} \int_0^T P_k\left(\log\frac{t}{2\pi}\right)\mathrm{d}t + O\left(T^{1/2+\epsilon}\right). \tag{1}$$

The leading coefficient of P_k, which is of degree k^2, equals $\frac{a_k g_k}{(k^2)!}$, with

$$a_k := \prod_p \left(1-\frac{1}{p}\right)^{k^2} {}_2F_1\left(k,k;1;\frac{1}{p}\right)$$

and the integer (which explains the normalization by $k^2!$)

$$g_k := \frac{k^2!}{\prod_{i,j=0}^{k-1}(i+j+1)},$$

so their conjecture implies an earlier conjecture of Keating and Snaith [KS00b, KS00a] at first order. That conjecture was based on a random matrix theory motivation for g_k.

The expressions given by CFKRS for P_K, however, are very complicated to use [CFK+05, CFK+08]. We have proved an alternative expression for the P_k [Deha].

Theorem 1. *The coefficients of* $P_k(x) = c_0(k)x^{k^2} + c_1(k)x^{k^2-1} + \cdots + c_{k^2}(k)$ *satisfy*

$$c_N(k) = \frac{1}{(k^2-N)!} \sum_{\substack{\kappa,\lambda \\ |\kappa|+|\lambda|=N}} (-1)^{|\lambda|} d_{\kappa\lambda} \dim(\lambda, S_k(\kappa)),$$

with the sum taken over pairs of partitions κ, λ *of combined weight* N*, and*

- $\dim(\mu,\nu)$*, an integer, the number of standard tableaux of skew shape* $\mu \setminus \nu$*;*
- $S_k(\kappa)$ *a partition of size* $k^2 - |\kappa|$*, obtained by removing the partition* κ^t *from the tip of the* $k \times k$ *Ferrers diagram (see Figure 1);*
- *the* $d_{\kappa\lambda}$ *given by explicit algebraic expressions of series summing over* $r \geq 1$ *the Taylor coefficients (of bounded degree) of the prime zeta function* $\sum_p \frac{1}{p^s}$ *at* r *(with logarithmic singularity removed for* $r = 1$*).*

Moreover, the $d_{\kappa\lambda}$ *depend simply on* k*, and*

$$\dim(\kappa, S_k(\lambda)) = \frac{g_k B(k)}{k^2 \cdot (k^2-1) \cdots (k^2-|\kappa|-|\lambda|+1)},$$

where $B(k)$ *is an explicitly computable polynomial in* k *of degree bounded by* $2(|\kappa|+|\lambda|)$.

This last statement makes it possible to obtain the analytic continuation of $c_N(k)$ in k, something that was impossible from the $2k$-fold contour integral definition in [CFK+05] or required interpolation in [CFK+08].

Moments as given in the CFKRS conjecture are useful to estimate various quantities tied to the behaviour of ζ on its critical line, such as its maximum values or the distribution of its (nontrivial) zeroes, all conjectured to lie on that same line by the Riemann Hypothesis. The Riemann zeta function hides here a vast zoo of other so-called

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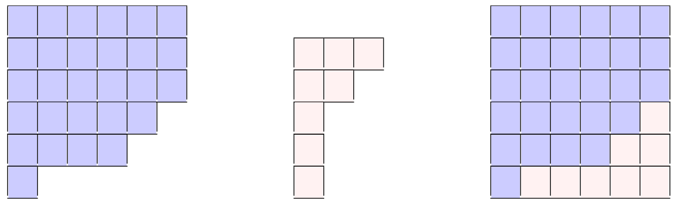

FIGURE 1. The partitions $\lambda = (6,6,6,5,4,1)$ (dark blue) and $\kappa = (3,2,1,1,1)$ (light red) satisfy $S_6(\kappa) = \lambda$ since their Ferrers diagrams can be assembled to a 6×6 square after flipping κ.

L-functions, for which similar conjectures have been formulated, either individually or in *families*. Some cases, traditionally more tractable, have even been proved [KS]. I am currently working on the extension of the results of Theorem 1 to these other cases.

We have alluded before to the earlier and weaker random matrix theory conjecture of Keating and Snaith (KS), which is only concerned with the leading coefficient of the polynomial P_k of the conjectured Equation (1). Keating and Snaith view the g_k as coming from moments of characteristic polynomials of Haar-distributed unitary matrices.

Hughes extended the KS ideas in a different direction than the CFKRS conjecture of Equation (1), and formulated a conjecture about the leading coefficient for the moments of the the derivative of the Riemann zeta function [CRS06, HKO00]. Moments of the derivative are important for instance in works of Hall [Hal04, Hal08] on extreme spacings between zeroes of ζ. In a direct parallel with the KS conjecture, Hughes' replacement for g_k requires computing moments of derivatives of characteristic polynomials, which has turned out to be very difficult.

Theorem 2 ([Deh10]). *For $N, r, k \in \mathbb{N}$ with $r \le 2k$, Λ_g the characteristic polynomial of a Haar-distributed unitary matrix, $H(\lambda)$ the product of the hook lengths of λ and $c(\square)$ the content of a box $\square$ in λ (see Figure 2), we have, summing over partitions of r,*

$$\frac{\int_{\mathrm{U}(N)} |\Lambda_g(1)|^{2k} \left(\mathrm{i}\frac{\Lambda_g'(1)}{\Lambda_g(1)}\right)^r dg}{\int_{\mathrm{U}(N)} |\Lambda_g(1)|^{2k}\, dg} = \sum_{\lambda \vdash r} \frac{r!}{H(\lambda)^2} \prod_{\square \in \lambda} \frac{(k + c(\square))(-N + c(\square))}{2k + c(\square)}. \tag{2}$$

The difficulty here comes from the loss of multiplicativity of the integrand in the eigenvalues (the derivative of a characteristic polynomial is a sum over eigenvalues, unlike the characteristic polynomial itself). One can envision many generalizations of this result: higher derivatives, other Lie groups as they are tied to other L-functions,...

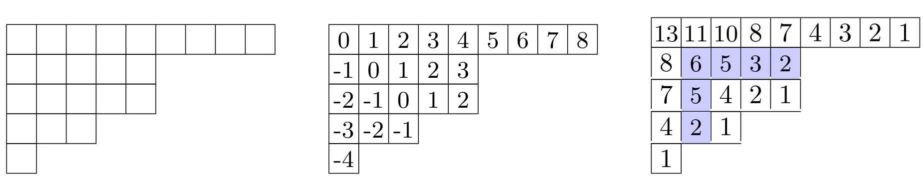

FIGURE 2. The Young diagram of $(9,5,5,3,1)$, its contents (linear statistic of the coordinates), and its hook lengths (length of the hook associated to that box), with the hook based at (2,2) highlighted.

In a broad sense, similar problems have motivated much of my work in [DH11, Deh11, Deh07, Deh08, Deh10, DZ, Dehc] and led me gradually to realize the relevance of integer

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partitions and their uses as a very general tool, either in pure random matrix theory or even number theory as in Theorem 1.

Plancherel averages

The identity $\sum_{\lambda \vdash N} \frac{N!}{H(\lambda)^2} = 1$ (a consequence of the hook formula of Frame, Robinson and Thrall), defines the *Plancherel measure* on integer partitions of N and justifies the name *Plancherel average* for sums of the type in RHS(2). Moreover, the contents of a partition λ are eigenvalues of Jucys-Murphy elements (some elements of the group ring of the symmetric group $\mathbb{C}[\mathcal{S}_n]$) on the Specht module associated to λ. This explains the particular importance of such sums in the work of Okounkov-Vershik for the study of the representation theory of $\mathcal{S}_n$ at finite and asymptotic n (see [OV96] and the following series, or [CSST10]) and in the work of Lassalle [Las08] on character values for symmetric groups.

Partitions have been extensively studied, but some recent results tie in very nicely with the Plancherel averages of the type introduced above.

Logan-Shepp and Vershik-Kerov have shown in a beautiful result that the Plancherel measure for partitions of increasing weight tends to a δ-measure, with support on partitions whose renormalized shape is close to a known limit shape (this is illustrated below for a partition of size 1000). This (law-of-large-numbers-like) result has been refined to a central limit theorem by Kerov and Ivanov-Olshanski [IO02], giving information about the (Gaussian) fluctuations from that limit shape.

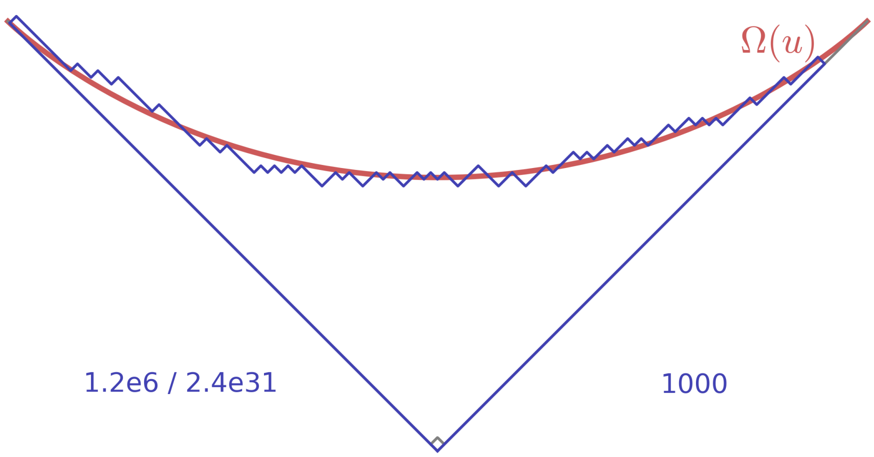

Figure 3. The *Russian* diagram (*i.e.* rotated 135°) of some partition of 1000 that is 1.2 10^6 times more likely under the Plancherel than under the uniform measure on the 2.4 10^{31} partitions of 1000. One can observe convergence to the limit shape Ω.

In addition, Stanley has proved [Sta10] a striking property for Plancherel averages, that in contrast concerns partitions of finite weight: if F is a symmetric polynomial, then

$$\sum_{\lambda \vdash r} \frac{r!}{H(\lambda)^2} F\left(\{c(\square) : \square \in \lambda\}\right) = P(r) \in \mathbb{Q}[r],$$

and he and others have made the polynomial P specific in some cases.

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In [NO06], Nekrasov and Okounkov have used the Logan-Shepp/Vershik-Kerov limit shape to compute some Plancherel averages that occur in Gromov-Witten theory (see also [Oko06]). Since the Plancherel average from Theorem 2 is more delicate, we have extended this technique to also exploit Stanley's polynomiality and the full central limit theorem of Kerov. Note also that the Nekrasov-Okounkov sums involves hooks, while Theorem 2 involves contents as well. We now state a result of [Dehc], in greater generality than is needed to give more information about the sum of Theorem 2.

Theorem 3. *Let* $\{\alpha_a\}_{a=1}^A$ *and* $\{\beta_b\}_{b=1}^B$ *be two sets of complex variables. Set* $C = \frac{1}{4}((\sum_a \alpha_a - \sum_b \beta_b)^2 - \sum_a \alpha_a^2 - \sum_b \beta_b^2)$. *Then, there exists a sequence* $f_j(r)$ *of polynomials in* r, *with leading coefficient* $\frac{C^j}{j!} r^{2j}$, *such that for each fixed integer* r

$$\sum_{\lambda \vdash r} \frac{r!}{H(\lambda)^2} \prod_{\Box \in \lambda} \frac{\prod_{i=1}^A (1 + \alpha_i \cdot z \cdot c(\Box))}{\prod_{i=1}^B (1 + \beta_i \cdot z \cdot c(\Box))} = 1 \cdot z^0 + f_1(r) \cdot z^2 + f_2(r) \cdot z^4 + \cdots$$

In addition, the method also gives the lower coefficients of the $f_j(r)$.

Plancherel averages are a relatively new tool that has very quickly led to results in random matrix theory (as described here), string theory, and combinatorics itself (see for instance all the works on Han's hook formulas [Han], including an extension due to Han and myself [DH11]). These averages thus certainly deserve further study themselves, and Theorem 3 is going in that direction.

A complement to these techniques is to use Fourier duality: the Plancherel measure on partitions of n is the Fourier transform of the Haar (*i.e.* uniform) measure on $\mathcal{S}_n$. Therefore, to Plancherel averages correspond sums over partitions weighted according to the cardinality of the associated conjugacy class in $\mathcal{S}_n$. Sometimes this will be an advantage as it might simplify the combinatorics involved. Together with Zeindler, a Ph.D. student of Nikeghbali at Universität Zürich, I have proved in [DZ] the following theorem (and other generalizations).

Theorem 4. *Given a permutation* $\sigma \in \mathcal{S}_n$, *let* $t(\sigma)$ *be the partition of* n *associated to the cycle-type of* σ *and set* $1/z_\lambda := \frac{\#\mathcal{C}_\lambda}{n!}$ *to be the proportion of permutations in* $\mathcal{S}_n$ *with that cycle-type. Let* $f(x) = \sum_i a_i x^i$ *be a polynomial such that* $f(1) = 1$, *and set* $f_\lambda(x) = \prod_i f(x^{\lambda_i})$. *Then, as formal generating series,*

$$\mathbb{E}_{\substack{\sigma \in \mathcal{S}_n \\ t(\sigma)=\lambda}} f_{t(\sigma)}(x) t^n = \sum_\lambda \frac{1}{z_\lambda} f_\lambda(x) t^{|\lambda|} = \prod_{k=0}^{\infty} (1 - x^k t)^{-a_k}.$$

The left equality follows from a trivial reindexing, but shows that the case $f(x) = 1 - x$ corresponds to the restriction of (2) to averages of characteristic polynomials of permutation matrices, a discrete subgroup of $U(n)$. This initially sparked our interest in this problem.

In [DZ], we actually state a vastly more general version of this theorem where f has randomized coefficients, and is multivariate and merely analytic. We also make the domain of convergence of the generating function explicit. Zeindler has now extended these results, and proved in his thesis a central limit theorem for such averages.

Other projects

In addition to working on the questions described above, I would like to work in the future on the following related problems, focusing in part on asymptotic partitions.

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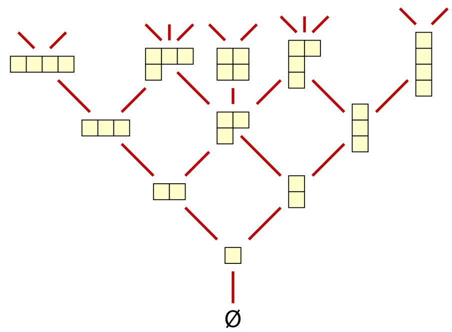

FIGURE 4. The truncation of the Young lattice of partitions consisting of partitions of weight less than 5. The invariant $\dim_k(\mu)$ counts the number of shortest paths from μ down to the lowest reachable partition, where at each step we are allowed to use k segments, all downwards. Two paths are equivalent iif they pause at the same partitions between steps. They do not need to use the same k intermediate segments to reach them. (image by David Eppstein for Wikipedia).

k-Plancherel averages. An alternative definition for the Plancherel measure of a partition μ of r is

$$Pl_1(\mu) = \frac{r!}{H(\mu)^2} = \frac{(\dim\mu)^2}{r!},$$

where $\dim(\phi,\mu) = \dim\mu$ is the number of Young tableaux of shape μ, or equivalently $\dim\mu = \dim_1\mu$ is the number of paths from μ to the empty partition in the *Young lattice*, the Hasse diagram for the partial order on partitions given by inclusion of their Young diagrams (see Figure 4). We generalize $\dim_1(\mu)$ to $\dim_k(\mu)$, which now counts the number of paths from μ down to a minimal partition (which, by non-obvious uniqueness, is thereby defined as its *k-core*), given that at each step we must keep a partition, and we are only allowed to remove the cells of a *k-hook*, *i.e.* a hook whose length is divisible by k (and we rejustify cells afterwards). For instance, from the partition $(9,5,5,3,1)$ pictured in Figure 1, three 3-hooks could be removed. Removal of the highlighted hook, based at $(2,2)$, would would lead us to partition $(9,4,2,1,1)$ as a first step, while removal of the hook based at $(4,2)$ would give $(9,4,3,3,1)$ and of the hook at $(7,1)$ to partition $(6,5,5,3,1)$. Whichever way we iterate this process from there, we will always be left with the 3-core $(6,4,2,1,1)$ from which no further 3-hook can be removed.

Given a partition μ which occurs as a k-core, we can then define a generalization $Pl_{\mu,k}$ of the Plancherel measure Pl_1 in much the same way as before. This uses Stanley's theory of k-differential posets [Sta88] for the renormalization to a probability measure. Both Kerov's central limit theorem and Stanley's polynomiality result can then be extended to this new measure [DB], and this leads to very interesting questions of congruence modulo k between the Stanley polynomials P for various k-cores and the polynomial Stanley had obtained for Pl_1. Those congruences would be of interest for computational purposes as well.

Mod-* behaviour in partitions. Mod-* convergence is a new type of convergence that was first uncovered in the context of asymptotics for moments of the Riemann zeta functions, by Jacod, Kowalski and Nikeghbali [JKN08]. Assume we have a random variable for which a central limit theorem is known, after renormalization. This could

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be for instance

$$(3)\quad X_T = \log\left|\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right| \text{ for } t \text{ uniform in } [T,2T] \quad \text{or}$$

$$Y_N = \log|\Lambda_g(1)| \text{ for } g \text{ Haar-distributed in } \mathrm{U}(N)$$

(with limits in T or N). Selberg's central limit theorem, for instance, says that

$$\frac{X_T}{\sqrt{\frac{1}{2}\log\log T}} \xrightarrow{d} \mathcal{N}(0,1).$$

In this setting, Jacod, Kowalski and Nikeghbali have shown that this renormalization is sometimes detrimental, in the sense that this central limit theorem but also deeper results can be derived from one (thus stronger, but sometimes only conjectural) statement on the characteristic function of the unnormalized random variable, *e.g.* a result of the form

$$\lim_{N\to\infty}\frac{\mathbb{E}e^{\mathrm{i}uY_N}}{f(N)} = \phi(u).$$

They also interpret the conjectures of Keating and Snaith in such a way.

In light of all the connections between random matrices and large partitions, it is thus natural to look for such convergence for statistics of large partitions.

Hook products and the prime number theorem. Define (as Erdos and Diamond did in [DE80]) an approximation to the Moebius μ-function by

$$\mu_T(n) = \begin{cases} \mu(n) & n < T \\ 0 & n > T \\ -T\cdot\sum_{i=1}^{T-1}\frac{\mu(i)}{i} & n = T \end{cases}$$

For a multiset M of integers, define the multiset $M_k = \left\{\frac{m}{k} : m \in M_k \text{ s.t. } k|m\right\}$ (multiple occurrence of the same m lead to multiple occurrence of the same $\frac{m}{k}$, and so for instance $M = M_1$).

For each T, we now define

$$\phi_T(M) = \prod_{k=1}^{\infty}\left(\prod_{i\in M_k} i\right)^{\mu_T(k)}.$$

Erdos and Diamond only studied the case where M is the very special multiset $[[n]] := \{1,\cdots,n\}$. In this case, they proved that for each $\epsilon_1,\epsilon_2 > 0$, there exists an (explicit) unbounbed sequence of Ts such that

$$|\log\phi_T([[n]]) - n| \le \epsilon_1 n \text{ for all } n \ge N_1(T),$$

and

$$|\log\phi_T([[n]]) - \psi(n)| \le \epsilon_2 n \text{ for all } n \ge N_2(T),$$

with $\psi(x) = \sum_{p^k\le x}\log p$. The first inequality is proved using what amounts to Stirling's formula. The second follows from bounding the valuation at every prime p of $\phi_T(M)$.

Together, these provide effective, asymptotically optimal bounds on $\psi(n)$ and thereby perfects Chebyshev's method to obtain effective bounds for $\pi(x)$, the prime-counting function (their method does require the Prime Number Theorem in the proof of the second inequality, so it is not elementary).

The (multi)sets $[[n]]$ are of course very special, but the only type Erdos and Diamond considered. Given λ a partition, let $\mathfrak{H}(\lambda)$ be the multiset of hook lengths of λ. These multisets (which reduce to the Erdos-Diamond type when λ has one row or column) are well-studied, and the Littlewood decomposition of partitions into k-cores and k-quotients gives operations on them with the flavor of modular arithmetic on integers. In

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fact, computer experiments[1] suggests for instance that precise asymptotic information can be obtained, in two different ways, for $\log \phi_T(\mathfrak{H}(n\lambda)$ where $n\lambda$ is the partition of $n|\lambda|^2$ obtained by scaling the diagram of λ in both directions by n.

Progress on this question will require a more delicate understanding of the hooks of the k-core and k-quotients of a partition, as well as some results I have already proved in the process of understanding k-Plancherel averages.

Ultimately, a double inequality as in the Erdos-Diamond work would lead to new effective, asymptotically optimal, bounds on weighted versions of the ψ function.

Since the Littlewood decomposition was introduced in the 50s to study the decomposition in characteristic p (prime) of the irreducible characters of $\mathcal{S}_{|\lambda|}$, and since the divisibilities by p of elements of $\mathfrak{H}(\lambda)$ also play a prominent role there, it will also be interesting to see if these questions can be translated back to group theory as well.

I have started working on these questions in [Dehb].

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