Strong bubbles and strict local martingales*

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Abstract

In a numéraire-independent framework, we study a financial market with N assets which are all treated in a symmetric way. We define the fundamental value $^*S$ of an asset S as its superreplication price and say that the market has a *strong bubble* if $^*S$ and S deviate from each other. None of these concepts needs any mention of martingales. Our main result then shows that under a weak absence-of-arbitrage assumption (basically NUPBR), a market has a strong bubble if and only if in all numéraires for which there is an equivalent local martingale measure (ELMM), asset prices are *strict local martingales* under *all* possible ELMMs. We show by an example that our bubble concept lies strictly between the existing notions from the literature. We also give an example where asset prices are strict local martingales under one ELMM, but true martingales under another, and we show how our approach can lead naturally to endogenous bubble birth.

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1 Introduction

This paper uses superreplication to define financial bubbles and analyses them in a numéraire-independent paradigm. For background, let us first discuss some basic ideas.

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The literature on bubbles is vast, diverse and impossible to survey here, even if only approximately. The Encyclopedia of Quantitative Finance has a 15-page entry "Bubbles and crashes" [26], with a list of more than 100 references. The recent survey by Scherbina/Schlusche [42] puts more emphasis on behavioural models and rational models with frictions, and also gives a brief overview on the history of bubbles. The books of Brunnermeier [2] or Shiller [43] are often quoted as early classics; and the recent paper "A mathematical theory of financial bubbles" by Protter [38] also contains around 160 references plus some discussions of literature. Here, we only recall some basic notions.

The standard formulation in financial economics says that a bubble appears when the *market value* of an asset differs from its *fundamental value*. More formally, we describe an asset (Δ, Y) by its cumulative dividend process $\Delta = (\Delta_t)$ and its ex-dividend price process $Y = (Y_t)$, both in the same fixed unit. If ${}^*Y_t$ denotes the asset's (undiscounted) fundamental value at time t, then ${}^*Y_t \neq Y_t$ (or ${}^*Y_t < Y_t$) means that the asset has a bubble, and the difference $Y_t - {}^*Y_t$ is usually called the (size of the) bubble.

There are different ideas for defining the notion of a "fundamental value", and we discuss them in detail in Section 6. One main school uses the risk-neutral value of discounted future payments; this raises the question of how one chooses a risk-neutral measure. The other main school uses the superreplication cost of the asset; this induces (in a more subtle and hidden way) a dependence on the chosen unit via the class of allowed trading strategies. We follow the second method, but make sure that we always keep track of, and as far as possible eliminate, the dependence on the unit of account.

One key feature of our approach is that all our definitions are economically motivated and use only *primal* quantities, i.e. assets and trading strategies. *Dual* objects like numéraires and martingale measures also appear, but only in a second step when we give dual characterisations of the introduced primal notions. In particular, we show (Theorem 3.7) that *strict local martingale measures arise naturally* in the context of modelling financial bubbles. Moreover, our concept of (strong) bubbles does not depend on any choice of a risk-neutral measure, but is robust in a sense made precise below. Finally, we provide many concrete and explicit examples; these include an incomplete market with a strong bubble (Example 5.3), an incomplete market where one risk-neutral measure sees a bubble while a second does not (Example 5.5), and a natural setup where bubble birth occurs endogenously (Example 5.4).

The paper is structured as follows. We introduce the main concepts of our approach in Section 2. Section 3 defines strong bubbles, maximal strategies, viability and efficiency, and presents in Theorem 3.7 the central characterisation of strong bubbles via strict local martingales. Section 4 gives dual characterisations of (dynamic) viability and efficiency and uses them to prove Theorem 3.7. It also introduces and characterises no dominance as the property that distinguishes efficiency from viability. We provide explicit examples in Section 5, and compare our definitions and results to the existing literature on bubble modelling in Section 6. Finally, the Appendix on superreplication prices and maximality collects some supplementary results used in the proofs in the main body of the paper.

1.1 Probabilistic setup and notation

We work throughout on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ satisfying the usual conditions of right-continuity and completeness, where $T > 0$ denotes a fixed and finite time horizon. We assume that $\mathcal{F}_0$ is $\mathbb{P}$-trivial.

We call $\mathcal{T}_{[0,T]}$ the set of all stopping times with values in $[0,T]$. For $\sigma \in \mathcal{T}_{[0,T]}$, set $\mathcal{T}_{[\sigma,T]} := \{\tau \in \mathcal{T}_{[0,T]} : \tau \ge \sigma\}$. For $\tau \in \mathcal{T}_{[0,T]}$, we denote by $\mathbf{L}^0_+(\mathcal{F}_\tau)$, $\overline{\mathbf{L}^0_+}(\mathcal{F}_\tau)$, $\mathbf{L}^0_{++}(\mathcal{F}_\tau)$ the set of all $\mathcal{F}_\tau$-measurable random variables taking $\mathbb{P}$-a.s. values in $[0,\infty)$, $[0,\infty]$, $(0,\infty)$, respectively. Finally, we denote by $e^i = (0,\dots,0,1,0,\dots,0)$ for $i = 1,\dots,N$ the i-th unit vector in $\mathbb{R}^N$ and set $\mathbf{1} = \sum_{i=1}^N e^i = (1,\dots,1) \in \mathbb{R}^N$.

A product-measurable process $\xi = (\xi_t)_{0\le t\le T}$ is *predictable on* $[\![\sigma,T]\!]$ if the random variable ξ_σ is $\mathcal{F}_\sigma$-measurable and the process $\xi \mathbb{1}_{]\!]\sigma,T]\!]}$ is predictable. So if ξ is predictable on $[\![\sigma,T]\!]$ and $A \in \mathcal{F}_\sigma$, also $\xi\mathbb{1}_A$ is predictable on $[\![\sigma,T]\!]$. For an $\mathbb{R}^N$-valued semimartingale $X = (X^1_t,\dots,X^N_t)_{0\le t\le T}$ and $\sigma \in \mathcal{T}_{[0,T]}$, we denote by ${}^\sigma L(X)$ the set of all $\mathbb{R}^N$-valued processes $\zeta = (\zeta^1_t,\dots,\zeta^N_t)_{0\le t\le T}$ which are predictable on $[\![\sigma,T]\!]$ and for which the stochastic integral process $\int_\sigma^t \zeta_s \,\mathrm{d}X_s := \int_{(0,t]} \zeta_s \mathbb{1}_{]\!]\sigma,t]\!]}(s)\,\mathrm{d}X_s$, $0 \le t \le T$, is defined in the sense of N-dimensional stochastic integration (see [22, Section III.6] for details).

2 Main concepts

We introduce here our model for the financial market and corresponding basic concepts.

2.1 Asset prices

Throughout this paper, we consider a financial market with $N > 1$ assets and denote by $\tilde{\mathbf{S}} = (\tilde{S}^1_t,\dots,\tilde{S}^N_t)_{0\le t\le T}$ the assets' price process in some fixed but not specified unit. This unit may be tradable (e.g. in the form of a bond) or not; we explicitly avoid assuming that one of the assets is constant 1, or that there exists a "riskless asset" B in the background. All we initially impose is that the process $\tilde{\mathbf{S}}$ is $\mathbb{R}^N$-valued, adapted, RCLL, and $\tilde{S}^i \ge 0$ $\mathbb{P}$-a.s. for each i since we have primary assets in mind. To exclude the case where all assets default and we are left with a nonexistent market, we also assume that the financial market is *nondegenerate* with $\tilde{\mathbf{S}} \cdot \mathbf{1}$ strictly positive, meaning that

$$\inf_{0\le t\le T} \tilde{\mathbf{S}}_t \cdot \mathbf{1} = \inf_{0\le t\le T} \sum_{i=1}^N \tilde{S}^i_t > 0 \qquad \mathbb{P}\text{-a.s.} \tag{2.1}$$

Condition (2.1) also appears in [48] and (in well hidden form) in the recent paper [11].

It is folklore in mathematical finance that in a reasonable financial market, relative prices should be semimartingales after some suitable discounting; see e.g. [29] and the references therein. To formalise this, we introduce the set $\tilde{\mathcal{D}}$ of one-dimensional adapted RCLL processes $\tilde{D} = (\tilde{D}_t)_{0\le t\le T}$ with

$$\inf_{0\le t\le T} \tilde{D}_t > 0 \qquad \mathbb{P}\text{-a.s.} \tag{2.2}$$

and call the elements of $\tilde{\mathcal{D}}$ *generalised numéraires*. We assume that

$$\text{there exists some } \bar{D} \in \tilde{\mathcal{D}} \text{ such that } \bar{D}\tilde{\mathbf{S}} \text{ is a semimartingale} \tag{2.3}$$

and **choose and fix one such $\bar{D}$ and the corresponding process $\mathbf{S} := \bar{D}\tilde{\mathbf{S}}$.** We call this particular, fixed $\mathbf{S}$ a *semimartingale representative* of the market described by $\tilde{\mathbf{S}}$.

It is economically clear that all prices are relative and that the basic qualitative properties of a model should not depend on the chosen unit. To make this precise, we call a process $\tilde{S}'$ *economically equivalent* to $\tilde{\mathbf{S}}$ if $\tilde{S}'$ is also $\mathbb{R}^N$-valued, adapted and RCLL, and if $\tilde{S}' = \tilde{D}\tilde{\mathbf{S}}$ for some $\tilde{D} \in \tilde{\mathcal{D}}$. In other words, two processes are economically equivalent if they describe the same assets in possibly different units.

Our first simple result shows that our modelling approach does not depend on the initial choice of $\tilde{\mathbf{S}}$ and that it has nice semimartingale properties, in the following sense.

Lemma 2.1. *If $\tilde{S}' = \tilde{D}\tilde{\mathbf{S}}$, with $\tilde{D} \in \tilde{\mathcal{D}}$, is economically equivalent to $\tilde{\mathbf{S}}$, then $\tilde{S}'$ also satisfies* (2.1) *and* (2.3)*. Any semimartingale representative $S' = \bar{D}'\tilde{S}'$ is then economically equivalent to $\mathbf{S}$, $S' = D\mathbf{S}$, with a numéraire $D \in \mathcal{D}$ which is even a semimartingale.*

Proof. Since $\tilde{D} > 0$ in the sense of (2.2), (2.1) directly transfers from $\tilde{\mathbf{S}}$ to $\tilde{S}' = \tilde{D}\tilde{\mathbf{S}}$. From (2.3) for $\tilde{\mathbf{S}}$, we have $\mathbf{S} = \bar{D}\tilde{\mathbf{S}}$; so $\mathbf{S} = \bar{D}(\tilde{S}'/\tilde{D}) = (\bar{D}/\tilde{D})\tilde{S}'$ is a semimartingale and $\bar{D}/\tilde{D}$ is in $\tilde{\mathcal{D}}$, and we see that $\tilde{S}'$ also satisfies (2.3). If $S' = \bar{D}'\tilde{S}'$ is a semimartingale, we use $\tilde{S}' = \tilde{D}\tilde{\mathbf{S}}$ to write $S' = D\mathbf{S}$ with $D := \bar{D}'\tilde{D}/\bar{D}$ which is clearly in $\tilde{\mathcal{D}}$. But (2.1) for $\tilde{\mathbf{S}}$ and for $\tilde{S}'$ implies that $\mathbf{S} \cdot \mathbf{1} > 0$ and $S' \cdot \mathbf{1} > 0$, and so we can write $D = (S' \cdot \mathbf{1})/(\mathbf{S} \cdot \mathbf{1})$ to see that D is also a semimartingale. $\square$

In the sequel, we always assume that (2.1) **and** (2.3) **are satisfied, and choose and fix a semimartingale representative S.** All other semimartingale representatives (denoted by S', S'' or $\bar{S}$) in $\mathcal{S}$ are then economically equivalent to $\mathbf{S}$ with a *semimartingale numéraire*, and we introduce the set of *numéraires*,

$$\begin{aligned}\mathcal{D} &:= \tilde{\mathcal{D}} \cap \{\text{semimartingales}\}\\ &= \{\text{all one-dimensional semimartingales } D = (D_t)_{0\le t\le T} \text{ with } \inf_{0\le t\le T} D_t > 0 \ \mathbb{P}\text{-a.s.}\},\end{aligned}$$

and the market generated by $\mathbf{S}$, which is

$$\mathcal{S} := \{S' = D\mathbf{S} : D \in \mathcal{D}\}.$$

The key difference between $\mathbf{S}$ and $\tilde{\mathbf{S}}$ is that $\mathbf{S}$ is a semimartingale, and we exploit this when we formalise trading and self-financing strategies with the help of stochastic integrals. Up to a change of unit, however, $\mathbf{S}$ and $\tilde{\mathbf{S}}$ agree; so we can view the choice of working with $\mathbf{S}$ as merely dictated by convenience, and we can always rewrite everything back into the units of $\tilde{\mathbf{S}}$ if that is preferred for some reason; see Remark 2.6 below for more details.

Remark 2.2. In Herdegen [18, 17], the elements of $\mathcal{D}$ are not called numéraires, but *exchange rate processes*. The most apt naming would probably be "unit converters", since multiplying by D_t converts prices $\mathbf{S}_t$ in one unit to prices $S'_t = D_t\mathbf{S}_t$ in another unit.

Example 2.3 (*Classic setup of mathematical finance*)**.** One particular case is what we call the *classic setup* of mathematical finance. Suppose there is one asset which has for $\mathbb{P}$-almost all ω a positive price. We then relabel all assets, call that particular asset B or *riskless asset* or *bond*, and the other $d := N - 1$ assets the *risky assets*. (More precisely, we need $\mathbb{P}[\inf_{0\le t\le T} B_t > 0] = 1$, so that $B \in \tilde{\mathcal{D}}$ is a generalised numéraire.) Then we can express all other assets in units of that special asset by defining $X^i := \tilde{S}^{i+1}/B$ for $i = 1, \dots, N-1$, and we call the X^i the risky assets discounted by the bond. For later use, we also introduce the vector process $Y := (\tilde{S}^2, \dots, \tilde{S}^N)$ so that $\tilde{\mathbf{S}} = (B, Y) = B\mathbf{S}$ with $\mathbf{S} := (1, X)$. Then we have $N = d + 1$ basic traded assets; but they are not symmetric because one is a bond which can never reach the value 0. Moreover, if there are several assets like B, the choice of the one we use for discounting is arbitrary. So concepts defined in terms of $X = Y/B$ depend implicitly on the chosen discounting process B, and it can become difficult to keep track of this dependence all the time.

The vast majority of papers in mathematical finance—with the obvious exception in the literature on interest rate modelling—works with the end result of the above procedure. Usually, papers start with an $\mathbb{R}^d$-valued process X and call this the (discounted) price process of d risky assets. Almost without exception, it is also assumed (but very often not mentioned explicitly) that there is in addition to X a traded riskless bond whose price is identically 1—and this assumption is exploited in the standard problem formulations. (Most papers also assume that X is a semimartingale, which corresponds to our choice of a semimartingale representative $\mathbf{S}$.)

As one can see, the classic setup is intrinsically asymmetric. This obscures a number of important phenomena, and so we want to start with a *symmetric* treatment of all traded assets. Since we make no assumptions on $\tilde{D}$ in (2.2) except strict positivity, all our results include the classic setup with nonnegative prices; but they do not exploit its assumptions and asymmetry, and hence they are both more general and in our view more natural. The simplest example of a model which cannot be formulated in the classic setup is one with two assets ($N = 2$); they are both nonnegative, but both can default, i.e. become 0. One of them hits 0 at some (maybe random) time on a set A only; the other hits 0 on A^c only. If $0 < \mathbb{P}[A] < 1$, this cannot be put into the form of the classic setup.

Remark 2.4. One can of course argue in the above example that introducing a third asset of the form $S^3 = \alpha S^1 + (1-\alpha)S^2$ with $\alpha \in (0, 1)$ would lead us back into the classic setup without changing the market, since S^3 adds no new trading opportunities. However, this easy way out is an ad hoc fix, and also raises the question how the resulting classic setup depends perhaps on the choice of α. Rather than trying to find a case-by-case approach, we prefer to deal with (2.1) and (2.3) in a general and systematic way.

Now let us return to our basic model. We want to describe (frictionless) continuous trading and work with self-financing strategies; so we need stochastic integrals, and therefore exploit below that $\mathbf{S}$ is a semimartingale. Again, this includes the classic setup.

In the sequel, we only want to work with notions which are independent of the choice of a specific semimartingale representative $S' \in \mathcal{S}$ (or a particular unit). More precisely, a notion should hold for our fixed semimartingale representative $\mathbf{S}$ if and only if it holds for each $S' \in \mathcal{S}$; then we say that the notion holds for the market $\mathcal{S}$ and call it *"numéraire-independent"*. Whenever this *numéraire independence* is not directly clear from the context or the definitions, we shall make a comment or give an explanation.

2.2 Self-financing strategies and strategy cones

In this section, we introduce trading strategies. This is almost standard, with small (but important) differences because we are not in the classic setup. Recall that $\mathbf{S} \in \mathcal{S}$ is our fixed semimartingale representative of the market $\mathcal{S}$.

Definition 2.5. Fix a stopping time $\sigma \in \mathcal{T}_{[0,T]}$. The space ${}^{\sigma}L^{\mathrm{sf}}(\mathcal{S}) =: {}^{\sigma}L^{\mathrm{sf}}$ of *self-financing strategies (for $\mathcal{S}$) on* $[\![\sigma, T]\!]$ consists of all N-dimensional processes ϑ which are predictable on $[\![\sigma, T]\!]$, in ${}^{\sigma}L(\mathbf{S})$, and such that the *value process* $V(\vartheta)(\mathbf{S})$ of ϑ (in the unit corresponding to $\mathbf{S}$) satisfies the self-financing condition

$$V(\vartheta)(\mathbf{S}) := \vartheta \cdot \mathbf{S} = \vartheta_\sigma \cdot \mathbf{S}_\sigma + \int_\sigma \vartheta_u \,\mathrm{d}\mathbf{S}_u \qquad \text{on } [\![\sigma, T]\!],\ \mathbb{P}\text{-a.s.} \tag{2.4}$$

It is not immediately obvious but true that the above concept is numéraire-independent. In fact, one can show for each $S' \in \mathcal{S}$ that if ϑ is in ${}^{\sigma}L(\mathbf{S})$ and satisfies (2.4), then ϑ is also in ${}^{\sigma}L(S')$ and satisfies (2.4) for S' instead of $\mathbf{S}$; see [48, Theorem 2.1] or [17, Lemma 2.8], and note that the proof in [17] for $\sigma \neq 0$ is verbatim the same as for $\sigma = 0$. In particular, writing ${}^{\sigma}L^{\mathrm{sf}}(\mathcal{S})$ and not ${}^{\sigma}L^{\mathrm{sf}}(\mathbf{S})$ is justified. Note that the value process of any strategy ϑ satisfies the *numéraire invariance property*

$$V(\vartheta)(D\mathbf{S}) = DV(\vartheta)(\mathbf{S}) \qquad \text{for every numéraire } D \in \mathcal{D}. \tag{2.5}$$

This means that when we change units from $\mathbf{S}$ to $S' = D\mathbf{S}$, the wealth from ϑ in new units is simply the old wealth multiplied by D, as it must be from basic financial intuition.

Remark 2.6. 1) Combining (2.5), (2.4) and the numéraire independence of (2.4) gives

$$V(\vartheta)(S') = V_\sigma(\vartheta)(S') + D \int_\sigma \vartheta_u \,\mathrm{d}(S'/D)_u \qquad \text{on } [\![\sigma, T]\!],\ \mathbb{P}\text{-a.s.,}$$

for all $\vartheta \in {}^{\sigma}L^{\mathrm{sf}}$ and semimartingale representatives $S' = D\mathbf{S}$. This *change-of-numéraire formula* has appeared, among others, in [13] or [46]. Since it *follows* from the definition (2.4) for any *semimartingale* representative S', it is natural to extend it *by definition* to all *other* representatives as well. So if we want to work with self-financing strategies not for $\mathbf{S}$, but the original (possibly non-semimartingale) $\tilde{\mathbf{S}} = \mathbf{S}/\bar{D}$, we rewrite the self-financing condition (2.4) in the unit corresponding to $\tilde{\mathbf{S}}$ by multiplying everything by $\bar{D}$ at the appropriate time, i.e., as

$$V(\vartheta)(\tilde{\mathbf{S}}) := \vartheta \cdot \tilde{\mathbf{S}} = \vartheta_\sigma \cdot \tilde{\mathbf{S}}_\sigma + \frac{1}{\bar{D}} \int_\sigma \vartheta_u \,\mathrm{d}(\tilde{\mathbf{S}}\bar{D})_u \qquad \text{on } [\![\sigma, T]\!],\ \mathbb{P}\text{-a.s.} \tag{2.6}$$

This avoids defining stochastic integrals with respect to $\tilde{\mathbf{S}}$ (which might be impossible).

2) In the classic setup with $N = d+1$, $\mathbf{S} = (1, X)$ and discounted asset prices given by the $\mathbb{R}^d$-valued semimartingale X, self-financing strategies on $[\![\sigma, T]\!]$ can be identified with pairs (v_σ, ψ) consisting of $\mathcal{F}_\sigma$-measurable random variables v_σ and $\mathbb{R}^d$-valued predictable X-integrable processes ψ; see [12, Remark 5.8] or [9, Lemma 2.2.1]. Indeed, setting $v_\sigma := V_\sigma(\vartheta)(\mathbf{S})$ and using that asset 0 has a constant price of 1, we can write (2.4) for a strategy $\vartheta = (\eta, \psi)$ in $\mathbf{S} = (1, X)$ as

$$\eta = V(\vartheta)(\mathbf{S}) - \psi \cdot X = v_\sigma + \int_\sigma \vartheta_u \,\mathrm{d}X_u - \psi \cdot X.$$

This identification of ϑ with (v_σ, ψ) is so familiar that it is done without mention in most papers. In our symmetric setup, such a simple identification is no longer possible; trading strategies must be treated as processes of dimension $N = d + 1$, and the self-financing condition (2.4) imposes a linear constraint on their coordinates.

Clearly, ${}^\sigma L^{\mathrm{sf}}(\mathcal{S})$ is a vector space. It is also closed under multiplication with $\mathcal{F}_\sigma$-measurable random variables; so we can scale a strategy on $[\![\sigma, T]\!]$ not only by a constant, but also by a random factor known at the beginning σ of the time period on which we trade.

To avoid doubling phenomena, one usually considers sub-cones of ${}^\sigma L^{\mathrm{sf}}$ for "allowed" trading. We first give the abstract definition.

Definition 2.7. For a stopping time $\sigma \in \mathcal{T}_{[0,T]}$, a *strategy cone (for $\mathcal{S}$) on $[\![\sigma, T]\!]$* is a nonempty subset ${}^\sigma\Gamma \subseteq {}^\sigma L^{\mathrm{sf}}(\mathcal{S})$ with the properties

1) if $\vartheta^1, \vartheta^2 \in {}^\sigma\Gamma$ and $c^1_\sigma, c^2_\sigma \in \mathbf{L}^0_+(\mathcal{F}_\sigma)$, then $c^1_\sigma \vartheta^1 + c^2_\sigma \vartheta^2 \in {}^\sigma\Gamma$;

2) if $(\vartheta^n)_{n\in\mathbb{N}}$ is a countable family in ${}^\sigma\Gamma$ and $(A_n)_{n\in\mathbb{N}}$ a partition of Ω into pairwise disjoint $\mathcal{F}_\sigma$-measurable sets, then $\sum_{n=1}^\infty \mathbb{1}_{A_n}\vartheta^n \in {}^\sigma\Gamma$.

A family of strategy cones $({}^\sigma\Gamma)_{\sigma\in\mathcal{T}_{[0,T]}}$, where each ${}^\sigma\Gamma$ is a strategy cone on $[\![\sigma, T]\!]$, is called *time-consistent* if ${}^{\sigma_1}\Gamma \subseteq {}^{\sigma_2}\Gamma$ for $\sigma_1 \le \sigma_2$ in $\mathcal{T}_{[0,T]}$.

The simplest example of a strategy cone on $[\![\sigma, T]\!]$ is ${}^\sigma L^{\mathrm{sf}}$ itself. Moreover, the family $({}^\sigma L^{\mathrm{sf}})_{\sigma\in\mathcal{T}_{[0,T]}}$ is clearly time-consistent. The main example used in this paper is given by the class of undefaultable strategies introduced below in Definition 2.9.

If ${}^\sigma\Gamma \subseteq {}^\sigma L^{\mathrm{sf}}(\mathcal{S})$ is a strategy cone on $[\![\sigma, T]\!]$, we set, for any norm $\|\cdot\|$ in $\mathbb{R}^N$,

$$\mathbf{b}^\sigma\Gamma := \Big\{\vartheta \in {}^\sigma\Gamma : \sup_{(\omega,t)\in\Omega\times[0,T]} \|\vartheta\mathbb{1}_{[\![\sigma,T]\!]}\| \le c_\sigma \ \mathbb{P}\text{-a.s., for some } c_\sigma \in \mathbf{L}^0_+(\mathcal{F}_\sigma)\Big\},$$

$$\mathbf{h}^\sigma\Gamma := \Big\{\vartheta \in {}^\sigma\Gamma : \vartheta\mathbb{1}_{[\![\sigma,T]\!]} = \vartheta_\sigma\mathbb{1}_{[\![\sigma,T]\!]}\Big\}.$$

Clearly, $\{0\} \subseteq \mathbf{h}^\sigma\Gamma \subseteq \mathbf{b}^\sigma\Gamma \subseteq {}^\sigma\Gamma$, and $\mathbf{h}^\sigma\Gamma$ and $\mathbf{b}^\sigma\Gamma$ are again strategy cones on $[\![\sigma, T]\!]$. We call $\vartheta \in \mathbf{b}^\sigma\Gamma$ a *bounded* strategy in ${}^\sigma\Gamma$ and $\vartheta \in \mathbf{h}^\sigma\Gamma$ a *buy-and-hold* strategy in ${}^\sigma\Gamma$. Note that if the family $({}^\sigma\Gamma)_{\sigma\in\mathcal{T}_{[0,T]}}$ is time-consistent, so are $(\mathbf{b}^\sigma\Gamma)_{\sigma\in\mathcal{T}_{[0,T]}}$ and $(\mathbf{h}^\sigma\Gamma)_{\sigma\in\mathcal{T}_{[0,T]}}$.

Remark 2.8. 1) Note that our buy-and-hold strategies always go up to the end T of the trading interval.

2) Calling strategies in $\mathbf{b}^\sigma\Gamma$ *bounded* may seem puzzling at first sight. But any $\vartheta \in \mathbf{b}^\sigma\Gamma$ is uniformly bounded on $[\![\sigma, T]\!]$ by an $\mathcal{F}_\sigma$-measurable random variable $c_\sigma \in \mathbf{L}^0_+(\mathcal{F}_\sigma)$, and the latter play the role of "constants" on $[\![\sigma, T]\!]$. (Recall that ${}^\sigma L^{\mathrm{sf}}$ is closed under multiplication with elements of $\mathbf{L}^0_+(\mathcal{F}_\sigma)$, and we impose the cone structure in Definition 2.7 for elements of $\mathbf{L}^0_+(\mathcal{F}_\sigma)$.) In particular, $\sigma = 0$ yields the usual concept of a bounded strategy.

3) We parametrise strategies in numbers of "shares", not wealth amounts or fractions of wealth. So for a bounded strategy, asset holdings are bounded but wealth need not be.

It is well known that to avoid undesirable phenomena in a financial market, one must exclude doubling-type strategies. The usual way to do that is to impose solvency constraints—strategies are allowable for trading only if their value processes are bounded below by some quantity. If this approach should not depend on a specific unit, the only possible choice for the lower bound is 0. This motivates the following definition.

Definition 2.9. Fix a stopping time $\sigma \in \mathcal{T}_{[0,T]}$. We call a strategy $\vartheta \in {}^\sigma L^{\mathrm{sf}}(\mathcal{S})$ an *undefaultable strategy on* $[\![\sigma, T]\!]$ and write $\vartheta \in {}^\sigma L^{\mathrm{sf}}_+(\mathcal{S})$ or just $\vartheta \in {}^\sigma L^{\mathrm{sf}}_+$ if

$$V(\vartheta)(\mathbf{S}) \geq 0 \qquad \text{on } [\![\sigma, T]\!],\ \mathbb{P}\text{-a.s.}$$

The notion of undefaultable is clearly numéraire-independent, due to (2.5). Moreover, each ${}^\sigma L^{\mathrm{sf}}_+$ is a strategy cone, and $({}^\sigma L^{\mathrm{sf}}_+)_{\sigma \in \mathcal{T}_{[0,T]}}$ is a time-consistent family.

Definition 2.10. A *numéraire strategy* (for the market $\mathcal{S}$) is a strategy $\eta \in {}^0 L^{\mathrm{sf}}(\mathcal{S})$ with $\inf_{0 \leq t \leq T} V_t(\eta)(\mathbf{S}) > 0$ $\mathbb{P}$-a.s., i.e., such that $V(\eta)(\mathbf{S}) \in \mathcal{D}$ is a numéraire. (Actually, $V(\eta)(\mathbf{S})$ might even be called a *tradable numéraire* since it is the value process of a self-financing strategy.) We call $\mathcal{S}$ a *numéraire market* if such an η exists.

Note that the above concept is numéraire-independent since $V(\eta)(\mathbf{S}) > 0$ holds for our fixed $\mathbf{S} \in \mathcal{S}$ if and only if it holds for all $S' \in \mathcal{S}$, due to the numéraire invariance (2.5). Note also that any numéraire strategy is automatically in ${}^0 L^{\mathrm{sf}}_+(\mathcal{S})$.

By our standing nondegeneracy assumption (2.1), the *market portfolio* $\eta^{\mathcal{S}} := \mathbf{1}$ of holding one unit of each asset is always a numéraire strategy; it even lies in $\mathbf{h}{}^0 L^{\mathrm{sf}}_+(\mathcal{S})$ and is bounded. Similarly, in a classic setup $\tilde{\mathbf{S}} = (B, Y)$ of $N = d{+}1$ assets, where Y denotes d undiscounted "risky assets" and $B \in \tilde{\mathcal{D}}$ an undiscounted bond, the buy-and-hold strategy e^1 of the bond is a bounded numéraire strategy, with $V(e^1)(\tilde{\mathbf{S}}) = B$.

Each numéraire strategy η naturally induces a $\mathbb{P}$-a.s. unique *numéraire representative* $S^{(\eta)} \in \mathcal{S}$ such that $V(\eta)(S^{(\eta)}) \equiv 1$. It is given explicitly by $V(\eta)$*-discounted prices*

$$S^{(\eta)} := \frac{\mathbf{S}}{V(\eta)(\mathbf{S})} = \frac{\tilde{\mathbf{S}}}{V(\eta)(\tilde{\mathbf{S}})}. \tag{2.7}$$

Because $V(\eta)$ satisfies (2.5), the middle term in (2.7) yields the same result for any other representative $S' = D\mathbf{S}$ of $\mathcal{S}$. In the classic setup $\tilde{\mathbf{S}} = (B, Y)$ as above with a bond B and $\eta = e^1$, (2.7) reduces to $S^{(e^1)} = \tilde{\mathbf{S}}/B = (1, X) = \mathbf{S}$ as in Example 2.3.

2.3 Contingent claims and superreplication prices

For defining our notion of bubbles and providing dual characterisations of primal notions, we need superreplication prices. They are also very useful for a valuation of financial contracts in a numéraire-independent way, but we do not address that aspect in the present paper. This section introduces or recalls some of the required concepts; more details and information can be found in [17, Section 2.5] or [16, Sections 4 and 5].

Definition 2.11. An *improper contingent claim at time* $\tau \in \mathcal{T}_{[0,T]}$ for the market $\mathcal{S}$ is a *map* $F : \mathcal{S} \to \overline{\mathbf{L}^0_+}(\mathcal{F}_\tau)$ satisfying the *numéraire invariance condition*

$$F(DS') = D_\tau F(S') \qquad \mathbb{P}\text{-a.s., for all } S' \in \mathcal{S} \text{ and all } D \in \mathcal{D}. \tag{2.8}$$

F is called a *contingent claim at time* τ if it is valued in $\mathbf{L}^0_+(\mathcal{F}_\tau)$, and *strictly positive* if it is valued in $\mathbf{L}^0_{++}(\mathcal{F}_\tau)$.

A contingent claim F in our setup assigns to each representative $S' \in \mathcal{S}$ (which corresponds to a choice of unit) a payoff $F(S')$ at time τ (in the same unit), which is an $\mathcal{F}_\tau$-measurable random variable. The simplest example is the value $V_\tau(\vartheta)$ at time τ of any self-financing strategy ϑ; (2.8) here follows from (2.5). For the canonical and most general example, we choose a pair $(g, \bar{S}) \in \mathbf{L}^0_+(\mathcal{F}_\tau) \times \mathcal{S}$ and define F by $F(S') = F(D\bar{S}) := D_\tau g$ for any $S' = D\bar{S}$ in $\mathcal{S}$; this clearly satisfies (2.8). Then g represents a payoff in the unit corresponding to $\bar{S}$, and we call $F =: F_{\tau,g,\bar{S}}$ the contingent claim at time τ induced by g with respect to $\bar{S}$. Like the self-financing condition in (2.6), we can extend (2.8) to arbitrary representatives $\tilde{S}' = S'/\bar{D}'$ by setting $F(\tilde{S}') := F(S')/\bar{D}'_\tau$.

Remark 2.12. It is important to distinguish between a contingent claim $F(\cdot)$ (in the above sense), which is a mapping with the property (2.8), and the corresponding payoff $F(S')$ (in the unit corresponding to S'), which is a random variable. In particular, the product of two contingent claims or a constant $c \geq 0$ are not contingent claims. (The contingent claim describing the constant payoff $c \geq 0$ at time τ in units of $\bar{S}$ is $F_{\tau,c,\bar{S}}$.) However, for every numéraire strategy η and every contingent claim F at time τ, we have

$$F(\cdot) = F(S^{(\eta)}) V_\tau(\eta)(\cdot). \tag{2.9}$$

Indeed, the identity (2.9) holds for $\bar{S} := S^{(\eta)}$ due to (2.8) because $V(\eta)(S^{(\eta)}) \equiv 1$ by (2.7), and then for general S' due to (2.8) because $V_\tau(\eta)$ is a contingent claim at time τ.

Definition 2.13. Let $\sigma \leq \tau \in \mathcal{T}_{[0,T]}$ be stopping times, ${}^\sigma\Gamma$ a strategy cone on $[\![\sigma, T]\!]$ and F a contingent claim at time τ. The *superreplication price of F at time σ for ${}^\sigma\Gamma$* is the mapping $\Pi_\sigma(F \,|\, {}^\sigma\Gamma) : \mathcal{S} \to \overline{\mathbf{L}^0_+}(\mathcal{F}_\sigma)$ defined by

$$\begin{aligned}\Pi_\sigma(F \,|\, {}^\sigma\Gamma)(S') := \operatorname{ess\,inf}\{v \in \overline{\mathbf{L}^0_+}(\mathcal{F}_\sigma) : \exists \vartheta \in {}^\sigma\Gamma \text{ such that } \mathbb{P}\text{-a.s. on } \{v < \infty\}, \\ V_\tau(\vartheta)(S') \geq F(S') \text{ and } V_\sigma(\vartheta)(S') \leq v\}.\end{aligned} \tag{2.10}$$

It is not difficult to check that $\Pi_\sigma(F \,|\, {}^\sigma\Gamma)$ is an improper contingent claim at time σ. The following result lists some other basic properties. Note that these are properties of functions on $\mathcal{S}$, and that they are all numéraire-independent in the (usual) sense that they hold for our fixed $\mathbf{S} \in \mathcal{S}$ if and only if they hold for all $S' \in \mathcal{S}$; this is due to the numéraire invariance property (2.8). The proofs are straightforward and hence omitted.

Proposition 2.14. *Let $\sigma \leq \tau \in \mathcal{T}_{[0,T]}$ be stopping times, ${}^\sigma\Gamma$ a strategy cone on $[\![\sigma, T]\!]$ and F, F_1, F_2, G contingent claims at time τ with $F \leq G$ $\mathbb{P}$-a.s. Let c_σ be a nonnegative $\mathcal{F}_\sigma$-measurable random variable. Then we have*

$$\begin{aligned} \Pi_\sigma(F \,|\, {}^\sigma\Gamma) &\leq \Pi_\sigma(G \,|\, {}^\sigma\Gamma) && \textit{(monotonicity)}, \\ \Pi_\sigma(c_\sigma F \,|\, {}^\sigma\Gamma) &= c_\sigma \Pi_\sigma(F \,|\, {}^\sigma\Gamma) && \textit{(positive } \mathcal{F}_\sigma\textit{-homogeneity)}, \\ \Pi_\sigma(F_1 + F_2 \,|\, {}^\sigma\Gamma) &\leq \Pi_\sigma(F_1 \,|\, {}^\sigma\Gamma) + \Pi_\sigma(F_2 \,|\, {}^\sigma\Gamma) && \textit{(subadditivity)}. \end{aligned}$$

Note that positive $\mathcal{F}_\sigma$-homogeneity gives $\Pi_\sigma(\mathbb{1}_{A_\sigma} F \,|\, {}^\sigma\Gamma) = \mathbb{1}_{A_\sigma} \Pi_\sigma(F \,|\, {}^\sigma\Gamma)$ for $A_\sigma \in \mathcal{F}_\sigma$. For conditional risk measures, this is called locality or the local property. However, in contrast to risk measures, the cash-additivity analogue $\Pi_\sigma(F + C_\sigma \,|\, {}^\sigma\Gamma) = C_\sigma + \Pi_\sigma(F \,|\, {}^\sigma\Gamma)$ for contingent claims C_σ at time σ does not hold in general.

3 Absence of arbitrage, strong bubbles, and strict local martingales

In this section, we first introduce our notion of bubbles, then present a concept of absence of arbitrage, and finally show how the combination of these ideas leads in a natural way to the appearance of strict local martingales. It is important to point out here that neither our bubbles nor our concept of absence of arbitrage make any mention of martingales.

3.1 Definition of strong bubbles via superreplication prices

The standard approach in financial economics is to define bubbles by comparing market prices to fundamental values. For the latter, we use here superreplication prices.

Definition 3.1. The *fundamental value* of asset $i \in \{1, \dots, N\}$ at time $\sigma \in \mathcal{T}_{[0,T]}$ in the unit corresponding to our fixed representative $\mathbf{S}$ is defined by

$$\begin{aligned} {}^*S^i_\sigma &:= \Pi_\sigma\Big(V_T(e^i)\,\Big|\,{}^\sigma L^{\mathrm{sf}}_+\Big)(\mathbf{S}) \\ &= \operatorname{ess\,inf}\{v \in \mathbf{L}^0_+(\mathcal{F}_\sigma) : \exists \vartheta \in {}^\sigma L^{\mathrm{sf}}_+ \text{ with } V_T(\vartheta)(\mathbf{S}) \geq S^i_T \\ &\qquad\qquad \text{and } V_\sigma(\vartheta)(\mathbf{S}) \leq v,\ \mathbb{P}\text{-a.s.}\} \end{aligned} \tag{3.1}$$

We set ${}^*\mathbf{S} := ({}^*S^1, \dots, {}^*S^N)$ and say that the market $\mathcal{S}$ *has a strong bubble* if ${}^*\mathbf{S}$ and $\mathbf{S}$ are not indistinguishable, i.e., if $\mathbb{P}[{}^*S^i_\sigma < S^i_\sigma] > 0$ for some asset $i \in \{1, \dots, N\}$ and $\sigma \in \mathcal{T}_{[0,T]}$.

If one accepts the idea that the fundamental value of an asset should be given by its superreplication price, the above definition clearly formalises the standard idea of a bubble from financial economics. However, it is more general than existing definitions (for example in [21, 34]) since we compare $\mathbf{S}$ and ${}^*\mathbf{S}$ not only at time 0. In particular, it may happen that $\mathcal{S}$ has a strong bubble, but ${}^*\mathbf{S}_0 = \mathbf{S}_0$. In that sense, our definition includes the possibility of "bubble birth". We give an explicit example in Example 5.4 below and provide a more detailed discussion in Section 6.4 below.

Remark 3.2. 1) The buy-and-hold strategy e^i of holding one unit of asset i is in ${}^\sigma L^{\mathrm{sf}}_+$ with $V_\sigma(e^i)(\mathbf{S}) = S^i_\sigma$ and $V_T(e^i)(\mathbf{S}) = S^i_T$. So ${}^*S^i_\sigma \leq S^i_\sigma$, and unlike in (2.10), it is enough in (3.1) to take the ess inf only over $\mathbf{L}^0_+(\mathcal{F}_\sigma)$. Like (2.10), the notion of having a strong bubble is numéraire-independent. Like the fundamental value, it depends on the class ${}^\sigma L^{\mathrm{sf}}_+$ of strategies used in Definition 3.1; but that dependence is quite weak, in view of 2).

2) One can also use the definition (3.1) with ${}^\sigma L^{\mathrm{sf}}_+$ replaced by a larger class ${}^\sigma\Gamma$ of strategies. But if one has a hedging duality as in (3.4) below, the essential supremum is attained with a strategy in ${}^\sigma L^{\mathrm{sf}}_+$, as argued in the comment just after (3.4). This will be important later when we compare our approach to the literature.

3.2 Maximal strategies

Suppose we are given a class Θ of possible strategies. A strategy $\vartheta \in \Theta$ can be considered a "reasonable investment" from that class only if it cannot be directly improved by another strategy from the same class. More precisely, using strategies in Θ with the same (or a lower) initial investment should not allow one to create more wealth at time T. It is natural to call such a strategy ϑ *maximal*; see Remark 3.4 below for more comments.

Definition 3.3. For a stopping time $\sigma \in \mathcal{T}_{[0,T]}$ and a strategy cone ${}^\sigma\Gamma$ on $[\![\sigma, T]\!]$, a strategy $\vartheta \in {}^\sigma\Gamma$ is *weakly maximal for* ${}^\sigma\Gamma$ if there is no pair $(f, \bar\vartheta) \in (\mathbf{L}^0_+(\mathcal{F}_T) \setminus \{0\}) \times {}^\sigma\Gamma$ with

$$V_T(\bar\vartheta)(\mathbf{S}) \geq V_T(\vartheta)(\mathbf{S}) + f \quad \text{ and } \quad V_\sigma(\bar\vartheta)(\mathbf{S}) \leq V_\sigma(\vartheta)(\mathbf{S}), \qquad \mathbb{P}\text{-a.s.}$$

It is *strongly maximal* or just *maximal for* ${}^\sigma\Gamma$ if there is no $f \in \mathbf{L}^0_+(\mathcal{F}_T) \setminus \{0\}$ such that for all $\varepsilon > 0$, there exists $\bar\vartheta \in {}^\sigma\Gamma$ with

$$V_T(\bar\vartheta)(\mathbf{S}) \geq V_T(\vartheta)(\mathbf{S}) + f \quad \text{ and } \quad V_\sigma(\bar\vartheta)(\mathbf{S}) \leq V_\sigma(\vartheta)(\mathbf{S}) + \varepsilon, \qquad \mathbb{P}\text{-a.s.} \tag{3.2}$$

Note above that f, which satisfies $f \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}[f > 0] > 0$, stands for the extra wealth (in the same units as $\mathbf{S}$) at time T, on top of what we get from ϑ, that we generate by $\bar\vartheta$. If ϑ is weakly maximal, no $\bar\vartheta$ achieves f without increasing the initial capital at time σ. If ϑ is (strongly) maximal, the improvement is asymptotically impossible even with a small but strictly positive increase of initial capital at σ. Both concepts are numéraire-independent; for strong maximality, this is best seen from its alternative description, in terms of superreplication prices, in the comment after Lemma A.1 in the Appendix.

Remark 3.4. 1) The terminology "maximal strategy" goes back at least to Delbaen/Schachermayer [4, 6, 7]. However, their setting is different from ours so that maximality has a different meaning. For a more detailed discussion, see [17, Remark 3.2].

2) Both above definitions of maximality are slightly different from those in Herdegen [17, Definitions 3.1 and 3.9]. In [17], a strategy is called maximal if it is maximal on $[\![0, \sigma]\!]$, for every stopping time $\sigma \in \mathcal{T}_{[0,T]}$. However, under a natural extra assumption on the strategy cone ([17, Definition 3.5]), a strategy which is (weakly or strongly) maximal in our sense (i.e. on $[\![0, T]\!]$) is also (weakly or strongly) maximal in the sense of [17]. The above assumption (which essentially amounts to predictable convexity) is in particular satisfied for the class ${}^0L^{\mathrm{sf}}_+$ of undefaultable strategies. Finally, Corollary A.3 below also shows that maximality in ${}^0L^{\mathrm{sf}}_+$ (i.e. on $[\![0, T]\!]$) is equivalent to maximality in each ${}^\sigma L^{\mathrm{sf}}_+$ (i.e. on $[\![\sigma, T]\!]$). All this discussion is relevant since we later use some of the results in [17].

3) It is clear that strong implies weak maximality, and [17, Example 3.13] shows that the converse does not hold in general. But if the zero strategy 0 is strongly maximal for ${}^\sigma L^{\mathrm{sf}}_+$, $\sigma \in \mathcal{T}_{[0,T]}$, then weak implies strong maximality for ${}^\sigma L^{\mathrm{sf}}_+$; see Lemma A.4 below.

3.3 Viability and efficiency criteria for markets

A financial market should behave in a reasonable way, and this should be reflected in the properties of its model description. Let us formalise this and then explain the intuition. Recall that maximal means strongly maximal.

Definition 3.5. A market $\mathcal{S}$ is called

- *statically viable* if the zero strategy 0 is maximal for $\mathbf{h}{}^\sigma L^{\mathrm{sf}}_+(\mathcal{S})$, for all $\sigma \in \mathcal{T}_{[0,T]}$.
- *dynamically viable* if the zero strategy 0 is maximal for ${}^\sigma L^{\mathrm{sf}}_+(\mathcal{S})$, for all $\sigma \in \mathcal{T}_{[0,T]}$.

Static viability means that at every time σ, just doing nothing cannot be improved by a self-financing buy-and-hold strategy. Dynamic viability is even stronger—one cannot improve on inactivity by trading even if one trades continuously in time. Of course, in both cases, one must obey the constraint (from ${}^{\sigma}L^{\mathrm{sf}}_{+}$) of keeping wealth nonnegative.

Dynamic viability by its definition implies static viability, but the converse is not true, as one can verify by easy examples even in a finite-state discrete-time setup (with more than one time period). For finite discrete time, we show later in Proposition 4.3 that dynamic viability is equivalent to the classic *no-arbitrage* condition NA. In general, Corollary A.3 below implies that a market $\mathcal{S}$ is dynamically viable if and only if the zero strategy is maximal for ${}^{0}L^{\mathrm{sf}}_{+}$; so it is enough to check maximality for the starting time 0 instead of all $\sigma \in \mathcal{T}_{[0,T]}$. Strong maximality of 0 in ${}^{0}L^{\mathrm{sf}}_{+}$ has been coined *numéraire-independent no-arbitrage* (NINA) and analysed in detail in Herdegen [17]. To summarise, dynamic viability can be viewed as a (weak and general) property of absence of arbitrage.

The next concept strengthens viability.

Definition 3.6. A market $\mathcal{S}$ is called

- *statically efficient* if each $\vartheta \in \mathbf{h}^{\sigma}L^{\mathrm{sf}}_{+}(\mathcal{S})$ is maximal for $\mathbf{h}^{\sigma}L^{\mathrm{sf}}_{+}(\mathcal{S})$, for all $\sigma \in \mathcal{T}_{[0,T]}$.
- *dynamically efficient* if each $\vartheta \in \mathbf{h}^{\sigma}L^{\mathrm{sf}}_{+}(\mathcal{S})$ is maximal for ${}^{\sigma}L^{\mathrm{sf}}_{+}(\mathcal{S})$, for all $\sigma \in \mathcal{T}_{[0,T]}$.

Viability means that one cannot improve the zero strategy of doing nothing. An efficient market has even more structure—all buy-and-hold strategies are good in the sense that they cannot be improved, in a certain class, without risk or extra capital. It is clear that dynamic implies static efficiency, and like for viability, easy examples in a two-period model show that the converse is not true in general.

The connection between viability and efficiency is more subtle. Clearly efficiency (dynamic or static) implies viability (of the same kind). At first sight, one might expect the converse as well—why should it matter whether one improves zero or a general buy-and-hold strategy? But there *is* a difference (see [17, Example 3.15]), and the reason is that we work with the class of *undefaultable* strategies, which is a cone but not a linear space. If we try to improve a strategy and subtract it to construct something better, this leads us outside that cone in general—except of course if we subtract zero.

Interestingly and notably, there is no difference between efficiency and viability in finite discrete time. For that setting, Proposition 4.3 below shows that dynamic efficiency is equivalent to dynamic viability, and one can show (see [18, Lemma VIII.1.19]) that the two static concepts are then also equivalent to each other. This reflects the well-known fact that if one can achieve arbitrage in finite discrete time with a general strategy, one can also achieve arbitrage with an undefaultable strategy. However, this is specific to finite discrete time because the proof relies on backward induction (see [9, Section 2.2]).

3.4 Strong bubbles and strict local martingales

In contrast to most of the existing literature, none of our definitions so far has involved any mention of martingales. But our first main result shows that (strict) local martingales appear automatically when we study strong bubbles.

Theorem 3.7. *For a market $\mathcal{S}$ satisfying* (2.1)*, the following are equivalent:*

1) *The zero strategy 0 is maximal in ${}^0L^{\mathrm{sf}}_+$, and $\mathcal{S}$ has a strong bubble.*

2) *$\mathcal{S}$ is dynamically viable, but not dynamically efficient.*

3) *There exist a representative $\bar{S} \in \mathcal{S}$ and $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ such that $\bar{S}$ is a local $\mathbb{Q}$-martingale; and for any such pair $(\bar{S}, \mathbb{Q})$, the process $\bar{S}$ is a strict local $\mathbb{Q}$-martingale.*

The proof of Theorem 3.7 is given in Section 4.4. For ease of formulation, we introduce the following terminology. Recall that an *equivalent local martingale measure (ELMM)* for a process Y is a probability measure $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ such that Y is a local $\mathbb{Q}$-martingale.

Definition 3.8. A *representative/ELMM pair* is a pair $(\bar{S}, \mathbb{Q})$, where $\bar{S} \in \mathcal{S}$ and $\mathbb{Q}$ is an ELMM for $\bar{S}$.

Note that we do not claim that our fixed representative $\mathbf{S}$ has any strict local martingale properties. This is not possible in general—$\mathbf{S}$ itself might fail to admit any ELMM.

Theorem 3.7 is remarkable for several reasons. Mathematically, it is very satisfactory because it gives necessary and sufficient conditions, in terms of local martingale properties, for a market to have a strong bubble. In particular, strict local martingales turn up naturally and automatically. Moreover, the strict local martingale property is *robust* in the sense that for each market representative $\bar{S}$ which admits an ELMM, we have the strict local martingale property *simultaneously* under *all* possible ELMMs $\mathbb{Q}$—it cannot happen that we "see" in $\bar{S}$ a bubble under one measure and no bubble under another. The reason is that our fundamental values are defined by superreplication prices, and like these, our bubble concept does not depend on a choice of a valuation/risk-neutral/martingale measure $\mathbb{Q}$. This is in contrast to the approach of Jarrow, Protter and Shimbo [24, 25, 38], where fundamental values and bubbles are defined in terms of some fixed ELMM $\mathbb{Q}$ for the basic assets. We illustrate below in Example 5.5 that the choice of $\mathbb{Q}$ matters—it can happen that an asset has a bubble under some $\mathbb{Q}$, but has no bubble under another $\mathbb{Q}'$.

Remark 3.9. 1) Our market has $N > 1$ traded primary assets. So a representative $\bar{S}$, which is an $\mathbb{R}^N$-valued process, is a strict local $\mathbb{Q}$-martingale iff there is at least one coordinate $\bar{S}^i$, $i \in \{1, \dots, N\}$, which has the local, but not the true $\mathbb{Q}$-martingale property. This reflects Corollary 4.7 below which says that the market fails to be dynamically efficient iff at least one of the buy-and-hold strategies e^i is not maximal for ${}^0L^{\mathrm{sf}}_+$.

2) Dynamically viable markets with a strong bubble can only appear in models with infinitely many trading dates. In finite discrete time, Proposition 4.3 below shows that dynamic viability, dynamic efficiency and the no-arbitrage property NA are all equivalent; so we cannot have there strong bubbles without arbitrage. We think that this dichotomy is natural and some phenomena inherently need an infinite set of trading dates.

3) Throughout this paper, our setting has a last date; we either work in continuous time on the (right-closed) interval $[0, T]$ or in discrete time on $\{0, 1, \dots, T\}$ (then with $T \in \mathbb{N}$). Results like those for $[0, T]$ can also be developed for dates in $[0, \infty)$ or in $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, but need extra care as time goes to ∞. This is left for future research.

To illustrate the strength of Theorem 3.7, we present a corollary for the classic setup.

Theorem 3.10. *Let $X \geq 0$ be an $\mathbb{R}^d$-valued semimartingale and define its fundamental value process ${}^*X$, for $i = 1, \ldots, d$ and each stopping time $\sigma \in \mathcal{T}_{[0,T]}$, as*

$$\begin{aligned} {}^*X^i_\sigma := \operatorname{ess\,inf}\{v \in \mathbf{L}^0_+(\mathcal{F}_\sigma) : \exists\ & v_\sigma \in \mathbf{L}^0(\mathcal{F}_\sigma) \text{ and } \mathbb{R}^d\text{-valued } \psi \in {}^\sigma L(X) \text{ with} \\ & v_\sigma + \textstyle\int_\sigma \psi_u \,\mathrm{d}X_u \geq 0 \text{ on } [\![\sigma, T]\!] \text{ and} \\ & v_\sigma + \textstyle\int_\sigma^T \psi_u \,\mathrm{d}X_u \geq X^i_T \text{ and } v_\sigma \leq v,\ \mathbb{P}\text{-a.s.}\}. \end{aligned} \tag{3.3}$$

Suppose that X satisfies NUPBR and ${}^*X$ is not indistinguishable from X. Then there exist a strictly positive semimartingale $D \in \mathcal{D}$ and a probability measure $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ such that the $(d+1)$-dimensional process (D, DX) is a $\mathbb{Q}$-local martingale, and for every such pair (D, Q), the process (D, DX) is actually a strict local $\mathbb{Q}$-martingale. In particular, X is a strict local $\mathbb{Q}'$-martingale under any ELMM $\mathbb{Q}'$ for X (whenever such $\mathbb{Q}'$ exist).

Proof. Consider the market $\mathcal{S}$ generated by $\mathbf{S} := (1, X)$. Via Remark 2.6, 2), we can identify pairs (v_σ, ψ) as in (3.3) with $\vartheta \in {}^\sigma L^{\mathrm{sf}}_+$ satisfying $V_T(\vartheta)(\mathbf{S}) \geq S^i_T$ and $V_\sigma(\vartheta)(\mathbf{S}) \leq v$. So ${}^*\mathbf{S} = ({}^*1, {}^*X)$ shows that ${}^*X \not\equiv X$ implies ${}^*\mathbf{S} \not\equiv \mathbf{S}$, and hence $\mathcal{S}$ has a strong bubble. Moreover, NUPBR implies by [17, Proposition 3.24 (b)] that 0 is maximal in ${}^0L^{\mathrm{sf}}_+$, hence in every ${}^\sigma L^{\mathrm{sf}}_+$ by Corollary A.3 below, and so we can apply Theorem 3.7. Writing $\bar{S} = D\mathbf{S}$ because $\bar{S}$ and $\mathbf{S}$ are economically equivalent then gives the result. □

The above short proof might suggest that Theorem 3.10 can also be proved quickly by classic arguments and theory. Let us sketch such a line of reasoning because it provides some important basic insight, but at the same time shows that things are more subtle than they look at first sight. First, because X satisfies NUPBR, it admits by [46, Theorem 2.6] a strict σ-martingale density, i.e., a strictly positive local $\mathbb{P}$-martingale Z such that ZX is a $\mathbb{P}$-σ-martingale. Since $X \geq 0$, (Z, ZX) is then a local $\mathbb{P}$-martingale so that we can take $D = Z$ and $\mathbb{Q} = \mathbb{P}$. This gives the first assertion.

Now take any pair $(D, \mathbb{Q})$ such that $\bar{S} := (D, DX)$ is a local $\mathbb{Q}$-martingale. Again by Remark 2.6, 2) and using (2.4) and the numéraire invariance (2.5), we can identify any pair $(v_\sigma, \psi) \in \mathbf{L}^0(\mathcal{F}_\sigma) \times {}^\sigma L(X)$ with a self-financing strategy for $\bar{S}$, i.e. an $\mathbb{R}^{d+1}$-valued $\vartheta \in {}^\sigma L(\bar{S})$ such that, with $\mathbf{S} := (1, X)$,

$$V(\vartheta)(\bar{S}) = \vartheta \cdot \bar{S} = \vartheta_\sigma \cdot \bar{S}_\sigma + \int_\sigma \vartheta_u \,\mathrm{d}\bar{S}_u = DV(\vartheta)(\mathbf{S}) = D_\sigma v_\sigma + \int_\sigma \psi_u \,\mathrm{d}X_u \qquad \text{on } [\![\sigma, T]\!],\ \mathbb{P}\text{-a.s.}$$

This gives by comparing (3.1) and (3.3) that ${}^*\bar{S}^{i+1}_\sigma = D_\sigma {}^*X^i_\sigma$ for $i = 1, \ldots, d$ and any $\sigma \in \mathcal{T}_{[0,T]}$, and since also $\bar{S}^{i+1} = DX^i$, we obtain from ${}^*X \not\equiv X$ that ${}^*\bar{S} \not\equiv \bar{S}$, as above.

To exploit now that $\bar{S}$ is a local $\mathbb{Q}$-martingale, we should like to use the classic hedging duality from [30, Theorem 3.2]. This says that for an $\mathbb{R}^m$-valued semimartingale $Y \geq 0$,

$$\begin{aligned} \hat{Y}^i_\sigma := \operatorname{ess\,inf}\{v \in \mathbf{L}^0_+(\mathcal{F}_\sigma) : \exists\ & v_\sigma \in \mathbf{L}^0(\mathcal{F}_\sigma) \text{ and } \mathbb{R}^m\text{-valued } \zeta \in {}^\sigma L(Y) \text{ with} \\ & v_\sigma + \textstyle\int_\sigma \zeta_u \,\mathrm{d}Y_u \geq 0 \text{ on } [\![\sigma, T]\!] \text{ and} \\ & v_\sigma + \textstyle\int_\sigma^T \zeta_u \,\mathrm{d}Y_u \geq Y^i_T \text{ and } v_\sigma \leq v,\ \mathbb{P}\text{-a.s.}\} \\ = \operatorname{ess\,sup}\{\mathbb{E}_R[Y^i_T \,|\, \mathcal{F}_\sigma] : R \text{ is an ELMM for } Y\}, & \end{aligned} \tag{3.4}$$

for $i = 1, \ldots, m$ and any stopping time $\sigma \in \mathcal{T}_{[0,T]}$, provided that there exists some ELMM R for Y. (Note that since Y is nonnegative, the self-financing strategies resulting from the optional decomposition theorem in [30] are actually undefaultable and not only a-admissible for some $a > 0$.) So if there is an ELMM R and if $Y \geq 0$, we have from (3.4) that $\hat{Y}^i_\sigma \leq Y^i_\sigma$ $\mathbb{P}$-a.s. for all i and σ, because Y is for each ELMM R an R-supermartingale. If in addition $\hat{Y} \not\equiv Y$, then (3.4) gives $\mathbb{P}[\hat{Y}^i_\sigma < Y^i_\sigma] > 0$ for some i and σ. But this means that Y is a strict local R-martingale, under each ELMM R for Y.

A careful look at (3.4) now shows that we cannot directly use this for proving Theorem 3.10. For $m = d$ and $Y = X$, we get ${}^*X^i_\sigma = \hat{Y}^i_\sigma$—but we have in general no ELMM $R \approx \mathbb{P}$ for X, unless we replace the assumption NUPBR on X by the stronger condition NFLVR. On the other hand, for $m = d + 1$ and $Y = \bar{S}$, we do have an ELMM $\mathbb{Q}$ for $\bar{S}$, and so the right-hand side of (3.4) makes sense. However, we also have from (3.1) that

$$\begin{aligned} {}^*\bar{S}^i_\sigma := \operatorname{ess\,inf}\{v \in \mathbf{L}^0_+(\mathcal{F}_\sigma) : \exists\ &\mathbb{R}^{d+1}\text{-valued } \vartheta \in {}^\sigma L(\bar{S}) \text{ with} \\ &\vartheta_\sigma \cdot \bar{S}_\sigma + \textstyle\int_\sigma \vartheta_u \,\mathrm{d}\bar{S}_u \geq 0 \text{ on } [\![\sigma, T]\!] \text{ and} \\ &\vartheta_\sigma \cdot \bar{S}_\sigma + \textstyle\int_\sigma^T \vartheta_u \,\mathrm{d}\bar{S}_u \geq \bar{S}^i_T \text{ and } \vartheta_\sigma \cdot \bar{S}_\sigma \leq v,\ \mathbb{P}\text{-a.s.}, \\ &\text{and } \vartheta \cdot \bar{S} = \vartheta_\sigma \cdot \bar{S}_\sigma + \textstyle\int_\sigma \vartheta_u \,\mathrm{d}\bar{S}_u \geq 0 \text{ on } [\![\sigma, T]\!],\ \mathbb{P}\text{-a.s.}\}, \end{aligned} \quad (3.5)$$

where the last condition is the self-financing property (2.4). Comparing (3.5) with (3.4) thus shows that we do not get ${}^*\bar{S}^i_\sigma = \hat{Y}^i_\sigma$, but only ${}^*\bar{S}^i_\sigma \geq \hat{Y}^i_\sigma$—and so we cannot exploit the hedging duality (3.4) as above to deduce that $\bar{S}$ should be a strict local $\mathbb{Q}$-martingale.

In summary, arguing as above looks very natural and makes it intuitively very clear where the robust strict local martingale properties for our strong bubbles come from. However, the actual proof of Theorem 3.7 will be different. Note that the above argument, even if it could be made to work, would only give one implication for Theorem 3.7. The converse from 3) to 1) is more delicate; see Example 5.4 and the discussion after it.

Remark 3.11. The proof of Theorem 3.7 shows that in the classic setup with $\mathbf{S} = (1, X)$, a strong bubble can arise in two ways. Maybe one of the *risky* assets in X can be dominated by dynamic trading in the other risky assets and the bond; then ${}^*X^i \not\equiv X^i$ and the risky asset i has a bubble. But alternatively, if ${}^*1 \not\equiv 1$, the *bond* itself can be dominated by trading in the other (risky) assets, and in that case, choosing it initially as numéraire was rash—discounting with such a bond is a bad idea from an economic perspective. If one assumes NFLVR as in [3, 24, 25, 38], then ${}^*1 \not\equiv 1$ is not possible and the bond has no bubble. In contrast, the setup of [21] does allow a bubble in the bond.

4 Efficiency, true martingale measures, and no dominance

In this section, we prove Theorem 3.7. To that end, we first provide dual characterisations, in terms of local martingale properties, of (dynamic) viability and efficiency. We also show that viability and efficiency are distinguished by a concept of no dominance, and this leads to another equivalent description of a market with a strong bubble.

4.1 Strong bubbles revisited

We first show that strong bubbles come from the difference between dynamic efficiency and dynamic viability. This almost gives the equivalence of 1) and 2) in Theorem 3.7.

Proposition 4.1. *Suppose that $\mathcal{S}$ is dynamically viable. Then $\mathcal{S}$ has a strong bubble if and only if it is not dynamically efficient.*

Proof. If $\mathcal{S}$ has a strong bubble, we have $\mathbb{P}[{}^*S^i_\sigma < S^i_\sigma] > 0$ for an asset $i \in \{1,\dots,N\}$ and $\sigma \in \mathcal{T}_{[0,T]}$. Then $C := V_\sigma(e^i) - \Pi_\sigma(V_T(e^i)\,|\,{}^\sigma L^{\mathrm{sf}}_+)$ is a nonzero contingent claim at time σ, and $F(\cdot) := C(S^{(\eta^{\mathcal{S}})})V_T(\eta^{\mathcal{S}})(\cdot)$ is by (2.9) a nonzero contingent claim at time T. Lemma A.1 below for $V_T(e^i)$ (at time T) and $V_\sigma(\eta^{\mathcal{S}})$ (at time $\sigma \le T$) thus yields for $\varepsilon > 0$ a strategy $\vartheta^\varepsilon \in {}^\sigma L^{\mathrm{sf}}_+$ with $V_T(\vartheta^\varepsilon) \ge V_T(e^i)$ and, by the definition of C and (2.9), $V_\sigma(\vartheta^\varepsilon)(\cdot) \le \Pi_\sigma(V_T(e^i)\,|\,{}^\sigma L^{\mathrm{sf}}_+)(\cdot) + \varepsilon V_\sigma(\eta^{\mathcal{S}})(\cdot) = V_\sigma(e^i)(\cdot) - C(S^{(\eta^{\mathcal{S}})})V_\sigma(\eta^{\mathcal{S}})(\cdot) + \varepsilon V_\sigma(\eta^{\mathcal{S}})(\cdot)$. Setting $\bar\vartheta^\varepsilon := \vartheta^\varepsilon + C(S^{(\eta^{\mathcal{S}})})\eta^{\mathcal{S}}\mathbb{1}_{[\![\sigma,T]\!]}$ gives a strategy which is also in ${}^\sigma L^{\mathrm{sf}}_+$ because $C \ge 0$, and by construction, it satisfies $V_\sigma(\bar\vartheta^\varepsilon) = V_\sigma(\vartheta^\varepsilon) + C(S^{(\eta^{\mathcal{S}})})V_\sigma(\eta^{\mathcal{S}}) \le V_\sigma(e^i) + \varepsilon V_\sigma(\eta^{\mathcal{S}})$ and $V_T(\bar\vartheta^\varepsilon) = V_T(\vartheta^\varepsilon) + C(S^{(\eta^{\mathcal{S}})})V_T(\eta^{\mathcal{S}}) \ge V_T(e^i) + F$, as in (3.2). Thus e^i is not maximal for ${}^\sigma L^{\mathrm{sf}}_+$ and $\mathcal{S}$ is not dynamically efficient.

Conversely, if $\mathcal{S}$ is not dynamically efficient, there is by Corollary 4.7 below an asset $i \in \{1,\dots,N\}$ such that e^i is not maximal for ${}^0L^{\mathrm{sf}}_+$. By way of contradiction, suppose that ${}^*S^i_\sigma = S^i_\sigma$ $\mathbb{P}$-a.s. for all $\sigma \in \mathcal{T}_{[0,T]}$. Because $\mathcal{S}$ is dynamically viable, [17, Theorem 4.19] gives the existence of a maximal strategy $\vartheta^* \in {}^0L^{\mathrm{sf}}_+$ with ${}^*S^i_0 = V_0(\vartheta^*)(\mathbf{S}) = V_0(e^i)(\mathbf{S}) = S^i_0$ and $V_T(\vartheta^*)(\mathbf{S}) \ge V_T(e^i)(\mathbf{S}) = S^i_T$ $\mathbb{P}$-a.s. Hence we get $V_\sigma(\vartheta^*)(\mathbf{S}) \ge {}^*S^i_\sigma$ $\mathbb{P}$-a.s. for all $\sigma \in \mathcal{T}_{[0,T]}$ by the definition in (3.1), it follows that $V_\sigma(\vartheta^*)(\mathbf{S}) - V_\sigma(e^i)(\mathbf{S}) \ge {}^*S^i_\sigma - S^i_\sigma = 0$ $\mathbb{P}$-a.s. for each $\sigma \in \mathcal{T}_{[0,T]}$, and so $\vartheta^* - e^i \in {}^0L^{\mathrm{sf}}_+$. As $V_0(\vartheta^* - e^i)(\mathbf{S}) = 0$, dynamic viability of $\mathcal{S}$ directly yields $V_\sigma(\vartheta^* - e^i)(\mathbf{S}) = 0$ $\mathbb{P}$-a.s. for each $\sigma \in \mathcal{T}_{[0,T]}$. Thus $V(\vartheta^*)(\mathbf{S}) = V(e^i)(\mathbf{S})$ $\mathbb{P}$-a.s., and so e^i is like ϑ^* maximal for ${}^0L^{\mathrm{sf}}_+$. This is a contradiction. □

4.2 Viability and absence of arbitrage

This section provides a dual characterisation of dynamic viability in terms of local martingale properties. This is our second step on the way to proving Theorem 3.7.

We first show that everything simplifies in finite discrete time. This is no surprise as that setting is well known to be easier than a model with infinitely many trading dates.

Definition 4.2. We say that the market $\mathcal{S}$ satisfies *no arbitrage (NA)* if no strategy $\vartheta \in {}^0L^{\mathrm{sf}}_+(\mathcal{S})$ satisfies

$$V_0(\vartheta)(\mathbf{S}) = 0, \quad V_T(\vartheta)(\mathbf{S}) \ge 0 \ \mathbb{P}\text{-a.s.} \quad \text{and} \quad \mathbb{P}[V_T(\vartheta)(\mathbf{S}) > 0] > 0. \tag{4.1}$$

For finite discrete time and the classic setup as in Example 2.3, this is the standard classic definition of absence of arbitrage; see [9, Definition 2.2.3 and the subsequent section]. Note that Definition 4.2 is numéraire-independent, and that requiring $V_T(\vartheta)(\mathbf{S}) \ge 0$ $\mathbb{P}$-a.s. is redundant since $\vartheta \in {}^0L^{\mathrm{sf}}_+$.

Proposition 4.3. *Let $\mathcal{S}$ be a market in finite discrete time and recall the nondegeneracy assumption* (2.1)*. Then the following are equivalent:*

1) $\mathcal{S}$ *satisfies NA.*

2) $\mathcal{S}$ *is dynamically viable.*

3) $\mathcal{S}$ *is dynamically efficient.*

4) *There exist a numéraire strategy* η *and a probability measure* $\mathbb{Q} \approx \mathbb{P}$ *on* $\mathcal{F}_T$ *such that the* $V(\eta)$*-discounted price process* $S^{(\eta)} = \frac{\mathbf{S}}{V(\eta)(\mathbf{S})}$ *is a* $\mathbb{Q}$*-martingale.*

5) *For each numéraire strategy* η*, there exists a probability measure* $\mathbb{Q} \approx \mathbb{P}$ *on* $\mathcal{F}_T$ *such that the* $V(\eta)$*-discounted price process* $S^{(\eta)} = \frac{\mathbf{S}}{V(\eta)(\mathbf{S})}$ *is a* $\mathbb{Q}$*-martingale.*

Proof. We show below in Theorem 4.6 that 4) implies 3), and it is clear that 5) implies 4) and that 3) implies 2). Next, 2) implies that 0 is weakly maximal for ${}^0L_+^{\mathrm{sf}}$, which is in turn clearly equivalent to $\mathcal{S}$ satisfying NA, and so we obtain 1). So it only remains to argue that 1) implies 5), and this is where we exploit the setting of finite discrete time.

Let η be any numéraire strategy; by (2.1), the market portfolio $\eta^{\mathcal{S}} = \mathbf{1}$ is one example. Then we have (4.1), with $\mathbf{S}$ replaced by $X := S^{(\eta)}$. We claim that for $t \in \{1, \dots, T\}$, there can be no $\mathcal{F}_{t-1}$-measurable $\mathbb{R}^N$-valued random vector ξ such that

$$\xi \cdot (X_t - X_{t-1}) \geq 0 \ \mathbb{P}\text{-a.s.} \quad \text{and} \quad \mathbb{P}[\xi \cdot (X_t - X_{t-1}) > 0] > 0. \tag{4.2}$$

Indeed, if we have such t and ξ, we can define an $\mathbb{R}^N$-valued predictable process ϑ by

$$\vartheta_k := \begin{cases} 0, & k \leq t-1, \\ \xi - (\xi \cdot X_{t-1})\eta_t, & k = t, \\ \xi \cdot (X_t - X_{t-1})\eta_k, & k > t. \end{cases}$$

(This is the usual strategy of investing ξ into the risky assets X from time $t-1$ to t and then putting the proceeds into the numéraire.) It is straightforward to check that ϑ is in ${}^0L_+^{\mathrm{sf}}$ due to (4.2), and because $V(\eta)(S^{(\eta)}) \equiv 1$, ϑ satisfies

$$V_0(\vartheta)(S^{(\eta)}) = 0 \quad \text{and} \quad \mathbb{P}[V_T(\vartheta)(S^{(\eta)}) > 0] = P[\xi \cdot (X_t - X_{t-1}) > 0] > 0,$$

contradicting (4.1). Thus, applying [12, Proposition 5.11 and Theorem 5.16] to the model $(1, X)$ gives $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ such that $X = S^{(\eta)}$ is a $\mathbb{Q}$-martingale, and we have 5). $\square$

Despite its simplicity, Proposition 4.3 illustrates a key difference to the classic setup of mathematical finance from Example 2.3; this is also discussed in [17, Section 4.1]. Primal objects are still self-financing strategies, parametrised by $\mathbb{R}^N$-valued predictable $\mathbf{S}$-integrable processes ϑ which satisfy the self-financing constraint (2.4). But dual objects are no longer just equivalent local martingale measures (ELMMs), but representative/ELMM pairs $(\bar{S}, \mathbb{Q})$ as introduced in Definition 3.8. This is analogous to the consistent price systems that appear in arbitrage theory under transaction costs.

Recall that $\mathbf{S}$ is nonnegative and describes a numéraire market due to (2.1). Hence we have from Herdegen [17] the following numéraire-independent version of the FTAP.

Theorem 4.4. *The following are equivalent:*

1) *$\mathcal{S}$ is dynamically viable.*

2) *$\mathcal{S}$ satisfies numéraire-independent no-arbitrage (NINA), i.e., the zero strategy* 0 *is maximal for* ${}^0L_+^{\mathrm{sf}}$.

3) *There exists a representative/ELMM pair* $(\bar{S}, \mathbb{Q})$.

Note that in general, $\bar{S}$ may fail to be a true $\mathbb{Q}$-martingale, and $\mathbf{S}$ itself need not admit an ELMM.

Proof. The equivalence of 2) and 3) follows from the equivalence of (a) and (d) in [17, Theorem 4.10]. 1) implies 2) by the Definition 3.5 of dynamic viability, and that 2) implies 1) is shown below in Corollary A.3. $\square$

Remark 4.5. We get local and not σ-martingales since our assets are nonnegative.

4.3 Efficiency and true martingale measures

This section gives a dual characterisation of dynamic efficiency in terms of true martingale properties. This first leads in turn to another characterisation on the purely primal side, and is then further exploited in the subsequent sections.

The next theorem follows by combining several results from Herdegen [17]. Recall again that $\mathbf{S}$ is nonnegative and describes a numéraire market due to (2.1).

Theorem 4.6. *The following are equivalent:*

1) *$\mathcal{S}$ is dynamically efficient.*

2) *For each bounded numéraire strategy η, there exists $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ such that $S^{(\eta)}$ is a (true) $\mathbb{Q}$-martingale.*

3) *There exists a representative/ELMM pair $(\bar{S}, \mathbb{Q})$ such that $\bar{S}$ is a (true) $\mathbb{Q}$-martingale.*

Proof. Since $\mathbf{S}$ is nonnegative, both the market portfolio $\eta^{\mathcal{S}} = \mathbf{1}$ and the corresponding representative $S^{(\eta^{\mathcal{S}})} = \mathbf{S}/(\mathbf{S} \cdot \mathbf{1})$ are bounded, as required for Corollaries 4.15 and 4.16 in [17]. If we have 1), then $\eta^{\mathcal{S}}$ is maximal for ${}^0L_+^{\mathrm{sf}}$ and 2) follows from Corollaries 4.16, (c) $\Rightarrow$ (a), and 4.15, (a) $\Rightarrow$ (c), in [17]. Since there exists a bounded numéraire strategy, it is clear that 2) implies 3).

Suppose we have 3). Fix any stopping time $\sigma \in \mathcal{T}_{[0,T]}$ and any $\vartheta \in \mathbf{h}^\sigma L_+^{\mathrm{sf}}$. As $\bar{S}$ is a $\mathbb{Q}$-martingale on $[\![\sigma, T]\!]$, so is $V(\vartheta)(\bar{S})$. For any $\bar{\vartheta} \in {}^\sigma L_+^{\mathrm{sf}}$, the process $V(\bar{\vartheta})(\bar{S})$ is a nonnegative stochastic integral of a $\mathbb{Q}$-martingale and hence a $\mathbb{Q}$-supermartingale. So if $V_T(\bar{\vartheta}) \geq V_T(\vartheta)$, we get $V_\sigma(\vartheta)(\bar{S}) = \mathbb{E}_\mathbb{Q}[V_T(\vartheta)(\bar{S}) \,|\, \mathcal{F}_\sigma] \leq \mathbb{E}_\mathbb{Q}[V_T(\bar{\vartheta})(\bar{S}) \,|\, \mathcal{F}_\sigma] \leq V_\sigma(\bar{\vartheta})(\bar{S})$, and this shows that ϑ is weakly maximal for ${}^\sigma L_+^{\mathrm{sf}}$. But 3) implies by Theorem 4.4 also that $\mathcal{S}$ is dynamically viable, and so weak maximality in ${}^\sigma L_+^{\mathrm{sf}}$ is equivalent to (strong) maximality in ${}^\sigma L_+^{\mathrm{sf}}$, by Lemma A.4 below. Hence we get 1) and the proof is complete. $\square$

With the help of the above dual characterisation, we can give some further equivalent primal descriptions of dynamically efficient markets.

Corollary 4.7. *The following are equivalent:*

1) *$\mathcal{S}$ is dynamically efficient.*

2) *The market portfolio $\eta^{\mathcal{S}} = \mathbf{1}$ (buy and hold one unit of each asset) is maximal for ${}^0L_+^{\mathrm{sf}}(\mathcal{S})$.*

3) *The strategy e^i (buy and hold one unit of asset i) is maximal for ${}^0L_+^{\mathrm{sf}}(\mathcal{S})$, for each $i = 1, \dots, N$.*

4) *For each stopping time $\sigma \in \mathcal{T}_{[0,T]}$, each $\vartheta \in \mathbf{b}{}^\sigma L_+^{\mathrm{sf}}(\mathcal{S})$ is maximal for ${}^\sigma L_+^{\mathrm{sf}}(\mathcal{S})$.*

Proof. 4) implies 1) because $\mathbf{h}{}^\sigma L_+^{\mathrm{sf}} \subseteq \mathbf{b}{}^\sigma L_+^{\mathrm{sf}}$, and 1) trivially implies 2), which is in turn equivalent to 3) by [17, Corollary 4.16]. To see that 3) implies 4), we first argue as in [17, Lemma 4.13] to show that $V(\vartheta)(S^{(\eta^{\mathcal{S}})})$ is a true martingale on $[\![\sigma, T]\!]$ for $\vartheta \in \mathbf{b}{}^\sigma L_+^{\mathrm{sf}}$, and then proceed as in the second part of the proof of Theorem 4.6, with $\vartheta \in \mathbf{h}{}^\sigma L_+^{\mathrm{sf}}$ there replaced by $\vartheta \in \mathbf{b}{}^\sigma L_+^{\mathrm{sf}}$. ☐

4.4 Proof of Theorem 3.7

With the above preparations, we are now ready for the

Proof of Theorem 3.7. If $\mathcal{S}$ is dynamically viable, the zero strategy is maximal in ${}^\sigma L_+^{\mathrm{sf}}$ for each $\sigma \in \mathcal{T}_{[0,T]}$, or equivalently in ${}^0L_+^{\mathrm{sf}}$, by Corollary A.3 below. The equivalence of 1) and 2) therefore follows from Proposition 4.1.

By Theorem 4.4, dynamic viability of $\mathcal{S}$ as in 2) is equivalent to the first property in 3). In combination with Theorem 4.6, dynamic efficiency of $\mathcal{S}$ then fails as in 2) if and only if the second property in 3) holds. This completes the proof. ☐

4.5 No dominance

The proof of Proposition 4.1 shows that understanding strong bubbles depends on the difference between dynamic efficiency and dynamic viability. In this section, we characterise this with the help of no dominance, a concept which in itself has some history. It is folklore in mathematical finance that simple risk-neutral valuation results need something more than just absence of arbitrage. This important insight goes back to R. Merton [35] who wrote that "a necessary condition for a rational option pricing theory is that the option be priced such that it is neither a dominant nor a dominated security" and explained that "security (portfolio) A is *dominant* over security (portfolio) B, if on some known date in the future, the return on A will exceed the return on B for some possible states of the world, and will be at least as large as on B, in all possible states of the world".

The above formulation is intuitive, but not very precise. None of "security", "portfolio" or "return" is exactly defined. Subsequent papers have developed different mathematical formulations for the idea, and the key difference lies precisely in those two terms.

The works of Jarrow, Protter and Shimbo [24, 25, 38] incorporate "return" by the assumption that each financial product (including basic assets and dynamic trading strategies) has a market price at each time. They do not explain where this comes from; results are obtained by imposing certain structural assumptions on market prices, including "no dominance". In contrast, Jarrow/Larsson [23] only talk about basic assets and compute the "return" from the value processes of self-financing strategies. This is more specific than the approach in [24, 25, 38], but also gives in our view potentially sharper results with weaker assumptions on the underlying market. In particular, one can try to impose "no dominance" only on basic assets and then try to deduce analogous properties for suitable valuations applied to complex assets, portfolios or derivatives. We therefore follow [23] in spirit when we introduce our numéraire-independent versions of no dominance.

Definition 4.8. The market $\mathcal{S}$ is said to satisfy

- *static no dominance* if the market portfolio $\eta^{\mathcal{S}} = \mathbf{1}$ is weakly maximal for $\mathbf{h}^{\sigma}L^{\mathrm{sf}}_{+}(\mathcal{S})$, for all $\sigma \in \mathcal{T}_{[0,T]}$.
- *dynamic no dominance* if the market portfolio $\eta^{\mathcal{S}} = \mathbf{1}$ is weakly maximal for ${}^{\sigma}L^{\mathrm{sf}}_{+}(\mathcal{S})$, for all $\sigma \in \mathcal{T}_{[0,T]}$.

Due to Corollary A.3 below, dynamic no dominance is equivalent to the market portfolio $\eta^{\mathcal{S}}$ being weakly maximal for ${}^{0}L^{\mathrm{sf}}_{+}$. Moreover, the latter holds if and only if for $i = 1, \dots, N$, the buy-and-hold strategies e^i for each primary asset i are weakly maximal for ${}^{0}L^{\mathrm{sf}}_{+}$. In fact, the "only if" part is clear since any improvement of an e^i will also improve $\eta^{\mathcal{S}}$, and the "if" part follows from [17, Corollary 3.8], which proves that the weakly maximal strategies in ${}^{0}L^{\mathrm{sf}}_{+}$ form a convex cone. This shows that our definition of dynamic no dominance is very close in spirit to the concept of no dominance in [23]. On the other hand, the concept of static no dominance seems to be new. It is more delicate to analyse, and we refer to [18, Section VIII.3.5] for some more comments.

Our next result shows that no dominance is precisely the extra ingredient that distinguishes efficiency from viability.

Proposition 4.9. *$\mathcal{S}$ is dynamically efficient if and only if it is dynamically viable and satisfies dynamic no dominance.*

Proof. Efficiency trivially implies viability and yields, in the dynamic case, that $\eta^{\mathcal{S}}$ is strongly (and a fortiori weakly) maximal for ${}^{\sigma}L^{\mathrm{sf}}_{+}$, for each $\sigma \in \mathcal{T}_{[0,T]}$. Conversely, we get from Lemma A.4 below that under dynamic viability, weak is equivalent to strong maximality of $\eta^{\mathcal{S}}$ for ${}^{0}L^{\mathrm{sf}}_{+}$. So dynamic efficiency follows from Corollary 4.7. $\square$

Remark 4.10. One can also prove the static analogue of Proposition 4.9 where "dynamic" is replaced by "static" in all three appearances. The arguments are a bit different (see [18, Proposition VIII.3.19]) and omitted for reasons of space. However, we mention that they actually show that static efficiency and static no dominance are equivalent.

One important result in the classic setup is that no dominance is the extra strengthening of "absence of arbitrage" required to obtain the existence of an equivalent true (as opposed to local or σ-) martingale measure; see [23, Theorem 3.2]. Our next result establishes the same connection in our numéraire-independent framework.

Corollary 4.11. *The following are equivalent:*

1) *$\mathcal{S}$ satisfies NINA and dynamic no dominance.*

2) *There exists a representative/ELMM pair $(\bar{S}, \mathbb{Q})$, where $\bar{S}$ is a (true) $\mathbb{Q}$-martingale. Moreover, we can choose $\bar{S} = S^{(\eta)}$ with a bounded numéraire strategy η.*

Proof. By Theorem 4.4, NINA or strong maximality of 0 in ${}^0L^{\mathrm{sf}}_+$ is equivalent to dynamic viability of $\mathcal{S}$. Together with dynamic no dominance, this is by Proposition 4.9 equivalent to dynamic efficiency of $\mathcal{S}$, and this in turn is equivalent to 2) by Theorem 4.6. □

Finally, combining Propositions 4.1 and 4.11 immediately gives

Corollary 4.12. *If $\mathcal{S}$ is dynamically viable, then it has a strong bubble if and only if it does not satisfy dynamic no dominance.*

5 Examples

This section illustrates our results and some of their subtleties by concrete examples. Examples 5.1 and 5.2 (complete markets) are well known and serve as warmup only. Example 5.3 (a CEV model with stochastic volatility) exhibits an incomplete market which has a strong bubble; connections to the literature are discussed after the example. Example 5.4 shows how bubble birth can occur endogenously in our framework, and also demonstrates that our notion of strong bubbles lies strictly between the existing bubble concepts from the literature; see Section 6.4 for a detailed comparison. Finally, Example 5.5 exhibits a market where one ELMM makes prices strict local martingales, while another makes them true martingales. This illustrates that bubbles *defined* from ELMMs strongly depend on the choice of the ELMM.

Example 5.1 (*Complete markets with a strong bubble*)**.** Let $\mathcal{S}$ be the market generated by $\mathbf{S} = (1, X)$, where X is a strict local $\mathbb{P}$-martingale. We suppose that $\mathbf{S}$ is complete, which means that X has the predictable representation property in the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ we work with. We claim that $\mathcal{S}$ is then dynamically viable, but not dynamically efficient; so this is a generic example of a market with a strong bubble.

First, dynamic viability follows from Theorem 4.4 for $\bar{S} := S^{(e^1)} = \mathbf{S}$, where $e^1 = (1, 0)$ is the buy-and-hold strategy of the first asset. Next, due to completeness and continuity of X, the only $\mathbb{P}$-martingales strongly $\mathbb{P}$-orthogonal to X are constants, since $\mathcal{F}_0$ is $\mathbb{P}$-trivial. Thus the density process Z of any ELMM $\mathbb{Q}$ for X must be constant, hence 1, so that $\mathbb{Q} \equiv \mathbb{P}$. But X is a strict local $\mathbb{P}$-martingale; so there cannot be any $\mathbb{Q} \approx \mathbb{P}$ which makes X a true $\mathbb{Q}$-martingale, 2) in Theorem 4.6 with $\eta = e^1$ fails, and $\mathcal{S}$ is not dynamically efficient.

Example 5.2. One *concrete example* of a strict local $\mathbb{P}$-martingale with the predictable representation property is the well-known example from [5], where $\mathcal{S}$ is generated by $S' = (Y, 1)$ and Y is a three-dimensional Bessel process BES^3. Then $\mathcal{S}$ is also generated by $\mathbf{S} := S'/Y = (1, 1/Y) =: (1, X)$, and the process X satisfies the SDE

$$\mathrm{d}X_t = -|X_t|^2 \,\mathrm{d}W^{\mathbb{P}}_t, \qquad X_0 = s_0 > 0, \tag{5.1}$$

with a $\mathbb{P}$-Brownian motion $W^{\mathbb{P}}$. (A more detailed discussion can be found in [18, Example VIII.2.3].) More generally, we could assume that X is a *constant elasticity of variance (CEV)* process, i.e., satisfies the SDE, with $\sigma > 0$ and $\beta > 1$,

$$\mathrm{d}X_t = \sigma |X_t|^\beta \,\mathrm{d}W_t^{\mathbb{P}}, \qquad X_0 = s_0 > 0. \tag{5.2}$$

It is well known that (5.2) has a unique strong solution X which is a positive continuous strict local $\mathbb{P}$-martingale; see [31, Section 9.8] for a detailed discussion of the CEV model. One can also check (see [18, Example VIII.4.2]) that the CEV process has the predictable representation property for its own filtration (which can equivalently be generated by $W^{\mathbb{P}}$). The process in (5.1) is the special case where $\beta = 2, \sigma = 1$.

For a complete market as above, there is essentially no difference between our strong bubbles and the definition of bubbles via strict local martingales; see Section 6.4.1. Most of the subtleties appear only in the incomplete case (see Section 6.4.2) where there is no unique candidate for an ELMM. So it is important to have examples like the next one.

Example 5.3 (*An incomplete market with a strong bubble*)**.** On $[0, T]$, consider two independent $\mathbb{P}$-Brownian motions $W^{\mathbb{P}}$ and W' with respect to a given filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. The market $\mathcal{S}$ is generated by $\mathbf{S} = (S^1, S^2) = (1, X)$, where X satisfies the SDE

$$\mathrm{d}X_t = V_t |X_t|^\beta \,\mathrm{d}W_t^{\mathbb{P}}, \qquad X_0 = x_0 > 0. \tag{5.3}$$

Here $\beta > 1$ is a constant and the stochastic volatility $V = (V_t)_{0\leq t\leq T}$ satisfies the SDE

$$\mathrm{d}V_t = \alpha(V_t - \overline{\sigma})(V_t - \underline{\sigma}) \,\mathrm{d}W_t', \qquad V_0 = v_0 \in (\underline{\sigma}, \overline{\sigma}), \tag{5.4}$$

for constants $\alpha > 0$ and $\overline{\sigma} > \underline{\sigma} > 0$. This can be interpreted as a CEV model (see Example 5.2) with stochastic volatility V and elasticity of variance $\beta > 1$. It is not difficult to check that (5.4) has a unique strong solution satisfying $\underline{\sigma} < V < \overline{\sigma}$ $\mathbb{P}$-a.s.; see e.g. [39, Section 3]. The exact form of V is not important for the argument that follows; we only use that V is a continuous $(\mathcal{F}_t)$-adapted strong Markov process uniformly bounded from above and below by positive constants.

To argue that (5.3) has a unique strong solution, we first show a more general result: If $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ is such that $W^{\mathbb{P}}$ is a $\mathbb{Q}$-Brownian motion on $[0, T]$, then (5.3) has a unique strong solution X with $\mathbb{E}_{\mathbb{Q}}[X_t] < x_0$, $0 < t \leq T$, i.e., X is a strict local $\mathbb{Q}$-martingale. Moreover, there exists $\varepsilon \in (0, T]$ which depends on x_0, but not on v_0, such that

$$\mathbb{E}_{\mathbb{Q}}[X_\varepsilon] > \frac{x_0}{2}. \tag{5.5}$$

Let us argue these claims, using [37, Chapter V] as reference. A solution to (5.3) under $\mathbb{Q}$ (up to a possible explosion time) is unique because $f : [0, \infty) \times \Omega \times [0, \infty) \to \mathbb{R}$, $f(t, \omega, x) := V_t(\omega)|x|^\beta$, is uniformly in t locally random Lipschitz in x, i.e., for each $n \in \mathbb{N}$, there is a finite random variable K_n with $\sup_{0\leq t\leq T} |f(t, \omega, x) - f(t, \omega, y|) \leq K_n(\omega)|x - y|$ for all $x, y \in [0, n]$. To establish existence and prove the remaining assertions, we use a time-change argument reducing (5.3) to the SDE of the standard CEV model. To simplify the presentation, we assume that after possibly enlarging the original probability space, there exists a $\mathbb{Q}$-Brownian motion $(W_t^{\mathbb{Q}})_{t\geq 0}$ with $(W_t^{\mathbb{Q}})_{0\leq t\leq T} = W^{\mathbb{P}}$. Denote by $\mathcal{N}$ the $\mathbb{Q}$-nullsets in $\mathcal{F}_T \vee \sigma(W_s^{\mathbb{Q}}; s\geq 0)$ and set $\tilde{\mathcal{F}}_t := \mathcal{F}_{t\wedge T} \vee \sigma(W_s^{\mathbb{Q}}; s\leq t) \vee \mathcal{N}$ for $t\geq 0$. Then $\tilde{V}_t := V_{t\wedge T}$ for $t\geq 0$ is a continuous $(\tilde{\mathcal{F}}_t)$-adapted process with $(\tilde{V}_t)_{0\leq t\leq T} = V$ and values in $(\underline{\sigma}, \overline{\sigma})$ $\mathbb{Q}$-a.s. We are going to construct a strong solution on $[0,\infty)$ of the SDE

$$\mathrm{d}\tilde{X}_t = \tilde{V}_t|\tilde{X}_t|^\beta \,\mathrm{d}W_t^{\mathbb{Q}}, \qquad \tilde{X}_0 = x_0 > 0, \tag{5.6}$$

and it is clear that $X_t = \tilde{X}_t$ for $0\leq t\leq T$ is then a strong solution to (5.3).

Define $\tilde{M}_t := \int_0^t \tilde{V}_s \,\mathrm{d}W_s^{\mathbb{Q}}$ and $\Lambda_t := \int_0^t |\tilde{V}_s|^2 \,\mathrm{d}s$ for $t\geq 0$. Then $\tilde{M}$ is under $\mathbb{Q}$ a continuous local $(\tilde{\mathcal{F}}_t)$-martingale null at 0, and Λ has $\mathbb{Q}$-a.s. continuous trajectories, is null at 0, strictly increasing, and satisfies $\mathbb{Q}$-a.s.

$$\overline{\sigma}^2 t > \Lambda_t > \underline{\sigma}^2 t \quad \text{ for } t\geq 0. \tag{5.7}$$

Setting $\tau_t := \inf\{s\geq 0 : \Lambda_s \geq t\}$, $t\geq 0$, gives an increasing continuous time change for $(\tilde{\mathcal{F}}_t)_{t\geq 0}$. Define $\hat{\mathcal{F}}_t := \tilde{\mathcal{F}}_{\tau_t}$ and $\hat{W}_t := \tilde{M}_{\tau_t}$ for $t\geq 0$. Then $\hat{W}$ is under $\mathbb{Q}$ a continuous local $(\hat{\mathcal{F}}_t)$-martingale with $\langle \hat{W}\rangle_t = \langle \tilde{M}\rangle_{\tau_t} = \Lambda_{\tau_t} = t$ $\mathbb{Q}$-a.s. and hence a $\mathbb{Q}$-Brownian motion for $(\hat{\mathcal{F}}_t)_{t\geq 0}$. In this time-changed filtration, consider the SDE for the standard CEV model,

$$\mathrm{d}\hat{X}_t = |\hat{X}_t|^\beta \,\mathrm{d}\hat{W}_t, \qquad \hat{X}_0 = x_0 > 0. \tag{5.8}$$

This has a unique strong solution $\hat{X}$ which is a positive continuous strict local $\mathbb{Q}$-martingale (cf. Example 5.2). Moreover, the explicit formula for the transition density (see [10, Equation (7)]) yields $\lim_{t\searrow 0} \mathbb{E}_{\mathbb{Q}}[\hat{X}_t] = x_0$. Define $\tilde{X}_t := \hat{X}_{\Lambda_t}$ for $0\leq t\leq T$, and note that $\tilde{M}_t = \hat{W}_{\Lambda_t}$, $t\geq 0$. Then $\tilde{X}$ is a positive continuous local $\mathbb{Q}$-martingale for the filtration $(\tilde{\mathcal{F}}_t)_{0\leq t\leq T}$, and plugging in the definitions and using (5.8) shows that it satisfies the SDE

$$\mathrm{d}\tilde{X}_t = |\tilde{X}_t|^\beta \,\mathrm{d}\tilde{M}_t = \tilde{V}_t|\tilde{X}_t|^\beta \,\mathrm{d}W_t^{\mathbb{Q}}, \qquad \tilde{X}_0 = x_0,$$

as desired for (5.6). Moreover, $\tilde{X}$ is under $\mathbb{Q}$ a positive $(\tilde{\mathcal{F}}_t)$-supermartingale by Fatou's lemma, and so by (5.7) and the properties of $\hat{X}$,

$$\mathbb{E}_{\mathbb{Q}}[\tilde{X}_t] = \mathbb{E}_{\mathbb{Q}}[\hat{X}_{\Lambda_t}] \leq \mathbb{E}_{\mathbb{Q}}[\hat{X}_{\underline{\sigma}^2 t}] < x_0.$$

By the same argument, $\mathbb{E}_{\mathbb{Q}}[\tilde{X}_t] \geq \mathbb{E}_{\mathbb{Q}}[\hat{X}_{\overline{\sigma}^2 t}]$, and since the right-hand side does not depend on v_0, this together with $\lim_{t\searrow 0} \mathbb{E}_{\mathbb{Q}}[\hat{X}_t] = x_0$ establishes (5.5).

To show that $\mathcal{S}$ has a strong bubble, we first note that $\mathbf{S} = S^{(e^1)}$ and $X = V(e^2)(S^{(e^1)})$. By the above result for $\mathbb{Q} = \mathbb{P}$, X is a local $\mathbb{P}$-martingale; so $\mathcal{S}$ is dynamically viable by Theorem 4.4. Next, according to Theorem 4.6, $\mathcal{S}$ is not dynamically efficient if there is no $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ such that X is a (true) $\mathbb{Q}$-martingale. But if $\mathbb{Q}$ is an ELMM for X, then $W^{\mathbb{P}} = \int V^{-1}|X|^{-\beta}\,\mathrm{d}X$, by (5.3) and strict positivity of X and V, is a continuous local $\mathbb{Q}$-martingale with quadratic variation $\langle W^{\mathbb{P}}\rangle_t = \int_0^t V_s^{-2}|X_s|^{-2\beta}\,\mathrm{d}\langle X\rangle_s = t$, $0\leq t\leq T$, and so $W^{\mathbb{P}}$ is also a $\mathbb{Q}$-Brownian motion. Again by the above result, X is therefore a strict local $\mathbb{Q}$-martingale, so $\mathcal{S}$ is not dynamically efficient, and $\mathcal{S}$ has a strong bubble.

Example 5.3 is of interest for several reasons. First, it is a CEV model with stochastic volatility and thus quite realistic from a practical perspective. In fact, if we replace the volatility process V from (5.4) by a geometric Brownian motion, we get the well-known SABR model (see [15]). Next, since we have a strong bubble, Theorem 3.7 tells us that for all representative/ELMM pairs $(\bar{S}, \mathbb{Q})$, $\bar{S}$ is under $\mathbb{Q}$ always a strict local martingale. So Example 5.3 gives a concrete incomplete market with a bubble which is robust towards the choice of the ELMM one wants to use. (This can also be seen from the above arguments—we show that X is a strict local $\mathbb{Q}$-martingale whenever we have (5.3), (5.4) under some $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$.) Apart from Jarrow/Larsson [23, Theorem 5.7], such a robust bubble model has not been presented in the literature so far. (Note that [23, Theorem 5.7] does not argue that one has strict local $\mathbb{Q}$-martingales on a *finite* time horizon.)

Example 5.4 (*Bubble birth*)**.** For incomplete markets, there can be situations where we see no bubble at time 0, but there is a bubble at some later time. This is called "bubble birth" by some authors (e.g. [24, 25, 1]), and it comes up in our framework in a very natural way. In abstract terms, we need a market generated by a process $\mathbf{S}$ which is a true martingale prior to a suitable stopping time τ, and a strict local martingale after τ. To construct an explicit example and show rigorously that it has the desired features, we work with the class of single-jump local martingales introduced and studied in [20, 19]. This yields an intuitive description, allows concrete computations, and gives at the same time a precise control over sufficiently many ELMMs.

On $(\Omega, \mathcal{F}, \mathbb{P})$, take a Brownian motion W and an independent random variable γ with values in $(0, 1]$ and $0 < \mathbb{P}[\gamma = 1] < 1$; so the distribution of γ has an atom at 1. We need no other conditions on the distribution of γ; so for example, adding the requirement $\mathbb{P}[\gamma \geq t_0 > 0] = 1$ would be allowed. Define filtrations $\mathbb{F}^W, \mathbb{F}^\gamma, \mathbb{F}$ for $0 \leq t \leq 1$ by $\mathcal{F}_t^W := \sigma(W_s : 0 \leq s \leq t)$, $\mathcal{F}_t^\gamma := \sigma(\mathbb{1}_{\{\gamma \leq s\}} : 0 \leq s \leq t)$ and $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^\gamma \vee \mathcal{N}$, where $\mathcal{N}$ denotes the $\mathbb{P}$-nullsets in $\mathcal{F}_1^W \vee F_1^\gamma$. The market $\mathcal{S}$ is generated by $\mathbf{S} = (1, X)$, where $X = (X_t)_{0 \leq t \leq 1}$ is the unique strong solution to the SDE

$$\mathrm{d}X_t = X_t\Big(\mu\,\mathrm{d}t + \sigma(t,\gamma)\,\mathrm{d}W_t\Big), \qquad X_0 = 1,$$

with $\mu \in \mathbb{R}$ and $\sigma : [0,1]^2 \to [\sigma_0, \infty)$ for some $\sigma_0 > 0$ given by

$$\sigma(t,v) = \sigma_0 \left(1 + \frac{1}{1-t}\mathbb{1}_{\{v \leq t < 1\}}\right).$$

The random time γ is a stopping time in $\mathbb{F}^\gamma \subseteq \mathbb{F}$, and X is before γ just a geometric Brownian motion. At time γ, there is a jump in the volatility which then blows up until time 1 in such a way that X converges to 0. Intuitively, γ can be interpreted as the time when "the bubble is born"; see below for a more precise discussion.

As in Definition 3.1, we denote by ${}^*\mathbf{S}_t$ the fundamental value or superreplication price of $\mathbf{S}$ at time t. We claim that this is given, for $t < 1$, by ${}^*\mathbf{S}_t = (1, {}^*X_t)$ with

$${}^*X_t = X_t \mathbb{1}_{\{\gamma > t\}}. \tag{5.9}$$

For "$\leq$" in (5.9), we note that $X_1 = 0$ on $\{\gamma \leq t\}$. The strategy $\vartheta := \mathbb{1}_{]\!]t,1]\!]}\mathbb{1}_{\{\gamma > t\}} e^2$ thus has $V_1(\vartheta)(\mathbf{S}) = \mathbb{1}_{\{\gamma > t\}} X_1 = X_1$ and $V_t(\vartheta)(\mathbf{S}) = \mathbb{1}_{\{\gamma > t\}} X_t$, and so we get ${}^*X_t \leq \mathbb{1}_{\{\gamma > t\}} X_t$.

To argue "$\geq$" in (5.9), we use the hedging duality from [30]; this says as in (3.4) that

$$\begin{aligned} {}^*\mathbf{S}_t &= \operatorname{ess\,sup}\{\mathbb{E}_\mathbb{Q}[\mathbf{S}_1 \,|\, \mathcal{F}_t] : \mathbb{Q} \text{ is an ELMM for } \mathbf{S}\} \\ &= \Big(1, \operatorname{ess\,sup}\{\mathbb{E}_\mathbb{Q}[X_1 \,|\, \mathcal{F}_t] : \mathbb{Q} \text{ is an ELMM for } X\}\Big). \end{aligned} \tag{5.10}$$

Hence it is enough to exhibit for each $\varepsilon > 0$ an ELMM $\mathbb{Q}$ for X with

$$\mathbb{E}_{\mathbb{Q}}[X_1 \,|\, \mathcal{F}_t] \geq (1-\varepsilon) X_t \mathbb{1}_{\{\gamma > t\}}. \tag{5.11}$$

Define the local $(\mathbb{P}, \mathbb{F}^W)$-martingale $Z^1 = (Z^1_s)_{0 \leq s \leq 1}$ by

$$\mathrm{d}Z^1_s = -Z^1_s \frac{\mu}{\sigma(s,\gamma)} \,\mathrm{d}W_s, \qquad Z^1_0 = 1.$$

As $\frac{|\mu|}{\sigma(s,\gamma)} \leq \frac{|\mu|}{\sigma_0}$, Z^1 satisfies Novikov's condition and hence is a true $(\mathbb{P}, \mathbb{F}^W)$-martingale on $[0,1]$. The change of measure corresponding to Z^1 eliminates the drift from X and turns X into a local martingale. Next, define the local $(\mathbb{P}, \mathbb{F}^\gamma)$-martingale $Z^2 = (Z^2_s)_{0 \leq s \leq 1}$ by

$$\begin{aligned} Z^2_s = &\left(\frac{1-\varepsilon}{\mathbb{P}[\gamma > s]} + \frac{\varepsilon}{\mathbb{P}[\gamma < 1]} \frac{\mathbb{P}[s < \gamma < 1]}{\mathbb{P}[\gamma > s]} \right) \mathbb{1}_{\{s < \gamma\}} \\ &+ \frac{\varepsilon}{\mathbb{P}[\gamma < 1]} \mathbb{1}_{\{\gamma \leq s, \gamma < 1\}} + \frac{1-\varepsilon}{\mathbb{P}[\gamma = 1]} \mathbb{1}_{\{s = 1 = \gamma\}}; \end{aligned} \tag{5.12}$$

the corresponding change of measure changes the distribution of γ, but leaves W unchanged and hence keeps X a local martingale. For the true martingale property of Z^2, note that $Z^2 = \mathcal{M}^G F$ in the notation of [20], where $F(t) = \frac{1-\varepsilon}{1-G(t)} + \frac{\varepsilon}{G(1-)} \frac{G(1-)-G(t)}{1-G(t)}$ and G is the distribution function of γ. Now F is clearly (locally) absolutely continuous with respect to G, we have $\Delta G(1) = \mathbb{P}[\gamma = 1] > 0$ by the assumption on γ, and

$$\mathcal{M}^G_1 F = Z^2_1 = \frac{\varepsilon}{\mathbb{P}[\gamma < 1]} \mathbb{1}_{\{\gamma < 1\}} + \frac{1-\varepsilon}{\mathbb{P}[\gamma = 1]} \mathbb{1}_{\{\gamma = 1\}} \tag{5.13}$$

from (5.12) is bounded, hence integrable. Therefore it follows from [20, Theorem 3.5 (c)] that $Z^2 = \mathcal{M}^G F$ is a true $(\mathbb{P}, \mathbb{F}^\gamma)$-martingale on $[0,1]$.

Define $\mathbb{Q} \approx \mathbb{P}$ by $\mathrm{d}\mathbb{Q} = Z^2_1 Z^1_1 \,\mathrm{d}\mathbb{P}$. This will be an ELMM for X in $\mathbb{F}$ if we can check that $Z^2 Z^1$ is a (true) $(\mathbb{P}, \mathbb{F})$-martingale and $Z^2 Z^1 X$ is a local $(\mathbb{P}, \mathbb{F})$-martingale. The first claim follows from [19, Lemma A.1 (a)] with $Y^1 = Z^1$ and $Y^2 = Z^2$ there (note that the result also holds for a completed filtration and for a general distribution for γ). For the second claim, we use that as $Z^2 Z^1 X$ is a continuous (hence special) $(\mathbb{P}, \mathbb{F})$-semimartingale on $[0,1]$, it suffices to show that it is a (local) $(\mathbb{P}, \mathbb{F})$-martingale on $[0,u]$ for each fixed $u < 1$. But $Z^2 Z^1 X$ is even a true $(\mathbb{P}, \mathbb{F})$-martingale on $[0,u]$ by [19, Lemma A.1 (a)] applied on $[0,u]$, with $Y^1 = Z^1 X = \mathcal{E}(\int_0^\cdot (\sigma(s,\gamma) - \frac{\mu}{\sigma(s,\gamma)}) \,\mathrm{d}W_s)$ and $Y^2 = Z^2$ there.

Now fix $t < 1$. On the set $\{\gamma > t\}$, we have $Z^1_t X_t \mathbb{1}_{\{\gamma > t\}} = \mathcal{E}((\sigma_0 - \frac{\mu}{\sigma_0})W)_t \mathbb{1}_{\{\gamma > t\}}$ because $\sigma(s,\gamma) \equiv \sigma_0$ up to time t. In the same way, $Z^1_1 X_1 \mathbb{1}_{\{\gamma = 1\}} = \mathcal{E}((\sigma_0 - \frac{\mu}{\sigma_0})W)_1 \mathbb{1}_{\{\gamma = 1\}}$, and $X_1 = 0$ on $\{\gamma < 1\}$. Combining this with (5.13) and $\mathbb{1}_{\{\gamma = 1\}} = \mathbb{1}_{\{\gamma = 1\}} \mathbb{1}_{\{\gamma > t\}}$ yields

$$Z^2_1 Z^1_1 X_1 = \frac{1-\varepsilon}{\mathbb{P}[\gamma = 1]} \mathbb{1}_{\{\gamma = 1\}} \mathbb{1}_{\{\gamma > t\}} Z^1_t X_t \frac{\mathcal{E}((\sigma_0 - \frac{\mu}{\sigma_0})W)_1}{\mathcal{E}((\sigma_0 - \frac{\mu}{\sigma_0})W)_t},$$

and the ratio is independent from γ. The Bayes rule and $\mathcal{F}_t = \mathcal{F}_t^\gamma \vee \mathcal{F}_t^W$ $\mathbb{P}$-a.s. then give

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[X_1 \,|\, \mathcal{F}_t] &= \frac{\mathbb{E}[Z_1^2 Z_1^1 X_1 \,|\, \mathcal{F}_t]}{Z_t^2 Z_t^1} \\
&= \frac{1}{Z_t^2 Z_t^1} \frac{1-\varepsilon}{\mathbb{P}[\gamma = 1]} Z_t^1 X_t \mathbb{1}_{\{\gamma > t\}} \, \mathbb{E}\left[\mathbb{1}_{\{\gamma=1\}} \frac{\mathcal{E}((\sigma_0 - \frac{\mu}{\sigma_0})W)_1}{\mathcal{E}((\sigma_0 - \frac{\mu}{\sigma_0})W)_t} \,\middle|\, \mathcal{F}_t \right] \\
&= \frac{1}{Z_t^2} \frac{1-\varepsilon}{\mathbb{P}[\gamma = 1]} X_t \mathbb{1}_{\{\gamma > t\}} \mathbb{P}[\gamma = 1 \,|\, \mathcal{F}_t^\gamma] \, \mathbb{E}\left[\frac{\mathcal{E}((\sigma_0 - \frac{\mu}{\sigma_0})W)_1}{\mathcal{E}((\sigma_0 - \frac{\mu}{\sigma_0})W)_t} \,\middle|\, \mathcal{F}_t^W \right] \\
&= \frac{1}{Z_t^2} \frac{1-\varepsilon}{\mathbb{P}[\gamma = 1]} X_t \mathbb{1}_{\{\gamma > t\}} \frac{\mathbb{P}[\gamma = 1]}{\mathbb{P}[\gamma > t]}, \qquad (5.14)
\end{aligned}
$$

because $\{\gamma > t\}$ is an atom of $\mathcal{F}_t^\gamma$ and the stochastic exponential is a $(\mathbb{P}, \mathbb{F}^W)$-martingale. On the other hand, we have from (5.12)

$$
Z_t^2 = \frac{1}{\mathbb{P}[\gamma > t]} \mathbb{1}_{\{\gamma > t\}} \left(1 - \varepsilon + \varepsilon \frac{\mathbb{P}[t < \gamma < 1]}{\mathbb{P}[\gamma < 1]} \right) \le \frac{1}{\mathbb{P}[\gamma > t]} \mathbb{1}_{\{\gamma > t\}}.
$$

Plugging this into (5.14) yields (5.11) and completes the proof of (5.9).

For any $t < 1$ with $\mathbb{P}[\gamma > t] < 1$, the relation ${}^*X_t = X_t \mathbb{1}_{\{\gamma > t\}}$ in (5.9) now implies that $\mathbb{P}[{}^*X_t < X_t] > 0$ so that we see a bubble at time t. On the other hand, ${}^*X_0 = X_0$ since $\gamma > 0$; so we see no bubble at time 0. Because X is strictly positive on $[0, 1)$, we also get

$$
{}^*\mathbf{S} = \mathbf{S} \text{ on } [\![0, \gamma]\!] \quad \text{and} \quad \gamma = \inf\{t \in [0, 1] : {}^*\mathbf{S}_t \neq \mathbf{S}_t\}.
$$

This shows that γ is indeed the time when “the bubble is born”.

Together with (5.10), the property $\mathbb{P}[{}^*X_t < X_t] > 0$ shows that $\mathbf{S}$ is on the interval $[0, 1]$ a *strict* local $\mathbb{Q}$-martingale under *every* ELMM $\mathbb{Q}$ for $\mathbf{S}$. But at the same time, we have ${}^*\mathbf{S} = \mathbf{S}$ on $[\![0, \gamma]\!]$ and in particular ${}^*\mathbf{S}_0 = \sup\{\mathbb{E}_{\mathbb{Q}}[\mathbf{S}_1] : \mathbb{Q} \text{ ELMM for } \mathbf{S}\} = \mathbf{S}_0$. If for instance γ is chosen to satisfy also $\gamma \ge t_0 > 0$ $\mathbb{P}$-a.s., the above means that no ELMM is able to detect a bubble before time t_0, and in particular not at time 0.

Example 5.4 illustrates that our approach (but not the one of [21, 34]) allows the possibility of “bubble birth”; but this can only occur in an incomplete setting. It also implies (see Section 6.4.2 for details) that our approach is strictly less demanding than the bubble concept in [21, 34]. Finally, it also shows why the implication 3) $\Rightarrow$ 1) in Theorem 3.7 is not completely straightforward to prove by arguments from the classic theory.

The next example gives a concrete model where $\mathbf{S}$ is under some ELMM $\mathbb{Q}$ a strict local martingale, but under another ELMM $\mathbb{Q}'$ a true martingale. Of course, by Theorem 3.7, the market generated by this model then does not have a strong bubble.

Example 5.5 (*A $\mathbb{Q}$-bubble which is not a $\mathbb{Q}'$-bubble*)**.** Start with two $\mathbb{Q}$-Brownian motions $W^i = (W_t^i)_{0 \le t \le T}$, $i = 1, 2$, with respect to a given filtration $(\mathcal{F}_t)_{0 \le t \le T}$; this need not be generated by (W^1, W^2). We assume that W^1 and W^2 are positively but not perfectly correlated: there is a constant $\lambda \in (0, 1)$ such that $\mathrm{d}\langle W^1, W^2 \rangle_t = \lambda \, \mathrm{d}t$ (we use ρ for something else below). The market $\mathcal{S}$ is generated by $\mathbf{S} = (1, X)$, where X satisfies the SDE, for some constant $\xi > 0$,

$$
\mathrm{d}X_t = \xi X_t V_t \, \mathrm{d}W_t^1, \qquad X_0 = x_0 > 0. \qquad (5.15)
$$

The volatility process $V = (V_t)_{0 \le t \le T}$ is stochastic and satisfies the SDE, for some $b > 0$,

$$\mathrm{d}V_t = bV_t \,\mathrm{d}W_t^2, \qquad V_0 = 1. \tag{5.16}$$

It is clear that (5.16) and (5.15) have unique strong solutions V and X. We claim that

(i) X is a strict local $\mathbb{Q}$-martingale on $[0, T]$;

(ii) there is a probability measure $\mathbb{Q}' \approx \mathbb{Q}$ on $\mathcal{F}_T$ such that X is a true $\mathbb{Q}'$-martingale.

To prove this, we use the results of [44]. Setting $a := (b\lambda, b\sqrt{1-\lambda^2})$, $\sigma := (\xi, 0)$ and $\rho := 0$, we are exactly in the setup of [44, Theorems 3.3 and 3.9] with $\alpha = 1$. Note that $a \cdot \sigma = \xi b\lambda > 0$, and a, σ are not parallel. So we immediately get the existence of $\mathbb{Q}'$ (called $\mathbb{Q}^a$ in [44, Theorem 3.9]) for (ii). The result (i) does not follow directly from [44, Theorem 3.3], since a strict local martingale on $[0, \infty)$ might still be a true martingale on a given finite interval. But X is a positive local $\mathbb{Q}$-martingale, hence a $\mathbb{Q}$-supermartingale, and so it suffices to show that $\mathbb{E}_{\mathbb{Q}}[X_T] < x_0$. For that, by [44, Lemma 4.2], it is enough to show that $\mathbb{Q}[\hat{\tau} < T] > 0$, where $\hat{\tau}$ is the explosion time of the SDE

$$\mathrm{d}\hat{V}_t = b\hat{V}_t \,\mathrm{d}\hat{W}_t + b\xi\lambda \hat{V}_t^2 \,\mathrm{d}t, \qquad \hat{V}_0 = 1, \tag{5.17}$$

with a generic $\mathbb{Q}$-Brownian motion $\hat{W} = (\hat{W}_t)_{t \ge 0}$. For the rest of the example, denote by $\hat{V}$ the canonical process on the path space $C([0, \infty); (0, \infty) \cup \{\Delta\})$, where Δ is an absorbing cemetery state, by $\mathbb{P}_v$ the distribution on the path space of the solution of (5.17) with initial value $v > 0$, and by ϑ the shift operator. It follows from [44, Lemma 4.3] that under each $\mathbb{P}_v$, $\hat{V}$ explodes in finite time with positive probability and is valued in $(0, \infty)$ before the explosion (the argument in [44] does not depend on the initial value v). With $T_v := \inf\{T \ge 0 : \mathbb{P}_v[\hat{\tau} < T] > 0\}$ for $v > 0$, this means that $T_v < \infty$. We claim that in fact $T_v = 0$ for all $v > 0$, and this will complete the proof, because we then have $\mathbb{P}_v[\hat{\tau} < T] > 0$ for all $T > 0$, as desired.

We first show that $v \mapsto T_v$ is decreasing. Indeed, if $T_{v_1} < T_{v_2}$ for $0 < v_1 < v_2$, there is $\varepsilon > 0$ with $\mathbb{P}_{v_1}[\hat{\tau} < T_{v_1} + \varepsilon] > 0$ and $\mathbb{P}_{v_2}[\hat{\tau} < T_{v_1} + \varepsilon < T_{v_2}] = 0$. With $\tau_y^{\uparrow} := \inf\{t \ge 0 : \hat{V}_t \ge y\}$, we can use $\hat{\tau} = \hat{\tau} \circ \vartheta_{\tau_{v_2}^{\uparrow}}$ and the strong Markov property to get

$$0 < \mathbb{P}_{v_1}[\hat{\tau} < T_{v_1} + \varepsilon] = \mathbb{P}_{v_1}\Big[\hat{\tau} \circ \vartheta_{\tau_{v_2}^{\uparrow}} < T_{v_1} + \varepsilon\Big] = \mathbb{E}_{v_1}\Big[\mathbb{P}_{v_2}[\hat{\tau} < T_{v_1} + \varepsilon]\Big] = 0,$$

a contradiction. So $v \mapsto T_v$ is decreasing and $T_\infty := \lim_{v \to \infty} T_v$ exists in $[0, \infty)$. If $T_\infty > 0$, there is $\varepsilon > 0$ with $\mathbb{P}_v[\hat{\tau} \le \varepsilon] = 0$ for all $v \in (0, \infty)$, and the Markov property gives

$$\mathbb{P}_v[\hat{\tau} \le 2\varepsilon] = \mathbb{P}_v[\hat{\tau} \le 2\varepsilon, \hat{\tau} > \varepsilon] = \mathbb{P}_v[\hat{\tau} \circ \vartheta_\varepsilon \le 2\varepsilon] = \mathbb{E}_v\Big[\mathbb{P}_{\hat{V}_\varepsilon}[\hat{\tau} \le \varepsilon]\Big] = 0$$

for all $v \in (0, \infty)$. Iterating this argument yields $\mathbb{P}_v[\hat{\tau} \le n\varepsilon] = 0$ for all $n \in \mathbb{N}$, $v \in (0, \infty)$, and we obtain a contradiction. So $T_\infty = 0$. Finally, we show that $T_v = 0$ for all $v > 0$. If this fails, there is $v_0 \in (0, \infty)$ with $T_{v_0} > 0$, and then there is $\varepsilon > 0$ with $\mathbb{P}_{v_0}[\hat{\tau} \le 2\varepsilon] = 0$. Using that $v \mapsto T_v$ is decreasing and $T_\infty = 0$, pick $v_1 > v_0$ large enough that $T_{v_1} < \varepsilon$; then $\mathbb{P}_v[\hat{\tau} \le \varepsilon] > 0$ for all $v \ge v_1$ since T_v is decreasing in v. Because $b\xi\lambda > 0$, a standard comparison argument for SDEs yields $\hat{V} \ge \tilde{V}$ $\mathbb{P}_{v_0}$-a.s., where $\tilde{V} = v_0\mathcal{E}(b\hat{W})$

satisfies $\mathrm{d}\tilde{V}_t = b\tilde{V}_t\,\mathrm{d}\hat{W}_t$, and so $\mathbb{P}_{v_0}[\hat{V}_\varepsilon \ge v_1] \ge \mathbb{P}_{v_0}[\tilde{V}_\varepsilon \ge v_1] > 0$ since $\tilde{V}_\varepsilon$ has a lognormal distribution. Using the Markov property then gives the contradiction

$$\begin{aligned}0 &= \mathbb{P}_{v_0}[\hat{\tau} \le 2\varepsilon] \ge \mathbb{P}_{v_0}[\hat{\tau} \le 2\varepsilon, \hat{V}_\varepsilon \ge v_1, \hat{\tau} > \varepsilon] = \mathbb{P}_{v_0}[\hat{\tau} \circ \vartheta_\varepsilon \le \varepsilon, \hat{V}_\varepsilon \ge v_1]\\ &= \mathbb{E}_{v_0}\Big[\mathbb{E}_{V_\varepsilon}[\hat{\tau} \le \varepsilon]\mathbb{1}_{\{V_\varepsilon \ge v_1\}}\Big] > 0.\end{aligned}$$

So $T_v = 0$ for all $v > 0$, and X is a strict local $\mathbb{Q}$-martingale on $[0, T]$, for each $T > 0$.

6 Comparison to the literature

Since the literature on bubbles is too large for a detailed overview, we are very modest and only compare our modelling approach to some seminal recent papers from the mathematical finance literature. For that, it is helpful to provide a unified framework within which different approaches can be analysed. A critical approach to mathematical models for bubbles can be found in the paper by Guasoni/Rásonyi [14].

6.1 Fundamental values

For a time horizon $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$, we describe an asset (Δ, Y) by its cumulative dividend $\Delta = (\Delta_t)_{0\le t\le T}$ and its ex-dividend price process $Y = (Y_t)_{0\le t\le T}$, both in the same unit. (We use Y so that the notation is consistent with Example 2.3.) We also include a bond $B = (B_t)_{0\le t\le T}$; so holding one unit of each asset over a time interval $(t, u]$ gives at time u a total cashflow or gain of

$$Y_u - \frac{B_u}{B_t}Y_t + B_u\int_t^u \frac{1}{B_s}\,\mathrm{d}\Delta_s.$$

Its equivalent discounted back to time t is

$$B_t\Big(\frac{Y_u}{B_u} - \frac{Y_t}{B_t} + \int_t^u \frac{1}{B_s}\,\mathrm{d}\Delta_s\Big) =: B_t\Big(X_u - X_t + \int_t^u \frac{1}{B_s}\,\mathrm{d}\Delta_s\Big) =: B_t(W_u - W_t),$$

and the discounted gains process from holding one unit of the asset is therefore

$$W_t = \frac{Y_t}{B_t} + \int_0^t \frac{1}{B_s}\,\mathrm{d}\Delta_s = X_t + \int_0^t \frac{1}{B_s}\,\mathrm{d}\Delta_s, \qquad 0 \le t \le T. \tag{6.1}$$

If Y and Δ are multidimensional, we add a superscript i for Y, Δ and W. Without dividends, (6.1) reduces to $W = Y/B = X$, the B-discounted asset price.

Denote by ${}^*Y_t$ the (undiscounted) fundamental value of the asset (Δ, Y) at time t. If ${}^*Y_t \ne Y_t$, it is natural to say that the asset has a bubble. But how do we define ${}^*Y_t$?

If we think axiomatically in linear valuation terms, one possible approach is to postulate a *fundamental value operator* which assigns fundamental values to assets or general financial products. With monotonicity, linearity and some continuity, this has the form

$$\begin{aligned}\Phi_t^Z(\Delta, Y) &:= B_t\mathbb{E}\Big[\frac{Z_u}{Z_t}\frac{Y_u}{B_u} + \int_t^u \frac{Z_s}{Z_t}\frac{1}{B_s}\,\mathrm{d}\Delta_s\,\Big|\,\mathcal{F}_t\Big] = \frac{B_t}{Z_t}\mathbb{E}\Big[Z_uX_u + \int_t^u \frac{Z_s}{B_s}\,\mathrm{d}\Delta_s\,\Big|\,\mathcal{F}_t\Big]\\ &= \frac{1}{\rho_t}\mathbb{E}\Big[\rho_uY_u + \int_t^u \rho_s\,\mathrm{d}\Delta_s\,\Big|\,\mathcal{F}_t\Big] =: \Phi_t^\rho(\Delta, Y), \qquad 0 \le t \le u \le T,\end{aligned} \tag{6.2}$$

where the positive adapted processes Z and $\rho := Z/B$ are often called a *deflator* or a *state price density*, respectively. For the bond asset $(0, B)$, this gives

$$\Phi_t^Z(0, B) = B_t \mathbb{E}\left[\frac{Z_u}{Z_t} \,\middle|\, \mathcal{F}_t\right].$$

If the bond has no bubble, then Z is a positive $\mathbb{P}$-martingale; more generally, Z is assumed to be a positive local $\mathbb{P}$-martingale with $Z_0 = 1$. Similarly, if (6.2) equals Y_t so that the asset (Δ, Y) has no bubble, the process $\rho Y + \int \rho \,\mathrm{d}\Delta = ZX + \int \frac{Z}{B} \,\mathrm{d}\Delta$ is a $\mathbb{P}$-martingale; more generally, this process is assumed to be a local $\mathbb{P}$-martingale.

In the case where Z is a martingale, setting $\mathrm{d}\mathbb{Q} = Z_T \,\mathrm{d}\mathbb{P}$ defines a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$. Using the Bayes rule and (6.1), we can then rewrite (6.2) as

$$\begin{aligned} \Phi_t^Z(\Delta, Y) &= B_t \mathbb{E}_{\mathbb{Q}}\left[\frac{Y_u}{B_u} + \int_t^u \frac{1}{B_s} \,\mathrm{d}\Delta_s \,\middle|\, \mathcal{F}_t\right] =: \Phi_t^{\mathbb{Q}}(\Delta, Y) \\ &= B_t \mathbb{E}_{\mathbb{Q}}\left[X_u + \int_t^u \frac{1}{B_s} \,\mathrm{d}\Delta_s \,\middle|\, \mathcal{F}_t\right] = B_t \mathbb{E}_{\mathbb{Q}}[W_u - W_t \,|\, \mathcal{F}_t] + Y_t. \end{aligned} \tag{6.3}$$

So if we set ${}^*Y_t := \Phi_t^{\mathbb{Q}}(\Delta, Y)$, having a bubble means that W (or $X = Y/B$ in the absence of dividends) is not a $\mathbb{Q}$-martingale. In the same way, if Z is not a $\mathbb{P}$-martingale and ${}^*Y_t := \Phi_t^Z(\Delta, Y)$, we have a bubble when $\rho Y + \int \rho \,\mathrm{d}\Delta = ZX + \int \frac{Z}{B} \,\mathrm{d}\Delta$ is not a $\mathbb{P}$-martingale. Since this depends on Z or $\mathbb{Q}$, one ought to call it a $\mathbb{Q}$-*bubble* or a Z-*bubble*.

As an alternative to the *linear* fundamental value operator, one can use the *superreplication price* for the asset (Δ, Y), which is defined by

$$\begin{aligned} \Psi_t(\Delta, Y \,|\, {}^t\Gamma) := \operatorname{ess\,inf}\{v \in \mathbf{L}_+^0(\mathcal{F}_t) :\ & \exists \text{ self-financing strategy in } (\Delta, Y) \text{ in a class } {}^t\Gamma \\ & \text{with initial wealth } v \le Y_t \text{ at time } t \\ & \text{and final wealth } V_T \ge Y_T + B_T \textstyle\int_t^T \frac{1}{B_s} \,\mathrm{d}\Delta_s\}. \end{aligned} \tag{6.4}$$

In general, Ψ is nonlinear. If $Y \ge 0$, $\Delta \ge 0$ is increasing, ${}^t\Gamma \supseteq {}^tL_+^{\mathrm{sf}}$ and there exists a deflator, the fundamental hedging duality (see [30, 45, 17] allows us to rewrite Ψ as a supremum of linear quantities, namely

$$\begin{aligned} \Psi_t(\Delta, Y \,|\, {}^t\Gamma) = \operatorname{ess\,sup}\{\Phi_t^Z(\Delta, Y) = \Phi_t^\rho(\Delta, Y) :\ & Z > 0 \text{ is a local } \mathbb{P}\text{-martingale} \\ & \text{with } Z_0 = 1 \text{ and and such that} \\ & \rho Y + \textstyle\int \rho \,\mathrm{d}\Delta = ZX + \int \frac{Z}{B} \,\mathrm{d}\Delta \\ & \text{is a local } \mathbb{P}\text{-martingale}\}. \end{aligned} \tag{6.5}$$

In a complete (and suitably arbitrage-free) model, Z or ρ (exist and) are unique, and then both approaches coincide because $\Psi_t(\Delta, Y \,|\, {}^t\Gamma) = \Phi_t^Z(\Delta, Y) = \Phi_t^\rho(\Delta, Y)$. In particular, Ψ then becomes linear. For incomplete markets, this is no longer true; see Section 6.4.2.

A process Z as in (6.5) is called a *local martingale density* or *local martingale deflator*, and its existence (for $\Delta \equiv 0$, i.e. without dividends) is equivalent to the no-arbitrage condition of *no unbounded profit with bounded risk* (NUPBR); see [28, 46]. This is in turn equivalent (see [28]) to absence of arbitrage of the first kind (NA_1) or absence of cheap thrills, and the latter condition appears also in [32]; see [17] for a discussion from a numéraire-independent perspective.

With this terminology, we now discuss some important papers from the literature.

6.2 Bubbles and equilibrium

In two seminal papers, Loewenstein and Willard [33, 34] start with a triple (B, Y, Δ) where B and Y are positive Itô processes in a Brownian filtration, $\Delta \geq 0$ is increasing and Y, Δ are multidimensional. They assume that there exists a local martingale deflator Z or ρ as in (6.5). In [33] but not in [34], they also impose completeness of the market by assuming that Z or ρ is unique. The main goal and result in both papers is a study of the additional restrictions on bubbles that result from market clearing in *equilibrium*.

Interestingly, there is a shift from [33] to [34] in the definition of fundamental values. While the first paper [33] takes the linear definition (6.2) via $\Phi_t^\rho(\Delta, Y)$, the second [34] uses the nonlinear superreplication price $\Psi_0(\Delta, Y \,|\, {}^0\Gamma)$ at time 0 from (6.4) and calls this "neoclassical economics". (The class ${}^0\Gamma$ of budget feasible strategies used in [34] contains ${}^0L_+^{\mathrm{sf}}$. So by Remark 3.2, 2), the superreplication prices in [34] for ${}^0\Gamma$ are the same as our ${}^*\mathbf{S}$ in Definition 3.1.) This follows in the footsteps of Heston et al. [21] which uses a setup $(1, X, 0)$ without dividends, where X is a one-dimensional local or stochastic volatility model. The authors in [21] say that "An asset with a nonnegative price has a "bubble" if there is a self-financing portfolio with pathwise nonnegative wealth that costs less than the asset and replicates the asset's price at a fixed future date. The bubble's value is the difference between the asset's price and the lowest cost replicating strategy" [21, Definition 2.1]. In terms of Section 6.1, [21, 34] hence use as fundamental value the superreplication price. The main focus of [21] is on relating the existence of bubbles to multiplicity (nonuniqueness) of solutions to the valuation PDEs of call and put options. But the authors also provide in their specific stochastic volatility framework necessary and sufficient conditions for various bubbles (on the bond or on the stock). All papers emphasise that their definition is in line with the economic literature (e.g. Diba/Grossman [8], Tirole [47], or Santos/Woodford [41]).

6.3 Bubbles and mathematics

Jarrow, Protter and Shimbo provide a detailed study of asset price bubbles in two papers—[24] for complete and [25] for incomplete markets. Their setup is a triple $(1, X, \Delta)$ as above, with $B \equiv 1$ and $X, \Delta \geq 0$ one-dimensional semimartingales. They work on a stochastic interval $[\![0, \tau[\![$ with a stopping time τ and a liquidation value X_τ at τ for the discounted stock X: this can be replaced by the final stock price X_T without changing the essence of the model. (We remark that their dividend process Δ should be increasing for some of their arguments.) Instead of the existence of a local martingale deflator Z or equivalently NUPBR, [24, 25] impose the stronger condition NFLVR for the gains process $W = X + \Delta$; so there exists an ELMM $\mathbb{Q}$ for W by the fundamental theorem of asset pricing. The paper [24] on complete markets assumes that $\mathbb{Q} = \mathbb{Q}^*$ is unique; [25] does not, and $\mathcal{M}_{\mathrm{e}}(W)$ denotes the nonempty set of ELMMs $\mathbb{Q}$ for W.

For the complete case [24], the fundamental value is defined as in (6.3) by

$$ {}^*X_t := \Phi_t^{\mathbb{Q}^*}(\Delta, X) := \mathbb{E}_{\mathbb{Q}^*}[X_T + (\Delta_T - \Delta_t) \,|\, \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}^*}[W_T - W_t \,|\, \mathcal{F}_t] + X_t, \tag{6.6} $$

and so an asset price bubble $X_t - {}^*X_t = W_t - \mathbb{E}_{\mathbb{Q}^*}[W_T \,|\, \mathcal{F}_t]$ appears if and only if the local $\mathbb{Q}^*$-martingale W is not a true $\mathbb{Q}^*$-martingale. Unlike in Loewenstein/Willard [33], a bubble in the bond is not possible in [24, 25] due to the assumption of NFLVR. As in [33], the link between bubbles and strict local martingales comes directly from the definition (6.6) of the fundamental value. [24] also introduces different types of bubbles (depending on the time horizon), provides a decomposition of bubbles and discusses the valuation of contingent claims. A more specific setup appears earlier in the paper by Cox/Hobson [3]; this has a model $(1, X, 0)$ without dividends, where $X \geq 0$ is a continuous semimartingale, and assumes NFLVR and completeness. The main focus of [3] is then on valuation of options in the presence of such bubbles, and in particular on the issue of put-call parity.

The incomplete case in [25] is more challenging. Since $\mathcal{M}_{\mathrm{e}}(W)$ is no longer a singleton, it is not clear which ELMM one should use to define a fundamental value as in (6.6). One can pick one $\mathbb{Q}^*$ and use that throughout; but this is ad hoc and would just lead back to the complete case results. To address this issue, [25] proposes a mechanism where "the market" chooses and sometimes (at random times σ_i) changes the measure used in (6.6), so that one works with $\mathbb{Q}^i \in \mathcal{M}_{\mathrm{e}}(W)$ for times t between σ_i and σ_{i+1}. In effect, this means that one uses a fundamental value of the form

$$ {}^*X_t := \Phi_t^{\mathbb{Q}_t^*}(\Delta, X) := \mathbb{E}_{\mathbb{Q}_t^*}[W_T - W_t \mid \mathcal{F}_t] + X_t, \tag{6.7} $$

where the measure $\mathbb{Q}_t^*$ used at time t now depends on t as well, and this makes the analysis of $X - {}^*X$ more involved. (For example, it becomes more complicated to bring in local martingales—with respect to which measure?) In the same spirit but a different setup[1], Biagini et al. [1] study the case where $\mathbb{Q}_t^*$ moves smoothly from one $\mathbb{Q}$ to another R in $\mathcal{M}_{\mathrm{e}}(W)$. In both cases, we personally find the choice of $\mathbb{Q}^i$, or $\mathbb{Q}$ and R, not fully convincing economically. For example, [25] assumes that there are enough liquidly traded derivatives in the market to *determine* the ELMM $\mathbb{Q}^*$, and that $\mathbb{Q}^*$ can be *identified* from market prices. But we are not aware of any well-established procedures to implement an identification of $\mathbb{Q}^*$ from market prices, and we find a $\mathbb{Q}^*$ determined from liquid derivative prices conceptually difficult to reconcile with possible violations, due to bubbles, of e.g. put-call parity. In [1], there is a detailed and balanced discussion of rationales for (6.7); but we personally still think that abandoning time-consistency for fundamental valuation, which is typically implied by (6.7), does represent a rather radical step.

In any case, the difference $X - {}^*X$ from (6.6) should be called a $\mathbb{Q}^*$-bubble, as W can be a true martingale under another ELMM $\mathbb{Q}'$ (see Example 5.5). Instead of having a notion which depends on the choice of an ELMM, we prefer to define bubbles by an approach using only basic assets.

6.4 Strong bubbles—our approach

To relate our work to the existing literature, we suppose for simplicity that there are no dividends ($\Delta \equiv 0$). Let $\mathcal{S}$ be generated by $\tilde{\mathbf{S}} = (B, Y)$ or $\mathbf{S} = (1, X) = (1, Y/B)$, where B is a bond and Y a vector of stocks as in Example 2.3. We then distinguish two cases.

[1] [25] needs a bigger filtration $\mathbb{G}$ to accommodate the σ_i (which are independent of $\mathbb{F}$), whereas [1] always stays within $\mathbb{F}$.

6.4.1 Complete markets

For the classic setup, completeness means that the local martingale deflator Z in (6.5) is unique; we call it Z^*. It exists under the assumption NUPBR, and since (6.5) gives $\Psi = \Phi^{Z^*}$, both approaches (6.2) and (6.4) from Section 6.1 give the same notion of bubbles. If we even have NFLVR, then Z^* is a true $\mathbb{P}$-martingale, hence defines a (unique) ELMM $\mathbb{Q}^*$, and we have a bubble if and only if discounted prices $X = Y/B$ form a strict local $\mathbb{Q}^*$-martingale. The complete case has been studied under NUPBR in [33], and under NFLVR in [3, 24], among others.

In our setup, completeness means that for each representative $\bar{S} \in \mathcal{S}$, there is at most one ELMM $\mathbb{Q}$ for the price process $\bar{S}$. If $\mathcal{S}$ is dynamically viable, there exists a representative/ELMM pair $(\bar{S}, \mathbb{Q})$ (Theorem 4.4). If $\mathcal{S}$ in addition has a strong bubble, the (unique, by completeness) ELMM $\mathbb{Q}$ for $\bar{S} = \tilde{D}'\tilde{\mathbf{S}}$ is such that $\bar{S}^j$ is a strict local $\mathbb{Q}$-martingale for at least one asset j (Theorem 3.7 and Remark 3.9, 1)). Let us call $Z^{\mathbb{Q}}$ the density process of $\mathbb{Q}$ with respect to $\mathbb{P}$; then (the vector) $Z^{\mathbb{Q}}\bar{S}$ is a local $\mathbb{P}$-martingale and $Z^{\mathbb{Q}}\bar{S}^j$ is a strict local $\mathbb{P}$-martingale. Setting $Z := Z^{\mathbb{Q}}\tilde{D}'B$ gives that Z and ZX are local $\mathbb{P}$-martingales, and either Z (if $j = 1$) or ZX^i (for $i = j - 1$ if $j > 1$) is a strict local $\mathbb{P}$-martingale. So under completeness and dynamic viability, if $\mathcal{S}$ has a strong bubble, either the bond (if $j = 1$) or one of the stocks (if $j > 1$) has a bubble in the sense of [33]. It is not difficult to check that also the converse is true.

In summary, for complete markets, all approaches essentially lead to the same concept. Assuming NFLVR or only NUPBR makes a small difference; but the difference between completeness versus incompleteness is much more significant, as we shall see now.

6.4.2 Incomplete markets

For incomplete markets, there is a genuine difference between the definitions (6.2) and (6.4) for fundamental values. As a consequence, the resulting bubble concepts are of different strength, and we claim that our notion lies strictly in the middle between the existing approaches from the literature.

Let us start with [21, 34]. In both papers, the authors only look at the fundamental value at time 0 and use this to define whether or not the asset has a bubble. In our terminology, the requirement for a bubble in [21, 34] is thus ${}^*\mathbf{S}_0 \neq \mathbf{S}_0$. If we assume NFLVR so that X admits an ELMM, the argument after Theorem 3.10 shows that X is then a *strict* local martingale under *every* ELMM $\mathbb{Q}$ for X, and so $\mathcal{S}$ has a strong bubble by Theorem 3.7. However, the converse is not true: Example 5.4 gives a market with a strong bubble and NFLVR, where X is a strict local martingale under all its ELMMs—but it can happen that this cannot be detected up to some time $t_0 > 0$ (which could even be very close to the horizon T if we choose γ like that). So our notion of a strong bubble is strictly weaker (or less restrictive) than the bubble concept in [21, 34]. (We remark that assuming NFLVR is not really crucial; if we only have NUPBR, then ${}^*\mathbf{S}_0 \neq \mathbf{S}_0$ implies that ZX becomes a strict local $\mathbb{P}$-martingale for every local martingale deflator Z.)

On the other end of the scale for incomplete markets, we have [25] (and also the survey [38]). Here, under the assumption NFLVR, $\mathbf{S}$ is said to have a bubble, for a fixed ELMM $\mathbb{Q}$ for X, if X is a strict local $\mathbb{Q}$-martingale. As one can see from the above discussion, this directly follows if $\mathcal{S}$ has a strong bubble, and Example 5.5 shows that the converse does not hold in general. So our notion of a strong bubble is strictly stronger (or more restrictive) than the bubble concept in [25].

6.5 Bubbles and other aspects

A lot of recent work has focused on models which satisfy NUPBR, but not NFLVR. This often brings up connections to strict local martingales and hence to bubbles (in the sense of $\mathbb{Q}$-bubbles). As some typical examples, we mention the benchmark approach (Platen/Heath [36]), stochastic portfolio theory (Karatzas/Fernholz [27]) or "hedging under arbitrage" (Ruf [40]). These topics are mostly tangential to our modelling here, and we refer to [18, Chapter VIII] for a more detailed discussion.

Despite its numéraire-dependence as discussed in Remark 3.4, 1), one important inspiration for many of our concepts has been the work of Delbaen and Schachermayer, especially [6] for numéraire changes and related topics and [7] for maximality. We emphasise again that a direct comparison is delicate because we use a different notion of admissible strategies. But there is no doubt that F. Delbaen is also well aware of the close connections between maximal elements/strategies, bubbles, and strict local martingales. This is for example illustrated by a presentation given in June 2012 at the QMF conference in Cairns, Australia. We quote from these slides that "A bubble is something that has a price that is too high or for the same amount of money you can get something better" and that "$H \cdot S$, acceptable, could be called a bubble if the price of $f = (H \cdot S)_\infty$ is strictly lower than 0". Delbaen also proposes some ideas to define non-bubbles; however, we have not seen any published work or preprint so far.

A Superreplication prices and maximality

This appendix collects some technical results used in the proofs. We first show how to approximate superreplication prices.

Lemma A.1. *Let $\sigma \le \tau \in \mathcal{T}_{[0,T]}$ be stopping times, ${}^\sigma\Gamma$ a strategy cone on $[\![\sigma, T]\!]$ and F a contingent claim at time τ with $\Pi_\sigma(F \,|\, {}^\sigma\Gamma) < \infty$ $\mathbb{P}$-a.s. (i.e., $\Pi_\sigma(F \,|\, {}^\sigma\Gamma)(\mathbf{S}) < \infty$ $\mathbb{P}$-a.s. or, equivalently by* (2.8)*, $\Pi_\sigma(F \,|\, {}^\sigma\Gamma)(S') < \infty$ $\mathbb{P}$-a.s. for all $S' \in \mathcal{S}$). Then for all $\varepsilon > 0$ and all strictly positive contingent claims C at time σ, there exists a strategy $\vartheta \in {}^\sigma\Gamma$ satisfying*

$$V_\tau(\vartheta) \ge F \quad \text{and} \quad V_\sigma(\vartheta) \le \Pi_\sigma(F \,|\, {}^\sigma\Gamma) + \varepsilon C, \qquad \mathbb{P}\text{-a.s.}$$

Proof. First, for any strictly positive contingent claim C at time σ, there exists an $S' \in \mathcal{S}$ with $C(S') = 1$ $\mathbb{P}$-a.s. Indeed, set $D_T := 1/C(\mathbf{S}) \in \mathbf{L}^0_{++}(\mathcal{F}_\sigma) \subseteq \mathbf{L}^0_{++}(\mathcal{F}_T)$, take $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ with $\mathbb{E}_\mathbb{Q}[D_T] < \infty$ and define $D \in \mathcal{D}$ as an RCLL version of the $\mathbb{Q}$-martingale $D_t = \mathbb{E}_\mathbb{Q}[D_T \,|\, \mathcal{F}_t]$, $0 \le t \le T$. Note that $D_\sigma = \mathbb{E}_\mathbb{Q}[D_T \,|\, \mathcal{F}_\sigma] = D_T$ $\mathbb{P}$-a.s. because D_T is $\mathcal{F}_\sigma$-measurable. For $S' := D\mathbf{S}$, the numéraire invariance (2.8) for C then gives $C(S') = C(D\mathbf{S}) = D_\sigma C(\mathbf{S}) = D_T/D_T = 1$ $\mathbb{P}$-a.s.

Now take $\varepsilon > 0$ and note that $\Pi_\sigma(F \,|\, {}^\sigma\Gamma)$ is a contingent claim at time σ and $C(S') = 1$. So by the numéraire invariance (2.8), it suffices to show that there is $\vartheta \in {}^\sigma\Gamma$ with

$$V_\tau(\vartheta)(S') \ge F(S') \quad \text{and} \quad V_\sigma(\vartheta)(S') \le \Pi_\sigma(F \,|\, {}^\sigma\Gamma)(S') + \varepsilon, \qquad \mathbb{P}\text{-a.s.} \tag{A.1}$$

The set $\mathcal{V} := \{v \in \mathbf{L}^0_+(\mathcal{F}_\sigma) : \exists \vartheta \in {}^\sigma\Gamma \text{ with } V_\tau(\vartheta)(S') \geq F(S') \text{ and } V_\sigma(\vartheta)(S') \leq v, \text{ } \mathbb{P}\text{-a.s.}\}$ is nonempty due to $\Pi_\sigma(F \,|\, {}^\sigma\Gamma) < \infty$, and also closed under taking minima. Indeed, if $v_i \in \mathcal{V}$ and $\vartheta^i \in {}^\sigma\Gamma$ have $V_\tau(\vartheta^i)(S') \geq F(S')$ and $V_\sigma(\vartheta^i)(S') \leq v_i$ for $i = 1, 2$, $\mathbb{P}$-a.s., then $\hat{\vartheta} := \vartheta^1 \mathbb{1}_{\{v_1 \leq v_2\}} + \vartheta^2 \mathbb{1}_{\{v_1 > v_2\}}$ is in ${}^\sigma\Gamma$, and $V_\tau(\hat{\vartheta})(S') \geq F(S')$ and $V_\sigma(\hat{\vartheta})(S') \leq v^1 \wedge v_2$, $\mathbb{P}$-a.s. So there is a sequence $(v_n)_{n\in\mathbb{N}}$ decreasing to $\operatorname{ess\,inf} \mathcal{V} = \Pi_\sigma(F \,|\, {}^\sigma\Gamma)(S')$ $\mathbb{P}$-a.s., and $B_n := \{v_n \leq \Pi_\sigma(F \,|\, {}^\sigma\Gamma)(S') + \varepsilon\}$, $B_0 := \emptyset$ and $A_n := B_n \setminus B_{n-1}$ yields a partition $(A_n)_{n\in\mathbb{N}}$ of Ω into pairwise disjoint sets in $\mathcal{F}_\sigma$. Take $(\vartheta^n)_{n\in\mathbb{N}}$ in ${}^\sigma\Gamma$ with $V_\tau(\vartheta^n)(S') \geq F(S')$ and $V_\sigma(\vartheta^n)(S') \leq v_n$, $\mathbb{P}$-a.s. for all n. Then $\vartheta := \sum_{n=1}^\infty \mathbb{1}_{A_n} \vartheta^n$ is in ${}^\sigma\Gamma$ and satisfies (A.1). $\square$

Recall that for $\sigma \in \mathcal{T}_{[0,T]}$ and a strategy cone ${}^\sigma\Gamma$ on $[\![\sigma, T]\!]$, a strategy $\vartheta \in {}^\sigma\Gamma$ is (strongly) maximal for ${}^\sigma\Gamma$ if there is no random variable $f \in \mathbf{L}^0_+(\mathcal{F}_T) \setminus \{0\}$ such that for all $\varepsilon > 0$, there exists a strategy $\bar{\vartheta} \in {}^\sigma\Gamma$ with $V_T(\bar{\vartheta})(\mathbf{S}) \geq V_T(\vartheta)(\mathbf{S}) + f$ and $V_\sigma(\bar{\vartheta})(\mathbf{S}) \leq V_\sigma(\vartheta)(\mathbf{S}) + \varepsilon$, $\mathbb{P}$-a.s. This can now be reformulated more compactly: $\vartheta \in {}^\sigma\Gamma$ is (strongly) maximal for ${}^\sigma\Gamma$ if and only if there is no nonzero contingent claim F at time T such that

$$\Pi_\sigma\left(V_T(\vartheta) + F \,\middle|\, {}^\sigma\Gamma\right) \leq V_\sigma(\vartheta) \qquad \mathbb{P}\text{-a.s.}$$

We next show that superreplication prices for undefaultable strategies are time-consistent, using that the family of all ${}^\sigma L^{\mathrm{sf}}_+ = {}^\sigma L^{\mathrm{sf}}_+(\mathcal{S})$ is itself time-consistent; see Section 2.2.

Proposition A.2. *Let $\sigma_1 \leq \sigma_2 \leq \tau \in \mathcal{T}_{[0,T]}$ be stopping times and F a contingent claim at time τ with $\Pi_{\sigma_2}(F \,|\, {}^{\sigma_2}L^{\mathrm{sf}}_+) < \infty$ $\mathbb{P}$-a.s. Then*

$$\Pi_{\sigma_1}(F \,|\, {}^{\sigma_1}L^{\mathrm{sf}}_+) = \Pi_{\sigma_1}\left(\Pi_{\sigma_2}(F \,|\, {}^{\sigma_2}L^{\mathrm{sf}}_+) \,\middle|\, {}^{\sigma_1}L^{\mathrm{sf}}_+\right) \qquad \mathbb{P}\text{-a.s.} \tag{A.2}$$

Proof. Denote the left- and right-hand sides of (A.2) by L and R respectively. For "$\leq$", it suffices to show the inequality for each (or equivalently for some) $S' \in \mathcal{S}$ on the set $A := \{R(S') < \infty\} \in \mathcal{F}_{\sigma_1} \subseteq \mathcal{F}_{\sigma_2}$. By positive $\mathcal{F}_{\sigma_i}$-homogeneity, we may thus replace F by $F I_{\{R<\infty\}}$, or equivalently assume without loss of generality that $R < \infty$ $\mathbb{P}$-a.s. Analogously, for proving "$\geq$", we may assume that $L < \infty$ $\mathbb{P}$-a.s.

"$\leq$": Fix a numéraire strategy η. Take $\varepsilon > 0$ and let $C^i := F_{\sigma_i, 1, S^{(\eta)}}$ be the contingent claim at time σ_i with $C^i(S^{(\eta)}) = 1$, $i = 1, 2$. Lemma A.1 gives $\vartheta^i \in {}^{\sigma_i}L^{\mathrm{sf}}_+$, $i = 1, 2$, with

$$V_{\sigma_2}(\vartheta^1) \geq \Pi_{\sigma_2}(F \,|\, {}^{\sigma_2}L^{\mathrm{sf}}_+) \quad \text{and} \quad V_{\sigma_1}(\vartheta^1) \leq R + \varepsilon C^1, \qquad \mathbb{P}\text{-a.s.,} \tag{A.3}$$

$$V_\tau(\vartheta^2) \geq F \quad \text{and} \quad V_{\sigma_2}(\vartheta^2) \leq \Pi_{\sigma_2}(F \,|\, {}^{\sigma_2}L^{\mathrm{sf}}_+) + \varepsilon C^2, \qquad \mathbb{P}\text{-a.s.} \tag{A.4}$$

By (2.7), the choice of C^2, (A.3) and (A.4),

$$V_{\sigma_2}(\vartheta^1 + \varepsilon\eta)(S^{(\eta)}) = V_{\sigma_2}(\vartheta^1)(S^{(\eta)}) + \varepsilon C^2(S^{(\eta)}) \geq V_{\sigma_2}(\vartheta^2)(S^{(\eta)}) \qquad \mathbb{P}\text{-a.s.} \tag{A.5}$$

Set $\vartheta := (\vartheta^1 + \varepsilon\eta)\mathbb{1}_{[\![\sigma_1, \sigma_2]\!]} + (\vartheta^2 + V_{\sigma_2}(\vartheta^1 + \varepsilon\eta - \vartheta^2)(S^{(\eta)})\eta)\mathbb{1}_{]\!]\sigma_2, T]\!]}$. From (A.5) and the fact that $\vartheta^i \in {}^{\sigma_i}L^{\mathrm{sf}}_+$ and $\eta \in {}^0L^{\mathrm{sf}}_+ \subseteq {}^{\sigma_i}L^{\mathrm{sf}}_+$, it is easy to check that $\vartheta \in {}^{\sigma_1}L^{\mathrm{sf}}_+$. Moreover, the definition of ϑ gives by (A.4), (A.3) and (A.5) that $V_\tau(\vartheta) \geq F$ and $V_{\sigma_1}(\vartheta) \leq R + 2\varepsilon C^1$, $\mathbb{P}$-a.s. Thus $\Pi_{\sigma_1}(F \,|\, {}^{\sigma_1}L^{\mathrm{sf}}_+) \leq R + 2\varepsilon C^1$ by (2.10), and letting $\varepsilon \searrow 0$ yields the claim.

"$\geq$": Fix $\varepsilon > 0$ and a strictly positive contingent claim C at time σ_1. By Lemma A.1, there exists $\vartheta \in {}^{\sigma_1}L^{\mathrm{sf}}_+$ satisfying $V_\tau(\vartheta) \geq F$ and $V_{\sigma_1}(\vartheta) \leq L + \varepsilon C$, $\mathbb{P}$-a.s. So the definition of superreplication prices gives first $\Pi_{\sigma_2}(F \,|\, {}^{\sigma_2}L^{\mathrm{sf}}_+) \leq V_{\sigma_2}(\vartheta)$ $\mathbb{P}$-a.s. and then

$$R = \Pi_{\sigma_1}\left(\Pi_{\sigma_2}(F \,|\, {}^{\sigma_2}L^{\mathrm{sf}}_+) \,\middle|\, {}^{\sigma_1}L^{\mathrm{sf}}_+\right) \leq V_{\sigma_1}(\vartheta) \leq L + \varepsilon C \qquad \mathbb{P}\text{-a.s.}$$

The claim follows by letting $\varepsilon \searrow 0$. $\square$

A useful consequence of Proposition A.2 is that for undefaultable strategies, (strong) maximality only needs to be tested from time 0, i.e. on $[\![0, T]\!]$.

Corollary A.3. *Let $\sigma_1 \le \sigma_2 \in \mathcal{T}_{[0,T]}$ be stopping times. If $\vartheta \in {}^{\sigma_1}L_+^{\mathrm{sf}}$ is maximal for ${}^{\sigma_1}L_+^{\mathrm{sf}}$, it is also maximal for ${}^{\sigma_2}L_+^{\mathrm{sf}}$. Hence any $\vartheta \in {}^{0}L_+^{\mathrm{sf}}$ is maximal for each ${}^{\sigma}L_+^{\mathrm{sf}}$, $\sigma \in \mathcal{T}_{[0,T]}$, if and only if it is maximal for ${}^{0}L_+^{\mathrm{sf}}$.*

An analogous statement holds for weak maximality.

Proof. If $\vartheta \in {}^{\sigma_1}L_+^{\mathrm{sf}} \subseteq {}^{\sigma_2}L_+^{\mathrm{sf}}$ fails to be maximal for ${}^{\sigma_2}L_+^{\mathrm{sf}}$, there is a nonzero contingent claim F at time T with $\Pi_{\sigma_2}(V_T(\vartheta) + F \,|\, {}^{\sigma_2}L_+^{\mathrm{sf}}) \le V_{\sigma_2}(\vartheta) < \infty$ $\mathbb{P}$-a.s. Proposition A.2, monotonicity and the definition of superreplication prices then give

$$\begin{aligned} \Pi_{\sigma_1}\Big(V_T(\vartheta) + F \,\Big|\, {}^{\sigma_1}L_+^{\mathrm{sf}}\Big) &= \Pi_{\sigma_1}\bigg(\Pi_{\sigma_2}\Big(V_T(\vartheta) + F \,\Big|\, {}^{\sigma_2}L_+^{\mathrm{sf}}\Big) \,\bigg|\, {}^{\sigma_1}L_+^{\mathrm{sf}}\bigg) \\ &\le \Pi_{\sigma_1}\Big(V_{\sigma_2}(\vartheta) \,\Big|\, {}^{\sigma_1}L_+^{\mathrm{sf}}\Big) \le V_{\sigma_1}(\vartheta) \qquad \mathbb{P}\text{-a.s.} \end{aligned}$$

So ϑ fails to be maximal for ${}^{\sigma_1}L_+^{\mathrm{sf}}$, and we arrive at a contradiction. □

Finally, we show that under dynamic viability, weak and strong maximality coincide.

Lemma A.4. *If $\mathcal{S}$ is dynamically viable and $\sigma \in \mathcal{T}_{[0,T]}$ any stopping time, then $\vartheta \in {}^{\sigma}L_+^{\mathrm{sf}}(\mathcal{S})$ is weakly maximal for ${}^{\sigma}L_+^{\mathrm{sf}}(\mathcal{S})$ if and only if it is (strongly) maximal for ${}^{\sigma}L_+^{\mathrm{sf}}(\mathcal{S})$.*

Proof. Strong clearly implies weak maximality. Conversely, let $\vartheta \in {}^{\sigma}L_+^{\mathrm{sf}}$ be weakly maximal. We first claim that for each $\bar{\vartheta} \in {}^{\sigma}L_+^{\mathrm{sf}}$ with $V_T(\bar{\vartheta}) \ge V_T(\vartheta)$ $\mathbb{P}$-a.s., we have $V(\bar{\vartheta}) \ge V(\vartheta)$ $\mathbb{P}$-a.s. on $[\![\sigma, T]\!]$, so that $\bar{\vartheta} - \vartheta \in {}^{\sigma}L_+^{\mathrm{sf}}$. Indeed, if $\tau \in \mathcal{T}_{[\sigma,T]}$ is a stopping time such that the set $A := \{V_\tau(\bar{\vartheta}) < V_\tau(\vartheta)\}$ has $\mathbb{P}[A] > 0$, we take a numéraire strategy η and set $\hat{\vartheta} := \vartheta \mathbb{1}_{[\![0,\tau]\!]} + (\mathbb{1}_{A^c}\vartheta + \mathbb{1}_A(\bar{\vartheta} + V_\tau(\vartheta - \bar{\vartheta})(S^{(\eta)})\eta))\mathbb{1}_{]\!]\tau,T]\!]}$. Then $\hat{\vartheta} \in {}^{\sigma}L_+^{\mathrm{sf}}$, we have $V_\sigma(\hat{\vartheta}) = V_\sigma(\vartheta)$ $\mathbb{P}$-a.s., and using that $V_T(\bar{\vartheta}) \ge V_T(\vartheta)$ $\mathbb{P}$-a.s. gives

$$\begin{aligned} V_T(\hat{\vartheta}) &= \mathbb{1}_{A^c} V_T(\vartheta) + \mathbb{1}_A\Big(V_T(\bar{\vartheta}) + V_\tau(\vartheta - \bar{\vartheta})(S^{(\eta)})V_T(\eta)\Big) \\ &\ge \mathbb{1}_{A^c} V_T(\vartheta) + \mathbb{1}_A\Big(V_T(\vartheta) + V_\tau(\vartheta - \bar{\vartheta})(S^{(\eta)})V_T(\eta)\Big) \\ &= V_T(\vartheta) + \mathbb{1}_A V_\tau(\vartheta - \bar{\vartheta})(S^{(\eta)})V_T(\eta) \qquad \mathbb{P}\text{-a.s.} \end{aligned}$$

Since $\mathbb{P}[A] > 0$ and $V_T(\hat{\vartheta}) > V_T(\vartheta)$ on A by the definition of A, this shows that ϑ fails to be weakly maximal, and we arrive at a contradiction which proves our claim.

To show that ϑ is (strongly) maximal, suppose to the contrary that there is a nonzero contingent claim F at time T with $\Pi_\sigma(V_T(\vartheta) + F \,|\, {}^{\sigma}L_+^{\mathrm{sf}}) \le V_\sigma(\vartheta)$ $\mathbb{P}$-a.s. Take $\varepsilon > 0$ and a strictly positive contingent claim C at time σ. Then by Lemma A.1, there exists $\bar{\vartheta} \in {}^{\sigma}L_+^{\mathrm{sf}}$ with $V_T(\bar{\vartheta}) \ge V_T(\vartheta) + F$ and $V_\sigma(\bar{\vartheta}) \le \Pi_\sigma(V_T(\vartheta) + F \,|\, {}^{\sigma}L_+^{\mathrm{sf}}) + \varepsilon C \le V_\sigma(\vartheta) + \varepsilon C$, $\mathbb{P}$-a.s. By the first step, $\vartheta' := \bar{\vartheta} - \vartheta$ is in ${}^{\sigma}L_+^{\mathrm{sf}}$. Moreover, $V_T(\vartheta') \ge F$ and $V_\sigma(\vartheta') \le \varepsilon C$, $\mathbb{P}$-a.s., so that we get $\Pi_\sigma(F \,|\, {}^{\sigma}L_+^{\mathrm{sf}}) \le \varepsilon C$ $\mathbb{P}$-a.s. Letting $\varepsilon \searrow 0$ gives $\Pi_\sigma(F \,|\, {}^{\sigma}L_+^{\mathrm{sf}}) = 0$ $\mathbb{P}$-a.s. and so 0 is not maximal for ${}^{\sigma}L_+^{\mathrm{sf}}$, in contradiction to dynamic viability of $\mathcal{S}$. □

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