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Convergence of Option Values under Incompleteness

by

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Abstract: We study the problem of convergence of discrete-time option values to continuous-time option values. While previous papers typically concentrate on the approximation of geometric Brownian motion by a binomial tree, we consider here the case where the model is incomplete in both continuous and discrete time. Option values are defined with respect to the criterion of local risk-minimization and thus computed as expectations under the respective minimal martingale measures. We prove that for a jump-diffusion model with deterministic coefficients, these values converge; this shows that local risk-minimization possesses an inherent stability property under discretization.

Key words: option pricing, incomplete markets, convergence, minimal martingale measure, locally risk-minimizing trading strategies, jump-diffusion

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0. Introduction

A major controversy in the modern theory of option pricing is the debate of *discrete-time versus continuous-time* modelling. Whereas a continuous-time formulation is often more amenable to analysis and thus tends to provide better insight, one also frequently hears the counter-argument that a description of real market behaviour can only be based on discrete observations. In view of these contrasting opinions, it is natural to look for connections between the two competing approaches. In particular, one should like to obtain convergence results as one passes from discrete time to continuous time by a limiting procedure. There is an abundant literature on this question and we mention here only a very few references. One of the starting points is the paper by Cox/Ross/Rubinstein (1979) which provides a derivation of the famous *Black-Scholes formula*, first obtained in Black/Scholes (1973) and Merton (1973), by a passage to the limit from a binomial model. The survey by Willinger/Taqqu (1991) contains an excellent overview of several convergence approaches used so far, as well as an extensive list of references. Among the more recent contributions, we mention Eberlein (1992), where pathwise approximations of geometric Brownian motion by piecewise constant processes are constructed, and Duffie/Protter (1992), who discuss the convergence of the process of cumulative gains from trade.

Perhaps the first result that one would like to establish in this context is the *convergence of option prices*. There are quite a few results in this direction when stock prices are given by diffusion processes and the discrete-time models are binomial or suitable multinomial trees; see for instance Cox/Ross/Rubinstein (1979), He (1990) or Duffie/Protter (1992). The reason for this very restrictive choice of model is the fact that one has *completeness* at the level of both continuous and discrete time. This allows perfect replication of any contingent claim, and so option prices are uniquely determined by the assumption of absence of arbitrage.

In this paper, we attack the same question of convergence in an *incomplete* market. In that case, one is immediately and simultaneously faced with *two* closely intertwined problems. Not only is there the difficulty of establishing a convergence result, but it is even not clear in the first place what the appropriate definition of an option price or *option value* should be. Intuitively, one feels of course that a reasonable valuation methodology should allow one to deduce convergence. We show here that the criterion of *local risk-minimization* introduced in Schweizer (1988, 1991) possesses this feature, at least for the particular example considered here. This means that local risk-minimization has an inherent *stability property* under discretization which may be regarded as an additional argument in favour of this approach. For other recent results in a similar direction, see also Dengler (1993).

More precisely, we study the preceding problem in the case where the price S of the underlying asset is given by a *jump-diffusion process* with deterministic coefficients; see Merton (1976), Jeanblanc-Picqué/Pontier (1990), Shirakawa (1990), Xue (1992) and Mercurio/Runggaldier (1993) for similar models. The discrete-time processes S^m are obtained by first approximating the coefficient functions by piecewise constant ones and then simply evaluating the resulting continuous-time process at the given discretization points. Note that this is rather straightforward and does not require an elaborate construction of the approximating processes S^m as for instance in Nelson/Ramaswamy (1990). Section 1 contains a detailed description of the model and the discretization procedure explained above. In the continuous and in each discrete model, we then apply the criterion of local risk-minimization to determine option values. By the results of Schweizer (1988, 1991, 1993), this means that for each S^m, we use the *minimal martingale measure* $\widehat{P}^m$ for S^m to compute the value of a contingent claim H^m as $\widehat{E}^m[H^m]$; the valuation for H is $\widehat{E}[H]$ with $\widehat{P}$ corresponding to

S. We should like to emphasize that these quantities are not necessarily option prices in the usual sense of the word. They give the initial capital required for the construction of a locally risk-minimizing strategy which duplicates the contingent claim under consideration, but this strategy is typically not self-financing, and it may well happen that a non-vanishing hedging cost appears. For these reasons, we use the more cautious terminology "value" rather than "price".

In section 2, we prove that in our situation, the densities $\frac{d\widehat{P}^m}{dP}$ converge to $\frac{d\widehat{P}}{dP}$ in $\mathcal{L}^p(P)$ for every $p \in [1, \infty)$. Although our method of proof relies crucially on the jump-diffusion structure and in particular on the assumption of deterministic coefficients, we feel that the theorem itself is likely to hold in more general situations as well. We remark that a related result was obtained by He (1990) who proved the weak convergence of the density processes in the case where S is a multidimensional diffusion process and S^m is a suitable multinomial process. However, this is not comparable to our result here since he assumed in addition that S as well as each S^m is complete and thus admits a unique equivalent martingale measure. In section 3, we discuss some applications of our convergence theorem. One immediate consequence is the convergence of the values $\widehat{E}^m[H^m]$ to $\widehat{E}[H]$ if H^m converges to H in $\mathcal{L}^q(P)$ for some $q > 1$, and the last condition is usually easy to verify. Since the values $\widehat{E}^m[H^m]$ correspond to a discrete-time model, they can in principle always be computed, but the computational burden may occasionally become rather heavy. As a by-product of our convergence approach, we also obtain an additional approximation result which in some cases allows an easier computation of the approximating values as simple averages of Black-Scholes-type formulae.

1. Model and problem formulation

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T > 0$ a fixed and finite time horizon. Let $W = (W_t)_{0 \le t \le T}$ be a Brownian motion and $N = (N_t)_{0 \le t \le T}$ a 1-variate point process with deterministic intensity $\nu(t)$. Thus N is a Poisson process, and W and N are independent by Theorem II.6.3 of Ikeda/Watanabe (1981). We shall assume that $\nu(t)$ is bounded away from 0, uniformly in $t \in [0, T]$. Finally, $I\!F = (\mathcal{F}_t)_{0 \le t \le T}$ denotes the P-augmentation of the filtration generated by W and N.

Now denote by $S = (S_t)_{0 \le t \le T}$ the unique strong solution of the stochastic differential equation

$$dS_t = S_{t-}\big(b(t)\,dt + v(t)\,dW_t + \varphi(t)\,dN_t\big) \qquad , \qquad S_0 > 0. \tag{1.1}$$

We shall assume that b, v, φ, ν are left-continuous functions with right limits from $[0, T]$ to $I\!R$ which are bounded uniformly in $t \in [0, T]$. Furthermore, we impose the conditions

$$\varphi(t) > -1 \qquad \text{for all } t \in [0, T]$$

and

$$v^2(t) + \varphi^2(t) \inf_{0 \le s \le T} \nu(s) \ge \varepsilon \qquad \text{for some } \varepsilon > 0, \text{ uniformly in } t \in [0, T]. \tag{1.2}$$

This implies that S is strictly positive and that the function

$$\varrho(t) := \frac{b(t) + \varphi(t)\nu(t)}{v^2(t) + \varphi^2(t)\nu(t)} \qquad , \qquad 0 \le t \le T$$

is also left-continuous with right limits and bounded uniformly in $t \in [0,T]$. Finally we assume that

$$\varphi(t)\varrho(t) \le 1-\delta \qquad \text{for some } \delta > 0, \text{ uniformly in } t \in [0,T]. \tag{1.3}$$

By Itô's formula, the solution of (1.1) is explicitly given by

$$S_t = S_0 \exp\left(\int_0^t \left(b(s) - \frac{1}{2} v^2(s) \right) ds + \int_0^t v(s)\, dW_s + \int_0^t \log\big(1+\varphi(s)\big)\, dN_s \right) \tag{1.4}$$

for $t \in [0,T]$. Due to the boundedness of all coefficients, one can then show that

$$\sup_{0\le t\le T} |S_t| \in \mathcal{L}^p(P) \qquad \text{for every } p \in [1,\infty);$$

see for instance Lemma III.2.1 of Xue (1992) or Lemma II.8.1 of Schweizer (1993). In particular, S is a special semimartingale with canonical decomposition $S = S_0 + M + A$, where

$$M_t = \int_0^t S_{u-}\Big(v(u)\, dW_u + \varphi(u)\big(dN_u - \nu(u)\, du\big)\Big) \quad , \qquad 0 \le t \le T$$

and

$$A_t = \int_0^t S_{u-}\big(b(u) + \varphi(u)\nu(u)\big)\, du = \int_0^t \alpha_u \, d\langle M\rangle_u \quad , \qquad 0 \le t \le T$$

with

$$\alpha_t = \frac{1}{S_{t-}} \frac{b(t)+\varphi(t)\nu(t)}{v^2(t)+\varphi^2(t)\nu(t)} = \frac{\varrho(t)}{S_{t-}} \quad , \qquad 0 \le t \le T.$$

Next we define the process $\widehat{Z} = (\widehat{Z}_t)_{0\le t\le T}$ by

$$\begin{aligned}\widehat{Z}_t := \exp\Bigg(&\int_0^t \left(\varphi(s)\varrho(s)\nu(s) - \frac{1}{2} v^2(s)\varrho^2(s) \right) ds \\ &- \int_0^t v(s)\varrho(s)\, dW_s + \int_0^t \log\big(1-\varphi(s)\varrho(s)\big)\, dN_s \Bigg) \quad , \qquad 0 \le t \le T.\end{aligned}$$

Similar estimates as for S then show, using (1.3), that

$$\sup_{0\le t\le T} |\widehat{Z}_t| \in \mathcal{L}^p(P) \qquad \text{for every } p \in [1,\infty). \tag{1.5}$$

Since $\widehat{Z}$ also solves the stochastic differential equation

$$d\widehat{Z}_t = -\widehat{Z}_{t-}\Big(v(t)\varrho(t)\, dW_t + \varphi(t)\varrho(t)\big(dN_t - \nu(t)\, dt\big)\Big) \quad , \qquad \widehat{Z}_0 = 1,$$

we see that $\widehat{Z}$ is a strictly positive martingale under P, and this allows us to define an equivalent probability measure $\widehat{P}$ on $(\Omega, \mathcal{F})$ by setting $\frac{d\widehat{P}}{dP} := \widehat{Z}_T$. It is easy to check that S is a $(\widehat{P}, I\!\!F)$-martingale, and since $\widehat{Z}$ can be written as $\mathcal{E}\left(-\int \alpha \, dM\right)$, $\widehat{P}$ is in fact the *minimal equivalent martingale measure* for S with respect to $I\!\!F$; this can be proved as in Theorem (3.5) of Föllmer/Schweizer (1991). Moreover, Girsanov's theorem shows that

$$\widehat{W}_t := W_t + \int_0^t v(s)\varrho(s)\, ds \qquad , \qquad 0 \leq t \leq T$$

is a Brownian motion under $\widehat{P}$ and that N has intensity $\widehat{\nu}(t) := \nu(t)\big(1 - \varphi(t)\varrho(t)\big)$ under $\widehat{P}$. Again using Theorem II.6.3 of Ikeda/Watanabe (1981), we conclude that N is a Poisson process under $\widehat{P}$ and that $\widehat{W}$ and N are independent under $\widehat{P}$. For future reference, we note that (1.4) can be rewritten as

$$S_t = S_0 \exp\left(-\int_0^t \left(\frac{1}{2}v^2(s) + \varphi(s)\widehat{\nu}(s)\right) ds + \int_0^t v(s)\, d\widehat{W}_s + \int_0^t \log\big(1 + \varphi(s)\big)\, dN_s \right)$$

for all $t \in [0, T]$, since S satisfies

$$dS_t = S_{t-}\Big(v(t)\, d\widehat{W}_t + \varphi(t)\big(dN_t - \widehat{\nu}(t)\, dt\big)\Big).$$

Now consider any $\mathcal{F}_T$-measurable random variable H. If $H \in \mathcal{L}^p(P)$ for some $p > 2$, then H admits a so-called Föllmer-Schweizer decomposition as

$$H = \widehat{E}[H] + \int_0^T \xi^H_u \, dS_u + L^H_T \qquad P\text{-a.s.,} \tag{1.6}$$

where $\xi^H = (\xi^H_t)_{0 \leq t \leq T}$ is an $I\!\!F$-predictable process satisfying

$$E\left[\int_0^T \left(\xi^H_u\right)^2 d\langle M\rangle_u + \left(\int_0^T \left|\xi^H_u \alpha_u\right| d\langle M\rangle_u \right)^2 \right] < \infty$$

and $L^H = (L^H_t)_{0 \leq t \leq T}$ is a P-square-integrable $(P, I\!\!F)$-martingale null at 0 which is strongly P-orthogonal to M. For a proof of this result, we refer to Theorem 3.4 of Monat/Stricker (1995) or Theorem II.8.3 of Schweizer (1993). Moreover, the argument given in Schweizer (1993) also shows that

$$\widehat{V}_t := \widehat{E}[H] + \int_0^t \xi^H_u \, dS_u + L^H_t = \widehat{E}[H|\mathcal{F}_t] \qquad , \qquad 0 \leq t \leq T.$$

If we now interpret S as the discounted price of some risky asset in a financial market where there also exists a riskless asset whose discounted price is identically 1, then the existence of the above decomposition of H implies the existence of a dynamic *trading strategy* which is

H-*admissible* and *locally risk-minimizing* in the sense of Schweizer (1991). More precisely, define an adapted process $\eta^H = (\eta^H_t)_{0\le t\le T}$ by setting

$$\eta^H_t := \widehat{V}_t - \xi^H_t S_t \qquad , \qquad 0 \le t \le T.$$

If we interpret ξ^H_t as the number of shares of S held at time t and η^H_t as the amount invested in the riskless asset, then the value of this portfolio $\varphi^H = (\xi^H, \eta^H)$ is clearly given by

$$V_t(\varphi^H) = \xi^H_t S_t + \eta^H_t = \widehat{V}_t \qquad , \qquad 0 \le t \le T,$$

so that $V_T(\varphi^H) = H$ P-a.s. The cumulative costs incurred by using φ^H are given by

$$C_t(\varphi^H) = V_t(\varphi^H) - \int_0^t \xi^H_u \, dS_u = \widehat{E}[H] + L^H_t \qquad , \qquad 0 \le t \le T.$$

Since this is a $(P, I\!\!F)$-martingale strongly P-orthogonal to M and since it is easy to verify that S satisfies assumptions (X1) – (X5) of Schweizer (1991), Proposition 2.3 of Schweizer (1991) implies that φ^H is indeed H-admissible and locally risk-minimizing with respect to $I\!\!F$. The value process $V(\varphi^H) = \widehat{V}$ can thus be viewed as a *valuation process* for the contingent claim H with respect to the criterion of local risk-minimization. In particular, $\widehat{V}_0 = \widehat{E}[H]$ can be interpreted as a *valuation* for H at time 0.

What happens now if we use the same criterion to value options along a sequence of discretizations of S? If the above valuation concept is reasonable, then economic intuition suggests that the sequence of discrete-time values should converge to the continuous-time value. However, this is not so clear from a mathematical point of view, since the valuation measures will usually be different in every discretization. The convergence result established in the next section thus shows that the criterion of local risk-minimization has a very appealing stability property; this will be discussed below in more detail.

To be more precise, fix a sequence $(\tau_m)_{m\in I\!\!N}$ of partitions of $[0, T]$, i.e., $\tau_m = \{t^m_0, t^m_1, \ldots, t^m_{n_m}\}$ with $0 = t^m_0 < t^m_1 < \ldots < t^m_{n_m} = T$, whose mesh size $|\tau_m| := \max\limits_{t_i, t_{i+1}\in\tau_m} |t_{i+1} - t_i|$ tends to 0 as $m \to \infty$. Define piecewise constant functions $\psi^m : [0, T] \to I\!\!R$ by setting

$$\psi^m(t) := \psi(0) I_{\{0\}}(t) + \sum_{k=1}^{n_m} \psi(t^m_{k-1}) I_{(t^m_{k-1}, t^m_k]}(t) \tag{1.7}$$

for $m \in I\!\!N$ and $\psi \in \{b, v, \varphi\}$. Then each ψ^m is clearly left-continuous with right limits, $\psi^m(t)$ is bounded by $\|\psi\|_\infty$ uniformly in m and t, and (1.2) also holds for the corresponding approximating functions v^m, φ^m. Note that the function

$$\varrho^m(t) := \frac{b^m(t) + \varphi^m(t)\nu(t)}{\big(v^m(t)\big)^2 + \big(\varphi^m(t)\big)^2 \nu(t)} \qquad , \qquad 0 \le t \le T$$

is also left-continuous with right limits, but in general not piecewise constant. We shall assume that for m large enough, ϱ^m also satisfies the condition (1.3), uniformly in m; this is for instance the case if $\nu(t)$ is constant or, more generally, if $\nu(t)$ is continuous.

If we define $X^m = (X^m_t)_{0\le t\le T}$ as the solution of the stochastic differential equation

$$dX^m_t = X^m_{t-}\big(b^m(t)\,dt + v^m(t)\,dW_t + \varphi^m(t)\,dN_t\big) \qquad , \qquad X^m_0 = S_0,$$

then (1.4) with ψ^m replacing ψ gives the explicit expression for X^m_t, and we get

$$\sup_{m\in I\!N} E\left[\sup_{0\le t\le T} |X^m_t|^p\right] < \infty \qquad \text{for every } p\in[1,\infty).$$

The discrete-time process corresponding to τ_m is now obtained by simply evaluating X^m at all discretization points $t^m_k \in \tau_m$, and we write

$$S^m_k := X^m_{t^m_k} \qquad \text{for } k = 0,1,\dots,n_m.$$

Finally, the discrete-time filtration $I\!F^m = (\mathcal{F}^m_k)_{k=0,1,\dots,n_m}$ is obtained by setting

$$\mathcal{F}^m_k := \mathcal{F}_{t^m_k} \qquad \text{for } k = 0,1,\dots,n_m,$$

so that $\mathcal{F}^m_k \supseteq \sigma(S^m_0,\dots,S^m_k)$ for every k.

In analogy to the continuous-time case, we now introduce the process $\widehat{Z}^m = (\widehat{Z}^m_k)_{k=0,1,\dots,n_m}$ defined by

$$\widehat{Z}^m_k := \prod_{j=1}^{k}\left(1 - \frac{E\left[\Delta S^m_j \middle| \mathcal{F}^m_{j-1}\right]}{\mathrm{Var}\left[\Delta S^m_j \middle| \mathcal{F}^m_{j-1}\right]}\left(\Delta S^m_j - E\left[\Delta S^m_j \middle| \mathcal{F}^m_{j-1}\right]\right)\right)$$

for $k = 0,1,\dots,n_m$, where $\Delta S^m_j := S^m_j - S^m_{j-1}$ denotes the increment of S^m between t^m_{j-1} and t^m_j. Using the explicit expression for $\widehat{Z}^m$ provided in (2.11) – (2.14) below, one readily verifies that $\widehat{Z}^m$ is a P-square-integrable $(P, I\!F^m)$-martingale and that the product $\widehat{Z}^m S^m$ is also a $(P, I\!F^m)$-martingale. For this reason, we call the signed measure $\widehat{P}^m$ on $(\Omega,\mathcal{F})$ defined by $\frac{d\widehat{P}^m}{dP} := \widehat{Z}^m_{n_m}$ the *minimal signed martingale measure* for S^m with respect to $I\!F^m$. More details can be found for instance in Schweizer (1993).

If H^m is now any $\mathcal{F}^m_{n_m}$-measurable random variable and in $\mathcal{L}^2(P)$, then H^m can be written as

$$H^m = \widehat{E}^m[H^m] + \sum_{j=1}^{n_m} \xi^m_j \Delta S^m_j + L^m_{n_m} \qquad P\text{-a.s.,} \tag{1.8}$$

where $\xi^m = (\xi^m_k)_{k=1,\dots,n_m}$ is an $I\!F^m$-predictable process with $\xi^m_k \Delta S^m_k \in \mathcal{L}^2(P)$ for every k, and $L^m = (L^m_k)_{k=0,1,\dots,n_m}$ is a P-square-integrable $(P, I\!F^m)$-martingale null at 0 which is strongly P-orthogonal to the martingale part M^m in the Doob decomposition $S^m = S^m_0 + M^m + A^m$ of S^m with respect to $I\!F^m$. Note that (1.8) is exactly the discrete-time counterpart of (1.6). For a proof of (1.8), see for instance Lemma 4.10 of Schäl (1994) or Proposition I.6.1 of Schweizer (1993). Moreover, Theorem I.9 of Schweizer (1988) implies that ξ^m determines a unique H^m-admissible discrete-time strategy $\varphi^m = (\xi^m, \eta^m)$ which is locally risk-minimizing with respect to the discrete-time filtration $I\!F^m$. Its value process is given by

$$V_k(\varphi^m) = \xi^m_k S^m_k + \eta^m_k = \widehat{E}^m[H^m] + \sum_{j=1}^{k} \xi^m_j \Delta S^m_j + L^m_k =: \widehat{V}^m_k \qquad \text{for } k = 0,1,\dots,n_m,$$

and the value of H^m at time 0 with respect to $I\!F^m$ is $\widehat{V}_0^m = \widehat{E}^m[H^m]$. Furthermore,

$$\widehat{V}_k^m = \widehat{E}^m\left[H^m \middle| \mathcal{F}_k^m\right] \qquad P\text{-a.s. for } k = 0, 1, \ldots, n_m$$

in the sense that $\widehat{V}_{n_m}^m = H^m$ P-a.s. and $\widehat{V}^m \widehat{Z}^m$ is a $(P, I\!F^m)$-martingale.

Now we can reformulate our question: How do these option values behave if $|\tau_m|$ tends to 0? If for instance $H = (S_T - K)^+$ is a European call option on S and $H^m = (S_T^m - K)^+$ is its discretized version, then we certainly expect H^m to converge to H, and we hope that this will imply the convergence of the values $\widehat{E}^m[H^m]$ to $\widehat{E}[H]$. The conclusion below will be that this is indeed true, and the essential step in the argument will be to prove that $\widehat{Z}_{n_m}^m$ converges to $\widehat{Z}_T$ in a sufficiently strong sense.

2. Convergence of the minimal densities

In this section, we prove that $\widehat{Z}_{n_m}^m$ converges to $\widehat{Z}_T$ as $|\tau_m|$ tends to 0. For that purpose, we first establish an auxiliary result. Define the left-continuous *piecewise constant* functions $\bar{\varrho}^m : [0, T] \to I\!R$ by

$$\bar{\varrho}^m(t) := \frac{1}{\Delta t_k^m} \int_{t_{k-1}^m}^{t_k^m} \varrho^m(s)\, ds \qquad \text{for } t \in \left(t_{k-1}^m, t_k^m\right]$$

and $\bar{\varrho}^m(0) := \varrho(0)$. Then $\bar{\varrho}^m(t)$ is bounded uniformly in m and t, $\bar{\varrho}^m$ satisfies (1.3) whenever ϱ^m does, and

$$\lim_{m \to \infty} \bar{\varrho}^m(t) = \varrho(t) \qquad \text{for almost every } t \in [0, T]. \tag{2.1}$$

Denote by $U^m = (U_t^m)_{0 \le t \le T}$ the process defined by

$$U_t^m := \exp\left(\int_0^t \left(\varphi^m(s)\bar{\varrho}^m(s)\nu(s) - \frac{1}{2}\big(v^m(s)\bar{\varrho}^m(s)\big)^2 \right) ds - \int_0^t v^m(s)\bar{\varrho}^m(s)\, dW_s + \int_0^t \log\big(1 - \varphi^m(s)\bar{\varrho}^m(s)\big)\, dN_s \right) \tag{2.2}$$

for $t \in [0, T]$. The same arguments as for (1.5) then show that

$$\sup_{m \in I\!N} E\left[\sup_{0 \le t \le T} |U_t^m|^p \right] < \infty \qquad \text{for every } p \in [1, \infty). \tag{2.3}$$

Lemma 1. *As $|\tau_m|$ tends to 0,*

$$U_T^m \longrightarrow \widehat{Z}_T \qquad \text{in } \mathcal{L}^p(P) \text{ for every } p \in [1, \infty) \tag{2.4}$$

and

$$X_T^m \longrightarrow S_T \qquad \text{in } \mathcal{L}^p(P) \text{ for every } p \in [1,\infty). \tag{2.5}$$

Proof. By the definition of ψ^m, $\psi^m(t)$ converges to $\psi(t)$ for every $t \in [0,T]$ and for $\psi \in \{b, v, \varphi\}$. Hence we conclude that

$$\int_0^T \left(\varphi^m(s)\bar{\varrho}^m(s)\nu(s) - \frac{1}{2}\big(v^m(s)\bar{\varrho}^m(s)\big)^2\right) ds \longrightarrow \int_0^T \left(\varphi(s)\varrho(s)\nu(s) - \frac{1}{2}v^2(s)\varrho^2(s)\right) ds$$

by (2.1) and the dominated convergence theorem. Furthermore,

$$\int_0^T \log\big(1 - \varphi^m(s)\bar{\varrho}^m(s)\big)\, dN_s \longrightarrow \int_0^T \log\big(1 - \varphi(s)\varrho(s)\big)\, dN_s \qquad P\text{-a.s.}$$

by (2.1) and dominated convergence, since $1 - \varphi^m(t)\bar{\varrho}^m(t)$ is bounded away from 0 uniformly in m and t for large m by assumption. Finally,

$$\int_0^T \big(v^m(s)\bar{\varrho}^m(s) - v(s)\varrho(s)\big)^2\, ds \longrightarrow 0$$

by (2.1) and dominated convergence and thus

$$\int_0^T v^m(s)\bar{\varrho}^m(s)\, dW_s \longrightarrow \int_0^T v(s)\varrho(s)\, dW_s \qquad \text{in } \mathcal{L}^2(P).$$

This implies that U_T^m converges to $\widehat{Z}_T$ in probability as $m \to \infty$, and combining this with (1.5) and (2.3) yields (2.4). The proof of (2.5) is perfectly analogous.

q.e.d.

Now we are ready to state and prove the main result of this section.

Theorem 2. *As $|\tau_m|$ tends to 0,*

$$\widehat{Z}_{n_m}^m \longrightarrow \widehat{Z}_T \qquad \text{in } \mathcal{L}^p(P) \text{ for every } p \in [1,\infty). \tag{2.6}$$

Proof. 1) By Lemma 1, it is enough to show that $\widehat{Z}_{n_m}^m - U_T^m$ converges to 0 in $\mathcal{L}^p(P)$ for every $p \in [1,\infty)$ or even only for every $p \in I\!N$. Since

$$\left\|\widehat{Z}_{n_m}^m - U_T^m\right\|_{\mathcal{L}^p(P)} \le \|U_T^m\|_{\mathcal{L}^{2p}(P)} \left\|\frac{\widehat{Z}_{n_m}^m}{U_T^m} - 1\right\|_{\mathcal{L}^{2p}(P)}$$

and $\big(\|U_T^m\|_{\mathcal{L}^{2p}(P)}\big)_{m\in\mathbb{N}}$ is bounded due to (2.3), we only need to show that

$$E\left[\left(\frac{\widehat{Z}^m_{n_m}}{U_T^m}-1\right)^{2p}\right]\longrightarrow 0 \qquad \text{as } m\to\infty \text{ for every } p\in\mathbb{N}.$$

But

$$E\left[\left(\frac{\widehat{Z}^m_{n_m}}{U_T^m}-1\right)^{2p}\right]=\sum_{\ell=0}^{2p}\binom{2p}{\ell}E\left[\left(\frac{\widehat{Z}^m_{n_m}}{U_T^m}\right)^{\ell}\right](-1)^{2p-\ell},$$

and so (2.6) will be proved once we show that

$$\lim_{m\to\infty}E\left[\left(\frac{\widehat{Z}^m_{n_m}}{U_T^m}\right)^{\ell}\right]=1 \qquad \text{for every } \ell\in\mathbb{N}_0. \tag{2.7}$$

2) Now we compute $\widehat{Z}^m_{n_m}$. Due to (1.4) and (1.7), S^m can be written recursively as

$$S_k^m=S_{k-1}^m\exp\left(\left(b_k^m-\frac{1}{2}(v_k^m)^2\right)\Delta t_k^m+v_k^m\Delta W_k^m+\Delta N_k^m\log(1+\varphi_k^m)\right) \tag{2.8}$$

with the shorthand notation

$$\psi_k^m:=\psi(t_{k-1}^m)=\psi^m(t_k^m) \qquad \text{for } \psi\in\{b,v,\varphi,W,N\}. \tag{2.9}$$

Using the fact that ΔW_k^m and ΔN_k^m are independent of each other and of $\mathcal{F}_{k-1}^m$ with respective distributions $\mathcal{N}(0,\Delta t_k^m)$ and $\mathcal{P}(\bar{\nu}_k^m\Delta t_k^m)$, where

$$\bar{\nu}_k^m:=\frac{1}{\Delta t_k^m}\int_{t_{k-1}^m}^{t_k^m}\nu(s)\,ds, \tag{2.10}$$

we can compute

$$E\left[\Delta S_k^m\middle|\mathcal{F}_{k-1}^m\right]=S_{k-1}^m\Big(\exp\big((b_k^m+\varphi_k^m\bar{\nu}_k^m)\Delta t_k^m\big)-1\Big)$$

and

$$\mathrm{Var}\left[\Delta S_k^m\middle|\mathcal{F}_{k-1}^m\right]=(S_{k-1}^m)^2\exp\big(2(b_k^m+\varphi_k^m\bar{\nu}_k^m)\Delta t_k^m\big)\Big(\exp\Big(\big((v_k^m)^2+(\varphi_k^m)^2\bar{\nu}_k^m\big)\Delta t_k^m\Big)-1\Big).$$

With the abbreviations

$$q_k^m:=\frac{\exp\big((b_k^m+\varphi_k^m\bar{\nu}_k^m)\Delta t_k^m\big)-1}{\exp\big((b_k^m+\varphi_k^m\bar{\nu}_k^m)\Delta t_k^m\big)\Big(\exp\Big(\big((v_k^m)^2+(\varphi_k^m)^2\bar{\nu}_k^m\big)\Delta t_k^m\Big)-1\Big)}, \tag{2.11}$$

$$\widetilde{R}_k^m:=\exp\left(v_k^m\Delta W_k^m+\Delta N_k^m\log(1+\varphi_k^m)-\left(\frac{1}{2}(v_k^m)^2+\varphi_k^m\bar{\nu}_k^m\right)\Delta t_k^m\right) \tag{2.12}$$

and

$$(2.13) \qquad R_k^m := q_k^m(\widetilde{R}_k^m - 1),$$

we thus obtain

$$(2.14) \qquad \widehat{Z}_{n_m}^m = \prod_{j=1}^{n_m}(1 - R_j^m).$$

By (2.2), this implies

$$\frac{\widehat{Z}_{n_m}^m}{U_T^m} = \prod_{j=1}^{n_m}(1 - R_j^m)\exp(-L_j^m)$$

with

$$(2.15) \qquad L_k^m := \left(\varphi_k^m \bar{\varrho}_k^m \bar{\nu}_k^m - \frac{1}{2}(v_k^m \bar{\varrho}_k^m)^2\right)\Delta t_k^m - v_k^m \bar{\varrho}_k^m \Delta W_k^m + \Delta N_k^m \log(1 - \varphi_k^m \bar{\varrho}_k^m);$$

note that we have used here the fact that $\varphi^m, v^m, \bar{\varrho}^m$ are all piecewise constant. Since the processes W and N are independent and have independent increments, the random variables $(1 - R_j^m)\exp(-L_j^m)$ are independent for $j = 1, \ldots, n_m$ and so

$$E\left[\left(\frac{\widehat{Z}_{n_m}^m}{U_T^m}\right)^\ell\right] = \prod_{j=1}^{n_m} E\left[(1 - R_j^m)^\ell \exp(-\ell L_j^m)\right] = \exp\left(\sum_{j=1}^{n_m} \log E\left[(1 - R_j^m)^\ell \exp(-\ell L_j^m)\right]\right).$$

Hence (2.7) will follow if we show that

$$(2.16) \qquad \lim_{m\to\infty} \sum_{j=1}^{n_m} \log E\left[(1 - R_j^m)^\ell \exp(-\ell L_j^m)\right] = 0 \qquad \text{for every } \ell \in I\!N_0.$$

3) Now fix $\ell \in I\!N_0$ and $m \in I\!N$, and drop the index m for the moment to ease the notation. From (2.13), we get

$$(1 - R_k)^\ell \exp(-\ell L_k) = \sum_{i=0}^{\ell} \binom{\ell}{i}(1 + q_k)^{\ell - i}(-q_k)^i \widetilde{R}_k^i \exp(-\ell L_k),$$

and using (2.12) and (2.15) gives

$$E\left[\widetilde{R}_k^i \exp(-\ell L_k)\right] = \exp\big(f_k(\ell, i)\Delta t_k\big)$$

with

$$f_k(\ell, i) := \frac{1}{2}v_k^2\big((i + \ell\bar{\varrho}_k)^2 - i\big) - \ell\left(\varphi_k\bar{\varrho}_k\bar{\nu}_k - \frac{1}{2}v_k^2\bar{\varrho}_k^2\right) + \bar{\nu}_k\left(\frac{(1+\varphi_k)^i}{(1 - \varphi_k\bar{\varrho}_k)^\ell} - 1 - i\varphi_k\right).$$

Expanding e^x into a power series, we get

$$E\left[\widetilde{R}_k^i \exp(-\ell L_k)\right] = 1 + f_k(\ell, i)\Delta t_k + O\left((\Delta t_k)^2\right),$$

and since we have uniform bounds on all coefficients for large m, the error term is $O\left((\Delta t_k)^2\right)$ uniformly in m for large m, i.e.,

$$\limsup_{m\to\infty} \frac{1}{(\Delta t_k)^2} O\left((\Delta t_k)^2\right) < \infty.$$

Summing over i yields

$$\begin{aligned} E\left[(1-R_k)^\ell \exp(-\ell L_k)\right] &= \sum_{i=0}^{\ell} \binom{\ell}{i} (1+q_k)^{\ell-i}(-q_k)^i \Big(1 + f_k(\ell,i)\Delta t_k + O\left((\Delta t_k)^2\right)\Big) \\ &= 1 + \Delta t_k \sum_{i=0}^{\ell} \binom{\ell}{i} (1+q_k)^{\ell-i}(-q_k)^i f_k(\ell,i) + O\left((\Delta t_k)^2\right), \end{aligned}$$

and expanding $\log(1+x)$ into a power series leads to

$$\log E\left[(1-R_k)^\ell \exp(-\ell L_k)\right] = \Delta t_k \sum_{i=0}^{\ell} \binom{\ell}{i} (1+q_k)^{\ell-i}(-q_k)^i f_k(\ell,i) + O\left((\Delta t_k)^2\right),$$

again with an error term which is uniform in m for large m; notice that q_k^m is bounded uniformly in m for large m.

4) Next we compute

$$\begin{aligned} &\sum_{i=0}^{\ell} \binom{\ell}{i} (1+q_k)^{\ell-i}(-q_k)^i f_k(\ell,i) \\ &= \frac{1}{2}\ell^2 v_k^2 \bar\varrho_k^2 - \ell\left(\varphi_k \bar\varrho_k \bar\nu_k - \frac{1}{2} v_k^2 \bar\varrho_k^2\right) - \bar\nu_k + \frac{\bar\nu_k}{(1-\varphi_k\bar\varrho_k)^\ell} \sum_{i=0}^{\ell} \binom{\ell}{i} (1+q_k)^{\ell-i}(-q_k - q_k\varphi_k)^i \\ &\quad + (\ell v_k^2 \bar\varrho_k - \bar\nu_k\varphi_k) \sum_{i=0}^{\ell} i \binom{\ell}{i} (1+q_k)^{\ell-i}(-q_k)^i + \frac{1}{2} v_k^2 \sum_{i=0}^{\ell} i(i-1) \binom{\ell}{i} (1+q_k)^{\ell-i}(-q_k)^i \\ &= \frac{1}{2}\ell^2 v_k^2 \bar\varrho_k^2 - \ell\left(\varphi_k \bar\varrho_k \bar\nu_k - \frac{1}{2} v_k^2 \bar\varrho_k^2\right) - \bar\nu_k + \bar\nu_k \left(\frac{1-q_k\varphi_k}{1-\varphi_k\bar\varrho_k}\right)^\ell \\ &\quad - \ell q_k(\ell v_k^2 \bar\varrho_k - \bar\nu_k\varphi_k) + \frac{1}{2} v_k^2 q_k^2 \ell(\ell-1) \\ &= \frac{1}{2}\ell^2 v_k^2 (\bar\varrho_k - q_k)^2 + \ell\left(\varphi_k\bar\nu_k(q_k - \bar\varrho_k) - \frac{1}{2} v_k^2 (q_k^2 - \bar\varrho_k^2)\right) + \bar\nu_k \left(\left(\frac{1-q_k\varphi_k}{1-\varphi_k\bar\varrho_k}\right)^\ell - 1\right) \\ &=: g_k(\ell). \end{aligned}$$

5) Now sum over k and reinstate the index m to obtain

$$\sum_{k=1}^{n_m} \log E\left[(1-R_k^m)^\ell \exp(-\ell L_k^m)\right] = \sum_{k=1}^{n_m} \Big(g_k^m(\ell)\Delta t_k^m + O\big((\Delta t_k^m)^2\big)\Big).$$

The sum of the error terms is $O(1)|\tau_m|$ and thus tends to 0. Furthermore, (2.11) shows that

$$\lim_{m\to\infty} q_k^m = \frac{b(t) + \varphi(t)\nu(t)}{v^2(t) + \varphi^2(t)\nu(t)} = \varrho(t)$$

for every t, and as $|\tau_m|$ tends to 0,

$$\psi_k^m \longrightarrow \psi(t) \qquad \text{for every } t \text{ and for } \psi \in \{v, \varphi\}$$

by (2.9) and

$$\bar{\psi}_k^m \longrightarrow \psi(t) \qquad \text{for almost every } t \text{ and for } \psi \in \{\nu, \varrho\}$$

by (2.1) and (2.10). Hence we conclude that (2.16) holds, and this completes the proof.

q.e.d.

3. Applications

As an immediate consequence of Theorem 2, we obtain

Theorem 3. *Suppose H^m is $\mathcal{F}^m_{n_m}$-measurable for every $m \in I\!N$, H is $\mathcal{F}_T$-measurable and H^m converges to H in $\mathcal{L}^q(P)$ for some $q > 1$. Then*

$$\lim_{m \to \infty} \widehat{E}^m[H^m] = \widehat{E}[H]. \tag{3.1}$$

From the perspective of possible applications, this is the central result of this paper. It tells us that even in incomplete markets, one can get *convergence* of discrete-time option values to continuous-time option values if these values are determined at each step with respect to the criterion of local risk-minimization. We emphasize once more that this is not a trivial result: local risk-minimization is defined with respect to a given filtration, and so we have a different optimization problem in each discretization. Theorem 3 then shows that local risk-minimization has an inherent *stability property* under discretization and thus provides a strong argument in favour of this criterion for valuing options under incompleteness.

Consider now briefly the case where $\varphi \equiv 0$, i.e., S has no jump component. Then S is just geometric Brownian motion with (time-dependent) drift $b(t)$ and volatility $v(t)$, and this implies that S is *complete*. Hence $\widehat{P}$ is the unique equivalent martingale measure for S with respect to $I\!F$, and $\widehat{E}[H]$ is the unique price for H which is consistent with absence of arbitrage opportunities. If H has a complicated form, then an explicit formula for $\widehat{E}[H]$ is in general not available and so one resorts to approximations by using discrete-time models. In most papers so far, these discrete models are binomial trees, and one major argument for this choice (apart from computational reasons) is the fact that this is the only discrete-time process which, like its continuous-time counterpart S, is complete and thus allows pricing by arbitrage. Theorem 3 shows that this very restrictive choice is not necessary; one can equally well take a simple (incomplete) discretization of S if one then uses the minimal signed martingale measure $\widehat{P}^m$ to compute the value of the approximating claim H^m.

Remark. Although our proof of Theorem 2 relies crucially on the explicit structure of our model and in particular on the assumption of deterministic coefficients, we conjecture that Theorem 3 is valid in more generality. Obviously, (3.1) will hold whenever H^m tends to H in $\mathcal{L}^q(P)$ and

$$\frac{d\widehat{P}^m}{dP} \longrightarrow \frac{d\widehat{P}}{dP} \qquad \text{in } \mathcal{L}^p(P) \tag{3.2}$$

with $\frac{1}{p} + \frac{1}{q} = 1$. It would be interesting to see a proof of (3.2) in a more general situation.

Note that we have assumed in Theorem 3 that H^m converges to H in $\mathcal{L}^q(P)$. In general, this condition is easy to verify; if for instance $(\mathcal{F}^m_{n_m})_{m\in I\!N}$ increases to $\mathcal{F}_T$, we can always choose

$$H^m := E\left[H\middle|\mathcal{F}^m_{n_m}\right]$$

by the martingale convergence theorem. For specific examples, however, other choices of H^m may be more natural.

Example 1. Suppose $H = (S_T - K)^+$ is a European call option on S with strike price K. If we denote by $H^m = (S^m_{n_m} - K)^+$ the corresponding call option in the discrete-time model, then Lemma 1 implies that H^m tends to H in $\mathcal{L}^q(P)$ for every $q \in [1, \infty)$ and so we can apply Theorem 3 to deduce the convergence of the corresponding values. More generally, the same arguments work with $H = f(S_T)$ and $H^m = f(S^m_{n_m})$ for every continuous function f which satisfies for instance a polynomial growth condition.

Example 2. If $H = \left(\frac{1}{T}\int_0^T S_u\,du - K\right)^+$ is a fixed strike Asian option, its natural discrete-time counterpart is

$$H^m = \left(\frac{1}{T}\sum_{j=1}^{n_m} S^m_{j-1}\Delta t^m_j - K\right)^+ .$$

It is then straightforward to check that H^m tends to H in $\mathcal{L}^q(P)$ for every $q \in [1, \infty)$. Hence $\widehat{E}^m[H^m]$ provides an approximation for $\widehat{E}[H]$ by Theorem 3, and this may be useful since the latter expectation is quite difficult to compute.

In general terms, Theorem 3 tells us that an approximation for the value $\widehat{E}[H]$ can be obtained by computing a suitable expectation in a suitable discretization. The great advantage of this lies in the fact that in a discrete-time model, every quantity of interest can in principle be computed explicitly. Let us illustrate this in our situation for a call option of the form $H^m = (S^m_{n_m} - K)^+$. By (2.14), $\widehat{E}^m[H^m]$ can be written as

$$\widehat{E}^m[H^m] = S_0 E\left[\prod_{j=1}^{n_m}(1 - R^m_j)\left(\prod_{k=1}^{n_m}\frac{S^m_k}{S^m_{k-1}} - \frac{K}{S_0}\right)^+\right]. \tag{3.3}$$

Now drop the index m for ease of notation and use (2.8), (2.12) and (2.13) to obtain

$$\begin{aligned}
&E\left[\prod_{j=1}^{n}(1 - R_j)\left(\prod_{k=1}^{n}\frac{S_k}{S_{k-1}} - \frac{K}{S_0}\right)^+\right] \\
&= E\Bigg[\prod_{j=1}^{n}\left(1 + q_j - q_j \exp\left(v_j\Delta W_j + \Delta N_j\log(1+\varphi_j) - \left(\frac{1}{2}v_j^2 + \varphi_j\bar{\nu}_j\right)\Delta t_j\right)\right) \\
&\qquad \left(\prod_{k=1}^{n}\exp\left(\left(b_k - \frac{1}{2}v_k^2\right)\Delta t_k + v_k\Delta W_k + \Delta N_k\log(1+\varphi_k)\right) - \frac{K}{S_0}\right)^+\Bigg].
\end{aligned}$$

With suitable constants f_j, g_j, this can be expressed as

$$E\Bigg[\prod_{j=1}^{n}\Big(f_j + g_j \exp\big(v_j \Delta W_j + \Delta N_j \log(1+\varphi_j)\big)\Big)$$

$$\left(\prod_{k=1}^{n} \exp\big(v_k \Delta W_k + \Delta N_k \log(1+\varphi_k)\big) - \bar{K}\right)^{+}\Bigg]$$

$$= E\Bigg[\Bigg(\prod_{j=1}^{n} f_j + \sum_{j=1}^{n} g_j \exp\big(v_j \Delta W_j + \Delta N_j \log(1+\varphi_j)\big)\prod_{k\neq j} f_k$$

$$+ \sum_{i,j=1}^{n} g_i g_j \exp\big(v_i \Delta W_i + v_j \Delta W_j + \Delta N_i \log(1+\varphi_i) + \Delta N_j \log(1+\varphi_j)\big)\prod_{k\neq i,j} f_k + \dots$$

$$+ \left(\prod_{j=1}^{n} g_j\right)\exp\left(\sum_{j=1}^{n}\big(v_j \Delta W_j + \Delta N_j \log(1+\varphi_j)\big)\right)\Bigg)$$

$$\left(\exp\left(\sum_{j=1}^{n}\big(v_j \Delta W_j + \Delta N_j \log(1+\varphi_j)\big)\right) - \bar{K}\right)^{+}\Bigg]$$

which, apart from deterministic constants, is a sum of terms of the form

$$E\left[\exp\left(\sum_{j=1}^{n}(c_j \Delta W_j + d_j \Delta N_j)\right)\left(\exp\left(\sum_{j=1}^{n}(\bar{c}_j \Delta W_j + \bar{d}_j \Delta N_j)\right) - \widetilde{K}\right)^{+}\right]. \tag{3.4}$$

Since W and N are independent, the expectation can be performed first with respect to the Poisson and then with respect to the Gaussian variables. Because the random variables ΔN_j are independent with respective distributions $\mathcal{P}(\lambda_j)$, where $\lambda_j = \bar{\nu}_j \Delta t_j$, (3.4) then becomes

$$\sum_{k_1,\dots,k_n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{\lambda_j^{k_j}}{k_j\,!}e^{-\lambda_j}\right)\exp\left(\sum_{j=1}^{n} d_j k_j\right)$$
$$E\left[\left(\exp\left(\sum_{j=1}^{n}\big((c_j+\bar{c}_j)\Delta W_j + \bar{d}_j k_j\big)\right) - \widetilde{K}\exp\left(\sum_{j=1}^{n} c_j \Delta W_j\right)\right)^{+}\right]. \tag{3.5}$$

The problem at this point is reduced to the computation of expectations of the form

$$E\left[\left(e^{G_1} - \widetilde{K}e^{G_2}\right)^{+}\right] = E\left[E\left[\left(e^{G_1} - \widetilde{K}e^{G_2}\right)^{+}\Big| G_2\right]\right], \tag{3.6}$$

where G_1 and G_2 are Gaussian random variables with given means m_1, m_2, variances σ_1^2, σ_2^2 and covariance $R = \varrho\sigma_1\sigma_2$. In (3.5), we have for instance

$$m_1 = \sum_{j=1}^{n} \bar{d}_j k_j, \quad m_2 = 0, \quad \sigma_1^2 = \sum_{j=1}^{n} (c_j + \bar{c}_j)^2 \Delta t_j, \quad \sigma_2^2 = \sum_{j=1}^{n} c_j^2 \Delta t_j, \quad R = \sum_{j=1}^{n} c_j (c_j + \bar{c}_j) \Delta t_j.$$

The inner expectation on the right-hand side of (3.6) is now given by a Black-Scholes-type formula corresponding to the case where the terminal value of the risky asset is lognormal with mean

$$m_1 + \varrho \frac{\sigma_2}{\sigma_1} G_2 = m_1 + \frac{R G_2}{\sigma_1^2}$$

and variance

$$\sigma_2^2 (1 - \varrho^2) = \sigma_2^2 - \frac{R^2}{\sigma_1^2}.$$

The value of the expression in (3.6) is thus an average of Black-Scholes-type formulae over a Gaussian distribution, and this average must in general be computed numerically. To obtain the option value $\widehat{E}^m[H^m]$ in (3.3), a further averaging over a Poisson measure is required according to (3.5). In summary, the option value in the discrete-time model can be obtained as an average of Black-Scholes-type formulae, where the averaging is first performed with respect to a Gaussian measure and then with respect to a Poisson measure.

In the case where the claim H is an Asian option, either with fixed strike, i.e.,

$$H = \left(\frac{1}{T} \int_0^T S_u \, du - K \right)^+ ,$$

or with average strike, i.e.,

$$H = \left(S_T - \frac{K}{T} \int_0^T S_u \, du \right)^+ ,$$

completely analogous computations can be made. Consider for instance the second case with

$$H^m = \left(S_{n_m}^m - \frac{K}{T} \sum_{j=1}^{n_m} S_{j-1}^m \Delta t_j^m \right)^+ .$$

Instead of (3.6), we then obtain

$$E\left[\left(e^G - \widetilde{K} \sum_{j=1}^{n} e^{G_j} \right)^+ \right] = E\left[E\left[\left(e^G - \widetilde{K} \sum_{j=1}^{n} e^{G_j} \right)^+ \middle| G_1, \dots, G_n \right] \right], \tag{3.7}$$

where $G, G_1, \dots, G_n$ are correlated Gaussian random variables with appropriate means and variances. Thus the value in (3.7) can again be obtained by repeated averaging of Black-Scholes-type formulae over Gaussian distributions, or equivalently by a single averaging over a multivariate Gaussian distribution. To obtain from there the option value $\widehat{E}^m[H^m]$ requires a further averaging over a Poisson measure.

Remark. Despite the fact that Theorem 3 allows us to reduce the computation of approximate option values in our framework to a discrete-time problem, it may sometimes be computationally advantageous to use a different approximation. Suppose that H is of the form $H = f(S_T)$ for a continuous function f satisfying a polynomial growth condition. Define the process $\widehat{U}^m = (\widehat{U}^m_t)_{0\le t\le T}$ by setting

$$\widehat{U}^m_t := \exp\left(\int_0^t \left(\varphi^m(s)\varrho^m(s)\nu(s) - \frac{1}{2}\big(v^m(s)\varrho^m(s)\big)^2 \right) ds - \int_0^t v^m(s)\varrho^m(s)\, dW_s + \int_0^t \log\big(1-\varphi^m(s)\varrho^m(s)\big)\, dN_s \right)$$

for all $t \in [0,T]$; note that $\widehat{U}^m$ differs from U^m in (2.2) by the fact that ϱ^m replaces $\bar{\varrho}^m$. The same arguments as in section 1 show that $\widehat{U}^m$ is the density process of the minimal martingale measure $\widehat{Q}^m$ for X^m with respect to $I\!\!F$. Furthermore, $\widehat{U}^m_T$ converges to $\widehat{Z}_T$ in $\mathcal{L}^p(P)$ for every $p \in [1,\infty)$ by an analogous argument as for (2.4), and so Lemma 1 implies that

$$\widehat{E}[H] = E\Big[\widehat{Z}_T f(S_T)\Big] = \lim_{m\to\infty} E\Big[\widehat{U}^m_T f(X^m_T)\Big] = \lim_{m\to\infty} E_{\widehat{Q}^m}\left[f(X^m_T)\right].$$

Now use again the arguments of section 1 to conclude that X^m_T is given by

$$\begin{aligned} &S_0 \exp\left(\int_0^T v^m(s)\, d\widehat{W}^m_s + \int_0^T \log\big(1+\varphi^m(s)\big)\, dN_s - \int_0^T \left(\frac{1}{2}\big(v^m(s)\big)^2 + \varphi^m(s)\widehat{\nu}^m(s) \right) ds \right) \\ &= S_0 \exp\left(\int_0^T v^m(s)\, d\widehat{W}^m_s - \int_0^T \left(\frac{1}{2}\big(v^m(s)\big)^2 + \varphi^m(s)\widehat{\nu}^m(s) \right) ds \right) \prod_{j=1}^{n_m} (1+\varphi^m_j)^{\Delta N^m_j}, \end{aligned}$$

since φ^m is piecewise constant. But $\widehat{W}^m$ and N are independent under $\widehat{Q}^m$ and N is a Poisson process with $\widehat{Q}^m$-intensity $\widehat{\nu}^m(t) = \nu(t)\big(1-\varphi^m(t)\varrho^m(t)\big)$, and so we obtain

$$\begin{aligned} (3.8)\quad & E_{\widehat{Q}^m}\left[f(X^m_T)\right] \\ &= \exp\left(-\int_0^T \widehat{\nu}^m(s)\, ds \right) \sum_{k_1,\dots,k_{n_m}=0}^{\infty} \left\{ \prod_{j=1}^{n_m} \frac{(\widehat{\lambda}^m_j)^{k_j}}{k_j\,!} \right. \\ &\qquad \left. E\left[f\left(S_0 e^G \prod_{j=1}^{n_m} (1+\varphi^m_j)^{k_j} \exp\left(-\int_0^T \left(\frac{1}{2}\big(v^m(s)\big)^2 + \varphi^m(s)\widehat{\nu}^m(s) \right) ds \right) \right) \right] \right\}, \end{aligned}$$

where

$$\widehat{\lambda}^m_k := \int_{t^m_{k-1}}^{t^m_k} \widehat{\nu}^m(s)\, ds$$

and G has a normal distribution with mean 0 and variance $\int_0^T \big(v^m(s)\big)^2\, ds$. In fact, (3.8) is obtained by conditioning on N and using the independence of N and $\widehat{W}^m$ under $\widehat{Q}^m$, exactly as in the proof of Theorem 2.1 of Mercurio/Runggaldier (1993). For the case where H is a European call option, i.e., $f(x) = (x - K)^+$, (3.8) simplifies to a Poisson average of Black-Scholes-type formulae as in Mercurio/Runggaldier (1993). Since no further averaging over a Gaussian distribution is required, (3.8) is therefore easier to compute than (3.3).

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