On Bermudan Options

Dedicated to Dieter Sondermann on the occasion of his 65th birthday

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Abstract: A Bermudan option is an American-style option with a restricted set of possible exercise dates. We show how to price and hedge such options by superreplication and use these results for a systematic analysis of the rollover option.

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0. Introduction

In the same way that the Bermuda islands are situated between Europe and America, Bermudan options take an intermediate place between American and European options. They are characterised by their possible payoffs and a region R of permitted dates at which they can be exercised. The two extremes are American options where R consists of all dates, and European ones where R contains just one single element. In most examples of Bermudan options, R is a finite set, but one can in principle admit as allowed exercise dates any subset R of trading dates. We study here how to price and hedge such options by superreplication.

A systematic analysis of Bermudan options in full generality does not seem to exist so far. Bensoussan (1984), Karatzas (1988) in a complete Itô process model and then Kramkov (1996) in a general incomplete semimartingale model showed how to deal with American options but did not address the Bermudan case. In developing a theory of generalised optimal stopping problems, Wong (1996) also examined Bermudan options with a fairly general set R, but still in the same setting as Karatzas (1988). We present here results for both an incomplete market and a general region R; this is done in section 1 where we see how one has to impose certain assumptions on R. It turns out that these must be made slightly stronger than those given by Wong (1996). Section 2 gives very explicit structural results for the practically important case where R is finite. One can then value and hedge a Bermudan option by successively working backward in time and combining elements from American and European option pricing techniques. As an illustration, we show in section 3 how to systematically derive a price and hedging strategy for the rollover option, thus answering a question raised by Bilodeau (1997). Section 4 contains the proof of a technical approximation result.

1. Background and general results

In this section, we introduce the basic problem of valuing a Bermudan option and present some general results. We start with the usual setup for a financial market. So $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ and $T \in (0, \infty]$ is a filtered probability space with the usual conditions, and $X = (X_t)_{0 \le t \le T}$ is an $\mathbb{R}^d$-valued semimartingale that describes the discounted prices of d risky assets. There is also a riskless asset with price 1 at all times. To have an arbitrage-free model, we assume that X satisfies the condition (NFLVR) of "no free lunch with vanishing risk" so that the set $\mathbb{M}^e_\sigma$ of equivalent σ-martingale measures Q for X is non-empty; see Theorem 1.1 of Delbaen/Schachermayer (1998).

Definition. A *Bermudan option* is a pair (U, R) where $R \subseteq [0, T]$ is the *region of permitted* *exercise dates* and $U = (U_t)_{0 \le t \le T}$ is a nonnegative adapted RCLL process called the *payoff process*. The holder of a Bermudan option can choose a stopping time τ with values in R; he will then obtain the payoff U_τ at time τ from the option writer. We denote by $\mathcal{S}_{t,T}$ the set of all stopping times τ with values in $[t,T]$ and define $\mathcal{S}_{t,T}(R) := \{\tau \in \mathcal{S}_{t,T} \,|\, \tau \text{ has values in } R\}$. We assume throughout that R contains the final date T.

Any European option with payoff $H \ge 0$ at time T can be viewed as a Bermudan option with $U = HI_{\{T\}}$ and $R = \{T\}$. An American option is obtained for $R = [0,T]$. In most practical examples, the set R contains finitely many possible exercise dates $t_1 < t_2 < \ldots < t_N$ for some $N \in I\!N$. This case will be analysed separately in the next section.

The problem of valuing a Bermudan option is to find a value for (U,R) at each time $t \in [0,T]$. This of course depends on the option writer's attitude towards risk, and we study here the (extreme) case where the goal is to find a price for superreplication. This question has been addressed by Kramkov (1996) for the case of American options, and his results suggest two possible ways of dealing with the Bermudan case:

1) In direct analogy to Theorem 3.3 of Kramkov (1996), we could study the process V^0 defined by

$$V_t^0 := \operatorname*{ess\,sup}_{Q \in I\!M_\sigma^e,\, \tau \in \mathcal{S}_{t,T}(R)} E_Q[U_\tau | \mathcal{F}_t] \qquad , \qquad 0 \le t \le T. \tag{1.1}$$

2) Since a Bermudan option (U,R) has a nonnegative payoff that is 0 outside of R because the option cannot be exercised there, it ought to be equivalent to the American option with payoff process

$$\widetilde{U}_t := U_t I_{\{t \in R\}} \qquad , \qquad 0 \le t \le T. \tag{1.2}$$

By Theorem 3.3 of Kramkov (1996), this could be valued by the process $\widetilde{V}^0$ defined by

$$\widetilde{V}_t^0 := \operatorname*{ess\,sup}_{Q \in I\!M_\sigma^e,\, \tau \in \mathcal{S}_{t,T}} E_Q[\widetilde{U}_\tau | \mathcal{F}_t] \qquad , \qquad 0 \le t \le T. \tag{1.3}$$

It seems intuitively clear that both approaches should lead to the same result, but this is not entirely straightforward. Already a glance at the proof of Theorem 3.3 in Kramkov (1996) shows that the latter needs the payoff process to be RCLL, and so it is not surprising that we shall have to look at the structure of the region R in more detail.

Remark. In the special case where the financial market is given by a complete multidimensional Itô process model, the pricing and hedging of Bermudan options has also been studied in Chapter 5 of Wong (1996) as an application of generalised optimal stopping theory. However, the setup there is considerably more restrictive since X is an Itô process, $I\!M_\sigma^e$ reduces by completeness to a singleton $\{P^*\}$ and the payoff process U has to satisfy additional regularity and integrability conditions. Moreover, some of the results on the existence of RCLL modifications are not clear; see the comments below.

Proposition 1. *The process V^0 defined by (1.1) is a nonnegative generalized $I\!M^e_\sigma$-supermartingale: V^0 is adapted, nonnegative and for each $Q \in I\!M^e_\sigma$,*

$$E_Q[V^0_t|\mathcal{F}_s] \le V^0_s \qquad Q\text{-a.s. for } s \le t.$$

If $\sup_{Q\in I\!M^e_\sigma,\, \tau\in\mathcal{S}_{0,T}(R)} E_Q[U_\tau] < \infty$, *then V^0 is also Q-integrable for each $Q \in I\!M^e_\sigma$ and therefore an $I\!M^e_\sigma$-supermartingale.*

Proof. This is a straightforward modification of the proof of Proposition 4.3 in Kramkov (1996). Note that all conditional expectations are well-defined in $[0,\infty]$ as U is nonnegative.

q.e.d.

Corollary 2. *The process $\widetilde{V}^0$ defined by (1.3) is a nonnegative generalized $I\!M^e_\sigma$-supermartingale, and even an $I\!M^e_\sigma$-supermartingale if* $\sup_{Q\in I\!M^e_\sigma,\, \tau\in\mathcal{S}_{0,T}(R)} E_Q[U_\tau] < \infty$.

Proof. Apply Proposition 1 to the pair $\big(\widetilde{U},[0,T]\big)$ with $\widetilde{U}$ defined by (1.2) and use that $\sup_{Q\in I\!M^e_\sigma,\, \tau\in\mathcal{S}_{0,T}([0,T])} E_Q[\widetilde{U}_\tau] = \sup_{Q\in I\!M^e_\sigma,\, \tau\in\mathcal{S}_{0,T}(R)} E_Q[U_\tau]$.

q.e.d.

Proposition 3. *If R is at most countable and increasingly ordered (for the natural order in $[0,T]$), then $\widetilde{V}^0$ is a version of V^0.*

Proof. Fix $t \in [0,T]$. For $\tau \in \mathcal{S}_{t,T}(R)$, we have $\widetilde{U}_\tau(\omega) = U_\tau(\omega)$ since $\tau(\omega) \in R$. Because $\mathcal{S}_{t,T} \supseteq \mathcal{S}_{t,T}(R)$, this yields $\widetilde{V}^0_t \ge V^0_t$ P-a.s. Conversely, suppose that $R = \{s_i \,|\, i \in I\!N\}$ with $s_1 < s_2 < \dots$. Fix $\tau \in \mathcal{S}_{t,T}$, set $s_0 := 0$ and define $\tau' := \sum_{i=0}^{\infty} s_{i+1} I_{\{s_i<\tau\le s_{i+1}\}}$. Then τ' is a stopping time with values in $R \cap [t,T]$, and since U is nonnegative,

$$\widetilde{U}_\tau(\omega) = \sum_{i=0}^{\infty} U_{s_{i+1}}(\omega) I_{\{\tau(\omega)=s_{i+1}\}} \le \sum_{i=0}^{\infty} U_{s_{i+1}}(\omega) I_{\{s_i<\tau(\omega)\le s_{i+1}\}} = U_{\tau'}(\omega).$$

Since $\tau' \in \mathcal{S}_{t,T}(R)$, we obtain

$$E_Q[\widetilde{U}_\tau|\mathcal{F}_t] \le V^0_t \qquad P\text{-a.s. for all } Q \in I\!M^e_\sigma,$$

and since $\tau \in \mathcal{S}_{t,T}$ was arbitrary, we get $\widetilde{V}_t^0 \le V_t^0$ P-a.s.

q.e.d.

The case where R is uncountable is more delicate. We need an additional assumption on R and a technical approximation result.

Definition. We say that $R \subseteq [0,T]$ satisfies *condition (URC)* if there exists a strictly decreasing sequence $(r_n)_{n\in I\!N}$ with $\lim_{n\to\infty} r_n = 0$ and such that

$$s + r_n \in R \qquad \text{for every } s \in R \setminus \{T\} \text{ and (sufficiently large) } n \in I\!N. \tag{1.4}$$

Remarks. 1) Obvious examples of regions R satisfying condition (URC) are $R = [a,b) \cup \{T\}$ where we can take $r_n = \frac{1}{n}$, or the set of all dyadic rational numbers in $[0,T]$ with $r_n = 2^{-n}$.

2) In Wong (1996), a region $R \subseteq [0,T]$ is called *right-continuous* if each $s \in R$ admits a strictly decreasing sequence $(s_n)_{n\in I\!N}$ in R with $\lim_{n\to\infty} s_n = s$. The uniform right-continuity condition (URC) is more restrictive since it imposes in addition that $r_n = s_n - s$ can be chosen independently of s. This becomes important when s depends on ω and we want to use (r_n) to construct an approximating sequence of stopping times, because we need to control how s_n depends on ω. A critical inspection of the proof of Theorem 2.6.4 in Wong (1996) shows that problems arise if R is only right-continuous.

Lemma 4. *Suppose that R satisfies condition (URC). Fix $t \in [0,T)$. For any stopping time $\tau \in \mathcal{S}_{t,T}$, there exists then a sequence $(\varrho_n)_{n\in I\!N}$ of stopping times such that $\varrho_n \in \mathcal{S}_{t+r_n,T}(R)$ for all (sufficiently large) n and*

$$\big(\varrho_n(\omega)\big)_{n\in I\!N} \subseteq \big(\tau(\omega), T\big] \text{ decreases to } \tau(\omega) \qquad \text{for all } \omega \text{ such that } \tau(\omega) \in R \setminus \{T\}.$$

Proof. See Appendix.

Proposition 5. *If R satisfies condition (URC), then $\widetilde{V}^0$ is a version of V^0.*

Proof. Since $T \in R$ and U is adapted, $\widetilde{V}_T^0 = \widetilde{U}_T = U_T = V_T^0$. So fix $t \in [0,T)$. As in the proof of Proposition 3, we immediately get $\widetilde{V}_t^0 \ge V_t^0$ P-a.s. For the converse inequality, fix $\tau \in \mathcal{S}_{t,T}$, $Q \in I\!M_\sigma^e$ and choose $(\varrho_n)_{n\in I\!N}$ as in Lemma 4. Since U is right-continuous and (ϱ_n) decreases to τ on $\{\tau \in R\}$, we get

$$\widetilde{U}_\tau = U_\tau I_{\{\tau \in R\}} = \lim_{n\to\infty} U_{\varrho_n} I_{\{\tau\in R\}} \le \liminf_{n\to\infty} U_{\varrho_n}$$

since U is nonnegative. Fatou's lemma and $\varrho_n \in \mathcal{S}_{t+r_n,T}(R) \subseteq \mathcal{S}_{t,T}(R)$ thus give

$$E_Q[\widetilde{U}_\tau | \mathcal{F}_t] \le \liminf_{n\to\infty} E_Q[U_{\varrho_n} | \mathcal{F}_t] \le V_t^0 \qquad P\text{-a.s.}$$

and therefore $\widetilde{V}_t^0 \le V_t^0$ P-a.s.

q.e.d.

We next study the question whether V^0 and $\widetilde{V}^0$ admit RCLL modifications. In view of Proposition 5, we focus on V^0. Fix $Q \in I\!M_\sigma^e$ and define the function

$$g(t) := E_Q[V_t^0] \qquad , \qquad 0 \le t \le T. \tag{1.5}$$

Since V^0 is a generalized Q-supermartingale by Proposition 1, g is decreasing and so

$$g(t+) := \lim_{s \to t, s > t} g(s) \le g(t) \qquad \text{for all } t \in [0,T). \tag{1.6}$$

Definition. A point $t \in [0,T]$ is called *isolated from the right in* R if there is some $\delta = \delta(t) > 0$ such that $(t, t+\delta)$ contains no points $r \in R$. If t is not isolated from the right in R, there exists a strictly decreasing sequence $(s_n)_{n \in I\!N}$ in R with $\lim\limits_{n \to \infty} s_n = t$ and we say that *R is right-continuous at t*.

Lemma 6. *The function g defined by (1.5) is right-continuous in every point $t \in [0,T)$ that is not simultaneously in R and isolated from the right in R.*

Proof. In view of (1.6), we have to show that $g(t) \le g(t+)$ for all $t \in [0,T)$. Fix t and any sequence $(t_n)_{n \in I\!N} \subseteq (t,T)$ decreasing to t, and denote by $\mathcal{Z}_t$ the family of all density processes Z' of some $Q' \in I\!M_\sigma^e$ with respect to Q (which has been fixed above) and such that $Z'_s = 1$ for $s \le t$. As in the proof of Proposition 4.3 in Kramkov (1996), we obtain

$$g(t) = E_Q[V_t^0] = \sup_{Z' \in \mathcal{Z}_t \,,\, \tau' \in \mathcal{S}_{t,T}(R)} E_Q[Z'_{\tau'} U_{\tau'}] \le E_Q[Z_\tau U_\tau] + \varepsilon$$

for a pair $(Z, \tau) \in \mathcal{Z}_t \times \mathcal{S}_{t,T}(R)$.

1) Suppose first that R is right-continuous at t with (s_n) strictly decreasing to t. We can assume that $s_n \ge t_n$ for all sufficiently large n, and then $\tau_n := \tau \vee s_n$ is in $\mathcal{S}_{s_n,T}(R) \subseteq \mathcal{S}_{t_n,T}(R)$ and $Z^n := I_{[\![0,s_n[\![} + \frac{Z}{Z_{s_n}} I_{[\![s_n,T]\!]}$ is in $\mathcal{Z}_{s_n} \subseteq \mathcal{Z}_{t_n}$. Because (s_n) decreases to t and $\tau \ge t$, the sequence (τ_n) decreases to τ, and so $Z_t = 1$ and right-continuity of Z imply that $Z^n_{\tau_n} = Z_{\tau \vee s_n}/Z_{s_n}$ converges to Z_τ P-a.s. Right-continuity of U and Fatou's lemma then give

$$E_Q[Z_\tau U_\tau] \le \liminf_{n \to \infty} \sup_{Z' \in \mathcal{Z}_{t_n} \,,\, \tau' \in \mathcal{S}_{t_n,T}(R)} E_Q[Z'_{\tau'} U_{\tau'}] \le g(t+)$$

and thus $g(t) \le g(t+)$ since $\varepsilon > 0$ was arbitrary.

2) Now suppose that t is isolated from the right in R. Since $\tau \ge t$ and $t \notin R$, we obtain $\tau \ge t+\delta$ and thus $\tau \ge t_n$ for large n, hence $\tau \in \mathcal{S}_{t_n,T}(R)$. Moreover, $Z^n := I_{[\![0,t_n[\![} + \frac{Z}{Z_{t_n}} I_{[\![t_n,T]\!]}$ is in $\mathcal{Z}_{t_n}$ and we can argue exactly as above to conclude again that $E_Q[Z_\tau U_\tau] \le g(t+)$.

q.e.d.

Remark. The observant reader will have noticed that Lemma 6 does not cover the case where R is finite and $t \in R$. It is easy to give an example of a process U such that g is then not right-continuous in the point t; hence we cannot hope to get more than Lemma 6.

The preceding results can now be put together to show how Bermudan options can be priced by superhedging.

Theorem 7. *Let (U, R) be a Bermudan option such that $\sup\limits_{Q \in \mathbb{M}^e_\sigma,\, \tau \in \mathcal{S}_{0,T}(R)} E_Q[U_\tau] < \infty$ and R satisfies the condition (URC). Then V^0 and $\widetilde{V}^0$ have a common RCLL version $V = (V_t)_{0 \le t \le T}$, and V is the smallest RCLL $\mathbb{M}^e_\sigma$-supermartingale which dominates (U, R) in the sense that $V_t \ge U_t$ P-a.s. for every $t \in R$. Moreover, (U, R) can be dynamically hedged in the sense that there exist an X-integrable $\mathbb{R}^d$-valued predictable process $\vartheta = (\vartheta_t)_{0 \le t \le T}$ and an increasing adapted RCLL process $C = (C_t)_{0 \le t \le T}$ with $C_0 = 0$ such that $V = V_0 + \int \vartheta \, dX - C$.*

Proof. By Proposition 5, $\widetilde{V}^0$ and V^0 are versions of each other. By Proposition 1, V^0 is an $\mathbb{M}^e_\sigma$-supermartingale. Because condition (URC) implies that R is right-continuous at every $t \in [0, T)$, Lemma 6 and a standard argument imply that V^0 has an RCLL version that is again an $\mathbb{M}^e_\sigma$-supermartingale. Minimality of V is proved as in Kramkov (1996), and the existence of ϑ and C follows from the optional decomposition theorem in the form given in Theorem 5.1 of Delbaen/Schachermayer (1999).

q.e.d.

As in Kramkov (1996), we can interpret the triple (V_0, ϑ, C) as a hedging strategy with consumption where V_0 denotes the initial capital, ϑ^i_t is the number of units of asset i held at time t, and C_t is the total (discounted) amount spent on consumption during the time interval $[0, t]$. Since V dominates (U, R), this strategy is safe for the option writer because even if he spends some money according to C, he still manages by trading via ϑ to remain on the safe side in that he is always able to pay out whenever the option holder decides (and is allowed) to exercise. Thus V_0 is a reasonable ask price for (U, R), and minimality of V implies that any price below V_0 is potentially no longer safe for the seller.

2. Bermudan options with a finite set of exercise dates

In view of the remark following Lemma 6, Theorem 7 does not cover the practically important case where $R = \{t_1, t_2, \ldots, t_N\}$ in which the option can only be exercised at one of finitely many dates. In this section, we show how this situation can be dealt with by piecing together finitely many subintervals.

Remark. Iwaki et al. (1995) also study options with a finite set of exercise dates, but only for the standard put option. Moreover, the authors impose a Markov assumption to use dynamic programming arguments and price by using one arbitrary martingale measure without giving a reason for their choice. They do not address the issue of hedging.

By Proposition 3, V^0 and $\widetilde{V}^0$ are versions of each other so that it is enough to study V^0. Suppose that $0 =: t_0 < t_1 < t_2 < \ldots < t_N = T$ and define the random variables $B_0, B_1, \ldots, B_N$ recursively by $B_N := U_{t_N}$ and

$$B_i := \max\Big(U_{t_i}, \operatorname*{ess\,sup}_{Q \in I\!M^e_\sigma} E_Q[B_{i+1}|\mathcal{F}_{t_i}]\Big) \qquad \text{for } i = 0, 1, \ldots, N-1.$$

Proposition 8. *We have* $B_i = V^0_{t_i}$ *P-a.s. for* $i = 0, 1, \ldots, N$.

Proof. Since U is adapted, $B_N = U_{t_N} = U_T = V^0_T = V^0_{t_N}$. Suppose that $B_{i+1} = V^0_{t_{i+1}}$ for some $i < N$. Then we get for any $Q \in I\!M^e_\sigma$ from the Q-supermartingale property of V^0 that $E_Q[B_{i+1}|\mathcal{F}_{t_i}] = E_Q[V^0_{t_{i+1}}|\mathcal{F}_{t_i}] \le V^0_{t_i}$ and therefore $B_i \le \max(U_{t_i}, V^0_{t_i}) = V^0_{t_i}$ since V^0 dominates (U, R). Conversely, fix $\tau \in \mathcal{S}_{t_i,T}(R)$ and define $A := \{\tau = t_i\} \in \mathcal{F}_{t_i}$ and $\varrho := \tau I_{A^c}$ so that $\varrho \in \mathcal{S}_{t_{i+1},T}(R)$. Then we obtain for any $Q \in I\!M^e_\sigma$ that

$$E_Q[U_\tau|\mathcal{F}_{t_i}] = I_A U_{t_i} + I_{A^c} E_Q[U_\varrho|\mathcal{F}_{t_i}] \le I_A U_{t_i} + I_{A^c} E_Q[V^0_{t_{i+1}}|\mathcal{F}_{t_i}]$$

by conditioning U_ϱ on $\mathcal{F}_{t_{i+1}}$ and using the definition of V^0 in (1.1). Since $V^0_{t_{i+1}} = B_{i+1}$ by assumption and since Q and τ were arbitrary, we conclude that

$$V^0_{t_i} = \operatorname*{ess\,sup}_{Q \in I\!M^e_\sigma,\, \tau \in \mathcal{S}_{t_i,T}(R)} E_Q[U_\tau|\mathcal{F}_{t_i}] \le \max\Big(U_{t_i}, \operatorname*{ess\,sup}_{Q \in I\!M^e_\sigma} E_Q[B_{i+1}|\mathcal{F}_{t_i}]\Big) = B_i.$$

This completes the proof.

q.e.d.

Proposition 8 says that for a Bermudan option with finite R, the values in the possible exercise dates $t_i \in R$ are obtained by forming the $I\!M^e_\sigma$-uniform Snell envelope $(B_i)_{i=0,1,\ldots,N}$ of the finite family $(U_{t_i})_{i=0,1,\ldots,N}$ of payoffs. This is the natural generalization of the standard recipe for complete markets where $I\!M^e_\sigma$ is a singleton $\{P^*\}$; see for instance Chapter 2 of Lamberton/Lapeyre (1996). The next result shows that between two possible exercise dates t_i and t_{i+1}, the price of a Bermudan option is the same as the price of a European option with payoff $V^0_{t_{i+1}}$ at time t_{i+1}. This is again completely intuitive.

Proposition 9. *For* $i = 0, 1, \ldots, N$, *we have*

(2.1) $$V^0_t = \operatorname*{ess\,sup}_{Q \in I\!M^e_\sigma} E_Q[V^0_{t_{i+1}}|\mathcal{F}_t] \qquad \textit{for } t \in (t_i, t_{i+1}].$$

In particular, V^0 is a generalized $\mathbb{M}^e_\sigma$-supermartingale on each $(t_i, t_{i+1}]$ and has on each interval (t_i, t_{i+1}) an RCLL version V.

Proof. We only have to show (2.1) because the other assertions follow from Proposition 1 and Lemma 6. The Q-supermartingale property of V^0 gives $V^0_t \geq E_Q[V^0_{t_{i+1}}|\mathcal{F}_t]$ for every $Q \in \mathbb{M}^e_\sigma$ and $t \in (t_i, t_{i+1}]$ and therefore "$\geq$" in (2.1). For the converse inequality, note that $t > t_i$ yields $\mathcal{S}_{t,T}(R) = \mathcal{S}_{t_{i+1},T}(R)$ and therefore $E_Q[U_\tau|\mathcal{F}_{t_{i+1}}] \leq V^0_{t_{i+1}}$ for all $Q \in \mathbb{M}^e_\sigma$ and $\tau \in \mathcal{S}_{t,T}(R)$ by the definition of V^0. Conditioning on $\mathcal{F}_t$ and taking the supremum over Q and τ then implies that

$$V^0_t = \operatorname*{ess\,sup}_{Q \in \mathbb{M}^e_\sigma,\, \tau \in \mathcal{S}_{t,T}(R)} E_Q[U_\tau|\mathcal{F}_t] \leq \operatorname*{ess\,sup}_{Q \in \mathbb{M}^e_\sigma} E_Q[V^0_{t_{i+1}}|\mathcal{F}_t]$$

for $t \in (t_i, t_{i+1}]$, and this completes the proof.

q.e.d.

Piecing things together now gives the desired valuation result.

Theorem 10. *Let (U, R) be a Bermudan option with $R = \{t_1, \ldots, t_N\}$ and $0 =: t_0 < t_1 < \ldots < t_N = T$ for some $N \in \mathbb{N}$ and such that $\sup_{Q \in \mathbb{M}^e_\sigma,\, i=1,\ldots,N} E_Q[U_{t_i}] < \infty$. Then V^0 and $\widetilde{V}^0$ have a common version $V = (V_t)_{0 \leq t \leq T}$ that is RCLL on each interval (t_i, t_{i+1}), and V is the smallest such $\mathbb{M}^e_\sigma$-supermartingale which dominates (U, R). Moreover, (U, R) can be dynamically hedged in the sense that there exist an X-integrable $\mathbb{R}^d$-valued predictable process $\vartheta = (\vartheta_t)_{0 \leq t \leq T}$ and an increasing adapted process $C = (C_t)_{0 \leq t \leq T}$ which has $C_0 = 0$, is RCLL on each interval (t_i, t_{i+1}) and such that $V = V_0 + \int \vartheta\, dX - C$.*

Proof. The existence of V follows from Proposition 1, Proposition 3 and Lemma 6. Minimality of V is first proved along the points $t_i \in R$ as in Proposition VI-1-2 of Neveu (1975) and then by using Proposition 9 as in Kramkov (1996). On each subinterval $(t_i, t_{i+1}]$, we can apply the optional decomposition theorem to obtain

$$V_t = V_{t_i+} + \int_{t_i}^{t} \vartheta^{(i)}_s\, dX_s - C^i_t \qquad \text{for } t \in (t_i, t_{i+1}]$$

with $C^i_{t_i} = 0$, and then we define

$$\vartheta := \sum_{i=0}^{N-1} \vartheta^{(i)} I_{]\!]t_i, t_{i+1}]\!]},$$

$$C_t := \sum_{i \text{ with } t_i \leq t} C^i_t + (V_{t_i} - V_{t_i+})$$

to obtain the representation $V = V_0 + \int \vartheta\, dX - C$. To show that C is increasing, it only remains to prove that $V_{t_i} - V_{t_i+} \geq 0$, i.e., that V can only jump downward at exercise dates t_i. Using the fact that V_{t_i+} is $\mathcal{F}_{t_i+}$-measurable, the right-continuity of $I\!\!F$, Fatou's lemma and the right-continuity of V on (t_i, t_{i+1}), and the Q-supermartingale property of V, we obtain

$$V_{t_i+} = E_Q[V_{t_i+}|\mathcal{F}_{t_i+}] = E_Q[V_{t_i+}|\mathcal{F}_{t_i}] \leq \liminf_{\substack{\delta\to 0,\\ \delta>0}} E_Q[V_{t_i+\delta}|\mathcal{F}_{t_i}] \leq V_{t_i}.$$

q.e.d.

Remark. By combining the techniques used for proving Theorem 7 and Theorem 10, we can also cover examples of regions R that do not satisfy the assumptions of either of the above results – for instance a union of some intervals and a finite set.

3. An example: The rollover option

As an application of the preceding results, we now show how to price and hedge the *rollover option* discussed by Bilodeau (1997). In the simplest case, one has a single underlying asset S^1, a finite time horizon T and an intermediate date $t_0 \in (0,T)$. The holder of the option can decide at time t_0 if he wants to obtain a payoff at t_0 of $\max(S^1_{t_0}, K)$ with a fixed guarantee K, or if he prefers to roll over the guarantee. In the latter case, he will get at time T a payoff of $\max\left(S^1_T, K\frac{S^1_{t_0}}{S^1_0}\right)$.

As in Bilodeau (1997), we consider this option in the simple situation where S^1 is a geometric Brownian motion with constant parameters μ, σ and where the short rate is a constant r. Under these assumptions, Bilodeau (1997) derives a value for the rollover option by a clever but slightly ad hoc argument comparing naive and optimal exercise behaviour for the option holder. She also notes that "determining how to hedge (i.e., replicate) the rollover option, if feasible, would be of value [...] it is not clear that hedging could effectively be done starting at time 0". We show here how Theorem 10 allows to systematically find the value for "optimal behaviour" and also the strategy one can use for (super)hedging.

In the notation of the preceding sections, the (one-dimensional) discounted asset price here is $X_t = S^1_t e^{-rt}$; the set $I\!\!M^e_\sigma$ is a singleton $\{P^*\}$ if the filtration $I\!\!F$ is generated by S^1 (or the Brownian motion driving S^1), and $X_t = S^1_0 \exp\left(\sigma W^*_t - \frac{1}{2}\sigma^2 t\right)$ is under P^* a geometric Brownian motion and a martingale. The Bermudan option is specified by $R = \{t_0, T\}$ and

$$U_{t_0} = \max\left(X_{t_0}, Ke^{-rt_0}\right) = X_{t_0} + \left(Ke^{-rt_0} - X_{t_0}\right)^+,$$

$$U_T = \max\left(X_T, \frac{X_{t_0}}{X_0} Ke^{-r(T-t_0)}\right) = X_T + \left(\frac{X_{t_0}}{X_0}Ke^{-r(T-t_0)} - X_T\right)^+; \tag{3.1}$$

note that we only need the values of U at the permitted exercise dates.

According to the results in section 2, we can price and superhedge the rollover option by successively working backward over the intervals $(t_0, T]$ and $[0, t_0]$. On $(t_0, T]$, we have to deal by (3.1) with the sum of one share and one put option. Since the discounted asset price for this valuation problem starts at X_{t_0} and the discounted strike price is $\frac{X_{t_0}}{X_0} K e^{-r(T-t_0)}$, we have in non-discounted terms a strike of $\frac{X_{t_0}}{X_0} K$ and a price at time t of the underlying of $X_{t_0} \exp\left(\sigma(W_t^* - W_{t_0}^*) + (r - \frac{1}{2}\sigma^2)(t - t_0)\right) = e^{-rt_0} S_t^1$, and so we find the hedging strategy

$$\vartheta_t^1 = \Phi\left(\frac{\log\left(\frac{S_t^1}{S_{t_0}^1}\frac{S_0^1}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \qquad \text{on } (t_0, T]$$

by standard calculations.

From Proposition 8, the discounted price of the option at time $t_0 \in R$ is given by $V_{t_0} = \max(U_{t_0}, E^*[U_T | \mathcal{F}_{t_0}])$. Now

$$\begin{aligned}
M_{t_0} &:= E^*[U_T | \mathcal{F}_{t_0}] \\
&= E^*\left[X_T + X_{t_0}\left(\frac{K}{X_0}e^{-r(T-t_0)} - \frac{X_T}{X_{t_0}}\right)^+ \middle| \mathcal{F}_{t_0}\right] \\
&= X_{t_0}\left(1 + E^*\left[\left(\frac{K}{X_0}e^{-r(T-t_0)} - \frac{X_T}{X_{t_0}}\right)^+\right]\right) \\
&=: X_{t_0}(1 + c^*)
\end{aligned}$$

because X is a P^*-martingale and $\frac{X_T}{X_{t_0}}$ is P^*-independent of $\mathcal{F}_{t_0}$. Hence we get

$$V_{t_0} = \max\left(X_{t_0}, Ke^{-rt_0}, (1+c^*)X_{t_0}\right) = (1+c^*)\left(X_{t_0} + \left(\frac{Ke^{-rt_0}}{1+c^*} - X_{t_0}\right)^+\right)$$

since $c^* \geq 0$, and this shows that on $[0, t_0]$, we have to hedge $1 + c^*$ times the sum of one share and one put with a discounted strike price of $\frac{K}{1+c^*}e^{-rt_0}$. Thus we obtain as above that the hedging strategy is

$$\vartheta_t^1 = (1+c^*)\Phi\left(\frac{\log\frac{(1+c^*)S_t^1}{K} + (r + \frac{1}{2}\sigma^2)(t_0 - t)}{\sigma\sqrt{t_0 - t}}\right) \qquad \text{on } [0, t_0].$$

The price of the rollover option at time 0 is then

$$V_0 = E^*[V_{t_0}] = (1+c^*)\Big(S_0^1\big(1 + \Phi(d_1^*)\big) - K\Phi(d_2^*)\Big) \tag{3.2}$$

with

$$d_{1,2}^* = \frac{\log\frac{(1+c^*)S_0^1}{K} + (r \pm \frac{1}{2}\sigma^2)t_0}{\sigma\sqrt{t_0}},$$

where

$$c^* = E^*\left[\left(\frac{K}{X_0}e^{-r(T-t_0)} - \frac{X_T}{X_{t_0}}\right)^+\right] = \frac{K}{S_0^1}\Phi(-d_2') - \Phi(-d_1') \tag{3.3}$$

with

$$d_{1,2}' = \frac{\log\frac{S_0^1}{K} + (r \pm \frac{1}{2}\sigma^2)(T-t_0)}{\sigma\sqrt{T-t_0}}$$

is the time 0 price of a put option with maturity $T - t_0$, discounted strike $\frac{K}{S_0^1}e^{-r(T-t_0)}$ and written on a stock with volatility σ and initial price 1. Shuffling terms around shows that (3.2) and (3.3) yield the same result as (4) in Bilodeau (1997) for a dividend rate δ of 0.

Finally, the discounted consumption process C from Theorem 10 is constant except for a single jump at time t_0 of

$$C_{t_0+} - C_{t_0} = V_{t_0} - V_{t_0+} = V_{t_0} - M_{t_0} = \left(Ke^{-rt_0} - (1+c^*)X_{t_0}\right)^+.$$

In undiscounted terms, this means that the option writer has at time t_0 a gain from superhedging of $\left(K - (1+c^*)S_{t_0}^1\right)^+$.

Remark. To put the rollover option into an insurance context, Bilodeau (1997) also considers the case where the payoff is to be made upon death of an insured person. But to price this product, she simply forms the average over the above prices by weighting them according to the death and survival probabilities. This is inconsistent with a superreplication approach and so we do not pursue this issue here.

4. Appendix: Proof of Lemma 4

Let (r_n) be the sequence from condition (URC) so that (r_n) strictly decreases to 0. If $\tau \in \mathcal{S}_{t,T}(R)$ so that τ has values in R, it is clear from (1.4) that $\varrho_n := (\tau + r_n) \wedge T$ satisfies the assertion. The difficult part is to construct (ϱ_n) when τ need not have values in R.

We start by discretising τ. Define $J_{k,N} := \big(t + (k-1)2^{-N}(T-t), t + k2^{-N}(T-t)\big]$, set $s_{0,N} := tI_{\{t\in R\}} + TI_{\{t\notin R\}}$ and choose for each $k \in \{1, 2, \ldots, 2^N\}$ an element $s_{k,N} \in R \cap J_{k,N}$ if the latter set is non-empty; otherwise take $s_{k,N} := T$. Since $T \in R$, all the $s_{k,N}$ are then in R. Now define

$$\varrho_n := \left(\left(s_{0,N_n}I_{\{\tau=t\}} + \sum_{k=1}^{2^{N_n}} s_{k,N_n}I_{\{\tau\in J_{k,N_n}\}}\right) + r_n\right) \wedge T$$

for N_n still to be chosen. Then ϱ_n has values in R due to (1.4), and ϱ_n is a stopping time for well-chosen N_n. In fact, we have for $u \in [t,T)$ that

$$\{\varrho_n \leq u\} = \{\tau = t, s_{0,N_n} + r_n \leq u, t \in R\} \cup \bigcup_{k=1}^{2^{N_n}} \{s_{k,N_n} + r_n \leq u, \tau \in J_{k,N_n}, R \cap J_{k,N_n} \neq \emptyset\}$$

and the first set on the right-hand side is in $\mathcal{F}_t \subseteq \mathcal{F}_u$. For the union over k, it is enough to show that $A_k := \{s_{k,N_n} + r_n \leq u, \tau \in J_{k,N_n}\}$ is in $\mathcal{F}_u$ for each k because $A_k = \emptyset$ if $R \cap J_{k,N_n} = \emptyset$. But due to the definition of J_{k,N_n}, A_k is in $\mathcal{F}_{t+k2^{-N_n}(T-t)}$ and therefore in $\mathcal{F}_u$ if $t + k2^{-N_n}(T-t) \leq u$, and since the choice of s_{k,N_n} also yields

$$u \geq s_{k,N_n} + r_n > t + (k-1)2^{-N_n}(T-t) + r_n,$$

it is clearly enough to choose N_n so large that $r_n \geq 2^{-N_n}(T-t)$. Then $\varrho_n \in \mathcal{S}_{t+r_n,T}(R)$.

In the above construction, ϱ_n will often take the value T so that the sequence (ϱ_n) is not necessarily decreasing. We still have to choose s_{k,N_n} in such a way that $\big(\varrho_n(\omega)\big)_{n\in\mathbb{N}}$ decreases to $\tau(\omega)$ for $\tau(\omega) \in R \setminus \{T\}$. So consider such an ω. If $\tau(\omega) = t$ (and thus $t \in R$), then $\varrho_n(\omega) = t + r_n > \tau(\omega)$ for large n and clearly $\big(\varrho_n(\omega)\big)$ decreases to $\tau(\omega)$. If $\tau(\omega) > t$, there is some $k(\omega) \in \{1, 2, \ldots, 2^{N_n}\}$ such that $\tau(\omega) \in R \cap J_{k(\omega),N_n}$, and $\varrho_n(\omega) = s_{k(\omega),N_n} + r_n$ for some $s_{k(\omega),N_n} \in R \cap J_{k(\omega),N_n} \neq \emptyset$. Moreover, the intervals $J_{k(\omega),N_n}$ shrink to $\tau(\omega)$ as $n \to \infty$, and so $\varrho_n(\omega) \to \tau(\omega)$. To ensure that this happens monotonically from the right of $\tau(\omega)$ whenever $\tau(\omega) \in R \setminus \{T\}$, we define

$$s^0_{k,N_n} := \sup\,(R \cap J_{k,N_n}) \tag{4.1}$$

if this set is non-empty and $s^0_{k,N_n} := T$ otherwise. Because s^0_{k,N_n} is not necessarily in R, we finally choose $s_{k,N_n} \in R \cap J_{k,N_n}$ such that

$$s_{k,N_n} \geq s^0_{k,N_n} - (r_n - r_{n+1}) \qquad \text{if } s^0_{k,N_n} \neq T; \tag{4.2}$$

this uses $r_n - r_{n+1} > 0$ as (r_n) is strictly decreasing. We claim that for this choice of s_{k,N_n},

$$\big(\varrho_n(\omega)\big) \subseteq \big(\tau(\omega), T\big] \text{ decreases to } \tau(\omega) \text{ if } \tau(\omega) \in R \setminus \{T\}, \tag{4.3}$$

and this will finish the proof.

To prove (4.3), we can assume that $\tau(\omega) \in J_{k(\omega),N_n}$ for some $k(\omega) \in \{1, 2, \ldots, 2^{N_n}\}$. Then (4.2) yields

$$\varrho_n(\omega) = s_{k(\omega),N_n} + r_n \geq s^0_{k(\omega),N_n} + r_{n+1}.$$

But we also have $\tau(\omega) \in J_{k'(\omega),N_{n+1}}$ for some $k'(\omega) \in \{1, 2, \ldots, 2^{N_{n+1}}\}$, and as $J_{k'(\omega),N_{n+1}} \subseteq J_{k(\omega),N_n}$, we obtain

$$s^0_{k(\omega),N_n} \geq s^0_{k'(\omega),N_{n+1}}$$

from (4.1). Hence we get from (4.2) and (4.1) that

$$\varrho_n(\omega) = s_{k(\omega),N_n} + r_n \geq s^0_{k(\omega),N_n} + r_{n+1} \geq s^0_{k'(\omega),N_{n+1}} + r_{n+1} \geq s_{k'(\omega),N_{n+1}} + r_{n+1} = \varrho_{n+1}(\omega).$$

Moreover, (4.2) and (4.1) also yield

$$\varrho_n(\omega) = s_{k(\omega),N_n} + r_n \geq s^0_{k(\omega),N_n} + r_{n+1} \geq \tau(\omega) + r_{n+1},$$

and this proves (4.3) and completes the proof.

q.e.d.

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