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# Hedging of Contingent Claims under Incomplete Information

by

Hans Föllmer *)
Martin Schweizer *)

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# Hedging of Contingent Claims under Incomplete Information

*Hans Föllmer and Martin Schweizer*

*Universität Bonn*
*Institut für Angewandte Mathematik*
*Wegelerstraße 6*
*D – 5300 Bonn 1*
*West Germany*

## 1. Introduction

Consider a financial market where the price fluctuation of stocks is given by a stochastic process $X = (X_t)_{0 \le t \le T}$ on some probability space $(\Omega, \mathcal{F}, P)$. If this market is *complete* then any contingent claim, viewed as a random variable on $(\Omega, \mathcal{F}, P)$, can be generated by a dynamic portfolio strategy based on the underlying stock process $X$. This was the basic economic insight behind the Black-Scholes formula for option pricing.

In the absence of arbitrage opportunities, there is an equivalent probability measure $P^* \approx P$ such that $X$ is a martingale under $P^*$; this implies that $X$ is a *semimartingale* under the basic measure $P$. From a mathematical point of view, completeness now means that any contingent claim $H$ can be represented as a stochastic integral of the semimartingale $X$. The integrand in such a representation provides a sequential hedging strategy which is self-financing, and which creates the random amount $H(\omega)$ at the terminal time $T$ without any risk; cf. [7], [8]. Thus, the a priori risk as measured for example by the variance of $H$, is reduced to 0 by a suitable dynamic strategy.

In the *incomplete* case, a general claim is not necessarily a stochastic integral of $X$. From an economic point of view, this means that such a claim will have an *intrinsic risk*. We can only hope to reduce the a priori risk to this minimal component. Thus, the problem is to characterize and to construct those strategies which *minimize* the risk. For the martingale case $P = P^*$, the notion of a risk-minimizing strategy was introduced in [6]. In this context, it was shown that a unique risk-minimizing strategy exists, and that it can be constructed using the Kunita-Watanabe projection technique in the space $\mathcal{M}^2$ of square-integrable martingales.

In this paper we consider the general case $P \approx P^*$ where $X$ is no longer a martingale, but only a semimartingale under the given measure $P$. Here the problem of reducing risk becomes more delicate. In [11], risk-minimization was defined in a *local* sense, and the construction of such strategies was reduced to a stochastic optimality equation. The problem of solving this equation is equivalent to a projection problem in the space $\mathcal{S}^2$ of semimartingales. Section 2 gives an introduction to these questions; it is based on the results in [6], [11], [12], [13]. Here we restrict the discussion to the case where the semimartingale $X$ has *continuous paths*, and we put more emphasis on the projection problem in $\mathcal{S}^2$.

It seems natural to use a Girsanov transformation in order to shift this problem back to the space $\mathcal{M}^2$ where the standard projection technique can be used. But this needs some care: In the incomplete case, the equivalent martingale measure is no longer unique, and one has to make an appropriate choice $\widehat{P}$ of the martingale measure in order to determine the optimal strategy in terms of $\widehat{P}$. In section 3 we introduce the notion of a *minimal* martingale measure $\widehat{P} \approx P$ and discuss its existence and uniqueness. The idea is that we want to preserve the structure of $P$ as far as possible under the constraint that $X$ becomes a martingale under $\widehat{P}$. In the class of all equivalent martingale measures, this minimal modification of the basic measure $P$ can also be characterized in terms of the relative entropy $H(.|P)$. We show how the optimal strategy can be computed in terms of the minimal martingale measure $\widehat{P}$.

In section 4 we discuss the case where incompleteness is due to *incomplete information*. We assume that the claim $H$ admits an Itô representation as a stochastic integral with respect to some large filtration $(\widetilde{\mathcal{F}}_t)_{0 \le t \le T}$. But in constructing a dynamic strategy, we can only use a smaller filtration $(\mathcal{F}_t)_{0 \le t \le T}$ , where $\mathcal{F}_t$ specifies the information available to us at time $t$. We show how the unique optimal strategy can be constructed by projecting the Itô integrand down to the filtration $(\mathcal{F}_t)$. We also discuss the question to which extent this strategy is *robust* under an equivalent change of measure.

In section 5 we analyze the special case where incompleteness is due to a random fluctuation in the variance. As an example, consider the standard Black-Scholes model where $X$ is a geometric Brownian motion, and suppose that there is a random jump of the variance at some fixed time $t_0$. This example was introduced in [8]; for a martingale measure $P = P^*$, the corresponding strategy was investigated in [10]. Our attempt to understand its probabilistic structure from a more general point of view led to the projection results described in sections 4 and 5.

## 2. Minimizing Risk in an Incomplete Market

Let $X = (X_t)_{0 \le t \le T}$ be a stochastic process with *continuous* paths on some probability space $(\Omega, \mathcal{F}, P)$ with a right-continuous filtration $(\mathcal{F}_t)_{0 \le t \le T}$. The process $X$ is supposed to describe the price fluctuation of a given stock. In order to keep the notation simple we assume that $X$ is real-valued, but the extension of what follows to the multi-dimensional case is straightforward. We assume that $X$ belongs to the *space* $\mathcal{S}^2$ *of semimartingales*; cf. [2]. For the Doob-Meyer decomposition

$$X = X_0 + M + A \tag{2.1}$$

of $X$ into a local martingale $M = (M_t)_{0 \le t \le T}$ and a predictable process $A = (A_t)_{0 \le t \le T}$ with paths of bounded variation, this amounts to the integrability condition

$$(2.2) \qquad E\left[X_0^2 + \langle X\rangle_T + |A|_T^2\right] < \infty.$$

Here $\langle X\rangle = \langle M\rangle$ denotes the pathwise defined quadratic variation process of $X$ resp. $M$, and $|A| = (|A|_t)_{0\le t\le T}$ is the total variation of $A$. In particular,

$$(2.3) \qquad M \text{ is a square-integrable martingale under } P.$$

Consider a *contingent claim* at time $T$ given by a random variable

$$(2.4) \qquad H \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P).$$

In order to hedge against this claim, we want to use a portfolio strategy which involves the stock $X$ and a riskless bond $Y \equiv 1$, and which yields the random payoff $H$ at the terminal time $T$. Let $\xi_t$ and $\eta_t$ denote the amounts of stock and bond, respectively, held at time $t$. We assume that the process $\xi = (\xi_t)_{0\le t\le T}$ is *predictable* while $\eta = (\eta_t)_{0\le t\le T}$ is allowed to be *adapted*; cf. [6] for the underlying motivation. The *value* of the resulting portfolio at time $t$ is given by

$$(2.5) \qquad V_t = \xi_t X_t + \eta_t \qquad (0 \le t \le T),$$

the *cost* accumulated up to time $t$ by

$$(2.6) \qquad C_t = V_t - \int_0^t \xi_s dX_s \qquad (0 \le t \le T).$$

We only admit strategies $(\xi, \eta)$ such that the processes $V = (V_t)_{0\le t\le T}$ and $C = (C_t)_{0\le t\le T}$ are square-integrable, have right-continuous paths and satisfy

$$(2.7) \qquad V_T = H \qquad P - a.s.$$

We also require the integrability condition

$$(2.8) \qquad E\left[\int_0^T \xi_s^2 d\langle X\rangle_s + \left(\int_0^T |\xi_s| d|A|_s\right)^2\right] < \infty;$$

this ensures that the process of stochastic integrals in (2.6) is well defined and belongs to the space $\mathcal{S}^2$ of semimartingales. Such strategies will be called *admissible*.

Suppose that our claim $H$ admits an *Itô representation* of the form

$$(2.9) \qquad H = H_0 + \int_0^T \xi_s^H dX_s \qquad P - a.s.$$

where $\xi^H$ satisfies (2.8). Then we can use the strategy defined by

$$\text{(2.10)}\quad \xi := \xi^H \quad , \quad \eta := V - \xi \cdot X \quad , \quad V_t := H_0 + \int_0^t \xi_s^H dX_s \qquad (0 \leq t \leq T).$$

This strategy is clearly admissible. Moreover, it is *self-financing*, i.e.,

$$\text{(2.11)}\qquad C_t = C_T = H_0 \qquad (0 \leq t \leq T).$$

Thus, the Itô representation (2.9) leads to a strategy which produces the claim $H$ from the initial amount $C_0 = H_0$. No further cost arises, and no risk is involved.

Let us now introduce the standard assumption which excludes arbitrage opportunities, namely the existence of an *equivalent martingale measure* $P^*$. More precisely, we assume that $P^* \approx P$ is a probability measure on $(\Omega, \mathcal{F})$ such that

$$\text{(2.12)}\qquad \frac{dP^*}{dP} \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$$

and

$$\text{(2.13)}\qquad X \text{ is a martingale under } P^*.$$

Then the strategy in (2.10) can be identified as follows:

$$\text{(2.14)}\qquad V_t = E^*[H|\mathcal{F}_t] \qquad (0 \leq t \leq T),$$

and $\xi^H$ is obtained as the Radon-Nikodym derivative

$$\text{(2.15)}\qquad \xi^H = \frac{d\langle V, X\rangle}{d\langle X\rangle}$$

where $\langle V, X\rangle$ is the covariance process associated to $V$ and $X$. Thus, the strategy can be identified in terms of $P^*$ and does not depend on the specific choice of the initial measure $P \approx P^*$.

So far we have summarized the well-known mathematical construction of hedging strategies in a *complete* financial market model where *every* contingent claim is *attainable*, i.e., admits a representation (2.9); cf. [7], [8] . In that ideal case, hedging allows complete elimination of the risk involved in handling an option. In the *incomplete* case this is no longer possible. A typical claim will carry an *intrinsic risk*, and the problem consists in finding a dynamic portfolio strategy which reduces the actual risk to that intrinsic component. Let us first consider the case $P = P^*$ where $X$ is already a martingale under the initial measure $P$.

In this context, the following criterion of *risk-minimization* was introduced in [6]: We look for an admissible strategy which minimizes, at each time $t$, the remaining risk

$$E[(C_T - C_t)^2 | \mathcal{F}_t] \tag{2.16}$$

over all admissible continuations of this strategy from time $t$ on; cf. [6] for a detailed definition. $H$ is attainable if and only if this remaining risk can be reduced to 0, since this is equivalent to (2.11). But for a general contingent claim (2.4), the cost process associated to a risk-minimizing strategy will no longer be self-financing. Instead, it will be *mean-self-financing* in the sense that

$$E[C_T - C_t | \mathcal{F}_t] = 0 \qquad (0 \le t \le T).$$

In other words, the cost process $C$ associated to a risk-minimizing strategy is a *martingale*. In [6] the existence of a unique risk-minimizing strategy is shown. In order to describe it, consider the *Kunita-Watanabe decomposition*

$$H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H \tag{2.17}$$

with $H_0 \in \mathcal{L}^2(\Omega, \mathcal{F}_0, P)$, where

(2.18) $\quad L^H = (L_t^H)_{0 \le t \le T}$ is a square-integrable martingale orthogonal to $X$.

The risk-minimizing strategy is now given by

$$\xi := \xi^H \quad , \quad \eta := V - \xi \cdot X \tag{2.19}$$

with

$$V_t := H_0 + \int_0^t \xi_s^H dX_s + L_t^H \qquad (0 \le t \le T). \tag{2.20}$$

In the present martingale case, the process $V$ can also be computed directly as a right-continuous version of the martingale

$$V_t = E[H | \mathcal{F}_t] \qquad (0 \le t \le T). \tag{2.21}$$

In particular, $\xi^H$ is given by (2.15). Thus, the problem is solved by using a well-known projection technique in the space $\mathcal{M}^2$ of square-integrable martingales: we simply project the martingale $V$ associated to $H$ on the martingale $X$.

Let us now consider the *general incomplete* case where $P \approx P^*$, but where $P$ itself is no longer a martingale measure. Here the situation becomes more subtle, and we are going to face a projection problem which is no longer standard. In [11] a criterion of local risk-minimization is introduced. A strategy is called *locally risk-minimizing* if, for any $t < T$, the remaining risk (2.16) is minimal under all infinitesimal perturbations of the strategy at time $t$. This definition is made precise in terms of the differentiation of semimartingales, and it is shown to be essentially equivalent to the following property of the associated cost process $C = (C_t)_{0 \leq t \leq T}$:

(2.22) $\qquad C$ is a square-integrable martingale orthogonal to $M$ under $P$;

cf. [11], [12], [13]. This motivates the following

(2.23) **Definition.** An admissible strategy $(\xi, \eta)$ is called *optimal* if the associated cost process $C$ satisfies condition (2.22).

In discrete time, a unique optimal strategy does exist, and it can be determined by a sequential regression procedure running backwards from time $T$ to time 0; cf. [11]. In continuous time, the construction of such a strategy becomes more difficult. We start with the observation that an optimal strategy corresponds to a decomposition (2.17) of the claim where $L^H$ is now orthogonal to the martingale component $M$ of $X$.

(2.24) **Proposition.** *The existence of an optimal strategy is equivalent to a decomposition*

$$H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H \tag{2.25}$$

*with $H_0 \in \mathcal{L}^2(\Omega, \mathcal{F}_0, P)$, where $\xi^H$ satisfies (2.8) and*

(2.26) $\quad L^H = (L_t^H)_{0 \leq t \leq T}$ *is a square-integrable martingale orthogonal to $M$.*

*For such a decomposition, the associated optimal strategy $(\xi, \eta)$ is given by (2.19) and (2.20).*

**Proof.** For a decomposition (2.25) with (2.26), the cost process associated to the strategy $(\xi, \eta)$ defined by (2.19) and (2.20) is given by

$$C_t = H_0 + L_t^H \qquad (0 \leq t \leq T),$$

and so $(\xi, \eta)$ is optimal. Conversely, an optimal strategy leads to the decomposition

$$\begin{aligned} H &= C_T + \int_0^T \xi_s dX_s \\ &= C_0 + \int_0^T \xi_s dX_s + (C_T - C_0), \end{aligned}$$

and so we have (2.25) with $\xi^H = \xi$, $L_t^H = C_t - C_0$ $(0 \le t \le T)$ and $H_0 = C_0$.

Thus, the problem of minimizing risk is reduced to finding the representation (2.25) and (2.26). This is of course analogous to (2.17) and (2.18). But if $X$ is not a martingale, we can no longer use directly the usual Kunita-Watanabe projection technique.

(2.27) **Remark.** One possible approach is to use as a starting point the Kunita-Watanabe decomposition

$$H = N_0^H + \int_0^T \mu_s^H dM_s + (N_T^H - N_0^H) \tag{2.28}$$

of $H$ with respect to the square-integrable martingale $M$, where $N^H = (N_t^H)_{0 \le t \le T}$ is a square-integrable martingale orthogonal to $M$. Consider a decomposition

$$H = H_0 + \int_0^T \xi_s dX_s + L_T$$

as in (2.25), and introduce the Kunita-Watanabe decomposition

$$\int_0^T \xi_s dA_s = N_0^\xi + \int_0^T \mu_s^\xi dM_s + (N_T^\xi - N_0^\xi)$$

where $N^\xi = (N_t^\xi)_{0 \le t \le T}$ is a square-integrable martingale orthogonal to $M$. This leads to

$$H = (H_0 + N_0^\xi) + \int_0^T (\xi_s + \mu_s^\xi) dM_s + (L_T + N_T^\xi - N_0^\xi).$$

But (2.28) is unique, and so an optimal strategy must satisfy the *optimality equation*

$$\xi + \mu^\xi = \mu^H. \tag{2.29}$$

One can now focus on this equation and analyze existence and uniqueness of its solution. This is the approach taken in [11], [12].

In the next section we use a different method to study the *uniqueness* of the decomposition (2.25) and of the corresponding optimal strategy, and at the same time its *robustness* under an equivalent change of measure. In the complete case, the optimal strategy can be computed in terms of the unique equivalent martingale measure $P^*$. Thus, it does not depend on the specific choice of the measure $P \approx P^*$. In our present situation, the question becomes obviously more delicate. To begin with, the martingale measure $P^*$ is no longer unique [9], and it has been shown in [6] how different martingale measures $P^*$ may lead to different strategies. But it turns out that there is a *minimal* martingale measure $\widehat{P} \approx P$ such that the optimal strategy for $P$ can be computed in terms of $\widehat{P}$. In this partial sense, robustness will extend to our present case.

## 3. The Minimal Martingale Measure

Recall that the notion of a martingale measure $P^* \approx P$ was defined by properties (2.12) and (2.13). Such a martingale measure is determined by the right-continuous square-integrable martingale $G^* = (G^*_t)_{0 \le t \le T}$ with

$$G^*_t = E\left[\frac{dP^*}{dP}\middle|\mathcal{F}_t\right] \qquad (0 \le t \le T).$$

Under $P^*$, the Doob-Meyer decomposition of $M$ is given by $M = X - X_0 + (-A)$. But the theory of the Girsanov transformation shows that the predictable process of bounded variation can also be computed in terms of $G^*$:

$$-A_t = \int_0^t \frac{1}{G^*_{s-}}\, d\langle M, G^*\rangle_s \qquad (0 \le t \le T);$$

cf. [2], VII.49. Since $\langle M, G^*\rangle \ll \langle M\rangle = \langle X\rangle$, the process $A$ must be absolutely continuous with respect to the variance process $\langle X\rangle$ of $X$, i.e.,

$$A_t = \int_0^t \alpha_s d\langle X\rangle_s \qquad (0 \le t \le T) \tag{3.1}$$

for some predictable process $\alpha = (\alpha_t)_{0 \le t \le T}$.

(3.2) **Definition.** A martingale measure $\widehat{P} \approx P$ will be called *minimal* if

$$\widehat{P} = P \quad \text{on } \mathcal{F}_0, \tag{3.3}$$

and if any square-integrable $P$-martingale which is orthogonal to $M$ under $P$ remains a martingale under $\widehat{P}$:

$$L \in \mathcal{M}^2 \text{ and } \langle L, M\rangle = 0 \quad \Longrightarrow \quad L \text{ is a martingale under } \widehat{P} \tag{3.4}$$

Let us now look at the question of existence and uniqueness.

(3.5) **Theorem. 1)** *The minimal martingale measure $\widehat{P}$ is uniquely determined.*

**2)** *$\widehat{P}$ exists if and only if*

$$\widehat{G}_t = \exp\left(-\int_0^t \alpha_s dM_s - \frac{1}{2}\int_0^t \alpha_s^2 d\langle X\rangle_s\right) \qquad (0 \le t \le T) \tag{3.6}$$

*is a square-integrable martingale under $P$; in that case, $\widehat{P}$ is given by $\dfrac{d\widehat{P}}{dP} = \widehat{G}_T$.*

**3)** *The minimal martingale measure preserves orthogonality: Any $L \in \mathcal{M}^2$ with $\langle L, M\rangle = 0$ under $P$ satisfies $\langle L, X\rangle = 0$ under $\widehat{P}$.*

**Proof.** 1) Let $G^* = (G_t^*)_{0\le t\le T}$ be the square-integrable martingale associated to a martingale measure $P^* \approx P$. Then

$$G_t^* = G_0^* + \int_0^t \beta_s \, dM_s + L_t \qquad (0 \le t \le T)$$

where $L$ is a square-integrable martingale under $P$ orthogonal to $M$, and $\beta = (\beta_t)_{0\le t\le T}$ is a predictable process with

$$E\left[\int_0^T \beta_s^2 \, d\langle M\rangle\right] < \infty. \tag{3.7}$$

Under $P^*$, the predictable process of bounded variation in the Doob-Meyer decomposition of $M$ is given by

$$\int_0^t \frac{1}{G_{s-}^*} \, d\langle G^*, M\rangle_s = \int_0^t \frac{1}{G_{s-}^*} \cdot \beta_s \, d\langle X\rangle_s.$$

But $X = X_0 + M + A$ is assumed to be a martingale under $P^*$, and so we get

$$\alpha = -\frac{\beta}{G_-^*}; \tag{3.8}$$

since $G^* > 0$ $P$-a.s. due to $P^* \approx P$ and since $\langle M\rangle = \langle X\rangle$, (3.7) implies

$$\int_0^T \alpha_s^2 \, d\langle X\rangle_s < \infty \qquad P-a.s. \tag{3.9}$$

Now suppose that $P^*$ is *minimal.* Then $G_0^* = 1$ due to (3.3), and $L$ is a martingale under $P^*$ due to (3.4). This implies $\langle L, G^*\rangle = 0$, and so we get

$$\langle L\rangle = \langle L, G^*\rangle = 0,$$

hence $L \equiv 0$. Thus, $G^*$ solves the stochastic equation

$$G_t^* = 1 + \int_0^t G_{s-}^* \cdot (-\alpha_s)\, dM_s. \tag{3.10}$$

Since $M$ is continuous and $\langle M \rangle = \langle X \rangle$, we obtain $G^* = \widehat{G}$, hence uniqueness.

2) Due to (3.9), the process $\widehat{G}$ is well-defined by (3.6). But in general, it is only a local martingale under $P$. If $\widehat{G}$ corresponds to a martingale measure, then this local martingale is in fact a square-integrable martingale. Conversely, suppose that the process $\widehat{G}$ has this property; we want to show that the associated martingale measure $\widehat{P}$ is indeed minimal. Consider a martingale $L \in \mathcal{M}^2$ with $\langle L, M \rangle = 0$ under $P$. Since $\widehat{G}$ solves (3.10), we get $\langle L, \widehat{G} \rangle = 0$, and so $L$ is a local martingale under $\widehat{P}$. But since $L$ is a square-integrable martingale under $P$, we have

$$\sup_{0 \le t \le T} |L_t| \in \mathcal{L}^2(\Omega, \mathcal{F}, P),$$

hence

$$\sup_{0 \le t \le T} |L_t| \in \mathcal{L}^1(\Omega, \mathcal{F}, \widehat{P}),$$

since $\widehat{G}_T \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$. Thus, the local martingale $L$ is in fact a martingale under $\widehat{P}$.

3) Let us show that $L$ from above also satisfies $\langle L, X \rangle = 0$ under $\widehat{P}$. Recall the definition of the process

$$[Y, Z] := \langle Y^c, Z^c \rangle + \sum_s \Delta Y_s \cdot \Delta Z_s$$

for two semimartingales $Y$ and $Z$; cf. [4], 12.6. Since $X$ and $A$ are continuous, we have

$$\begin{aligned}
\langle L, X \rangle &= \langle L^c, X \rangle + \langle L^d, X \rangle \\
&= \langle L^c, X \rangle \\
&= [L, X] \\
&= [L, M] + [L, A] \\
&= [L, M]
\end{aligned}$$

under $\widehat{P}$. But since $M$ is continuous,

$$[L, M] = \langle L^c, M \rangle = \langle L, M \rangle = 0$$

under $P$, and this implies that $[L, M] = 0$ also under $\widehat{P}$; cf. [2], Theorem VIII.20.

Definition (3.2) means that $\widehat{P}$ preserves the martingale property as far as possible under the restriction (2.13). This minimal departure from the given measure $P$ can also be expressed in terms of the *relative entropy*

$$H(Q|P) = \begin{cases} \int \log \frac{dQ}{dP}\, dQ & \text{if } Q \ll P \\ +\infty & \text{otherwise.} \end{cases}$$

Recall that the relative entropy is always nonnegative, and that $H(Q|P) = 0$ is equivalent to $Q = P$.

(3.11) **Theorem.** *In the class of all martingale measures $P^*$, the minimal martingale measure $\widehat{P}$ is characterized by the fact that it minimizes the functional*

$$H(P^*|P) - \frac{1}{2}\cdot E^*\left[\int_0^T \alpha_s^2\, d\langle X\rangle_s\right]. \tag{3.12}$$

*In particular, $\widehat{P}$ minimizes the relative entropy $H(.|P)$ among all martingale measures $P^*$ with fixed expectation*

$$E^*\left[\int_0^T \alpha_s^2\, d\langle X\rangle_s\right]. \tag{3.13}$$

**Proof.** If $P^*$ is a martingale measure, then $M$ has the Doob-Meyer decomposition

$$M_t = X_t - X_0 + \left(-\int_0^t \alpha_s\, d\langle X\rangle_s\right)$$

under $P^*$. Due to (2.12), we have

$$G_T^* := \frac{dP^*}{dP} \in \mathcal{L}^2(\Omega, \mathcal{F}, P);$$

in particular, the relative entropy is finite:

$$H(P^*|P) = \int G_T^* \cdot \log G_T^*\, dP < \infty.$$

Now suppose that $\widehat{P} \approx P \approx P^*$ is the minimal martingale measure. Then

$$\begin{aligned} H(P^*|P) &= H(P^*|\widehat{P}) + \int \log \widehat{G}_T\, dP^* \\ &= H(P^*|\widehat{P}) + \int \left(-\int_0^T \alpha_s\, dM_s - \frac{1}{2}\cdot\int_0^T \alpha_s^2\, d\langle X\rangle_s\right) dP^* \\ &= H(P^*|\widehat{P}) + \frac{1}{2}\cdot E^*\left[\int_0^T \alpha_s^2\, d\langle X\rangle_s\right] \end{aligned}$$

(localize first, then pass to the limit using $H(P^*|P) < \infty$). In particular, the expectation in (3.13) is finite. Thus,

$$H(P^*|P) - \frac{1}{2} \cdot E^* \left[ \int_0^T \alpha_s^2 \, d\langle X \rangle_s \right] = H(P^*|\widehat{P}) \geq 0,$$

and the minimal value 0 is assumed if and only if $P^* = \widehat{P}$.

Let us now return to the problem of computing the optimal strategy in (2.23). Let $\mathcal{P}$ be the $\sigma$-field of predictable sets on $\overline{\Omega} = \Omega \times [0, T]$ associated to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. We denote by $\overline{P}$ the finite measure on $\mathcal{P}$ defined by $\overline{P}(d\omega, dt) = P(d\omega) d\langle X \rangle_t(\omega)$; $\overline{\widehat{P}}$ is defined in the same manner.

(3.14) **Theorem.** *The optimal strategy, hence also the corresponding decomposition (2.25), is uniquely determined. It can be computed in terms of the minimal martingale measure $\widehat{P}$: If $V = (V_t)_{0 \leq t \leq T}$ denotes a right-continuous version of the martingale*

$$V_t := \widehat{E}[H|\mathcal{F}_t] \qquad (0 \leq t \leq T), \tag{3.15}$$

*then the optimal strategy $(\xi, \eta)$ is given by (2.19) where*

$$\xi^H = \frac{d\langle V, X \rangle}{d\langle X \rangle} \tag{3.16}$$

*is obtained by projecting the $\widehat{P}$-martingale $V$ on the $\widehat{P}$-martingale $X$.*

**Proof.** 1) (2.4) and (2.12) imply $H \in \mathcal{L}^1(\Omega, \mathcal{F}_T, \widehat{P})$, and so the martingale $V$ in (3.15) is well-defined. Now suppose that we have a decomposition (2.25) with (2.26). By (3.4), $L^H$ is a martingale under $\widehat{P}$. The integrability argument in part 2) of the proof of theorem (3.5) shows that the process

$$\int_0^t \xi_s^H dX_s \qquad (0 \leq t \leq T)$$

is also a martingale under $\widehat{P}$. This implies that the process $V$ in (2.20) is given by (3.15).

2) By part 3) of theorem (3.5), we have $\langle L^H, X \rangle = 0$ $\overline{P}$-a.e., hence $\overline{\widehat{P}}$-a.e. Thus, the representation

$$V_t = H_0 + \int_0^t \xi_s^H dX_s + L_t^H \qquad (0 \leq t \leq T)$$

of the $\widehat{P}$-martingale $V$ implies that $\xi^H$ can be identified as the Radon-Nikodym derivative in (3.16).

The representation (3.15) and (3.16) of the optimal strategy in terms of $\widehat{P}$ also provides a natural approach to its *existence*. We can start with the Kunita-Watanabe decomposition of $H$ under the minimal martingale measure $\widehat{P}$. Then we have to ensure that the resulting processes $\xi^H$ and $V$ resp. $\eta$ fit into the preceding framework. If we insist on the space $\mathcal{S}^2$, then we need additional integrability conditions on $H$ and $A$. A more flexible alternative would be to *localize* both the definitions and the arguments in the preceding discussion.

In the next section we consider a situation where a different approach can be used. We suppose that the model is complete with respect to some larger filtration, and we show how the decomposition (2.25) with respect to the given filtration can be derived by projection. In particular, we obtain existence and uniqueness of the optimal strategy.

## 4. Incomplete Information

In this section we consider a situation which would be complete if we had more information. The information accessible to us is described by the filtration $(\mathcal{F}_t)_{0\le t\le T}$. We are going to suppose that the claim $H$ is attainable with respect to some larger filtration. Only at the terminal time $T$, but not at times $t < T$, all the information relevant to the claim will be available to us. So let $(\widetilde{\mathcal{F}}_t)_{0\le t\le T}$ be a right-continuous filtration such that

$$\mathcal{F}_t \subseteq \widetilde{\mathcal{F}}_t \subseteq \mathcal{F} \qquad (0\le t\le T). \tag{4.1}$$

Our basic assumption is that the Doob-Meyer decomposition

$$X = X_0 + M + A \tag{4.2}$$

of the semimartingale $X \in \mathcal{S}^2$ in (2.1) with respect to $(\mathcal{F}_t)$ is still valid with respect to $(\widetilde{\mathcal{F}}_t)$. In other words, we assume that

$$M \text{ is a martingale with respect to } (\widetilde{\mathcal{F}}_t)_{0\le t\le T}, \tag{4.3}$$

although it is adapted to the smaller filtration $(\mathcal{F}_t)$. A class of examples will be given in section 5.

Now suppose that $H$ in (2.4) is attainable with respect to the large filtration $(\widetilde{\mathcal{F}}_t)$, i.e.,

$$H = \widetilde{H}_0 + \int_0^T \widetilde{\xi}^H_s\, dX_s \tag{4.4}$$

where $\widetilde{H}_0$ is $\widetilde{\mathcal{F}}_0$-measurable, and where the process $\widetilde{\xi}^H = (\widetilde{\xi}^H_t)_{0\le t\le T}$ is predictable with respect to $(\widetilde{\mathcal{F}}_t)$. Let us specify our integrability assumptions. We assume that the $(\widetilde{\mathcal{F}}_t)$-semimartingale

$$\widetilde{H}_0 + \int_0^t \widetilde{\xi}^H_s dX_s \qquad (0 \le t \le T)$$

associated to $H$ belongs to the space $\mathcal{S}^2$. This amounts to the condition

$$E\left[\widetilde{H}_0^2 + \int_0^T \left(\widetilde{\xi}^H_s\right)^2 d\langle X\rangle_s + \left(\int_0^T |\widetilde{\xi}^H_s| d|A|_s\right)^2\right] < \infty. \tag{4.5}$$

Let $\widetilde{\mathcal{P}}$ denote the $\sigma$-field of predictable sets on $\overline{\Omega}$ associated to the filtration $(\widetilde{\mathcal{F}}_t)$. Recall from section 3 the measure $\overline{P}$, and note that (4.5) implies $\widetilde{\xi}^H \in \mathcal{L}^2(\overline{\Omega}, \widetilde{\mathcal{P}}, \overline{P})$.

(4.6) **Theorem.** *Suppose that $H$ satisfies (4.4) and (4.5). Then $H$ admits the representation*

$$H = H_0 + \int_0^T \xi^H_s dX_s + L^H_T \tag{4.7}$$

*with $H_0 := E\big[\widetilde{H}_0 \big| \mathcal{F}_0\big]$, where*

$$\xi^H := \overline{E}\left[\widetilde{\xi}^H \middle| \mathcal{P}\right] \tag{4.8}$$

*is the conditional expectation of $\widetilde{\xi}^H$ with respect to $\mathcal{P}$ and $\overline{P}$, and where $L^H = (L^H_t)_{0\le t\le T}$ is the square-integrable $(\mathcal{F}_t)$-martingale orthogonal to $M$ associated to*

$$L^H_T := \widetilde{H}_0 - H_0 + \int_0^T (\widetilde{\xi}^H_s - \xi^H_s) dX_s \ \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P). \tag{4.9}$$

**Proof.** 1) Let us first show that all components in (4.7) are square-integrable. Since $\xi^H \in \mathcal{L}^2(\overline{\Omega}, \mathcal{P}, \overline{P})$, we obtain

$$\int_0^T \xi^H_s dM_s \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P). \tag{4.10}$$

(4.5) implies $\widetilde{\xi}^H \cdot \alpha \in \mathcal{L}^1(\overline{\Omega}, \widetilde{\mathcal{P}}, \overline{P})$, hence $\xi^H \cdot \alpha \in \mathcal{L}^1(\overline{\Omega}, \mathcal{P}, \overline{P})$, and so we have

$$\int_0^T |\xi^H_s| \cdot |\alpha_s| d\langle X\rangle_s \in \mathcal{L}^1(\Omega, \mathcal{F}_T, P), \tag{4.11}$$

and in particular

$$\int_0^T \xi_s^H dA_s \in \mathcal{L}^1(\Omega, \mathcal{F}_T, P). \tag{4.12}$$

In order to obtain

$$\int_0^T \xi_s^H dA_s \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P),$$

we show that

$$\left| E\left[ Z_T \int_0^T \xi_s^H dA_s \right] \right| \leq c \cdot \|Z_T\|_2 \tag{4.13}$$

for any bounded $\mathcal{F}_T$-measurable $Z_T$ with $\mathcal{L}^2$-norm $\|Z_T\|_2$. Let $(Z_t)_{0\leq t\leq T}$ denote a right-continuous version with left limits of the martingale $E[Z_T|\mathcal{F}_t]$ $(0 \leq t \leq T)$, and put $Z^* := \sup\limits_{0\leq t\leq T} |Z_t|$. By predictable projection (cf. [2], VI.45, VI.57),

$$\begin{aligned}
\left| E\left[ Z_T \int_0^T \xi_s^H dA_s \right] \right| &= \left| E\left[ \int_0^T Z_{s-} \cdot \xi_s^H \cdot \alpha_s d\langle X\rangle_s \right] \right| \\
&= \left| E\left[ \int_0^T Z_{s-} \cdot \widetilde{\xi}_s^H \cdot \alpha_s d\langle X\rangle_s \right] \right| \\
&\leq E\left[ Z^* \int_0^T |\widetilde{\xi}_s^H| \cdot |\alpha_s| d\langle X\rangle_s \right] \\
&\leq \|Z^*\|_2 \cdot \left\| \int_0^T |\widetilde{\xi}_s^H| \cdot |\alpha_s| d\langle X\rangle_s \right\|_2 \\
&\leq c \cdot \|Z_T\|_2;
\end{aligned}$$

in the last step we use (4.5) and Doob's inequality for the supremum of a square-integrable martingale.

2) Clearly, $\widetilde{H}_0 - H_0 \in \mathcal{L}^2(\Omega, \widetilde{\mathcal{F}}_0, P)$ is orthogonal to all square-integrable stochastic integrals of $M$ with respect to the filtration $(\widetilde{\mathcal{F}}_t)$, hence in particular with respect to the filtration $(\mathcal{F}_t)$. Thus, it only remains to show

$$E\left[ \left( \int_0^T (\widetilde{\xi}_s^H - \xi_s^H) dX_s \right) \cdot \left( \int_0^T \mu_s dM_s \right) \right] = 0,$$

resp.

$$E\left[ \left( \int_0^T \widetilde{\xi}_s^H dX_s \right) \cdot \left( \int_0^T \mu_s dM_s \right) \right] = E\left[ \left( \int_0^T \xi_s^H dX_s \right) \cdot \left( \int_0^T \mu_s dM_s \right) \right], \tag{4.14}$$

say for all bounded $\mathcal{P}$-measurable processes $\mu = (\mu_t)_{0\le t\le T}$; this will imply that the martingale $L^H$ is orthogonal to $M$. We decompose the left side of (4.14) into

$$E\left[\left(\int_0^T \widetilde{\xi}_s^H dM_s\right)\cdot\left(\int_0^T \mu_s dM_s\right)\right] = E\left[\int_0^T \widetilde{\xi}_s^H \cdot \mu_s d\langle X\rangle_s\right]$$

and

$$E\left[\left(\int_0^T \widetilde{\xi}_s^H dA_s\right)\cdot\left(\int_0^T \mu_s dM_s\right)\right] = E\left[\int_0^T \widetilde{\xi}_s^H \cdot\left(\int_0^s \mu_u dM_u\right)\cdot \alpha_s d\langle X\rangle_s\right];$$

the second identity follows by predictable projection. But it is now clear that $\widetilde{\xi}^H$ can be replaced by $\xi^H$ in each of the two parts, and this yields (4.14).

The representation (4.7), together with the equivalence in proposition (2.24), leads to the following

(4.15) **Corollary.** *There exists a unique optimal strategy given by (2.19) and (2.20).*

We have seen in (3.14) how the optimal strategy $(\xi,\eta)$ is determined by the minimal martingale measure $\widehat{P}$. In our present context, its component $\xi = \xi^H$ can also be computed directly from $\widetilde{\xi}^H$ as a conditional expectation with respect to $\overline{\widehat{P}}$.

(4.16) **Theorem.** *The optimal strategy is given by (2.19) and (2.20) where*

$$\xi^H = \overline{\widehat{E}}\left[\widetilde{\xi}^H \middle| \mathcal{P}\right] \tag{4.17}$$

*is the conditional expectation of $\widetilde{\xi}^H$ with respect to $\mathcal{P}$ under the measure $\overline{\widehat{P}}$ associated to the minimal martingale measure $\widehat{P}$.*

**Proof.** Decomposing if necessary, we may assume $\widetilde{\xi}^H \ge 0$. We want to show

$$\widehat{E}\left[\int_0^T \widetilde{\xi}_s^H \cdot \vartheta_s d\langle X\rangle_s\right] = \widehat{E}\left[\int_0^T \xi_s^H \cdot \vartheta_s d\langle X\rangle_s\right] \tag{4.18}$$

for any nonnegative $\mathcal{P}$-measurable process $\vartheta = (\vartheta_t)_{0\le t\le T}$. The left side equals

$$E\left[\widehat{G}_T \int_0^T \widetilde{\xi}_s^H \cdot \vartheta_s d\langle X\rangle_s\right],$$

and by predictable projection, this is equal to

$$E\left[\int_0^T \widehat{G}_s \cdot \widetilde{\xi}_s^H \cdot \vartheta_s d\langle X\rangle_s\right].$$

But here we can replace $\widetilde{\xi}^H$ by $\xi^H$ since $(\widehat{G}_t \cdot \vartheta_t)_{0\le t\le T}$ is $\mathcal{P}$-measurable, and this implies (4.18).

## 5. Incompleteness due to a Random Variance

As our basic probability space we take a standard diffusion model on $C[0,T]$ together with an additional source of randomness, given by a probability space $(S,\mathcal{S},\mu)$, which will affect the variance of the diffusion. Let $\mathcal{F}$ be the natural product $\sigma$-field on

$$\Omega = C[0,T]\times S,$$

define $X_t(\omega)=\omega_0(t)$ for $\omega=(\omega_0,\eta)\in\Omega$, and let $(\mathcal{F}_t)_{0\le t\le T}$ denote the right-continuous filtration generated by the process $X=(X_t)_{0\le t\le T}$. Let $\beta=(\beta_t)_{0\le t\le T}$ and $\sigma(\eta)=(\sigma_t(\cdot,\eta))_{0\le t\le T}$ $(\eta\in S)$ be measurable adapted processes on $C[0,T]$ such that the stochastic differential equation

$$dX_t = \sigma_t(X,\eta)dW_t + \beta_t(X)dt, \tag{5.1}$$

with fixed initial value $X_0 = x_0$, where $W=(W_t)$ is a Wiener process, has a unique weak solution for any $\eta\in S$. Let $P_\eta$ be the corresponding distribution on $C[0,T]$; we assume that the mapping $\eta\mapsto P_\eta$ is measurable with respect to $\mathcal{S}$. Under suitable bounds on the processes $\sigma(\eta)$ and $\beta$, the diffusion model

$$(C[0,T],P_\eta) \text{ is } \textit{complete} \tag{5.2}$$

for any $\eta\in S$, i.e., any square-integrable contingent claim can be written as a stochastic integral of the coordinate process.

Let $P$ be the probability measure on $(\Omega,\mathcal{F})$ associated to $\mu$ and to the map $\eta\mapsto P_\eta$ by $P(d\omega_0,d\eta)=\mu(d\eta)P_\eta(d\omega_0)$. We assume that $X$ is a semimartingale of class $\mathcal{S}^2$ under $P$; this amounts to the condition

$$\int_S \mu(d\eta)E_\eta\left[\int_0^T \sigma_s^2(\cdot,\eta)ds + \left(\int_0^T |\beta_s|\cdot\sigma_s^{-2}ds\right)^2\right] < \infty. \tag{5.3}$$

In particular,

$$M_t := X_t - X_0 - \int_0^t \beta_s(X)ds \qquad (0 \le t \le T) \tag{5.4}$$

is a square-integrable martingale under $P$, with variance process $\langle M \rangle = \langle X \rangle$ given by

$$\langle X \rangle_t(\omega) = \int_0^t \sigma_s^2(X(\omega), \eta)ds \qquad (0 \le t \le T). \tag{5.5}$$

$P$ is concentrated on the measurable set of paths in $C[0,T]$ which admit a quadratic variation along a suitable sequence of partitions of $[0,T]$; cf. [5]. This shows that the left side of (5.5) can be defined to be $\mathcal{F}_t$-measurable. Thus, the $\sigma$-field $\mathcal{F}_t$ reveals through (5.5) the partial information $\sigma_s^2(X,\eta)$ $(0 \le s \le t)$ about $\eta$. In particular, (5.4) yields the Doob-Meyer decomposition

$$X_t = X_0 + M_t + \int_0^t \alpha_s d\langle X \rangle_s \qquad (0 \le t \le T) \tag{5.6}$$

of $X$ with respect to the filtration $(\mathcal{F}_t)$, where $\alpha = (\alpha_t)_{0 \le t \le T}$ can be chosen as a $\mathcal{P}$-measurable version of the process $(\beta_t \cdot \sigma_t^{-2})_{0 \le t \le T}$, due to the pathwise reconstruction of the process $\sigma^2$.

In general, the model $(\Omega, \mathcal{F}, P)$ is *incomplete* with respect to the filtration $(\mathcal{F}_t)$; cf. the explicit example below. But consider the larger right-continuous filtration $(\widetilde{\mathcal{F}}_t)_{0 \le t \le T}$ obtained by adding the full information about the second coordinate $\eta$ already at the initial time 0. The conditional model $(\Omega, P[\cdot|\widetilde{\mathcal{F}}_0](\eta))$ can be identified with the complete model $(C[0,T], P_\eta)$. This implies that

(5.7) $P$ is *complete* with respect to $(\widetilde{\mathcal{F}}_t)$,

i.e., any square-integrable $H \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$ can be written as

$$H = \widetilde{H}_0 + \int_0^T \widetilde{\xi}_s^H dX_s \tag{5.8}$$

with $\widetilde{H}_0 \in \mathcal{L}^2(\Omega, \widetilde{\mathcal{F}}_0, P)$ and a $\widetilde{\mathcal{P}}$-measurable process $\widetilde{\xi}^H = (\widetilde{\xi}_t^H)_{0 \le t \le T}$. Since the martingale $M$ defined in (5.4) is also a martingale with respect to the larger filtration $(\widetilde{\mathcal{F}}_t)$, condition (4.3) is satisfied, and we can apply the results of section 4. Under the integrability condition (4.5) on $\alpha$, theorem (4.6) leads to the representation

$$H = H_0 + \int_0^T \xi_s^H dX_s + L_T^H \tag{5.9}$$

with $H_0 := E[\widetilde{H}_0]$, where

$$\xi^H := \overline{E}\left[\widetilde{\xi}^H \middle| \mathcal{P}\right] \tag{5.10}$$

is the conditional expectation of $\widetilde{\xi}^H$ with respect to $\mathcal{P}$ and $\overline{P}$, and where $L^H = (L_t^H)_{0\le t\le T}$ is the square-integrable $(\mathcal{F}_t)$-martingale associated to

$$L_T^H := \widetilde{H}_0 - H_0 + \int_0^T (\widetilde{\xi}_s^H - \xi_s^H)dX_s \ \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P). \tag{5.11}$$

Due to corollary (4.15), we obtain in particular the following

(5.12) **Theorem.** *The strategy defined by (5.10) and (2.19) is optimal.*

Under further bounds on $\sigma(\eta)$ and $\beta$, the process

$$\widehat{G}_t = \exp\left(-\int_0^t \alpha_s dM_s - \frac{1}{2}\int_0^t \alpha_s^2 d\langle X\rangle_s\right) \qquad (0 \le t \le T)$$

is a square-integrable martingale, and so the *minimal* martingale measure $\widehat{P}$ is well defined. In that case, theorem (4.16) shows that the optimal strategy can also be computed in terms of $\widehat{P}$. In particular, it does not depend on our initial choice of the drift process $\beta$.

Since our model is incomplete, the martingale measure is not unique. In fact, any distribution $\nu \approx \mu$ on $(S, \mathcal{S})$ composed with the conditional distribution of $\widehat{P}$ with respect to $\eta$ induces a martingale measure; the minimal martingale measure is characterized by the condition $\nu = \mu$.

(5.13) **Example.** In order to illustrate the general projection method, let us consider the standard Black-Scholes model, but with a random jump of the diffusion parameter at time $t_0$. In the martingale case $P = P^*$, this example was introduced by Harrison and Pliska in [8] and analyzed in detail by S. Müller in [10]. In the notation of this section, we take $S = \{+, -\}$ and $\mu(\{+\}) = p$. For $\eta \in S$ and $t_0 \in (0, T)$, define the piecewise constant function

$$\sigma_t(\eta) = \sigma^0 I_{[0,t_0)}(t) + \sigma^\eta I_{[t_0,T]}(t) \qquad (0 \le t \le T)$$

with fixed parameters $\sigma^0, \sigma^+, \sigma^- > 0$. Let $P_\eta$ be the distribution of the solution of the stochastic differential equation

$$dX_t = \sigma_t(\eta) \cdot X_t dW_t + \gamma \cdot X_t\, dt$$

with some drift parameter $\gamma \in I\!R$. Any contingent claim can be written as a stochastic integral with respect to the larger filtration $(\widetilde{\mathcal{F}}_t)$:

$$H = H_0^+ I_B + H_0^- I_{B^c} + \int_0^T (\xi_s^+ I_B + \xi_s^- I_{B^c}) dX_s \tag{5.14}$$

where $H_0^{\pm}$ and $\xi^{\pm}$ denote the usual Black-Scholes values and strategies for a *known* variance $\sigma(\pm)$, and where $B = \{\eta = +\}$. The decomposition (5.9) with respect to the smaller filtration $(\mathcal{F}_t)$ is given by

$$H_0 = E[\widetilde{H}_0] = pH_0^+ + (1-p)H_0^-,$$

$$\xi = (p\xi^+ + (1-p)\xi^-)I_{[0,t_0]} + (\xi^+ I_B + \xi^- I_{B^c})I_{(t_0,T]}$$

and

$$L_t^H = (H_0^+ - H_0^-) \cdot L_t \qquad (0 \leq t \leq T),$$

where $L = (L_t)_{0 \leq t \leq T}$ denotes the martingale

$$L_t = (I_B - p) \cdot I_{[t_0,T]}(t) \qquad (0 \leq t \leq T).$$

This determines the optimal strategy, due to theorem (5.12). As pointed out in [8], the model *becomes* complete if we add the martingale $L$ to the market and extend $X$ to the two-dimensional stock process $\widetilde{X} = (X, L)$. In the present incomplete case, there is an *intrinsic risk* described by the risk process

$$E[(C_T - C_t)^2 | \mathcal{F}_t] = p(1-p)(H_0^+ - H_0^-)^2 I_{[0,t_0)}(t) \qquad (0 \leq t \leq T).$$

The optimal strategy depends explicitly on $p$ but not on the drift parameter $\gamma$. In fact, it can be computed in terms of the *minimal* martingale measure $\widehat{P}$ which eliminates the drift but does not change the parameter $p$. Under the assumption $P = P^*$, this optimal strategy was already obtained in [10].

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