A monetary value for initial information in portfolio optimization

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Abstract. We consider an investor maximizing his expected utility from terminal wealth with portfolio decisions based on the available information flow. This investor faces the opportunity to acquire some additional initial information $\mathcal{G}$. His subjective fair value of this information is defined as the amount of money that he can pay for $\mathcal{G}$ such that this cost is balanced out by the informational advantage in terms of maximal expected utility. We study this value for common utility functions in the setting of a complete market modeled by general semimartingales. The main tools are a martingale preserving change of measure and martingale representation results for initially enlarged filtrations.

Key Words: Initial enlargement of filtrations, utility maximization, value of information, martingale preserving measure, predictable representation property.

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1 Introduction

In this paper, we consider an investor who trades in a financial market so as to maximize his expected utility of wealth at a prespecified time. The investor faces the opportunity to acquire, in addition to the common information flow $\mathbb{F}$, some extra information $\mathcal{G}$ at a certain cost, e.g. by hiring a good analyst or by doing more research about companies he can invest in. Buying the information $\mathcal{G}$ reduces his initial capital but at the same time enlarges the information flow to $\mathbb{G} = \mathbb{F} \vee \mathcal{G}$ on which the investor can then base his portfolio decisions. Our basic question is then: At what cost is the reduction of the investor's initial wealth offset by the increase in the set of available portfolio strategies? To be more precise, let $u^{\mathbb{F}}(y)$ and $u^{\mathbb{G}}(y)$ be the maximal expected utilities from terminal wealth that can be obtained with initial capital y and portfolio decisions based on the information flow $\mathbb{F}$ and $\mathbb{G}$. If our investor has initial capital x, the *utility indifference value* π of the additional information $\mathcal{G}$ is defined as the solution $\pi = \pi(x)$ of the equation $u^{\mathbb{F}}(x) = u^{\mathbb{G}}(x - \pi)$. The quantity π can be interpreted as the investor's subjective fair (purchase) value of the additional information $\mathcal{G}$. Our aim is to calculate π for common utility functions in the situation of a complete market and to study the dependence of π on $\mathcal{G}$, on x, and on the utility function.

Pikovsky and Karatzas [22] gave the first rigorous account of a utility maximization problem under additional initial information, posing the problem in the mathematical framework of an initially enlarged filtration. Subsequent papers include Elliott et al. [7], Grorud and Pontier [9, 10] and Amendinger et al. [1, 2]. They all examined the maximal expected utility under additional information $u^{\mathbb{G}}(x)$ or the expected utility gain $u^{\mathbb{G}}(x) - u^{\mathbb{F}}(x)$. In comparison, the present indifference approach quantifies the informational advantage in terms of money, not utility. Similar ideas have been previously used by Hodges and Neuberger [13] and many others for the valuation of options instead of information.

The outline of this paper is as follows. Section 2 provides the mathematical framework and discusses the central assumptions, which ensure the existence of the so-called martingale preserving probability measure (MPPM) $\widetilde{Q}$ corresponding to a given probability Q. The main property of the MPPM is that it decouples $\mathbb{F}$ and $\mathcal{G}$ in such a way that $\mathbb{F}$-martingales under Q remain $\mathbb{G}$-martingales under $\widetilde{Q}$. This significantly facilitates the analysis of our problem. After giving a simplified approach to the MPPM, we transfer in Section 3 the strong predictable representation property for local martingales from $\mathbb{F}$ to the initially enlarged filtration $\mathbb{G}$. This extends prior work of Pikovsky [21], Grorud and Pontier [9] and Amendinger [1] to the general unbounded semimartingale case and in addition contributes a conceptually new proof. In Section 4, standard duality arguments are applied to solve the utility maximization problem in a general complete model when the initial information is non-trivial. We combine this in Section 5 with our martingale representation results to derive the utility indifference value for common utility functions. In Section 6, closed form expressions for this value are provided in an Itô process model where the additional information consists of a noisy signal about the terminal stock price.

2 Framework and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ satisfying the usual conditions and a finite time horizon $T > 0$. For simplicity we assume $\mathcal{F}_0$ to be trivial. As in Jacod and Shiryaev [17], we use a generalized notion of conditional expectation which is defined for all real-valued random variables. All semimartingales adapted to a complete and right-continuous filtration are taken to have right-continuous paths with left limits.

The initially enlarged filtrations framework

Let the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ be an initial enlargement of $\mathbb{F}$ by some σ-field $\mathcal{G} \subseteq \mathcal{F}$, i.e. $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{G}$, $t \in [0,T]$. We assume that $\mathcal{G}$ is generated by some random variable G taking values in a general measurable space $(X, \mathcal{X})$, i.e. $\mathcal{G} = \sigma(G)$. This causes no loss of generality since we can choose $(X, \mathcal{X}) := (\Omega, \mathcal{G})$ and take $G : (\Omega, \mathcal{F}) \to (\Omega, \mathcal{G})$, $\omega \mapsto \omega$. In most parts of this paper we shall assume that $\mathcal{G}$ satisfies the following decoupling condition.

Assumption 2.1 (D): *There exists a probability measure $R \sim P$ such that $\mathcal{F}_T$ and $\mathcal{G} = \sigma(G)$ are R-independent.*

The implications and significance of this decoupling assumption for our problem will be discussed in Remark 2.9. At the moment, let us just emphasize that it is a pure existence condition and should be viewed as an assumption on $\mathcal{G}$. A generic example is given in Section 6. For our analysis of the effects of additional information, the construction of a specific decoupling measure will be a crucial step.

Lemma 2.2 *Suppose Assumption 2.1 (D) is satisfied. Then $\mathbb{G}$ satisfies under P the usual conditions of completeness and right-continuity.*

Proof: Th. 1 in He and Wang [11] shows that if $(\mathcal{F}^1_t)_{t\in[0,T]}$ and $(\mathcal{F}^2_t)_{t\in[0,T]}$ are independent filtrations and satisfy the usual conditions, then so does $(\mathcal{F}^1_t \vee \mathcal{F}^2_t)_{t\in[0,T]}$. Assumption 2.1 (D) implies that $\mathbb{F}$ and the constant filtration given by the P-completion of $\mathcal{G}$ are independent under some $R \sim P$. Hence the claim follows. $\square$

We shall see below that Assumption 2.1 (D) is equivalent to

Assumption 2.3 (E): *A regular conditional distribution of G given $\mathcal{F}_T$ exists and is P-a.s. equivalent to the law of G, i.e. $P[G \in \cdot \,|\, \mathcal{F}_T](\omega) \sim P[G \in \cdot\,]$ for P-a.a. $\omega \in \Omega$.*

Assumption 2.3 (E) and Th. V.58 in [5] imply the existence of an $\mathcal{X} \otimes \mathcal{F}_T$-measurable function $p : X \times \Omega \to (0, \infty)$ such that for P-a.a. ω and for all $B \in \mathcal{X}$

$$P[G \in B \mid \mathcal{F}_T](\omega) = \int_B p(x,\omega) P[G \in dx]\,. \tag{2.1}$$

We define $p^G(\omega) := p(G(\omega), \omega)$ and $p^x(\omega) := p(x, \omega)$ for $\omega \in \Omega$, $x \in X$. Note that each p^x is $\mathcal{F}_T$-measurable and p^G is $\mathcal{G}_T$-measurable.

The martingale preserving probability measure

This section shows that Assumptions 2.1 (D) and 2.3 (E) are equivalent and ensure the existence of the martingale preserving probability measure. This has several useful consequences. First, note that both marginals of the decoupling measure R from Assumption 2.1 (D) can be chosen freely:

Lemma 2.4 *Let P_1, P_2 be probability measures on $\mathcal{F}_T$ and $\mathcal{G}$ respectively which are equivalent to P. If Assumption 2.1 (D) holds, there is a unique probability measure $\mu \sim P$ on $\mathcal{F}_T \vee \mathcal{G}$ such that $\mu = P_1$ on $\mathcal{F}_T$, $\mu = P_2$ on $\mathcal{G}$, and $\mathcal{F}_T$ and $\mathcal{G}$ are μ-independent. We write $\mu =: P_{\mathrm{dec}}(P_1, P_2)$.*

Proof: This is like Rem. 2.1 in [10]. In fact, by Assumption 2.1 (D) there exists $R \sim P$ such that $\mathcal{F}_T$ and $\mathcal{G}$ are R-independent. Defining $d\mu := Z_T^1 Z_T^2\, dR$ with $Z_T^1 := (dP_1/dR)|_{\mathcal{F}_T}$ and $Z_T^2 := (dP_2/dR)|_{\mathcal{G}}$, it is easy to show $E_\mu[1_A 1_B] = P_1[A]P_2[B]$ for $A \in \mathcal{F}_T$ and $B \in \mathcal{G}$, using the R-independence of $\mathcal{F}_T$ and $\mathcal{G}$. This yields the three properties of μ and determines a probability measure on $\mathcal{F}_T \vee \mathcal{G}$. $\square$

We shall see that a specific decoupling measure plays a key role in our problem:

Definition 2.5 *Let $Q \sim P$ and let $Q|_{\mathcal{F}_T}$, $P|_{\mathcal{G}}$ denote the restrictions on $\mathcal{F}_T$ and $\mathcal{G}$, respectively. The measure $\widetilde{Q} := P_{\mathrm{dec}}(Q|_{\mathcal{F}_T}, P|_{\mathcal{G}})$ is called* martingale preserving probability measure *(corresponding to Q).*

By Lemma 2.4, $\widetilde{Q}$ is the unique measure on $\mathcal{F}_T \vee \mathcal{G}$ with the following three properties: 1) $\widetilde{Q} = Q$ on $\mathcal{F}_T$, 2) $\widetilde{Q} = P$ on $\mathcal{G}$, and 3) $\mathcal{F}_T$ and $\mathcal{G}$ are $\widetilde{Q}$-independent. A consequence of 1) is that integrability properties of $\mathcal{F}_T$-measurable random variables under Q still hold under $\widetilde{Q}$. In particular, $(Q, \mathbb{F})$-martingales remain $(\widetilde{Q}, \mathbb{F})$-martingales. More importantly, the martingale property is preserved under $\widetilde{Q}$ in $\mathbb{G}$, i.e., under an initial enlargement of the filtration and a simultaneous measure change to $\widetilde{Q}$. Guided by ideas of Föllmer and Imkeller [8], this motivated the terminology *martingale preserving probability measure.* We summarize some useful properties of $\widetilde{Q}$ for further reference:

Corollary 2.6 *Suppose Assumption 2.1 (D) is satisfied. Let Q be a probability measure equivalent to P and denote by $\widetilde{Q}$ the corresponding martingale preserving measure. Then*

1. *We have $\mathcal{M}_{(\mathrm{loc})}(Q, \mathbb{F}) = \mathcal{M}_{(\mathrm{loc})}(\widetilde{Q}, \mathbb{F}) \subseteq \mathcal{M}_{(\mathrm{loc})}(\widetilde{Q}, \mathbb{G})$, and every semimartingale with respect to $(Q, \mathbb{F})$ is also a semimartingale with respect to $(Q, \mathbb{G})$.*

2. *Any $\mathbb{F}$-adapted process L has the same distribution under $\widetilde{Q}$ and Q. If L has in addition $(Q, \mathbb{F})$-independent increments, i.e. $L_t - L_s$ is Q-independent of $\mathcal{F}_s$ for $0 \leq s \leq t \leq T$, then L has also $(\widetilde{Q}, \mathbb{G})$-independent increments, and the semimartingale characteristics of L are the same for $(Q, \mathbb{F})$ and $(\widetilde{Q}, \mathbb{G})$. In particular, a $(Q, \mathbb{F})$-Lévy process (Brownian motion, Poisson process) is also a $(\widetilde{Q}, \mathbb{G})$-Lévy process (Brownian motion, Poisson process).*

3. *Let S be a multidimensional $(P, \mathbb{F})$-semimartingale. Then an $\mathbb{F}$-predictable process H is S-integrable with respect to $\mathbb{F}$ if and only if H is S-integrable with respect to $\mathbb{G}$, and the stochastic integrals of H with respect to S coincide for both filtrations.*

Proof: 1. The first assertion was shown in [2], Th. 2.5, and implies the second as $\widetilde{Q} \sim Q$.
2. The first statement is clear since $\widetilde{Q} = Q$ on $\mathcal{F}_T$. If the $\mathcal{F}_T$-measurable random variable $L_t - L_s$ is Q-independent from $\mathcal{F}_s$, it is also $\widetilde{Q}$-independent from $\mathcal{F}_s$ since $Q = \widetilde{Q}$ on $\mathcal{F}_T$; therefore it is also $\widetilde{Q}$-independent from $\mathcal{G}_s = \mathcal{F}_s \vee \mathcal{G}$ since $\mathcal{G}$ is $\widetilde{Q}$-independent from $\mathcal{F}_T$. The distribution of a process with independent increments is determined by its characteristics which are unique and non-random ([17], Th. II.5.2), and hence remain unaltered if we go from $(Q, \mathbb{F})$ to $(\widetilde{Q}, \mathbb{G})$. This also yields the last assertion.
3. As stochastic integrals are unaffected by an equivalent change of measure, we may consider the problem under $\widetilde{Q}$. Under $\widetilde{Q}$, every $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale and so the claim follows by Th. 7 and Prop. 8 (plus subsequent remark) from Jacod [15]. $\square$

By part 3, we need not distinguish between stochastic integrals under $\mathbb{F}$ and $\mathbb{G}$ in the sequel. Next, we see that the P-density of the MPPM $\widetilde{Q}$ on $\mathcal{G}_T$ can be constructed via p^G:

Proposition 2.7 *Let Q be a probability measure equivalent to P and denote by Z_T its $\mathcal{F}_T$-density with respect to P. If Assumption 2.3 (E) is satisfied then $\widetilde{Q} = P_{\mathrm{dec}}(Q, P)$ exists and is given via $d\widetilde{Q}/dP = Z_T/p^G$.*

We just sketch the proof. One argues almost exactly as in [9], Lemma 3.1, or as in the proof of Prop. 2.3 in [2] (replacing p_t^G there by our p^G and P by Q) to show that because of (2.1), Z_T/p^G is like Z_T a P-density, strictly positive, $d\mu := (Z_T/p^G)\, dP = (1/p^G)\, dQ$ defines a probability $\mu \sim P$, and that $\mu[A \cap \{G \in B\}] = E_P\left[Z_T I_A P[G \in B]\right] = Q[A]P[G \in B]$ for $A \in \mathcal{F}_T$, $B \in \mathcal{X}$. This yields the claim of Proposition 2.7, and thereby one implication of

Corollary 2.8 *Assumption 2.1 (D) and Assumption 2.3 (E) are equivalent.*

Since the range space X of our random variable G need not be Polish, the converse implication does not follow directly from Lemma 3.4 in [10]: we have to prove the existence of a regular conditional expectation. The proof is relegated to the Appendix.

Remark 2.9 1. What is the relevance of Assumption 2.1? Technically, it allows us to work on an *implicit* product model by switching from P to R. Loosely speaking, we can argue under R "as if" $(\Omega, \mathcal{F}_T \vee \mathcal{G}, R)$ equals $(\Omega \times X, \mathcal{F}_T \otimes \mathcal{X}, R|_{\mathcal{F}_T} \otimes R[G \in \cdot])$. But why not work with an *explicit* product model and assume independence under P? The simple answer is that this would be unnatural for our applications; see Section 6. In fact, neither the decoupling measure R nor an explicit product structure are given a priori in general.
2. We present here a simplified approach to the martingale preserving probability measure. Our method avoids the use of a conditional density process from Jacod [16], which is a crucial tool in [1, 2, 9, 10] but causes technical measurability problems. In comparison to related work, we also need no assumptions on the space $(X, \mathcal{X})$ where G takes its values.

3 Strong predictable representation property

Throughout this section let $S = (S^1, \ldots, S^d)^{tr}$ be a d-dimensional $\mathbb{F}$-semimartingale. Our aim is to show that under Assumption 2.1 (D) the martingale representation property of S with respect to $\mathbb{F}$ and some measure Q implies the same property with respect to the initially enlarged filtration $\mathbb{G}$ and the corresponding martingale preserving measure $\widetilde{Q}$. We give a conceptually new proof of this result without any further assumptions on the semimartingale S, thereby extending a previous result in [1] to full generality.
First, recall a classical martingale representation result. For $d = 1$ this is almost Th. 13.9 in He et al. [12], and the multidimensional case can be proved along the lines of Ch. XI.1.a in Jacod [14] with some modifications for the situation of a non-trivial initial σ-field.

Proposition 3.1 *Suppose the filtration $\mathbb{H}$ satisfies the usual conditions and there is a probability measure $Q^{\mathbb{H}} \sim P$ such that $S \in \mathcal{M}_{\text{loc}}(Q^{\mathbb{H}}, \mathbb{H})$. Denote*

$$(3.1)\quad \Gamma^{\mathbb{H}} := \left\{ Q \sim Q^{\mathbb{H}} \;\middle|\; \frac{dQ}{dQ^{\mathbb{H}}} \text{ is } \mathcal{H}_T\text{-measurable},\ Q = Q^{\mathbb{H}} \text{ on } \mathcal{H}_0,\ S \in \mathcal{M}_{\text{loc}}(Q, \mathbb{H}) \right\}.$$

Then the following statements are equivalent:

1. $\Gamma^{\mathbb{H}} = \{Q^{\mathbb{H}}\}$.
2. *The set $\mathcal{M}_{0,\text{loc}}(Q^{\mathbb{H}}, \mathbb{H})$ of local $(Q^{\mathbb{H}}, \mathbb{H})$-martingales null at 0 is equal to the set*

$$\left\{ \theta \cdot S \;\middle|\; \theta \text{ is } S\text{-integrable w.r.t. } (Q^{\mathbb{H}}, \mathbb{H}) \text{ in the sense of local martingales} \right\}$$

of stochastic integrals with respect to S.

We say that S has the *strong predictable representation property with respect to* $(Q^{\mathbb{H}}, \mathbb{H})$ (for short:$(Q^{\mathbb{H}}, \mathbb{H})$-PRP) if one of these statements is valid. Our main result in this section is the subsequent martingale representation transfer theorem.

Theorem 3.2 *Suppose Assumption 2.1 (D) is satisfied and S has the strong predictable representation property with respect to $(Q^{\mathbb{F}}, \mathbb{F})$ for some $Q^{\mathbb{F}} \sim P$. Let $Q^{\mathbb{G}} = \widetilde{Q^{\mathbb{F}}}$ denote the martingale preserving probability measure corresponding to $Q^{\mathbb{F}}$. Then S has the strong predictable representation property with respect to $(Q^{\mathbb{G}}, \mathbb{G})$. For short: $\Gamma^{\mathbb{F}} = \{Q^{\mathbb{F}}\}$ implies $\Gamma^{\mathbb{G}} = \{Q^{\mathbb{G}}\}$, or*

$$(3.2)\qquad \left(Q^{\mathbb{F}}, \mathbb{F}\right)\text{-PRP} \quad \text{implies} \quad \left(\widetilde{Q^{\mathbb{F}}}, \mathbb{G}\right)\text{-PRP}.$$

Proof: Let $Q' \in \Gamma^{\mathbb{G}}$. Without loss of generality we can assume that $dQ'/dQ^{\mathbb{G}} \in L^\infty(Q^{\mathbb{G}})$ (see [12], Th. 13.9). We prove that $Q' = Q^{\mathbb{G}}$ and thus show the claim by Proposition 3.1. Define $Z'_t := \left(dQ'/dQ^{\mathbb{G}}\right)\big|_{\mathcal{G}_t}$, $t \in [0,T]$. For all $t \in [0,T]$ the density Z'_t is $\mathcal{F}_t \vee \sigma(G)$-measurable and thus of the form $Z'_t(\cdot) = z_t(\cdot, G(\cdot))$ for an $\mathcal{F}_t \otimes \mathcal{X}$-measurable function $z_t(\omega, x)$. If ν denotes the distribution of G under $Q^{\mathbb{G}}$, the process $(z_t(\cdot, x))_{t \in [0,T]}$ is RCLL in t for ν-a.a. x because $(Z'_t)_{t\in[0,T]}$ is an RCLL process. Since $S \in \mathcal{M}_{\mathrm{loc}}(Q^{\mathbb{F}}, \mathbb{F})$ there is a localizing sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times such that for all n the stopped process S^{τ_n} is a uniformly integrable $(Q^{\mathbb{F}}, \mathbb{F})$-martingale, hence also a uniformly integrable $(Q^{\mathbb{G}}, \mathbb{G})$-martingale by Corollary 2.6. Because $S \in \mathcal{M}_{\mathrm{loc}}(Q', \mathbb{G})$ and Z' is bounded we conclude that the local $(Q^{\mathbb{G}}, \mathbb{G})$-martingale $Z'S^{\tau_n}$ is of class (D) under $Q^{\mathbb{G}}$ and therefore a uniformly integrable $(Q^{\mathbb{G}}, \mathbb{G})$-martingale. With $E^{\mathbb{G}}[\cdot]$ denoting expectations with respect to $Q^{\mathbb{G}}$, Lemma 3.3 below implies for $t \in [0,T]$ and $n \in \mathbb{N}$ that

$$z_t(\cdot, G)S_t^{\tau_n} \;=\; E^{\mathbb{G}}\left[z_T(\cdot, G)S_T^{\tau_n} \mid \mathcal{F}_t \vee \sigma(G)\right] \;=\; E^{\mathbb{G}}\left[z_T(\cdot, x)\, S_T^{\tau_n} \mid \mathcal{F}_t\right]\Big|_{x=G}\,.$$

Since $z_t(\cdot,\cdot)$ is $\mathcal{F}_t \otimes \mathcal{X}$-measurable and $\mathcal{F}_T$ and $\sigma(G)$ are $Q^{\mathbb{G}}$-independent this implies that $\left(Q^{\mathbb{G}} \otimes\ \nu\right)$-a.e.

$$z_t(\omega, x)S_t^{\tau_n}(\omega) = E^{\mathbb{G}}\left[z_T(\cdot, x)\, S_T^{\tau_n} \mid \mathcal{F}_t\right](\omega)\,. \tag{3.3}$$

In the same way we obtain for each $t \in [0,T]$ that $\left(Q^{\mathbb{G}} \otimes \nu\right)$-a.e.

$$z_t(\omega, x) = E^{\mathbb{G}}[z_T(\cdot, x) \mid \mathcal{F}_t](\omega)\,. \tag{3.4}$$

Thus (3.3) and (3.4) hold $\left(Q^{\mathbb{G}} \otimes \nu\right)$-a.e. simultaneously for all rational $t \in [0,T]$ and then by right-continuity in t of $z_t(\cdot, x)$ and $z_t(\cdot, x)S_t^{\tau_n}$ even simultaneously for all $t \in [0,T]$. Hence both $(z_t(\cdot, x))_{t\in[0,T]}$ and $(z_t(\cdot, x)S_t^{\tau_n})_{t\in[0,T]}$ are $(Q^{\mathbb{G}}, \mathbb{F})$-martingales and thus $(Q^{\mathbb{F}}, \mathbb{F})$-martingales by Corollary 2.6 for ν-a.a. x and $n \in \mathbb{N}$. But now $(Q^{\mathbb{F}}, \mathbb{F})$-PRP implies by Proposition 3.1 for ν-a.a. x that $z_T(\cdot, x) = 1$ $Q^{\mathbb{G}}$-a.s., and the $Q^{\mathbb{G}}$-independence of $\mathcal{F}_T$ and $\sigma(G)$ then yields $Z'_T = z_T(\cdot, G(\cdot)) = 1$. $\square$

In comparison to Th. 4.7 in [1] where S is assumed to be locally in $\mathcal{H}^2(Q^{\mathbb{F}}, \mathbb{F})$, our result is more general and also the proof is different. In [1] an L^2-approximation argument shows that any (local) $(Q^{\mathbb{G}}, \mathbb{G})$-martingale null at 0 can be represented as a stochastic integral of a $\mathbb{G}$-predictable integrand with respect to S if such a representation holds under $(Q^{\mathbb{F}}, \mathbb{F})$. Our argument proves that the uniqueness of the equivalent $\mathbb{F}$-martingale measure for S implies the uniqueness (modulo $\mathcal{G}_0$) of the equivalent $\mathbb{G}$-martingale measure in the sense of Proposition 3.1. We refer to [1] for a discussion of related work on martingale representation theorems in more special cases of Brownian and Brownian-Poissonian models in [6, 9, 21]. In a model of the latter type, Grorud and Pontier [10] use a weak martingale representation result to describe the sets of equivalent local martingale measures for S with respect to $\mathbb{F}$ and $\mathbb{G}$ when these sets are not singletons.
The missing argument needed in previous proof of Theorem 3.2 is provided by

Lemma 3.3 *Suppose $\mathcal{F}_T$ and $\mathcal{G}$ are R-independent under some measure $R \sim P$. Let $f : \Omega \times X \to \mathbb{R}$ be $\mathcal{F}_T \otimes \mathcal{X}$-measurable, $f(\cdot, G) \in L^1(R)$ and $t \le T$. Then there exists an $\mathcal{F}_t \otimes \mathcal{X}$-measurable function $g : \Omega \times X \to \mathbb{R}$ such that*

$$g(\cdot, G) \quad \textit{is a version of} \quad E_R\left[f(\cdot, G) | \mathcal{F}_t \vee \mathcal{G}\right] \tag{3.5}$$

and for $R[G \in \cdot]$-a.a. $x \in X$ we have $f(\cdot, x) \in L^1(R)$ and

$$g(\cdot, x) \quad \text{is a version of} \quad E_R\left[f(\cdot, x)|\mathcal{F}_t\right]. \tag{3.6}$$

Using the versions $g(\cdot, G)$ and $g(\cdot, x)$, $x \in X$, of the conditional expectations in (3.5) and (3.6), we have $E_R[f(\cdot, G)|\mathcal{F}_t \vee \mathcal{G}] = E_R[f(\cdot, x)|\mathcal{F}_t]\Big|_{x=G}$ as needed in the previous argument. The proof of Lemma 3.3 is given in the Appendix.

4 Utility maximization with non-trivial initial information

In this section, $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ denotes a filtration satisfying the usual conditions, and describes the information flow of an investor who maximizes his expected utility by dynamically trading in a complete continuous-time security market. We emphasize that $\mathcal{H}_0$ need not be trivial. The results are applied in Section 5 to $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$ to quantify the informational advantage from the additional information $\mathcal{G}$. They are similar but more general than in [1]. Since we use a different definition of admissible strategies, our setting covers utility functions defined on all of $\mathbb{R}$, not just on $\mathbb{R}^+$. We also need no local square integrability for the (discounted) risky asset prices; they are given by a general d-dimensional $\mathbb{H}$-semimartingale S. Throughout this section, the financial market is supposed to be $\mathbb{H}$-complete and free of arbitrage in the sense of

Assumption 4.1 ($\mathbb{H}$-C): *There is a* unique *probability measure $Q^{\mathbb{H}} \sim P$ with $dQ^{\mathbb{H}}/dP$ $\mathcal{H}_T$-measurable, $Q^{\mathbb{H}} = P$ on $\mathcal{H}_0$, and $S \in \mathcal{M}_{\mathrm{loc}}(Q^{\mathbb{H}}, \mathbb{H})$. In other words: $\Gamma^{\mathbb{H}} = \{Q^{\mathbb{H}}\}$.*

We denote by $Z^{\mathbb{H}} = (Z^{\mathbb{H}}_t)_{t\in[0,T]}$ the $\mathbb{H}$-density process of $Q^{\mathbb{H}}$ with respect to P and by $E^{\mathbb{H}}[\cdot]$ the expectation with respect to $Q^{\mathbb{H}}$. A measure Q is called *equivalent local martingale measure for S* if $S \in \mathcal{M}_{\mathrm{loc}}(Q, \mathbb{H})$ and $Q \sim P$. The existence of such a measure implies the absence of arbitrage (cf. [4] for details). Proposition 3.1 shows that under Assumption 4.1 ($\mathbb{H}$-C) S has the predictable representation property with respect to $(Q^{\mathbb{H}}, \mathbb{H})$. Such a financial market is called *$\mathbb{H}$-complete.* To formulate the investor's optimization problem, we now introduce admissible trading strategies.

Definition 4.2 *$\vartheta \in L(S, \mathbb{H})$ is called an* admissible strategy *if*

$$\textstyle\int_0^t \vartheta\, dS =: (\vartheta \cdot S)_t = E^{\mathbb{H}}\left[(\vartheta \cdot S)_T \middle| \mathcal{H}_t\right] \quad \text{for all } t \in [0, T]. \tag{4.1}$$

The set of admissible strategies is denoted by $\Theta^{\mathbb{H}}_{Adm}$.

Note that (4.1) requires that the right-hand side is well-defined and finite; this is satisfied for all t if and only if $E^{\mathbb{H}}[|(\vartheta \cdot S)_T|\,|\mathcal{H}_0] < \infty$. This property is part of the definition. However, our use of generalized conditional expectations as in [17], I.1, does not require that $(\vartheta \cdot S)_T$ is $Q^{\mathbb{H}}$-integrable. See Remark 4.4 for a more detailed discussion.
A *utility function* is a strictly increasing, strictly concave C^1 function $U : (\ell, \infty) \to \mathbb{R}$, $\ell \in [-\infty, 0]$, which satisfies $\lim_{x\uparrow\infty} U'(x) = 0$ and $\lim_{x\downarrow\ell} U'(x) = +\infty$. We use the

convention $U(x) := -\infty$ for $x \leq \ell$. The wealth process of a strategy $\vartheta \in \Theta^{I\!H}_{Adm}$ with initial capital x is given by $V_t = x + (\vartheta \cdot S)_t = x + \int_0^t \vartheta \, dS$, $0 \leq t \leq T$. An investor with information flow $I\!H$ and initial capital $x > \ell$ who wants to maximize his expected utility from terminal wealth faces the optimization problems of finding

$$u^{I\!H}(x) \quad := \sup_{V \in \mathcal{V}^{I\!H}(x)} E_P[U(V_T)] \tag{4.2}$$

or

$$\operatorname*{ess\,sup}_{V \in \mathcal{V}^{I\!H}(x)} E_P[U(V_T)|\mathcal{H}_0] \tag{4.3}$$

with $\mathcal{V}^{I\!H}(x) := \left\{ V \,\middle|\, V = x + \vartheta \cdot S,\ \vartheta \in \Theta^{I\!H}_{Adm},\ U^-(V_T) \in L^1(P) \right\}$. Clearly, an element of $\mathcal{V}^{I\!H}(x)$ attains the supremum (4.2) if it attains the (ω-wise) supremum (4.3).
Our next goal is to show that our definition of admissibility is quite natural in the present context. Using Proposition 3.1, the proof of the following lemma is fairly straightforward and is left to the reader; see [3] for details.

Lemma 4.3 *1. A measure $Q \sim P$ is a local martingale measure for S if and only if its density with respect to P on $\mathcal{H}_T$ is of the form $\left(\frac{dQ}{dP}\Big|_{\mathcal{H}_0} \right) Z^{I\!H}_T$.*

2. A process V satisfies $V_t = E^{I\!H}[V_T|\mathcal{H}_t]$ for $t \in [0,T]$ if and only if V is a martingale with respect to some equivalent local martingale measure Q for S. In particular, $\vartheta \in L(S, I\!H)$ is an admissible strategy if and only if there exists an equivalent local martingale measure Q for S such that $\vartheta \cdot S$ is a $(Q, I\!H)$-martingale.

3. If the lower bound ℓ for the domain of the utility function U is finite, we could replace $\Theta^{I\!H}_{Adm}$ by $\{\vartheta \in L(S, I\!H) \,|\, \vartheta \cdot S \geq c \text{ for some } c \in I\!R\}$ without changing either the supremum in (4.2) (or (4.3)) or the optimal solution, if this exists.

Remark 4.4 Let us now discuss Definition 4.2 in more detail. If $\mathcal{H}_0$ is not trivial, there is no unique measure $Q \sim P$ on $\mathcal{H}_T$ with $S \in \mathcal{M}_{\mathrm{loc}}(Q, I\!H)$ since there is complete freedom in the choice of Q on the initial σ-field $\mathcal{H}_0$. At first sight, Definition 4.2 seems to involve the particular measure $Q^{I\!H}$ via (4.1) in a crucial way. But part 2 of Lemma 4.3 shows that we could equally well require (4.1) with any equivalent local martingale measure Q for S. Moreover, by part 3 our definition is consistent with the classical setting for utility functions whose domain is bounded from below (cf. [18] or [19]). Nielsen ([20], Sect. 4.6) adopts a notion of admissibility that is equivalent to ours if the initial σ-field is trivial. Hence our concept of admissibility is quite natural in the context of a general complete market and a utility function whose domain might be unbounded from below. In comparison to [1], note also that Definition 4.2 ensures $\Theta^{I\!F}_{Adm} \subseteq \Theta^{I\!G}_{Adm}$ in the subsequent section, which is desirable for the interpretation.

To solve problem (4.3), let $I : (0, \infty) \to (\ell, \infty)$ be as usual the inverse of the derivative U'.

Proposition 4.5 *Suppose Assumption 4.1 ($\mathbb{H}$-C) is satisfied and there exists an $\mathcal{H}_0$-measurable random variable $\Lambda^{\mathbb{H}}(x):\Omega \to (0,\infty)$ with*

$$E^{\mathbb{H}}\left[I\left(\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}\right)\Big|\mathcal{H}_0\right]=x \tag{4.4}$$

and such that the process $V_t^{\mathbb{H}}:=E^{\mathbb{H}}\left[I\left(\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}\right)\Big|\mathcal{H}_t\right]$, $t\in[0,T]$, satisfies $U^-(V_T^{\mathbb{H}})\in L^1(P)$. Then $V^{\mathbb{H}}$ is the solution to the optimization problem (4.3), i.e. $V^{\mathbb{H}}\in\mathcal{V}^{\mathbb{H}}(x)$ and $E_P\left[U\left(V_T^{\mathbb{H}}\right)\Big|\mathcal{H}_0\right]=\operatorname*{ess\,sup}_{V\in\mathcal{V}^{\mathbb{H}}(x)} E_P[U(V_T)|\mathcal{H}_0]$.

Proof: By part 2 of Lemma 4.3, $V^{\mathbb{H}}$ is an $\mathbb{H}$-martingale with respect to some equivalent local martingale measure Q for S. It follows from Assumption 4.1 ($\mathbb{H}$-C) and Proposition 3.1 that S has the $(Q,\mathbb{H})$-PRP and $V^{\mathbb{H}}$ is in $\mathcal{V}^{\mathbb{H}}(x)$. Using the concavity of U, we obtain by standard arguments (cf. [1], Th. 5.1)

$$U(V_T^{\mathbb{H}})\geq U(V_T)+\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}(V_T^{\mathbb{H}}-V_T)\,,\quad V\in\mathcal{V}^{\mathbb{H}}(x)\,. \tag{4.5}$$

Even if $\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}\left(V_T^{\mathbb{H}}-V_T\right)$ is not integrable, we can take generalized conditional expectations (as in [17], I.1) to obtain $E_P\left[\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}(V_T^{\mathbb{H}}-V_T)\Big|\mathcal{H}_0\right]=0$, using that $\Lambda^{\mathbb{H}}(x)$ is $\mathcal{H}_0$-measurable. In combination with (4.5) this yields

$$E_P\left[U\left(V_T^{\mathbb{H}}\right)\Big|\mathcal{H}_0\right]\geq E_P[U(V_T)\,|\,\mathcal{H}_0]\,,\quad V\in\mathcal{V}^{\mathbb{H}}(x)\,. \tag{4.6}$$

Note that both conditional expectations in (4.6) are well-defined in the usual sense due to the definition of $\mathcal{V}^{\mathbb{H}}(x)$ and the assumption that $U^-(V_T^{\mathbb{H}})\in L^1(P)$. □

Remark 4.6 Sufficient conditions for the existence of the "Lagrange multiplier" $\Lambda^{\mathbb{H}}$ in Proposition 4.5 can be given similarly as in Lemma 5.2 of [1]; see [3]. In particular, one sufficient condition is that $E^{\mathbb{H}}\left[\left|I(\lambda Z^{\mathbb{H}})\right|\right]<\infty$ for all $\lambda>0$; this also appears in the classical theory for trivial $\mathcal{H}_0$, see Karatzas et al. [18].

For specific utility functions, $\Lambda^{\mathbb{H}}$ and $V^{\mathbb{H}}$ can often be calculated explicitly in terms of $Z^{\mathbb{H}}$ and x. The following result provides such formulae for common utility functions.

Corollary 4.7 *Suppose Assumption 4.1 ($\mathbb{H}$-C) is satisfied. Then the optimal wealth process $V^{\mathbb{H}}(x)$ and the expected utility function $u^{\mathbb{H}}$ for the utility functions U below are given as follows:*

1. *Logarithmic utility $U(x)=\log x$ on $(0,\infty)$: We have $u^{\mathbb{H}}(x)=\log x+E_P\left[\log\frac{1}{Z_T^{\mathbb{H}}}\right]=\log x+H\left(P|Q^{\mathbb{H}}\right)$. $u^{\mathbb{H}}$ is finite if and only if the relative entropy $H\left(P|Q^{\mathbb{H}}\right)$ is finite.*
2. *Power utility $U(x)=x^\gamma/\gamma$ on $(0,\infty)$ with $\gamma\in(0,1)$: If $E_P\left[\left(Z_T^{\mathbb{H}}\right)^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{H}_0\right]<\infty$ then $u^{\mathbb{H}}(x)=\frac{x^\gamma}{\gamma}E_P\left[E_P\left[\left(Z_T^{\mathbb{H}}\right)^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{H}_0\right]^{1-\gamma}\right]$, and $u^{\mathbb{H}}$ is finite if $E_P\left[\left(Z^{\mathbb{H}}\right)^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{H}_0\right]^{1-\gamma}$ is P-integrable.*

3. *Exponential utility $U(x) = -e^{-\alpha x}$ on $I\!R$ with $\alpha > 0$: If $H\left(Q^{I\!H}|P\right) < \infty$, then $u^{I\!H}$ is finite and we have $u^{I\!H}(x) = -e^{-\alpha x} E_P\left[Z_T^{I\!H} \exp\left(-E_P\left[Z_T^{I\!H} \log Z_T^{I\!H} \middle| \mathcal{H}_0\right]\right)\right]$.*

Proof: In order to apply Proposition 4.5 we calculate $\Lambda^{I\!H}$ and verify the required assumptions. Note that a solution $\Lambda^{I\!H}$ to (4.4) is unique if it exists. Proposition 4.5 then yields the optimal wealth process $V^{I\!H}$, and the formulae for $u^{I\!H}$ follow by calculations of $E_P[U(V_T^{I\!H})]$ which are left to the reader.

1. $I(y) = 1/y$ and $\Lambda^{I\!H}(x) = 1/x$ is the solution to (4.4). Obviously $x/Z^{I\!H}$ is a $(Q^{I\!H}, I\!H)$-martingale. Moreover, $E_P\left[\log\left(1/Z_T^{I\!H}\right)\right] = E^{I\!H}\left[\left(1/Z_T^{I\!H}\right)\log\left(1/Z_T^{I\!H}\right)\right]$ is the relative entropy of P with respect to $Q^{I\!H}$ on $\mathcal{H}_T$ and well-defined in $[0, \infty]$. By Proposition 4.5, we obtain $V_t^{I\!H}(x) = x/Z_t^{I\!H}$, $t \in [0, T]$, and thereby $u^{I\!H}$.

2. $I(y) = y^{\frac{1}{\gamma-1}}$, $E^{I\!H}\left[\left(Z_T^{I\!H}\right)^{\frac{1}{\gamma-1}} \middle| \mathcal{H}_0\right] = E_P\left[\left(Z_T^{I\!H}\right)^{\frac{\gamma}{\gamma-1}} \middle| \mathcal{H}_0\right]$ is finite by assumption, and the solution to (4.4) is given by $\Lambda^{I\!H}(x) = x^{\gamma-1}\left(E^{I\!H}\left[Z_T^{I\!H\,\frac{1}{\gamma-1}} \middle| \mathcal{H}_0\right]\right)^{-(\gamma-1)}$. Proposition 4.5 yields $V_t^{I\!H}(x) = x\, E^{I\!H}\left[\left(Z_T^{I\!H}\right)^{\frac{1}{\gamma-1}} \middle| \mathcal{H}_t\right]\left(E^{I\!H}\left[\left(Z_T^{I\!H}\right)^{\frac{1}{\gamma-1}} \middle| \mathcal{H}_0\right]\right)^{-1}$, $t \in [0, T]$, and thus $u^{I\!H}$.

3. $I(y) = -\frac{1}{\alpha}\log\frac{y}{\alpha}$, $E_P\left[Z_T^{I\!H}\log Z_T^{I\!H}\right] < \infty$ by assumption, and the solution to (4.4) is

$$\Lambda^{I\!H}(x) = \alpha \exp\left(-\alpha x - E_P\left[Z_T^{I\!H} \log Z_T^{I\!H} \middle| \mathcal{H}_0\right]\right).$$

Proposition 4.5 yields $V_t^{I\!H}(x) = x + \frac{1}{\alpha}E_P\left[Z_T^{I\!H}\log Z_T^{I\!H}\middle|\mathcal{H}_0\right] - \frac{1}{\alpha}E^{I\!H}\left[\log Z_T^{I\!H}\middle|\mathcal{H}_t\right]$, $t \in [0, T]$, and thereby $u^{I\!H}$. Since $E_P\left[Z_T^{I\!H}\log Z_T^{I\!H}|\mathcal{H}_0\right]$ is bounded from below, $u^{I\!H}$ is finite. □

5 Utility indifference value of initial information

We now consider an investor with information flow $I\!F$ trading in a complete financial market where the discounted prices of the risky assets are given by an $I\!F$-semimartingale $S = (S^i)_{i=1,\dots,d}$. This section introduces and studies the subjective monetary value of the additional initial information $\mathcal{G}$ for the investor. Closed form expressions for this value in concrete examples will be given in Section 6. Throughout the present section, we impose

Assumption 5.1 *Suppose Assumption 2.1 (D) is satisfied and Assumption 4.1 ($I\!F$-C) is satisfied with respect to $I\!F$.*

It follows by Theorem 3.2 that Assumption 4.1 ($I\!\!G$-C) is also satisfied with respect to $I\!\!G$ and the martingale preserving measure $Q^{I\!\!G}$ corresponding to $Q^{I\!F}$. We denote by $Z_T^{I\!F}$ and $Z_T^{I\!\!G}$ the densities of $Q^{I\!F}$ and $Q^{I\!\!G}$ with respect to P. Recall from Proposition 2.7 that $Z_T^{I\!F}/Z_T^{I\!\!G} = p^G$ since Assumption 2.3 (E) holds by Corollary 2.8. For both filtrations we are therefore within the framework of the previous section with $I\!H \in \{I\!F, I\!\!G\}$ and can use the corresponding results and notations.

Definition 5.2 *The* utility indifference value *of the additional initial information $\mathcal{G}$ is defined as a solution $\pi = \pi(x)$ of the equation*

$$u^{I\!F}(x) = u^{I\!\!G}(x - \pi)\,. \tag{5.1}$$

Equation (5.1) means that an investor with the goal to maximize his expected utility from terminal wealth is indifferent between the two alternatives to a) invest the initial capital x optimally by using the information flow $\mathbb{F}$, or b) acquire the additional information $\mathcal{G}$ by paying π and then invest the remaining capital $x - \pi$ optimally on the basis of the enlarged information flow $\mathbb{G}$. It is easy to see that the utility indifference value exists and is unique if the functions $u^{\mathbb{F}}$ and $u^{\mathbb{G}}$ are finite, continuous, strictly increasing and satisfy $\lim_{y \downarrow \ell} u^{\mathbb{G}}(y) < u^{\mathbb{F}}(x)$. Those conditions are satisfied in all subsequent examples. Because of $\Theta^{\mathbb{F}}_{Adm} \subseteq \Theta^{\mathbb{G}}_{Adm}$, we have $u^{\mathbb{F}}(x) \leq u^{\mathbb{G}}(x)$ for all x and hence π is nonnegative.
We now provide π for common utility functions. Recall that $\mathcal{G}_0 = \mathcal{G}$ up to P-nullsets.

Theorem 5.3 *Suppose Assumption 5.1 is satisfied. Then the utility indifference values π for the utility functions below are given as follows.*

1. *Logarithmic utility $U(x) = \log x$: If $H\left(P|Q^{\mathbb{G}}\right) = E_P\left[\log \frac{1}{Z_T^{\mathbb{G}}}\right] < \infty$ then*

$$\pi = x\left(1 - \exp\left(-E_P\left[\log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\right]\right)\right). \tag{5.2}$$

2. *Power utility $U(x) = x^\gamma/\gamma$, $\gamma \in (0,1)$: If $E_P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\right] < \infty$ then*

$$\pi = x\left(1 - \frac{E_P\left[\left(Z_T^{\mathbb{F}}\right)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{1-\gamma}{\gamma}}}{E_P\left[E_P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{G}\right]^{1-\gamma}\right]^{\frac{1}{\gamma}}}\right). \tag{5.3}$$

3. *Exponential utility $U(x) = -e^{-\alpha x}$, $\alpha > 0$: If $H(Q^{\mathbb{G}}|P) = E_P\left[Z_T^{\mathbb{G}} \log Z_T^{\mathbb{G}}\right] < \infty$ then*

$$\pi \;=\; -\frac{1}{\alpha}\log E_P\left[Z_T^{\mathbb{G}} \exp\left(E_P\left[Z_T^{\mathbb{G}} \log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\Big|\mathcal{G}\right]\right)\right]. \tag{5.4}$$

Note that we can replace $Z_T^{\mathbb{F}}/Z_T^{\mathbb{G}}$ with p^G.

Proof: By Theorem 3.2, Assumption 5.1 implies Assumption 4.1 ($\mathbb{G}$-C). For each part we can also show that the integrability assumptions from Corollary 4.7 are satisfied in both $\mathbb{G}$ and $\mathbb{F}$. Then $u^{\mathbb{F}}$ and $u^{\mathbb{G}}$ are given explicitly by Corollary 4.7 and we just have to verify (5.1). Since $u^{\mathbb{G}}$ is strictly increasing, the solution π is unique.
1. Jensen's inequality yields $E_P\left[\log \frac{1}{Z_T^{\mathbb{F}}}\right] = E_P\left[\log \frac{1}{E_P[Z_T^{\mathbb{G}}|\mathcal{F}_T]}\right] \leq E_P\left[\log \frac{1}{Z_T^{\mathbb{G}}}\right] < \infty$.
Part 1 of Corollary 4.7 implies $u^{\mathbb{G}}(x-\pi) - u^{\mathbb{F}}(x) = \log(x-\pi) + E_P\left[\log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\right] - \log x$, and inserting π given by (5.2) gives (5.1).
2. $E_P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\right] < \infty$ yields $E_P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{G}\right]^{1-\gamma} \in L^1(P)$ since $\gamma \in (0,1)$. Moreover, by Jensen's inequality we obtain

$$E_P\left[\left(Z_T^{\mathbb{F}}\right)^{\frac{\gamma}{\gamma-1}}\right] = E_P\left[E_P\left[Z_T^{\mathbb{G}}|\mathcal{F}_T\right]^{\frac{\gamma}{\gamma-1}}\right] \leq E_P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\right] < \infty.$$

Thus part 2 of Corollary 4.7 shows that (5.1) is equivalent to

$$\frac{x^\gamma}{\gamma} E_P\left[E_P\left[(Z_T^{\mathbb{F}})^{\frac{\gamma}{\gamma-1}} \right]^{1-\gamma} \right] = \frac{(x-\pi)^\gamma}{\gamma} E_P\left[E_P\left[(Z_T^{\mathbb{G}})^{\frac{\gamma}{\gamma-1}} \Big| \mathcal{G} \right]^{1-\gamma} \right],$$

and solving for π leads to (5.3).

3. Again Jensen's inequality shows $E_P\left[Z_T^{\mathbb{F}} \log Z_T^{\mathbb{F}}\right] \leq E_P\left[Z_T^{\mathbb{G}} \log Z_T^{\mathbb{G}}\right] < \infty$. Using the properties that define the martingale preserving measure $Q^{\mathbb{G}}$, we calculate

$$-E_P\left[Z_T^{\mathbb{G}} \log Z_T^{\mathbb{G}} \Big| \mathcal{G} \right] = E^{\mathbb{G}}\left[\log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}} - \log Z_T^{\mathbb{F}} \Big| \mathcal{G} \right] = E_P\left[Z_T^{\mathbb{G}} \log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}} \Big| \mathcal{G} \right] - E_P\left[Z_T^{\mathbb{F}} \log Z_T^{\mathbb{F}} \right].$$

In combination with part 3 of Corollary 4.7 we get for $y \in \mathbb{R}$

$$u^{\mathbb{G}}(y) \;=\; -\exp\Big(-\alpha y - E_P\left[Z_T^{\mathbb{F}} \log Z_T^{\mathbb{F}} \right] \Big) E^{\mathbb{G}}\left[\exp\left(E_P\left[Z_T^{\mathbb{G}} \log \frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}} \Big| \mathcal{G} \right] \right) \right].$$

For $y = x - \pi$ with π from (5.4) this yields $u^{\mathbb{G}}(x-\pi) = u^{\mathbb{F}}(x)$ by Corollary 4.7. $\square$

6 Examples: Terminal information distorted by noise

The canonical example for our setup is a situation where we have additional information about future values of the asset prices, distorted by some extra noise. In this section, we calculate closed form expressions for the utility indifference value for logarithmic and exponential utility when the complete financial market is given by the standard multi-dimensional Itô process model and the initial information consists of a noisy signal about the terminal value of the Brownian motion driving the asset prices. For power utility preferences, similar computations of formula (5.3) would be more cumbersome and are omitted for reasons of length.

Let the discounted prices of the risky assets be given by the SDEs

$$\text{(6.1)} \qquad \frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{j=1}^d \sigma_t^{ij}\, dW_t^j, \quad S_0^i > 0 \quad \text{for } i = 1, \ldots, d,$$

where $W = (W^j)_{j=1,\ldots,d}$ is a d-dimensional Brownian motion and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the P-augmentation of the filtration generated by W. The excess return vector $\mu = (\mu^i)_{i=1,\ldots,d}$ and the volatility matrix $\sigma = (\sigma^{ij})_{i,j=1,\ldots,d}$ are assumed predictable with $\int_0^T \left(|\mu_t| + |\sigma_t|^2\right) dt < \infty$ P-a.s., and σ_t has full rank P-a.s. for all $t \in [0,T]$. The *relative risk process* is $\lambda_t := \sigma_t^{-1}\mu_t$. We suppose that $\int_0^T |\lambda_t|^2\, dt < \infty$ and that $Z^{\mathbb{F}} := \mathcal{E}(-\int \lambda\, dW)$ is a $(P, \mathbb{F})$-martingale. Then $dQ^{\mathbb{F}} := Z_T^{\mathbb{F}}\, dP$ is the unique equivalent local martingale measure for S on $\mathcal{F}_T$ so that Assumption 4.1 ($\mathbb{F}$-C) is satisfied.

Suppose the additional information $\mathcal{G} = \sigma(G)$ is a noisy signal about the outcome of W_T, i.e. $G := (\delta_i W_T^i + (1-\delta_i)\varepsilon_i)_{i=1,\ldots,d}$, where the ε_i are i.i.d. $\mathcal{N}(0,1)$-distributed and independent of $\mathcal{F}_T$ and the δ_i are constants in $[0,1)$. If all δ_i are strictly positive, the additional information is also generated by $\widetilde{G} := \left(W_T^i + \frac{1-\delta_i}{\delta_i}\varepsilon_i\right)_{i=1,\dots,d}$ which is an unbiased signal for W_T. A regular conditional distribution of G given $\mathcal{F}_t$ exists for all $t \in [0,T]$ and is multivariate normal with mean vector $\left(\delta_i W_t^i\right)_{i=1,\dots,d}$ and covariance matrix $\operatorname{diag}\left(\delta_i^2(T-t) + (1-\delta_i)^2\right)_{i=1,\dots,d}$. Hence the conditional distribution of G given $\mathcal{F}_T$ is a.s. equivalent to the distribution of G which is also normal. Assumption 2.3 (E) is therefore satisfied and a straightforward computation gives

$$p^G = \prod_{i=1}^d \sqrt{\frac{\delta_i^2 T + (1-\delta_i)^2}{(1-\delta_i)^2}} \exp\left(\frac{1}{2}\left(\frac{G_i^2}{\delta_i^2 T + (1-\delta_i)^2} - \frac{(G_i - \delta_i W_T^i)^2}{(1-\delta_i)^2}\right)\right). \tag{6.2}$$

As in previous sections we denote by $Q^{\mathbb{G}}$ the martingale preserving measure corresponding to $Q^{\mathbb{F}}$ and by $Z_T^{\mathbb{G}}$ its density with respect to P. We recall the relation $Z_T^{\mathbb{G}} = Z_T^{\mathbb{F}}/p^G$.

Logarithmic utility indifference value

Consider first the logarithmic utility function $U(x) = \log x$ and assume $E_P\left[\int_0^T |\lambda_t|^2\,dt\right]$ is finite so that $H\left(P|Q^{\mathbb{F}}\right) = E_P\left[\log(1/Z_T^{\mathbb{F}})\right] = \frac{1}{2}E_P\left[\int_0^T |\lambda_t|^2\,dt\right] < \infty$. Together with

$$E_P\left[\log p^G\right] = \frac{1}{2}\sum_{i=1}^d \log\frac{\delta_i^2 T + (1-\delta_i)^2}{(1-\delta_i)^2} < \infty\,,$$

this leads to $E_P\left[\log\frac{1}{Z_T^{\mathbb{G}}}\right] = E_P\left[\log\frac{1}{Z_T^{\mathbb{F}}}\right] + E_P\left[\log p^G\right] < \infty$. By (5.2) we obtain that the logarithmic utility indifference value is given by

$$\pi = x\left(1 - \prod_{i=1}^d \sqrt{\frac{(1-\delta_i)^2}{\delta_i^2 T + (1-\delta_i)^2}}\right),$$

and we can analyze the behavior of this quantity as the parameters vary. If all δ_i converge to zero, then π tends to zero and in particular $\pi = 0$ if $\delta_i = 0$ for all i. Intuitively this shows that the information obtained from G becomes useless when increasing noise hides all information about W_T. Furthermore π is increasing in T, increasing in each δ_i and converges to x if $\delta_i \uparrow 1$ for one i with all other parameters fixed. For very small noise, the additional information thus intuitively almost offers an arbitrage opportunity; this is best seen in the case of constant coefficients μ and σ where S_T is a function of W_T. In fact, the value of the information for the ordinary investor then comes close to his total initial capital x and the investor cannot pay more than x since the logarithmic utility function enforces a strictly positive remaining initial capital $x - \pi$ by requiring an a.s. strictly positive final wealth. Note that the limiting case $\delta_i = 1$ is not included in our framework since it would violate Assumption 2.3 (E).

Exponential utility indifference value

Now consider the case where the investor's utility function is given by $U(x) = -\exp(-\alpha x)$ with $\alpha > 0$ and assume $H(Q^{\mathbb{F}}|P) < \infty$. By Girsanov's theorem, $\widetilde{W} := W + \int \lambda_t\,dt$ is a

$(Q^{\mathbb{F}},\mathbb{F})$-Brownian motion. The relative entropy $H(Q^{\mathbb{F}}|P)$ is given by

$$E_P\left[Z_T^{\mathbb{F}}\log Z_T^{\mathbb{F}}\right]=E^{\mathbb{F}}\left[-\int_0^T\lambda_t\,d\widetilde{W}_t+\frac{1}{2}\int_0^T|\lambda_t|^2\,dt\right]=\frac{1}{2}E^{\mathbb{F}}\left[\int_0^T|\lambda_t|^2\,dt\right]$$

and finite by assumption. From $E^{\mathbb{F}}\left[\left(\int_0^T\lambda_s^i\,ds\right)^2\right]\le TE^{\mathbb{F}}\left[\int_0^T|\lambda_s^i|^2\,ds\right]<\infty$ we conclude that W_T^i is in $L^2(Q^{\mathbb{F}})$ and hence in $L^2(Q^{\mathbb{G}})$ for all $i=1,\dots,d$. Since $Q^{\mathbb{G}}=P$ on $\sigma(G)$, the random variables G_i are independent and normally distributed under $Q^{\mathbb{G}}$. By (6.2) we obtain $\log p^G\in L^1(Q^{\mathbb{G}})$ and $E_P\left[\frac{Z_T^{\mathbb{F}}}{p^G}\log\frac{Z_T^{\mathbb{F}}}{p^G}\right]=E_P\left[Z_T^{\mathbb{F}}\log Z_T^{\mathbb{F}}\right]+E^{\mathbb{G}}\left[\log p^G\right]<\infty$. Straightforward calculation yields

$$\exp\left(E^{\mathbb{G}}\left[\log p^G\middle|\mathcal{G}\right]\right)=\exp\left(E^{\mathbb{G}}\left[\log p^g\right]\Big|_{g=G}\right)=$$

$$\prod_{i=1}^d\exp\left(\frac{1}{2}\left(\log\frac{\delta_i^2T+(1-\delta_i)^2}{(1-\delta_i)^2}+\frac{G_i^2}{\delta_i^2T+(1-\delta_i)^2}-\frac{E^{\mathbb{G}}\left[(g_i-\delta_iW_T^i)^2\right]\big|_{g=G}}{(1-\delta_i)^2}\right)\right).\tag{6.3}$$

As $Q^{\mathbb{G}}=Q^{\mathbb{F}}$ on $\mathcal{F}_T$, we obtain

$$E^{\mathbb{G}}\left[(g_i-\delta_iW_T^i)^2\right]\big|_{g=G}\;=\;G_i^2-2\delta_iE^{\mathbb{F}}\left[W_T^i\right]G_i+\delta_i^2E^{\mathbb{F}}\left[\left(W_T^i\right)^2\right].\tag{6.4}$$

Further calculation yields

$$E^{\mathbb{G}}\left[\exp\left(\frac{-\delta_i^2T}{2\left(\delta_i^2T+(1-\delta_i)^2\right)(1-\delta_i)^2}G_i^2+\frac{\delta_iE^{\mathbb{F}}[W_T^i]}{(1-\delta_i)^2}G_i-\frac{\delta_i^2E^{\mathbb{F}}\left[(W_T^i)^2\right]}{2(1-\delta_i)^2}\right)\right]$$

$$=\sqrt{\frac{(1-\delta_i)^2}{\delta_i^2T+(1-\delta_i)^2}}\exp\left(\frac{-\delta_i^2}{2(1-\delta_i)^2}\mathrm{Var}_{Q^{\mathbb{F}}}[W_T^i]\right).\tag{6.5}$$

The d factors in the product (6.3) are $Q^{\mathbb{G}}$-independent since $Q^{\mathbb{G}}=P$ on $\sigma(G)$. Hence $E^{\mathbb{G}}\left[\exp\left(E^{\mathbb{G}}[\log p^G|\mathcal{G}]\right)\right]$ is equal to the product of the $Q^{\mathbb{G}}$-expectations of the d single factors; computing $E^{\mathbb{G}}\left[(g_i-\delta_iW_T^i)^2\right]\big|_{g=G}$ by (6.4) and using (6.5) then leads to

$$E^{\mathbb{G}}\left[\exp\left(E^{\mathbb{G}}\left[\log p^G|\mathcal{G}\right]\right)\right]=\prod_{i=1}^d\exp\left(\frac{-\delta_i^2}{2(1-\delta_i)^2}\mathrm{Var}_{Q^{\mathbb{F}}}[W_T^i]\right).$$

By (5.4) we obtain that the exponential utility indifference value is given by

$$\pi\;=\;\frac{1}{2\alpha}\sum_{i=1}^d\frac{\delta_i^2}{(1-\delta_i)^2}\mathrm{Var}_{Q^{\mathbb{F}}}[W_T^i]$$

$$\;=\;\frac{1}{2\alpha}\sum_{i=1}^d\frac{\delta_i^2}{(1-\delta_i)^2}\left(T-2\mathrm{Cov}_{Q^{\mathbb{F}}}\left[\widetilde{W}_T^i,\int_0^T\lambda_s^i\,ds\right]+\mathrm{Var}_{Q^{\mathbb{F}}}\left[\int_0^T\lambda_s^i\,ds\right]\right).\tag{6.6}$$

We see that π is decreasing in the risk-aversion coefficient α, increasing in δ_i and tends to zero if all δ_i converge to zero. π tends to infinity if $\delta_i\uparrow 1$ for one i, ceteris paribus. Again this is precisely what intuition suggests should happen. If the relative risk process λ is deterministic, (6.6) yields the closed form solution

$$\pi=\frac{T}{2\alpha}\sum_{i=1}^d\frac{\delta_i^2}{(1-\delta_i)^2}.$$

7 Conclusion

Under our (equivalent) assumptions in Section 2, the initial enlargement framework for a financial market possesses an implicit product structure. This allows to transfer both existence of a local martingale measure (i.e. a no-arbitrage condition) and its uniqueness (in the sense of Proposition 3.1, i.e. completeness of the market) from a smaller filtration $\mathbb{F}$ to the initially enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathcal{G}$, and to study the utility indifference value of the additional information $\mathcal{G}$ by using a conditional density function.

A Appendix

This section contains two proofs omitted from the main body of the paper.

Proof of Corollary 2.8: If we choose $R = \widetilde{P}$ as the martingale preserving measure corresponding to P, i.e. $dR := 1/p^G \, dP$, Proposition 2.7 yields that Assumption 2.3 (E) implies Assumption 2.1 (D). Conversely, assume that there exists $R \sim P$ such that $\mathcal{F}_T$ and $\mathcal{G} = \sigma(G)$ are R-independent. Since the laws of (ω, G) on $\mathcal{F}_T \otimes \mathcal{X}$ under P and R are equivalent, there is a strictly positive Radon-Nikodým derivative f of $P \circ (\omega, G)^{-1}|_{\mathcal{F}_T \otimes \mathcal{X}}$ with respect to $R \circ (\omega, G)^{-1}|_{\mathcal{F}_T \otimes \mathcal{X}} = R|_{\mathcal{F}_T} \otimes R[G \in \cdot\,]$; the last equality uses the decoupling property of R. The function $f(x|\omega) := f(\omega, x) \left(\int_X f(\omega, x) \, R[G \in dx] \right)^{-1}$ with $x \in X$ and $\omega \in \Omega$ is strictly positive and it is straightforward to verify that $E_P \left[1_A P[G \in B | \mathcal{F}_T] \right] = E_P \left[1_A 1_{\{G \in B\}} \right] = E_P \left[1_A \int_B f(x|\cdot) \, R[G \in dx] \right]$ holds for $A \in \mathcal{F}_T$ and $B \in \mathcal{X}$. Hence a regular conditional P-distribution of G given $\mathcal{F}_T$ exists and is given by $P[G \in B | \mathcal{F}_T](\omega) = \int_B f(x|\omega) \, R[G \in dx]$, $B \in \mathcal{X}\,, \omega \in \Omega$. This implies that for P-a.a. ω we have $P[G \in \cdot\, | \mathcal{F}_T](\omega) \sim R[G \in \cdot\,] \sim P[G \in \cdot\,]\,.$ $\square$

Proof of Lemma 3.3: Let us first show the claim for functions of the form $f(\omega, x) = 1_A(\omega) 1_B(x)$ with $A \in \mathcal{F}_T$ and $B \in \mathcal{X}$. We fix a finite version h of $E_R[1_A | \mathcal{F}_t]$ and define $g(\omega, x) := 1_B(x) h(\omega)$. Then we have $E_R[1_A 1_B(x) | \mathcal{F}_t] = 1_B(x) E_R[1_A | \mathcal{F}_t] = g(\cdot, x)$, $x \in X$, and the R-independence of $\mathcal{F}_T$ and $\mathcal{G}$ yields for any $D \in \mathcal{F}_t \vee \mathcal{G}$ that $E_R[1_A 1_B(G) 1_D] = E_R[E_R[1_A | \mathcal{F}_t] 1_B(G) 1_D] = E_R[g(\cdot, G) 1_D]$. Hence, both (3.5) and (3.6) hold for such functions f. By monotone class arguments, the claim of Lemma 3.3 therefore holds for all bounded $\mathcal{F}_T \otimes \mathcal{X}$-measurable functions f, and (3.6) then even holds for all $x \in X$ instead of only $R[G \in \cdot\,]$-a.a. $x \in X$. Now assume only that $f(\cdot, G)$ is R-integrable. Using the R-independence of $\mathcal{F}_T$ and $\mathcal{G}$ and Fubini's theorem, we conclude that $f(\cdot, x) \in L^1(R)$ for $R[G \in \cdot\,]$-a.a. x. Let $f^n(\omega, x) := \min(\max(f(\omega, x), -n), n)$, $n \in \mathbb{N}$. By the above argument, there are $\mathcal{F}_t \otimes \mathcal{X}$-measurable functions g^n such that

$$g^n(\cdot, G) = E_R \left[f^n(\cdot, G) | \mathcal{F}_t \vee \mathcal{G} \right] \,, \tag{A.1}$$

$$g^n(\cdot, x) = E_R \left[f^n(\cdot, x) | \mathcal{F}_t \right] \,, \quad x \in X\,. \tag{A.2}$$

Define $g(\omega, x) := \liminf_{n\to\infty} g^n(\omega, x)$ where this is finite and $g(\omega, x) := 0$ elsewhere. By dominated convergence, the right hand sides (RHSs) of (A.1) converge for $n \to \infty$ a.s. to the RHS of (3.5). Analogously, for $R[G \in \cdot\,]$-a.a. x the random variable $g(\cdot, x)$ is in $L^1(R)$ and the RHSs of (A.2) converge a.s. to the RHS of (3.6). Because these limits are a.s. finite, we obtain (3.5) and (3.6). $\square$

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