On Savings Accounts in Semimartingale Term Structure Models

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Abstract: This paper is concerned with term structure models generated by semimartingales in general filtrations. Our main focus is on implied savings accounts for which we provide a simple existence proof and a precise formulation of a uniqueness result previously stated (with an incomplete proof) in a Brownian filtration. We also show that a continuous implied savings account can be replicated by a roll-over strategy in just maturing bonds. Finally, we give a necessary and a sufficient condition for the existence of forward rates and illustrate our results in a number of examples.

Key words: term structure of interest rates, semimartingales, supermartingale term structure model, multiplicative decomposition, implied savings account

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0. Introduction

This paper studies models for the term structure of interest rates with zero coupon bond prices given by

$$B(t,T) = E^Q\left[\frac{Z_T}{Z_t}\bigg|\mathcal{F}_t\right] \qquad , \qquad 0 \le t \le T \le T'.$$

The process Z is a strictly positive semimartingale with $Z_- > 0$ and the pair (Q,Z) is said to generate the term structure model B. This generalizes the well-known situation of a short-rate based model with

$$B(t,T) = E^Q\left[\exp\left(-\int_t^T r_s\,ds\right)\bigg|\mathcal{F}_t\right] = E^Q\left[\frac{1/\beta_T}{1/\beta_t}\bigg|\mathcal{F}_t\right]$$

by replacing the discount factor $1/\beta = \exp\left(-\int r_s\,ds\right)$ with Z. In comparison to earlier studies by Flesaker/Hughston (1996), Musiela/Rutkowski (1997a) or Rogers (1997), our contribution here is both of a theoretical and pedagogical nature. We no longer impose the usually made assumption of a Brownian filtration, but systematically use results from the general theory of stochastic processes to streamline and simplify the presentation as far as possible. Our main focus is on (implied and classical) savings accounts, but we also give conditions for the existence of forward rates and easy constructions for martingale measures. The primary technical tool used is the multiplicative decomposition of semimartingales.

The paper is structured as follows. Section 1 recalls basic terminology and preliminary results and establishes a bijection between term structure models with "nonnegative interest rates" and those generated by a Q-supermartingale Z. The idea for this result is due to Musiela/Rutkowski (1997a) and Schmidt (1996). Section 2 studies implied savings accounts, a concept introduced by Rutkowski (1996) and studied more systematically in Musiela/Rutkowski (1997a). A savings account implied by a term structure model B is a strictly positive predictable process A of finite variation such that

$$B(t,T) = E^R\left[\frac{1/A_T}{1/A_t}\bigg|\mathcal{F}_t\right] = E^R\left[\frac{A_t}{A_T}\bigg|\mathcal{F}_t\right] \qquad , \qquad 0 \le t \le T \le T'$$

for some R equivalent to P. Thus we want to generate B by using $(R, 1/A)$ instead of (Q,Z) because A has (like β in a short rate model) simpler path properties than Z. We provide a simple general existence proof and a precise formulation of a uniqueness theorem for implied savings accounts; this extends and clarifies results by Musiela/Rutkowski (1997a). Using infinite-dimensional trading strategies, we then prove under some technical conditions that a continuous implied savings account can be replicated by a roll-over strategy in just maturing bonds. This generalizes a result of Björk/Di Masi/Kabanov/Runggaldier (1997) to a situation where forward rates need not exist. Section 3 provides a necessary and a sufficient condition for the existence of forward rates and shows how the latter are given explicitly in terms of the implied savings account. Finally, section 4 illustrates the theory by a number of examples.

1. Model and preliminaries

This section lays out some terminology and recalls a number of basic results. We start with a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $I\!F = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions of right-continuity and completeness. We also assume that $\mathcal{F}_0$ is trivial. Whenever Q is a probability measure on $(\Omega, \mathcal{F})$, we use the notation W^Q to denote a standard Brownian motion with respect to Q and $I\!F$. We choose all semimartingales to be RCLL.

A *zero coupon bond with maturity* T is a security which pays at time T the certain amount 1. Its price at time t is denoted by $B(t,T)$, where $t \leq T \leq T'$ and T' is the fixed time horizon of our model. We always assume that each $B(\cdot, T)$ is strictly positive. A family of bond price processes $B(\cdot,T)$, $T \leq T'$, is called a *term structure model*; for brevity, we often write B for the entire collection $B(t,T)_{0\leq t\leq T\leq T'}$.

For sufficiently regular bond price processes, the *forward rates* $f(t,T)$ are defined via

$$B(t,T) = \exp\left(-\int_t^T f(t,s)\,ds \right) \qquad , \qquad t \leq T$$

and the *short rate* $r = (r_t)_{0\leq t\leq T'}$ is then defined by $r_t := f(t,t)$. If $\int_0^{T'} |r_s|\,ds < \infty$ P-a.s., we define the *(classical) savings account* β by

$$\beta_t := \exp\left(\int_0^t r_s\,ds \right) \qquad , \qquad t \leq T'. \tag{1.1}$$

In many cases, one starts with a model for the short rate r and assumes that bond prices discounted by the savings account β are martingales under some probability measure Q equivalent to P. Then one has

$$B(t,T) = E^Q\left[\frac{\beta_t}{\beta_T} \middle| \mathcal{F}_t \right] = E^Q\left[\exp\left(-\int_t^T r_s\,ds \right) \middle| \mathcal{F}_t \right] = E^Q\left[\frac{1/\beta_T}{1/\beta_t} \middle| \mathcal{F}_t \right] \tag{1.2}$$

since $B(T,T) = 1$. In order to study more general models, we replace the finite variation process $1/\beta$ by a semimartingale Z. Note that $1/\beta$ is decreasing (hence a supermartingale) if and only if the short rate r is nonnegative.

Definition. Let B be a term structure model, Q a probability measure equivalent to P and Z a strictly positive semimartingale with $Z_0 = 1$ and $Z_- > 0$. We say that B is a *semimartingale term structure model generated by* (Q, Z) if

$$B(t,T) = E^Q\left[\frac{Z_T}{Z_t}\middle|\mathcal{F}_t\right] \quad , \quad t \le T \le T'. \tag{1.3}$$

If Z is a Q-supermartingale, we call B a *supermartingale term structure model.*

Definition. Let N be a strictly positive process on $[0, T_0]$ where $T_0 \le T'$ is arbitrary but fixed. A probability measure Q is called *equivalent martingale measure for* B *with respect to* N if Q is equivalent to P and $B(\cdot, T)/N$ is a Q-martingale on $[0, T \wedge T_0]$ for all $T \le T'$.

Intuitively, one should think of N as the price of some tradable asset used for discounting; the typical example is the savings account β as in (1.2). Thus we see that a semimartingale term structure model B generated by (Q, Z) is *arbitrage-free* in the sense that it admits an equivalent martingale measure with respect to $N = 1/Z$. If we ask in addition for "non-negative interest rates" in the sense that $B(t, S) \ge B(t, T)$ for $S \le T$, we land exactly in the class of supermartingale term structure models. This is shown by the next result due independently to Musiela/Rutkowski (1997a) and Schmidt (1996).

Proposition 1. *Let* Q *be a probability measure equivalent to* P *and* Z *a strictly positive semimartingale on* $[0, T']$ *with* $Z_0 = 1$. *If* $Z_- > 0$, *then the bond price defined by (1.3) is for each* $T \le T'$ *a strictly positive semimartingale with the following properties:*

(1.4) $B(T, T) = 1$ *for all* $T \le T'$.

(1.5) *There exists a strictly positive semimartingale* N *on* $[0, T']$ *with* $N_- > 0$ *and such that the given measure* Q *is an equivalent martingale measure for* B *with respect to* N.

If Z *is a strictly positive* Q*-supermartingale, we have in addition that*

(1.6) $B(t, S) \ge B(t, T)$ *for all* $t \le S \le T \le T'$.

Conversely, every term structure model B *satisfying (1.4) and (1.5) is generated by* Q *and a strictly positive semimartingale* Z *with* $Z_0 = 1$ *and* $Z_- > 0$*; one can take* $Z = N_0/N$. *If* B *satisfies (1.4) – (1.6), this* Z *is even a strictly positive* Q*-supermartingale.*

Proof. Let Z be a strictly positive semimartingale with $Z_0 = 1$ and define $B(t, T)$ by (1.3). If $Z_- > 0$, $N := 1/Z$ is a strictly positive semimartingale with $N_- > 0$ and $B(\cdot, T)$ is a strictly positive semimartingale. (1.4) is immediate from (1.3) and $B(\cdot, T)/N$ is by (1.3) a Q-martingale on $[0, T]$ so that we have (1.5). If Z is a strictly positive Q-supermartingale, we have automatically $Z_- > 0$ by Proposition 6.20 of Jacod (1979). Moreover, (1.6) then holds because we obtain for $t \le S \le T$ that

$$\frac{B(t,T)}{B(t,S)} = \frac{E^Q\left[E^Q[Z_T|\mathcal{F}_S]\middle|\mathcal{F}_t\right]}{E^Q[Z_S|\mathcal{F}_t]} \le 1.$$

For the converse, we start with the process N from (1.5) and define $Z := N_0/N$. Then Z is a strictly positive semimartingale with $Z_0 = 1$ and $Z_- > 0$. Moreover, (1.5) and (1.4) imply

$$Z_t B(t,T) = N_0 \frac{B(t,T)}{N_t} = E^Q\left[\frac{N_0 B(T,T)}{N_T}\bigg|\mathcal{F}_t\right] = E^Q[Z_T|\mathcal{F}_t] \tag{1.7}$$

so that the bond prices admit the representation (1.3). If we additionally have (1.6), then (1.7) combined with (1.6) for $S = t$ shows that Z is a Q-supermartingale.

q.e.d.

Remark. The above proof shows that for fixed $S \leq T$, the property $B(t,S) \geq B(t,T)$ is equivalent to $E^Q[Z_T|\mathcal{F}_S] \leq Z_S$. Hence we conclude that, loosely speaking, arbitrage-free term structure models with nonnegative interest rates are those generated by supermartingales.

We next recall the *multiplicative decomposition* for strictly positive semimartingales after introducing some more terminology. If R is any probability measure, we denote by $\mathcal{M}^+_{1,\mathrm{loc}}(R)$ the set of strictly positive RCLL local R-martingales M with $M_0 = 1$ and by $\mathcal{A}^+_1$ the set of strictly positive predictable RCLL processes C of finite variation with $C_0 = 1$. The following well-known result can be found in Jacod (1979), Propositions 6.19 and 6.20.

Proposition 2 (Multiplicative decomposition of semimartingales)

Let R be any probability measure such that $\mathbb{F}$ satisfies the usual conditions under R. Then:

1) *Any strictly positive special R-semimartingale X with $X_- > 0$ and $X_0 = 1$ admits a unique multiplicative decomposition $X = MC$ with $M \in \mathcal{M}^+_{1,\mathrm{loc}}(R)$ and $C \in \mathcal{A}^+_1$. Uniqueness means that if we have two such decompositions $X = MC = M'C'$, then M and M' as well as C and C' are R-indistinguishable.*

2) *Any strictly positive R-supermartingale X with $X_0 = 1$ is a strictly positive special R-semimartingale satisfying $X_- > 0$ and the process C in its multiplicative decomposition is decreasing.*

The following terminology will be useful in the sequel.

Definition. Let R be a probability measure equivalent to P and L a strictly positive R-semimartingale with $L_- > 0$. We call (R, L) *good* if L is a special R-semimartingale and if the local R-martingale M in its multiplicative decomposition $L = MC$ is a true R-martingale.

2. Implied savings accounts

This section is concerned with implied savings accounts, a generalization of the classical savings account β from (1.1). To admit forward rates, hence a short rate and a savings account, the bond price family $B = B(t,T)$ must be sufficiently smooth in the maturity parameter T. In our general semimartingale setting, this is not always guaranteed and so there is possibly no savings account in the classical sense. Its role is then played by the implied savings account, a concept introduced by Rutkowski (1996) in a HJM setting and studied by Musiela/Rutkowski (1997a) in more general situations. The idea behind this is very simple: we take (1.2) and replace β by a predictable process A of finite variation.

Definition. Let B be a term structure model. A process $A \in \mathcal{A}_1^+$ is called *savings account implied by* B if there exists a probability measure Q equivalent to P such that B is generated by $(Q, 1/A)$; this means that

$$(2.1) \qquad B(t,T) = E^Q\left[\frac{A_t}{A_T}\middle|\mathcal{F}_t\right] = E^Q\left[\frac{1/A_T}{1/A_t}\middle|\mathcal{F}_t\right] \qquad \text{for } t \le T.$$

We then also call A an *implied savings account for* B *with respect to* Q.

In any term structure model with a short rate r, the classical savings account β is of course an implied savings account due to (1.2). If r is in addition nonnegative, β is clearly increasing and uniformly bounded from below. The next result, a version of Corollary 2.3 of Musiela/Rutkowski (1997a), is a simple generalization of this observation.

Lemma 3. *Suppose that B is a supermartingale term structure model. Any implied savings account A for B is then increasing, satisfies $1/A_- > 0$ and is uniformly bounded from below.*

Proof. By assumption, there exists Q equivalent to P such that $(Q, 1/A)$ generates B and this includes in particular $1/A_- > 0$. Since B as a supermartingale term structure model satisfies (1.4) – (1.6), the converse half of Proposition 1 shows that $C = 1/A$ is a strictly positive Q-supermartingale. But because C is also predictable and of finite variation, part 2) and the uniqueness result in Proposition 2 imply that C is decreasing. As $A_0 = 1$, the assertions follow.

q.e.d.

2.1. Existence of an implied savings account

The existence of an implied savings account was first established in Rutkowski (1996) in the specific context of a HJM model satisfying a number of regularity conditions. It was then more generally proved by Musiela/Rutkowski (1997a) in the setting of a Brownian filtration and quite generally is basically a direct application of the multiplicative decomposition. Recall from the proof of Proposition 1 that if B is generated by (Q, Z), then Q is an equivalent martingale measure for B with respect to $N := 1/Z$. To obtain an implied savings account, we study the multiplicative decompositions of the inverse of a bond price $B(\cdot, T^*)$ with fixed maturity T^* and of $1/N = Z$.

Definition. If $T^* \leq T'$ is a fixed maturity, a T^**-forward measure* is an equivalent martingale measure for B with respect to $B(\cdot, T^*)$.

Lemma 4. *Let B be a semimartingale term structure model generated by (Q, Z). Fix an arbitrary $T^* \in [0, T']$ and let Q^* be a T^*-forward measure. Then:*

1) *If Z is a special Q-semimartingale, we have the unique multiplicative decomposition*

$$1/N := Z = M^N C^N$$

where $M^N \in \mathcal{M}^+_{1,\mathrm{loc}}(Q)$ and $C^N \in \mathcal{A}^+_1$.

2) *If $B(\cdot, T^*)^{-1}$ is a special Q^*-semimartingale, we have the unique multiplicative decomposition*

$$B(\cdot, T^*)^{-1} = B(0, T^*)^{-1} M^* C^*$$

where $M^* \in \mathcal{M}^+_{1,\mathrm{loc}}(Q^*)$ and $C^* \in \mathcal{A}^+_1$.

3) *If Z is a Q-supermartingale, $B(\cdot, T^*)^{-1}$ is a Q^*-supermartingale and C^N and C^* are decreasing.*

Proof. Since Z_- is strictly positive, so is

$$B(t-, T) = \frac{\lim_{s \nearrow t} E^Q[Z_T | \mathcal{F}_s]}{Z_{t-}}$$

by the minimum principle for supermartingales; see VI.17 of Dellacherie/Meyer (1982). The first two assertions then follow from Proposition 2. The assertions in part 3) about C^N and C^* follow immediately from part 2) of Proposition 2 and it only remains to show that $B(\cdot, T^*)^{-1}$ is a Q^*-supermartingale under any T^*-forward measure Q^*. But since $B(\cdot, T)/B(\cdot, T^*)$ is then a Q^*-martingale for any $T \leq T'$, we obtain that B (or at least its restriction to $[0, T^*]$) is also generated by the pair $\big(Q^*, B(\cdot, T^*)^{-1}\big)$ instead of (Q, Z). Because B as a supermartingale term structure model satisfies (1.4) – (1.6), we conclude from the converse half of Proposition 1 that $B(\cdot, T^*)^{-1}$ is indeed a Q^*-supermartingale and this completes the proof.

q.e.d.

Theorem 5 (Existence of an implied savings account)

Suppose that B is a semimartingale term structure model generated by (Q,Z). Then:

1) *If the pair (Q,Z) is good, the probability measure Q^{C^N} defined by*

$$\left.\frac{dQ^{C^N}}{dQ}\right|_{\mathcal{F}_t} = M_t^N \qquad , \qquad t \le T'$$

is an equivalent martingale measure for B with respect to C^N. In particular, $A^N = 1/C^N$ is an implied savings account.

2) *Let Q^* be a T^*-forward measure. If the pair $(Q^*, B(\cdot,T^*)^{-1})$ is good, Q^{C^*} defined by*

$$\left.\frac{dQ^{C^*}}{dQ^*}\right|_{\mathcal{F}_t} = M_t^* \qquad , \qquad t \le T^*$$

is an equivalent martingale measure for B with respect to C^*. Hence $A^* = 1/C^*$ is also an implied savings account (on $[0,T^*]$, to be accurate).

Proof. Applying Bayes' rule yields by Lemma 4

$$B(t,T) = E^Q\left[\frac{Z_T}{Z_t}\middle|\mathcal{F}_t\right] = E^Q\left[\frac{M_T^N C_T^N}{M_t^N C_t^N}\middle|\mathcal{F}_t\right] = E^{Q^{C^N}}\left[\frac{C_T^N}{C_t^N}\middle|\mathcal{F}_t\right]$$

so that the process $B(\cdot,T)/A^N$ is a Q^{C^N}-martingale on $[0,T]$. The same reasoning for the pair $(Q^*, B(\cdot,T^*)^{-1})$ instead of (Q,Z) proves part 2).

q.e.d.

Remark. Musiela/Rutkowski (1997a) consider a term structure model satisfying

$$B(t,T) = E^Q\left[\frac{B(t,T')}{B(T,T')}\middle|\mathcal{F}_t\right] \qquad , \qquad t \le T \le T'$$

and $B(t,S) \ge B(t,T)$ for all $t \le S \le T$ where $I\!\!F$ is a Brownian filtration and Q is a probability measure equivalent to P. Thus Q is an equivalent martingale measure for B with respect to $N = B(\cdot,T')$ and Proposition 1 tells us that B is a supermartingale term structure model. Hence Theorem 5 contains as a special case the existence result (Proposition 2.2) of an implied savings account in Musiela/Rutkowski (1997a). But Theorem 5 is at the same time more general and has a simpler proof: We need no Brownian filtration and use no martingale representation theorem in our argument.

2.2. Uniqueness of the implied savings account

In this subsection, we give a general uniqueness result for the implied savings account which extends work by Rutkowski (1996) and in particular Musiela/Rutkowski (1997a) to the case of a general filtration. We start with an auxiliary result used again later, but first we fix some notation. A partition of $[0,T']$ is a finite set $\pi_n = \left\{t_0^n, t_1^n, \dots, t_{k_n}^n\right\}$ with $0 = t_0^n < t_1^n < \dots < t_{k_n}^n = T'$. The mesh size of π_n is $|\pi_n| := \max\limits_{i=1,\dots,k_n} (t_i^n - t_{i-1}^n)$ and a sequence $(\pi_n)_{n\in I\!N}$ of partitions is called increasing if $\pi_n \subseteq \pi_{n+1}$ for all n.

Lemma 6. *Suppose that the RCLL process C of finite variation is of class (D) under P and that G is a bounded adapted RCLL process. Let $(\pi_n)_{n\in I\!N}$ be an arbitrary increasing sequence of partitions of $[0,T']$ with $\lim\limits_{n\to\infty} |\pi_n| = 0$. If $\tau \le T'$ is any stopping time such that C^τ is of P-integrable variation, the sequence*

$$U_n := \sum_{t_i^n, t_{i+1}^n \in \pi_n} I_{\{t_i^n \le \tau\}} E\left[G_{t_i^n} \left(C_{t_{i+1}^n} - C_{t_i^n}\right) \Big| \mathcal{F}_{t_i^n} \right]$$

converges weakly in $L^1(P)$ to $\int_0^\tau G_{s-}\, dC_s^p$ where C^p denotes the dual predictable projection of C under P (which exists at least on $[\![0,\tau]\!]$). If C is predictable, then of course $C^p = C$.

Proof. For the case where $\tau = T'$ is the endpoint of the time interval under consideration, this is easy. In fact, let Y be any bounded random variable and consider an RCLL version of the bounded martingale $Y_s := E[Y|\mathcal{F}_s]$. Then we have

$$\begin{aligned} E[YU_n] &= E\left[\sum_{t_i^n, t_{i+1}^n \in \pi_n} Y_{t_i^n} E\left[G_{t_i^n} \left(C_{t_{i+1}^n} - C_{t_i^n} \right) \Big| \mathcal{F}_{t_i^n} \right] \right] \\ &= E\left[\sum_{t_i^n, t_{i+1}^n \in \pi_n} Y_{t_i^n} G_{t_i^n} \left(C_{t_{i+1}^n} - C_{t_i^n} \right) \right]. \end{aligned}$$

Since $C = C^{T'}$ is of P-integrable variation and G is bounded, we can use the dominated convergence theorem and the martingale property of $C - C^p$ to obtain

$$\lim_{n\to\infty} E[YU_n] = E\left[\int_0^{T'} Y_{u-} G_{u-}\, dC_u \right] = E\left[\int_0^{T'} Y_{u-} G_{u-}\, dC_u^p \right] = E\left[Y \int_0^{T'} G_{u-}\, dC_u^p \right]$$

by VI.61 of Dellacherie/Meyer (1982) because $\int G_-\, dC^p$ is predictable and of integrable variation. For a stopping time τ instead of T', the argument is slightly more delicate because we cannot simply replace C by C^τ. But the only tricky term is $\left(C_{t^n_{i+1}} - C^\tau_{t^n_{i+1}}\right) I_{\{t^n_i \le \tau < t^n_{i+1}\}}$ and this goes to 0 strongly in $L^1(P)$ since C is RCLL and of class (D) under P. For a more detailed proof, we refer to Proposition 6 of Döberlein/Schweizer/Stricker (2000).

q.e.d.

Remark. Lemma 6 is a slight variation of well-known results; see for instance VII.21 of Dellacherie/Meyer (1982), Lemma 2.14 of Jacod (1984) or Lemma 2.3 of Musiela/Rutkowski (1997a). However, all these results are formulated for $\tau = T'$ and the second assumes in addition that C is continuous.

The key to the uniqueness of the implied savings account is the following general theorem of independent interest. Because a rigorous proof is somewhat lengthy, we only state the result here and refer to Döberlein/Schweizer/Stricker (2000) for more details. We just mention that Lemma 6 is used both in that proof and in later results in the present paper.

Theorem 7. *Let C, C' be in $\mathcal{A}_1^+$ and Q, Q' equivalent probability measures. If C, C' are of class (D) under Q, Q' respectively and if*

$$E^Q\left[\frac{C_T}{C_t}\middle|\mathcal{F}_t\right] = E^{Q'}\left[\frac{C'_T}{C'_t}\middle|\mathcal{F}_t\right] \qquad \text{for } t \le T \le T',$$

then C and C' are indistinguishable.

The main result of this subsection now follows easily.

Theorem 8 (Uniqueness of the implied savings account)

Let A and A' be implied savings accounts for the semimartingale term structure model B with respect to Q and Q' respectively. If $1/A, 1/A'$ are of class (D) under Q, Q' respectively, then A and A' coincide. In particular, any two implied savings accounts for a supermartingale term structure model coincide.

Proof. Set $C = 1/A$ and $C' = 1/A'$. By assumption, B is generated by both (Q, C) and (Q', C') so that

$$E^Q\left[\frac{C_T}{C_t}\middle|\mathcal{F}_t\right] = B(t,T) = E^{Q'}\left[\frac{C'_T}{C'_t}\middle|\mathcal{F}_t\right] \qquad \text{for } t \le T \le T'.$$

Hence C and C' must coincide by Theorem 7. For a supermartingale term structure model, both C and C' are uniformly bounded by Lemma 3 and so the assertion again follows from Theorem 7.

q.e.d.

Remarks. **1)** If we have two implied savings accounts A and A' with respect to the *same* measure Q, the conclusion of Theorem 8 holds without any further assumptions. In fact, Theorem 7 then follows directly from the uniqueness of the multiplicative decomposition. This observation is due to Musiela/Rutkowski (1997a).

2) Both Theorem 7 and the first part of Theorem 8 already appear in Musiela/Rutkowski (1997a) in a context where $I\!\!F$ is a Brownian filtration. Our results no longer need this restriction. In addition, there is also a gap in the arguments by Musiela/Rutkowski (1997a) because they use a result like Lemma 6 in a situation where its assumptions are not satisfied.

2.3. Replication of the implied savings account

If we have a term structure model with a short rate r, the savings account β plays by (1.2) the role of a numeraire: all bond prices become Q-martingales when discounted by β. Hence it would be useful if β were available as a traded asset and so one asks if β can be replicated by trading in the given assets, i.e., the zero coupon bonds. It has been shown in Björk/Di Masi/Kabanov/Runggaldier (1997) (BDKR for short) that this is possible by using a roll-over strategy in just maturing bonds where the total amount at each instant is invested for an infinitesimal amount of time in a just maturing bond. Because this involves investing in zero coupon bonds with infinitely many different maturities even over a finite time period, one has to deal with infinite-dimensional trading strategies and use stochastic integration for predictable measure-valued processes with respect to processes with values in the space of continuous functions. The required theory has been developed in BDKR. This subsection shows under some technical conditions that the same replication result is true for a continuous implied savings account which is not assumed to be absolutely continuous. The key to this is the convergence result in Lemma 6.

In order not to overload this paper, we do not fully explain the infinite-dimensional stochastic integration theory. We just recall those concepts and results we have to use below and refer to BDKR for more details. Throughout this subsection, the process $B = \big(B(t,\cdot)\big)_{0\le t\le T'}$ is assumed to have values in the set of continuous functions on $[0,T']$ so that the bond price curve $T \mapsto B(t,T)$ at each instant t is a continuous function of maturity T. To have $B(\cdot,T)$ defined on all of $[0,T']$, we set $B(t,T) := B(T,T')^{-1}B(t,T')$ for $t \in [T,T']$; upon expiration of a bond, we thus switch to the long-term bond $B(\cdot,T')$. We denote by $I\!\!M$ the space of signed measures on $\big([0,T'],\mathcal{B}([0,T'])\big)$ and for a process φ of the form

$$\varphi_t(\omega) = \sum_{i=0}^{n-1} m_i\, I_{C_i\times(t_i,t_{i+1}]}(\omega,t)$$

with $C_i \in \mathcal{F}_{t_i}$, $m_i \in I\!\!M$ and $0 \le t_0 < t_1 < \ldots < t_n \le T'$, the integral of φ with respect to B

is

$$(\varphi \cdot B)_t := \sum_{i=0}^{n-1} I_{C_i} \int_0^{T'} \big(B(t_{i+1} \wedge t, T) - B(t_i \wedge t, T) \big)\, m_i(dT) \qquad , \qquad t \leq T'.$$

Intuitively, such a φ describes a trading strategy which holds $m_i(dx)$ bonds with maturity in $(x, x + dx]$ during the interval $(t_i, t_{i+1}]$ if ω is in C_i. With this interpretation, $\varphi \cdot B$ models the gains from trade arising from the strategy φ.

To extend the above integral to a larger class of integrands, one needs additional assumptions on the integrator B. As in BDKR, we assume that B is $\mathbb{M}$-regular in the sense that for any $m \in \mathbb{M}$, the process $\left(\int_0^{T'} B(t,T)\, m(dT) \right)_{0 \leq t \leq T'}$ has P-a.s. RCLL paths. We also assume that B is a controlled $C([0,T'])$-valued process with a control pair (κ, b); see BDKR for precise definitions and a class of examples. As shown in BDKR, these conditions allow one to define the integral with respect to B for integrands in a space $\mathcal{L}^2_{\text{loc}}(\kappa, b)$. The only properties of the resulting integral used below are that

(2.2) $\quad (\varphi \cdot B)^\tau = \big(\varphi I_{]\!]0,\tau]\!]} \big) \cdot B$ for any stopping time τ and any $\varphi \in \mathcal{L}^2_{\text{loc}}(\kappa, b)$

and that

(2.3) $\quad$ for a sequence (φ^n) converging to φ in $\mathcal{L}^2_{\text{loc}}(\kappa, b)$, the integrals $(\varphi^n \cdot B)_t$ converge to $(\varphi \cdot B)_t$ uniformly in t on compacts in P-probability.

We now define a *trading strategy* as a pair (V_0, φ) with $V_0 \in \mathbb{R}$ and a predictable $\mathbb{M}$-valued process φ in $\mathcal{L}^2_{\text{loc}}(\kappa, b)$;

$$V_t^\varphi := V_0 + (\varphi \cdot B)_t \qquad , \qquad t \leq T'$$

is its *value process*. A *roll-over strategy in just maturing bonds* is a strategy (V_0, φ) whose value process satisfies

$$V^\varphi = V_0 + (V_-^\varphi\, \delta) \cdot B$$

where V_-^φ is the process of left-hand limits of V^φ and δ_t is the Dirac measure in the point t. Existence and uniqueness of such a strategy are proved in BDKR under the assumption that there exists a short rate r for B. Moreover, they show that the value process of this strategy then coincides with the classical savings account β.

Theorem 10 below extends the preceding results to our general framework where forward rates need not exist. The value process of the roll-over strategy in this case turns out to coincide with the implied savings account and this illustrates once again the role of the implied savings account as the natural generalization of the classical savings account. Moreover, the implied savings account A is seen to be a tradable numeraire and so the pair $(Q, 1/A)$ generating the term structure model B has a clear economic interpretation. Note that situations without a short rate come up quite naturally in some contexts; as shown by Karatzas/Lehoczky/Shreve (1991), this happens for instance in equilibrium models with agents having finite marginal utility from consumption at the origin.

Proposition 9. *Suppose that B is an $I\!M$-regular controlled process and that the function b in the control pair (κ, b) has the form*

$$b(\omega, t, u, m) = \left| \int g(\omega, t, u, \vartheta)\, m(d\vartheta) \right|$$

where $g(\omega, t, u, \vartheta)$ is bounded and right-continuous in ϑ. Let A be an implied savings account with respect to Q and assume that A is continuous and $C := 1/A$ is of class (D) under Q. For any continuous real-valued process X, we then have

$$\big((X\delta) \cdot B\big)_t = -\int_0^t X_s A_s \, dC_s \qquad , \qquad t \le T'. \tag{2.4}$$

Proof. Let $(\tau_m)_{m \in I\!N}$ be a localizing sequence of stopping times such that for each m, X^{τ_m} and A^{τ_m} are bounded and C^{τ_m} is of Q-integrable variation. Fix $m \in I\!N$ and an increasing sequence $(\pi_n)_{n \in I\!N}$ of partitions of $[0, T']$ with $\lim\limits_{n\to\infty} |\pi_n| = 0$. Inspection of the proof of Lemma 3.3 of BDKR shows that the processes

$$\varphi^n_s(d\vartheta) := \sum_{t^n_i, t^n_{i+1} \in \pi_n} I_{\{t^n_i \le t \wedge \tau_m\}} X_{t^n_i} \delta_{t^n_{i+1}}(d\vartheta)\, I_{(t^n_i, t^n_{i+1}]}(s)$$

are an approximating sequence for the process $\varphi := X\delta I_{]\!]0,\tau_m]\!]}$ in $\mathcal{L}^2_{\mathrm{loc}}(\kappa, b)$; this uses that X is left-continuous. Moreover, the definition of the integral with respect to B yields

$$\begin{aligned}
(\varphi^n \cdot B)_t &= \sum_{t^n_i, t^n_{i+1} \in \pi_n} I_{\{t^n_i \le t \wedge \tau_m\}} \int_0^{T'} X_{t^n_i} \big(B(t^n_{i+1}, T) - B(t^n_i, T)\big)\, \delta_{t^n_{i+1}}(dT) \\
&= \sum_{t^n_i, t^n_{i+1} \in \pi_n} I_{\{t^n_i \le t \wedge \tau_m\}} X_{t^n_i} \big(1 - B(t^n_i, t^n_{i+1})\big) \\
&= -\sum_{t^n_i, t^n_{i+1} \in \pi_n} I_{\{t^n_i \le t \wedge \tau_m\}} X_{t^n_i} A_{t^n_i} E^Q\left[C_{t^n_{i+1}} - C_{t^n_i} \middle| \mathcal{F}_{t^n_i} \right];
\end{aligned}$$

the third equality uses via (2.1) that A is an implied savings account with respect to Q. Applying Lemma 6 implies for each $t \in [0, T']$ that

$$\lim_{n\to\infty} (\varphi^n \cdot B)_t = -\int_0^{t \wedge \tau_m} X_s A_s \, dC_s \qquad \text{weakly in } L^1(Q);$$

in particular, $((\varphi^n \cdot B)_t)_{n \in I\!N}$ is uniformly Q-integrable. On the other hand, φ^n converges to φ in $\mathcal{L}^2_{\mathrm{loc}}(\kappa, b)$ so that for each t, the integrals $(\varphi^n \cdot B)_t$ converge by (2.3) to $(\varphi \cdot B)_t$ in P-probability, hence also in Q-probability and therefore in $L^1(Q)$. Thus we get with (2.2)

$$\big((X\delta) \cdot B\big)_{t \wedge \tau_m} = \big((X\delta I_{]\!]0,\tau_m]\!]}) \cdot B\big)_t = (\varphi \cdot B)_t = -\int_0^{t \wedge \tau_m} X_s A_s \, dC_s \qquad \text{for each } t \in [0, T']$$

and letting m tend to infinity completes the proof.

q.e.d.

Theorem 10 (Replication of the implied savings account)

Under the assumptions of Proposition 9, we have

$$A_t = 1 + \big((A\delta) \cdot B\big)_t \qquad , \qquad t \leq T'. \tag{2.5}$$

In particular, there exists a roll-over strategy in just maturing bonds and its value process equals the implied savings account A.

Proof. Take $X := A$ and use Proposition 9 to obtain

$$\big((A\delta) \cdot B\big)_t = -\int_0^t A_s^2 \, d\left(\frac{1}{A}\right)_s = A_t - 1.$$

q.e.d.

The replication of the implied savings account also provides a further proof for its uniqueness, although under stronger assumptions than before.

Theorem 11 (Uniqueness of the implied savings account II)

Let the assumptions of Proposition 9 be satisfied. If A and A' are continuous implied savings accounts with respect to Q and Q' respectively and such that $1/A, 1/A'$ are of class (D) under Q, Q' respectively, they must coincide.

Proof. Proposition 9 for Q', A' instead of Q, A shows that (2.4) and (2.5) also hold for the implied savings account A'. Since A and A' are locally bounded, the assertion follows from Lemma 3.2 in BDKR which states that (2.5) has at most one locally bounded solution.

q.e.d.

3. Existence of a classical savings account

We have already seen that the implied savings account usually exists and that the classical savings account β is always an implied savings account, but need not always exist. To address this issue, we derive now sufficient conditions for the forward rates

$$f(t,T) = -\frac{\partial}{\partial T} \log B(t,T)$$

to exist. This problem has also been studied in Baxter (1997) and Musiela/Rutkowski (1997a) in the setting of a Brownian filtration. In our general framework, it turns out that forward rates exist essentially if and only if the implied savings account is absolutely continuous and in that case, the classical and the implied savings accounts coincide.

Example 1. To illustrate the relation of our semimartingale models to models admitting forward rates, we borrow an example from Musiela/Rutkowski (1997a). Let A be an implied savings account for B with respect to Q, set $C := 1/A$ and suppose that for each $T \in [0, T']$, the strictly positive Q-martingale $L_t := E^Q[C_T|\mathcal{F}_t]$ is of the form

$$L_t = E^Q[C_T|\mathcal{F}_t] = E^Q[C_T]\,\mathcal{E}\Big(\int \sigma(s,T)\,dW_s^Q\Big)_t \tag{3.1}$$

for some predictable process $\sigma(\cdot, T)$ with $\int\limits_0^T \sigma^2(s,T)\,ds < \infty$ P-a.s. This is for instance the case if $I\!F$ is generated by the Q-Brownian motion W^Q. Since $B(\cdot, T) = LA$ by (2.1), the product rule as in VIII.19 of Dellacherie/Meyer (1982) and (3.1) yield

$$dB(t,T) = L_t\,dA_t + A_t\,dL_t = B(t,T)\left(\frac{dA_t}{A_t} + \sigma(t,T)\,dW_t^Q\right) \tag{3.2}$$

Thus the process $\sigma(\cdot, T)$ implicitly given by (3.1) is just the volatility of the bond price $B(\cdot, T)$. Furthermore, (3.2) again indicates that the implied savings account A is intimately related to the classical savings account β. We shall indeed see that we have

$$\frac{dA_t}{A_t} = r_t\,dt$$

under regularity assumptions so that A then coincides with β.

Proposition 12. *Let B be a semimartingale term structure model generated by (Q, Z). Let A be an implied savings account for B with respect to Q and suppose that $C = 1/A = 1 + \int \varphi_s\,ds$ with an adapted process φ satisfying*

$$\int\limits_0^{T'} |\varphi_s|\,ds \in L^1(Q) \tag{3.3}$$

so that $1/A$ is of Q-integrable variation. Then the forward rates and the short rate exist and are given by

$$f(t,T) = -\frac{E^Q[\varphi_T|\mathcal{F}_t]}{E^Q\left[1/A_T\middle|\mathcal{F}_t\right]} \qquad , \qquad t \le T \le T' \tag{3.4}$$

and

$$(3.5)\qquad r_t = -\varphi_t A_t = \frac{\partial}{\partial t}\log A_t \qquad , \qquad t \le T'.$$

In particular, we then have $\beta = A$. If B is a supermartingale term structure model, the forward rates and the short rate are all nonnegative.

Proof. It is enough to prove (3.4) since (3.5) follows by taking $T = t$. Since A is an implied savings account with respect to Q, we have $B(t,T) = A_t E^Q[C_T|\mathcal{F}_t]$ by (2.1). Moreover, $\log B(t,T)$ is differentiable in T if and only if $B(t,T)$ is, with

$$\frac{\partial}{\partial T}\log B(t,T) = \frac{\frac{\partial}{\partial T}B(t,T)}{B(t,T)} = \frac{\frac{\partial}{\partial T}E^Q[C_T|\mathcal{F}_t]}{E^Q[C_T|\mathcal{F}_t]}.$$

Now fix $t \le T'$ and define the filtration $\mathbb{G}$ by $\mathcal{G}_s := \mathcal{F}_t$ for $t \le s \le T'$. If we denote by ^{o} the $\mathbb{G}$-optional projection under Q, then $\left(\int \varphi_u\,du\right)^o = \int \varphi_u^o\,du$ by using VI.57 and VI.59 of Dellacherie/Meyer (1982) and because $\int \varphi_u\,du$ is of Q-integrable variation by (3.3). For any $T \in (t,T']$, we thus obtain

$$E^Q[C_T|\mathcal{F}_t] - 1 = E^Q\left[\int_0^T \varphi_u\,du\middle|\mathcal{G}_T\right] = \left(\int \varphi_u\,du\right)_T^o = \int_0^T \varphi_u^o\,du$$

and this implies that $E^Q[C_T|\mathcal{F}_t]$ is λ-a.e. differentiable in T with derivative

$$\frac{\partial}{\partial T}E^Q[C_T|\mathcal{F}_t] = \varphi_T^o = E^Q[\varphi_T|\mathcal{G}_T] = E^Q[\varphi_T|\mathcal{F}_t].$$

This proves (3.4). If B is a supermartingale term structure model, Lemma 3 implies that any implied savings account for B is increasing. Thus C is decreasing, so φ must be nonpositive and hence the final assertion follows from (3.4) and (3.5).

q.e.d.

Remark. For the case where $\mathbb{F}$ is generated by a Brownian motion, a version of this result is due to Baxter (1997). The above simple argument using optional projections has been communicated to us by C. Stricker.

Example 2. Suppose that B satisfies

$$\frac{dB(t,T)}{B(t,T)} = a(t,T)\,dt + b(t,T)\,dW_t^{Q^*}$$

where a, b are bounded and Q^* is a T^*-forward measure. Itô's formula yields

$$B(\cdot,T)^{-1} = B(0,T)^{-1}\mathcal{E}\left(-\int b(s,T)\,dW_s^{Q^*}\right)\exp\Big(-\int \left(a(s,T) - b^2(s,T)\right)\,ds\Big)$$

and so the multiplicative decomposition of $B(\cdot,T^*)^{-1}$ under Q^* is given by

$$M^* = \mathcal{E}\left(-\int b(s,T^*)\,dW_s^{Q^*}\right) \qquad , \qquad C^* = \exp\left(-\int c(s,T^*)\,ds\right)$$

with $c(t,T) := a(t,T) - b(t,T)b(t,T^*)$ as in Musiela/Rutkowski (1997a). Since b is bounded, M^* is a Q^*-martingale and so $\left(Q^*, B(\cdot,T^*)^{-1}\right)$ is good. Because c is also bounded, we can apply Theorem 5 and Proposition 12 with $Q = Q^*$ and $C = C^*$ to obtain the short rate as

$$r_t = -\frac{\partial}{\partial t}\log C_t^* = c(t,T^*).$$

This result has also been obtained in Musiela/Rutkowski (1997a) under the additional assumption that $I\!F$ is a Brownian filtration.

The basic message of Proposition 12 is that under regularity assumptions, the absolute continuity of the implied savings account implies the existence of forward rates. The next result is a sort of converse; it shows essentially that a model with a short rate can always be generated by an absolutely continuous process.

Proposition 13. *Let B be a semimartingale term structure model generated by (Q,Z). Suppose that the short rate exists and satisfies $\int_0^{T'} |r_s|\,ds < \infty$ P-a.s. If the process $M := \beta Z$ is a local Q-martingale, then Z is a special Q-semimartingale and the process C^N from its multiplicative decomposition in Lemma 4 is given by*

$$C_t^N = \frac{1}{\beta_t} = \exp\left(-\int_0^t r_s\,ds\right).$$

In particular, C^N is then absolutely continuous and $\beta = 1/C^N$. If M is even a Q-martingale, (Q,Z) is good and B is generated by $\left(Q^{C^N}, C^N\right) = \left(Q^{C^N}, 1/\beta\right)$.

Proof. Since $M/\beta = Z = M^N C^N$ by Lemma 4, the first assertion follows from the uniqueness of the multiplicative decomposition. This also implies $M^N = M$ and so the last assertion follows from Theorem 5.

q.e.d.

Remark. The assumptions in Proposition 13 are economically reasonable in the following sense. If the short rate exists and is P-a.s. Lebesgue-integrable, the savings account β exists. If we then augment our market by adding β as a traded asset, the resulting larger economy should still admit no arbitrage and this is guaranteed by the condition that βZ is like all processes $B(\cdot,T)Z$ a local Q-martingale.

4. Further examples

In this section, we illustrate the preceding theory by some examples that show in particular how easily one can obtain a risk-neutral and a T^*-forward measure with the help of multiplicative decompositions. The next result provides the main tool for this.

Definition. A *risk-neutral measure* is an equivalent martingale measure for B with respect to the classical savings account β.

Proposition 14. *Let B be a semimartingale term structure model generated by (Q, Z) and define a T^*-forward measure Q^* by*

$$\frac{dQ^*}{dQ}\bigg|_{\mathcal{F}_t} := \frac{Z_t B(t,T^*)}{B(0,T^*)} \qquad , \qquad t \le T^*. \tag{4.1}$$

Assume that (Q, Z) is good so that $A^N = 1/C^N$ is by Theorem 5 an implied savings account. Moreover, suppose that $B(\cdot, T^*)^{-1}$ is a special Q^*-semimartingale and recall from Lemma 4 the processes C^* and M^* from its multiplicative decomposition. Then:

1) *$\big(Q^*, B(\cdot,T^*)^{-1}\big)$ is good and the implied savings accounts A^N and $A^* = 1/C^*$ are indistinguishable. In particular, the probability measure R defined by*

$$\frac{dR}{dQ^*}\bigg|_{\mathcal{F}_t} := M_t^* \qquad , \qquad t \le T^* \tag{4.2}$$

is an equivalent martingale measure for B with respect to A^N and coincides with Q^{C^N} from Theorem 5. If A^N coincides with the classical savings account β, R is also a risk-neutral measure.

2) *The density of Q^* with respect to Q can also be written as*

$$\frac{dQ^*}{dQ}\bigg|_{\mathcal{F}_t} = \frac{M_t^N}{M_t^*} \qquad , \qquad t \le T^*.$$

Proof. 1) A standard change of numéraire argument using Bayes' formula shows that Q^* defined by (4.1) is a T^*-forward measure. Because (Q, Z) is good, Theorem 5 implies that

$$Z_t = M_t^N C_t^N = \frac{dQ^{C^N}}{dQ}\bigg|_{\mathcal{F}_t} C_t^N$$

where Q^{C^N} is an equivalent martingale measure for B with respect to A^N. Combining this with (4.1) yields

$$M_t^* C_t^* = \frac{B(0,T^*)}{B(t,T^*)} = Z_t \frac{dQ}{dQ^*}\bigg|_{\mathcal{F}_t} = \frac{dQ^{C^N}}{dQ^*}\bigg|_{\mathcal{F}_t} C_t^N$$

and the uniqueness of the multiplicative decomposition implies that $A^* = A^N$ and

$$M_t^* = \frac{dQ^{C^N}}{dQ^*}\bigg|_{\mathcal{F}_t}.$$

Hence M^* is a Q^*-martingale, $\big(Q^*, B(\cdot,T^*)^{-1}\big)$ is good and (4.2) shows that R coincides with Q^{C^N} from above and so is an equivalent martingale measure for B with respect to A^N.

2) By the definition of Q^*, we have

$$\frac{dQ^*}{dQ}\bigg|_{\mathcal{F}_t} = \frac{Z_t B(t,T^*)}{B(0,T^*)} = \frac{M_t^N C_t^N}{M_t^* C_t^*}.$$

By part 1), A^N and A^* coincide and so the assertion follows.

q.e.d.

Example 3. We first consider a Gaussian term structure model with bond prices

$$\frac{dB(t,T)}{B(t,T)} = r_t\,dt + \sigma(t,T)\,dW_t^Q \tag{4.3}$$

where σ is a deterministic bounded real-valued function on $\left\{(s,t) \in I\!R^2 \,\middle|\, 0 \le s \le t \le T'\right\}$ and r is the short rate. We also assume that the initial term structure $B(0,\cdot)$ is continuous.

Because $Z := B(0,\cdot)\,\mathcal{E}\left(\int \sigma(s,\cdot)\,dW_s^Q\right) = 1/\beta$ is a continuous semimartingale, it is clear that B is generated by (Q,Z) and that Q is a risk-neutral measure. Our goal is to derive the density between a forward measure Q^* and the risk-neutral measure Q via part 2) of Proposition 14. Since this involves finding the multiplicative decomposition of $B(\cdot,T^*)^{-1}$ under Q^*, we seem at first sight to be led into a vicious circle. But it turns out that this is not the case because we only need the local martingale part under Q^* and this has the same structure as under Q. In fact, Girsanov's theorem implies that switching from Q to the equivalent measure Q^* will change the drift r and replace W^Q by a Q^*-Brownian motion W^{Q^*} in (4.3) and hence we conclude that $M^* = \mathcal{E}\left(-\int \sigma(s,T^*)\,dW_s^{Q^*}\right)$. Because $1/\beta$ is continuous and of finite variation, we have $M^N \equiv 1$ by the uniqueness of the multiplicative decomposition of Z. Hence $(Q, 1/\beta)$ is good and part 2) of Proposition 14 yields

$$\frac{dQ}{dQ^*}\bigg|_{\mathcal{F}_t} = \frac{M_t^*}{M_t^N} = \mathcal{E}\left(-\int \sigma(s,T^*)\,dW_s^{Q^*}\right)_t \qquad , \qquad t \le T^*.$$

This ends the example.

In the above situation, it is of course well known that $B(\cdot,T^*)/\beta$ and hence the volatility structure σ determines the density between a T^*-forward measure and a risk-neutral measure.

But sometimes Theorem 5 really simplifies the computation of densities between different equivalent martingale measures. The next example illustrates this point.

Example 4. Consider a *rational model* as introduced by Flesaker/Hughston (1996) and later studied by Burnetas/Ritchken (1997) and Goldberg (1998) or in Section 16.5 of Musiela/Rutkowski (1997b). Such a model is generated by (Q, Z) with Q equivalent to P and

$$Z_t = h(t) + g(t)M_t \tag{4.4}$$

where M is a strictly positive Q-martingale with $M_0 = 1$ and the functions g, h are C^1 and strictly positive on $[0, T']$. We also assume that $h(0) + g(0) = 1$ to have $Z_0 = 1$.

To get the multiplicative decomposition of Z, we note that (4.4) and Itô's formula yield

$$\frac{dZ_t}{Z_{t-}} = \frac{h'(t) + g'(t)M_t}{h(t) + g(t)M_t}\,dt + \frac{g(t)}{h(t) + g(t)M_{t-}}\,dM_t$$

or equivalently

$$Z = \mathcal{E}\left(\int \frac{g(s)}{h(s) + g(s)M_{s-}}\,dM_s\right)\mathcal{E}\left(\int \frac{h'(s) + g'(s)M_s}{h(s) + g(s)M_s}\,ds\right) = M^N C^N. \tag{4.5}$$

Under integrability assumptions on g, h and M, the local Q-martingale M^N is a true Q-martingale. Then the pair (Q, Z) is good and Theorem 5 implies that

$$\left.\frac{dQ^{C^N}}{dQ}\right|_{\mathcal{F}_t} := M_t^N = \mathcal{E}\left(\int \frac{g(s)}{h(s) + g(s)M_{s-}}\,dM_s\right)_t \qquad , \qquad t \le T'$$

defines an equivalent martingale measure for B with respect to $A^N = 1/C^N$; in particular, A^N is an implied savings account. Moreover, the explicit expression for C^N in (4.5) shows that under integrability assumptions on g, h and M, the process A^N satisfies the conditions of Proposition 12 so that the classical savings account β exists and coincides with A^N. Hence Q^{C^N} is in fact a risk-neutral measure for B and can be obtained directly by looking at the multiplicative decomposition of Z. In the same way, we can use part 2) of Proposition 14 to obtain a T^*-forward measure Q^*. Proposition 12 also tells us that B admits a short rate

$$r_t = -\frac{h'(t) + g'(t)M_t}{h(t) + g(t)M_t};$$

this is clearly nonnegative if g, h are decreasing. For the special case where M satisfies the stochastic differential equation $dM_t = \sigma_t M_t\,dW_t^Q$ with a predictable process σ, we recover in this way the results of Section 16.5.1 in Musiela/Rutkowski (1997b).

The main point of the above computations is to show how easily the multiplicative decomposition of Z gives at the same time the short rate and a risk-neutral measure R.

Alternatively, one could obtain R by computing explicitly its density process $\beta/N = \beta Z = \exp\left(\int r_u \, du\right) Z$ with respect to Q. This would not be difficult, but due to the form of Z and r still require more computations than we need here. On the other hand, the other results can be easily obtained by directly computing the bond prices from (1.3) with (4.4). This gives

$$B(t,T) = E^Q\left[\frac{Z_T}{Z_t}\middle|\mathcal{F}_t\right] = \frac{h(T)+g(T)M_t}{h(t)+g(t)M_t}$$

and therefore B admits forward rates

$$f(t,T) = -\frac{\frac{\partial}{\partial T}B(t,T)}{B(t,T)} = -\frac{h'(T)+g'(T)M_t}{h(t)+g(t)M_t}$$

and a short rate

$$r_t = f(t,t) = -\frac{h'(t)+g'(t)M_t}{h(t)+g(t)M_t}$$

because g and h are in C^1. Of course, this agrees with the results obtained via Proposition 12. The terminology "rational model" comes from the fact that all these expressions are rational functions of the driving martingale M. If g, h are decreasing, Z is a Q-supermartingale and r is then obviously nonnegative.

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