HEDGING BY SEQUENTIAL REGRESSION : AN INTRODUCTION TO THE MATHEMATICS OF OPTION TRADING

by H. Föllmer and M. Schweizer
ETH Zürich

0. Introduction

It is widely acknowledged that there has been a major breakthrough in the mathematical theory of option trading. This breakthrough, which is usually summarized by the Black-Scholes formula, has generated a lot of excitement and a certain mystique. On the mathematical side, it involves advanced probabilistic techniques from martingale theory and stochastic calculus which are accessible only to a small group of experts with a high degree of mathematical sophistication; hence the mystique. In its practical implications, it offers exciting prospects. Its promise is that, by a suitable choice of a trading strategy, the risk involved in handling an option can be eliminated completely.

Since October 1987, the mood has become more sober. But there are also mathematical reasons which suggest that expectations should be lowered. This will be the main point of the present expository account. We argue that, typically, the risk involved in handling an option has an irreducible intrinsic part. This intrinsic risk may be much smaller than the a priori risk, but in general one should not expect it to vanish completely. In this more sober perspective, the mathematical technique behind the Black-Scholes formula does not lose any of its importance. In fact, it should be seen as a sequential regression scheme whose purpose is to reduce the a priori risk to its intrinsic core.

We begin with a short introduction to the Black-Scholes formula in terms of currency options. Then we develop a general regression scheme in discrete time, first in an elementary two-period model, and then in a multiperiod model which involves martingale considerations and sets the stage for extensions to continuous time. Our method is based on the interpretation and extension of the Black-Scholes formula in terms of martingale theory. This was initiated by D. M. Kreps

and J. M. Harrison; see, for example, the excellent survey of Harrison and Pliska [6], [7]. The idea of embedding the Black-Scholes approach into a sequential regression scheme goes back to joint work of the first author with D. Sondermann. In continuous time and under martingale assumptions, this was worked out in [12] and [5]. The second author's thesis [13] deals with these problems in a general semimartingale model.

The present paper is a written version, with some extensions, of an expository talk given at the annual meeting of the Vereinigung Schweizerischer Versicherungs-mathematiker in September 1987. As in the talk, our purpose is to provide an elementary introduction to some key features of the mathematical theory of option pricing, with special emphasis on the use of linear regression.

1. From Huygens and Bernoulli to the formula of Black and Scholes

Consider a call option on US dollars against Swiss francs. This option gives the right to buy a specified amount of dollars, say $100, at a specified time T at a predetermined exchange rate K. If X_t denotes the exchange rate, i.e., the value in SFR of $100 at time $t \in [0, T]$, then the value V_T of the option at the terminal time T will be

$$H = (X_T - K)^+ = \begin{cases} X_T - K & \text{if } X_T \geq K \\ 0 & \text{otherwise.} \end{cases}$$

It is now natural to ask: What is a fair price for this option? In other words: What is the value V_0 of the option at the initial time 0 when the final outcome H is still uncertain?

Until 1973, when the fundamental papers of Black and Scholes [3] and Merton [10] appeared, there seemed to be an obvious answer. This answer could have been given already by Chr. Huygens [8] and J. Bernoulli [2]. To begin with, the exchange rate X_T and, consequently, the return H of the option should be viewed as random variables on some probability space $(\Omega, \mathcal{F}, P)$ which describes the possible time evolutions of the exchange rate and their respective probabilities.

Given such a stochastic model, the fair price of the option should be equal to the expected value $E[H]$ of the random variable H:

(1.1)
$$V_0 = E[H].$$

One could think of several modifications. For example, one could take into account an interest rate r and replace (1.1) by

(1.2)
$$V_0 = \frac{1}{1+r} \cdot E[H].$$

Also, the price could include a risk premium, and it would seem reasonable to compute such a premium in terms of the variance $\mathrm{Var}\,[H]$ of the random variable H, since this variance would appear to be a natural measure of the risk involved in handling the option. In essence, however, the problem would have seemed to be reduced to the choice of a suitable probability measure P.

Such a probability measure P on the space Ω of all continuous time evolutions $\omega : [0,1] \to \mathbf{R}$ was proposed in 1900 by L. Bachelier in his thesis "Théorie de la Spéculation" [1]; a rigorous construction was given by N. Wiener in 1923 [15]. Under this Wiener measure P, the stochastic process $X_t(\omega) := \omega(t) \quad (0 \le t \le T)$ behaves like a Brownian motion. This stochastic model has turned out to be of basic importance, not only for its fundamental connection to physics, but also on purely mathematical grounds. It is, for example, the natural reference model for functional versions of the central limit theorem.

In view of applications to the stock market one might want to use a modified version. For example, P. Samuelson [11] has proposed to model the stochastic process $X_t \; (0 \le t \le T)$ as the solution of a stochastic differential equation

(1.3)
$$dX_t = \sigma \cdot X_t \, dB_t + \mu \cdot X_t \, dt$$

where (B_t) is a Brownian motion. Here $\mu \cdot X_t \, dt$ is a trend forecast with drift parameter μ, and $\sigma \cdot X_t \, dB_t$ is a random fluctuation with volatility parameter σ. Suppose now that we accept (1.3) as a mathematical model for the stochastic time evolution of our exchange rate $X_t \; (0 \le t \le T)$. In this model, the random variable X_T has a log-normal distribution, and the Huygens-Bernoulli prescription (1.1) would lead to the formula

(1.4) $\quad V_0 = E[H]$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \left(X_0 \cdot \exp\left(\sigma \cdot \sqrt{T} \cdot u + \left(\mu - \frac{1}{2} \cdot \sigma^2 \right) \cdot T \right) - K \right)^+ \cdot \exp\left(-\frac{u^2}{2} \right) du.$$

But in the same model (1.3), the Black-Scholes formula gives a quite different answer. It tells you that the drift parameter μ is completely irrelevant, that you might as well replace P by the measure P^* corresponding to $\mu^* = 0$ which makes the exchange rate behave like a *fair game*, and that the fair price of the option should be computed as the expected value of H in this new model, even though the experts' forecast is given by P and not by P^*:

$$(1.5) \quad V_0 \;=\; E^*[H]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \left(X_0 \cdot \exp\left(\sigma \cdot \sqrt{T} \cdot u - \frac{1}{2} \cdot \sigma^2 \cdot T \right) - K \right)^{+} \cdot \exp\left(-\frac{u^2}{2} \right) du$$

$$= X_0 \cdot \Phi\left(\frac{1}{\sigma \cdot \sqrt{T}} \cdot \left(\log \frac{X_0}{K} + \frac{1}{2} \cdot \sigma^2 \cdot T \right) \right) - K \cdot \Phi\left(\frac{1}{\sigma \cdot \sqrt{T}} \cdot \left(\log \frac{X_0}{K} - \frac{1}{2} \cdot \sigma^2 \cdot T \right) \right),$$

where Φ denotes the cumulative distribution function of a standard normal distribution $N(0,1)$. Also, it is claimed that there is a trading strategy which requires the initial investment $V_0 = E^*[H]$ and then duplicates the contingent claim $H(\omega)$ without any additional cost, no matter which time evolution $\omega \in \Omega$ is realized by the random mechanism described by P. In particular, there is no reason to modify the fair price $V_0 = E^*[H]$ by a risk premium.

If we include interest rates in the model, then (1.5) is replaced by

$$(1.6) \qquad V_0 \;=\; E^*[(X_T - K \cdot e^{-\varrho \cdot T})^{+}]$$

$$= X_0 \cdot \Phi\left(\frac{1}{\sigma \cdot \sqrt{T}} \cdot \left(\log \frac{X_0}{K} + \left(\varrho + \frac{1}{2} \cdot \sigma^2 \right) \cdot T \right) \right)$$

$$- K \cdot e^{-\varrho \cdot T} \cdot \Phi\left(\frac{1}{\sigma \cdot \sqrt{T}} \cdot \left(\log \frac{X_0}{K} + \left(\varrho - \frac{1}{2} \cdot \sigma^2 \right) \cdot T \right) \right).$$

Here ϱ denotes the continuously compounded rate of return on a Swiss franc account.

A rigorous mathematical justification of this counter-intuitive prescription involves rather advanced tools, e.g., the Cameron-Martin-Maruyama-Girsanov formula and a deep representation theorem of K. Itô [9]. This explains some of the mystique around the Black-Scholes formula. But from an economic point of view, its crucial feature can already be explained in a very elementary setting which does not require any mathematical sophistication. To this end, we employ the well-known didactic device of using an example with binary structure.

Suppose that the current exchange rate is given by $X_0 = 135$. Consider a call option with a strike of $K = 145$ at time T. We assume the following binary scenario: the exchange rate at time T will be 175 with probability p or 80 with probability $1 - p$. Correspondingly, the return H of the option will be 30 with probability p and 0 with probability $1 - p$. Taking into account an interest rate r, the Huygens-Bernoulli prescription (1.2) would compute the fair price of the option as

$$(1.7) \qquad V_0 = E\left[\frac{H}{1+r}\right] = \frac{1}{1+r}\cdot p\cdot 30 \; ;$$

for $p = 0.5$ and $r = 0.05$ we would get $V_0 = 14.3$. The Black-Scholes prescription, however, would be the following. First replace p by p^* so that the exchange rate, properly discounted, behaves like a fair game:

$$X_0 = E^*\left[\frac{X_T}{1+r}\right]$$

or, more explicitly,

$$135 = \frac{1}{1+r}\cdot\left(p^*\cdot 175 + (1 - p^*)\cdot 80\right) .$$

Now compute the fair price as the expected value of the return H, properly discounted, in this new model:

$$V_0 = E^*\left[\frac{H}{1+r}\right] = \frac{1}{1+r}\cdot p^*\cdot 30 \; ;$$

for $r = 0.05$ we would get $p^* = 0.65$ and $V_0 = 18.6$.

At first sight, this change of the model seems completely arbitrary, just as in the more intricate model above. But in the present simple case we can give a direct economic justification. Suppose that at time 0 you sell the option. Then you can prepare for the resulting contingent claim at time T by using the following strategy:

Sell the option at the Black-Scholes price 18.6	+ 18.6
Buy \$31.6 at the present exchange rate of 1.35	− 42.7
Take a loan of SFR 24.1 with interest rate $r = 0.05$	+ 24.1
	0

Thus, the balance at time 0 is 0. At time T we have to distinguish two cases:

i)   The dollar has risen :   Option is exercized -30

                                        Sell dollars at 1.75 $+55.3$

                                          Pay back loan with interest -25.3

                                                                      $\overline{}$

                                                           0

ii)   The dollar has fallen : Option expires 0

                                        Sell dollars at 0.80 $+25.3$

                                          Pay back loan with interest -25.3

                                                                       $\overline{}$

                                                           0

This demonstrates that the Black-Scholes price 18.6 is just the right amount which is needed in order to hedge the option without any risk. Any option price different from the Black-Scholes price would enable either the option seller or the option buyer to make a sure profit without any risk : There would be an arbitrage opportunity.

In the next section, we explain how the correct hedging strategy can be found in a systematic way. In the preceding example it is clear that the model is too simplistic : there is no reason to restrict our attention to a binary scenario. Therefore, we are going to admit more general models. But this will force us to lower our expectations to a more realistic level. In fact, the situation becomes less pleasant as soon as we admit a third possibility for the value of X_T : It is no longer possible to reduce the risk to 0.

From the point of view of the continuous-time model (1.3), a binary situation as above should only serve as an infinitesimal building stone and not be taken seriously in itself. But also on the mathematically much more advanced level (1.3) there are good reasons to believe that the model, and in particular its promise of risk-free option trading, is too nice to be realistic.

2. Hedging in a two-period model: an exercise in linear regression

Let us first consider a simple two-period model where the exchange rates X_k at the initial time $k = 0$ and the terminal time $k = 1$ are random variables on some probability space $(\Omega, \mathcal{F}, P)$. At time 0 the exchange rate X_0 is known and can be treated as a constant, i.e., we assume that $P[X_0 = x_0] = 1$ for some $x_0 > 0$. The option, or rather the resulting contingent claim at time 1, is described by a random variable H defined on the same probability space. For a call option with a strike of K, we would have $H = (X_1 - K)^+$. To keep the exposition as simple as possible, we leave interest rates aside for the moment.

Let us now assume that we have sold the option. At time 1 we will have to pay the random amount $H(\omega)$. We should like to insure ourselves against this event; we want to *hedge* the option. To this end, we buy $100 \cdot \xi$ dollars and put aside η_0 Swiss francs. This initial portfolio at time 0 has the value

$$(2.1) \qquad\qquad V_0 = \xi \cdot X_0 + \eta_0 .$$

At the terminal time 1, we want a portfolio whose value is exactly equal to H. The value of the dollar account will be $\xi \cdot X_1$, and if we then adjust the Swiss franc account from η_0 to $\eta_1 = H - \xi \cdot X_1$, the value

$$(2.2) \qquad\qquad V_1 = \xi \cdot X_1 + \eta_1$$

of the resulting portfolio at time 1 will satisfy our condition

$$(2.3) \qquad\qquad V_1 = H .$$

For a given H, such a strategy will be determined by our initial choice of the constants ξ and V_0.

Let us examine the costs induced by such a strategy (ξ, V_0). If C_k denotes the cumulative cost up to time k, then we have

$$(2.4) \qquad\qquad C_0 = V_0 ,$$

and the additional cost due to our adjustment of the Swiss franc account at time 1 is given by

$$(2.5) \qquad\qquad \begin{aligned} C_1 - C_0 &= \eta_1 - \eta_0 \\ &= (V_1 - \xi \cdot X_1) - (V_0 - \xi \cdot X_0) \\ &= V_1 - V_0 - \xi \cdot \Delta X \end{aligned}$$

where we put $\Delta X := X_1 - X_0$. Let us now choose our trading strategy (ξ, V_0) in such a way that the remaining *risk* at time 0, measured by the expected quadratic cost

$$(2.6) \qquad \begin{aligned} R \; &:= \; E\left[\left(C_1 - C_0\right)^2\right] \\ &= \; E\left[\left(H - V_0 - \xi \cdot \Delta X\right)^2\right], \end{aligned}$$

is minimized. This is, of course, a well-known problem : we are simply looking for the best linear estimate of H based on ΔX. Thus, the optimal constants ξ and V_0 are given by

$$(2.7) \qquad \xi \; = \; \frac{\mathrm{Cov}\left(H, \Delta X\right)}{\mathrm{Var}\left[\Delta X\right]} \; = \; \frac{\mathrm{Cov}\left(H, X_1\right)}{\mathrm{Var}\left[X_1\right]}$$

and

$$(2.8) \qquad V_0 \; = \; E[H] - \xi \cdot E[\Delta X].$$

This optimal value V_0 of the initial portfolio may be regarded as a *fair price* of the option. In particular we obtain the condition

$$(2.9) \qquad C_0 \; = \; E[C_1].$$

This means that the optimal strategy is *mean-self-financing* : once we have determined the initial value $V_0 = C_0$, the additional cost $C_1 - C_0$ is a random variable with expectation $E[C_1 - C_0] = 0$. By this optimal trading strategy, the remaining risk is reduced to the minimal mean square prediction error

$$(2.10) \qquad \begin{aligned} R_{\min} \; &= \; \mathrm{Var}\left[H\right] - \xi^2 \cdot \mathrm{Var}\left[X_1\right] \\ &= \; \mathrm{Var}\left[H\right] \cdot \left(1 - \left(\rho(H, X_1)\right)^2\right) \end{aligned}$$

where ρ denotes the correlation coefficient. This value $R_{\min}$ may be viewed as the *intrinsic risk* of the option H. It is this intrinsic risk, and not the *a priori risk* measured by the variance $\mathrm{Var}\left[H\right]$ of H, on which any adjustment of the fair price V_0 by a suitable risk premium should be based.

In a model with an interest rate r on the Swiss franc account, we would have to replace (2.5) by

$$\begin{aligned} C_1 - C_0 \; &= \; \eta_1 - (1+r) \cdot \eta_0 \\ &= \; (1+r) \cdot \left(\frac{V_1}{1+r} - V_0 - \xi \cdot \left(\frac{X_1}{1+r} - X_0\right)\right). \end{aligned}$$

The optimal choice of ξ would still be (2.7); V_0, however, is then given by

$$(2.11) \qquad\qquad V_0 \;=\; E\left[\frac{H}{1+r}\right] - \xi\cdot E\left[\frac{X_1}{1+r} - X_0\right] \;.$$

Typically, the crucial quantities ξ, V_0 and $R_{\min}$ all depend on the underlying probability measure P. The intrinsic risk $R_{\min}$, although strictly less than the a priori risk $\mathrm{Var}\,[H]$, is still strictly positive and cannot be neglected. Only in the following binary scenario can we eliminate the risk completely, in analogy to what is promised by the Black-Scholes formula.

Suppose that only two cases appear with positive probability: Either the exchange rate goes up to some level x^+ and the contingent claim H assumes a corresponding value h^+, or the exchange rate goes down to some level x^- and H assumes the value h^-. Thus,

$$(2.12) \qquad\qquad\qquad P\,[X_1 = x^+, H = h^+] \;=\; p$$
$$P\,[X_1 = x^-, H = h^-] \;=\; 1 - p$$

with $x^- < (1+r)\cdot x_0 < x^+$ and some $p \in (0,1)$. This allows us to determine two constants ξ and V_0 such that

$$(2.13) \qquad\qquad P\left[H = (1+r)\cdot V_0 + \xi\cdot\big(X_1 - (1+r)\cdot X_0\big)\right] = 1\;.$$

In fact, (2.12) reduces (2.13) to two linear equations for the two unknowns ξ and V_0, and the solution is given by

$$(2.14) \qquad \xi \;=\; \frac{h^+ - h^-}{x^+ - x^-}$$

$$V_0 \;=\; \frac{1}{1+r}\cdot\left(h^+\cdot\frac{(1+r)\cdot x_0 - x^-}{x^+ - x^-} + h^-\cdot\frac{x^+ - (1+r)\cdot x_0}{x^+ - x^-}\right)\;.$$

Thus, there is a linear dependence between the two random variables H and $X_1 - (1+r)\cdot X_0$, and the linear regression becomes perfect: $R_{\min} = 0$. In this case, there is no need to adjust the fair price V_0 by a risk premium because the risk has completely disappeared. In particular, we have

$$(2.15) \qquad\qquad\qquad C_1 - C_0 \;=\; 0\;,$$

i.e., the strategy becomes *self-financing*. The random variable η_1 is now known in advance and coincides with the constant $(1+r)\cdot\eta_0$. For the specific values in our introductory example, we recover the strategy which was described in section 1.

Note that the optimal values ξ and V_0 in (2.14) do not involve the probability parameter p; they are the same for any measure $\widetilde{P}$ which preserves the binary structure (2.12) with some parameter $\widetilde{p} \in (0,1)$. In particular, we are free to switch from P to the measure P^* with

$$p^* = \frac{(1+r)\cdot x_0 - x^-}{x^+ - x^-}$$

so that

(2.16) $$x_0 = E^*\left[\frac{X_1}{1+r}\right].$$

Thus, the discounted exchange rate behaves like a *fair game* under P^*. In this new model, the fair price V_0 can now be computed directly, without going through the exercise of computing ξ and V_0 from two linear equations. In fact, (2.15) implies

$$E^*\left[V_1 - \xi\cdot\left(X_1 - (1+r)\cdot X_0\right)\right] = (1+r)\cdot V_0 ,$$

and by (2.3) and (2.16) we obtain

(2.17) $$V_0 = E^*\left[\frac{H}{1+r}\right],$$

as prescribed by Chr. Huygens and J. Bernoulli.

The preceding discussion of the binary scenario, viewed as a special case of the general two-period model, is just an elementary remark on the linear regression problem. But (2.17) is the exact analogue to the Black-Scholes formula. In both cases, the model is *complete* in the sense that any contingent claim H can be generated by a suitable strategy as in (2.13). This allows us to reduce the risk to 0, and to compute the value V_0 by an appropriate change of measure.

3. Hedging by sequential regression

Let us now consider a multiperiod model where the evolution of the exchange rate is given by a stochastic process X_k $(k = 0, \ldots, T)$ on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_k$ denote the σ-field of events which are observable up to and including time k. We assume that X_k is $\mathcal{F}_k$-measurable and square-integrable. In order to avoid complicated notations, we work again without interest rates. This is no restriction since we can always start by discounting the original price processes.

An *option* is described by a square-integrable random variable $H \in L^2(P)$; for example, a call option with strike K would correspond to $H = (X_T - K)^+$. A *trading strategy* is given by two stochastic processes ξ_k $(k = 1, \ldots, T)$ and η_k $(k = 0, \ldots, T)$. ξ_k is the amount of US dollars held in period k and has to be fixed at the beginning of that period, i.e., we assume that

(3.1) ξ_k is $\mathcal{F}_{k-1}$-measurable $(k = 1, \ldots, T)$.

The amount η_k of Swiss francs in period k can be chosen at the end of this period, i.e., we assume that

(3.2) η_k is $\mathcal{F}_k$-measurable $(k = 0, \ldots, T)$.

For such a trading strategy, the value of the portfolio at time k is given by

(3.3) $V_k = \xi_k \cdot X_k + \eta_k$ $(k = 1, \ldots, T)$,

and $V_0 = \eta_0$. We admit only strategies such that each V_k is square-integrable and such that the contingent claim H is produced in the end, i.e., we require

(3.4) $V_T = H$.

Due to the flexibility allowed by (3.2), this can always be achieved by a suitable choice of η_T. The cumulative cost at time k is given by

(3.5) $C_k = V_k - \sum_{j=1}^{k} \xi_j \cdot \Delta X_j$

with $\Delta X_j := X_j - X_{j-1}$, and $C_0 = V_0 = \eta_0$. Going backwards from time T, we can now apply the argument of the preceding section step by step to determine our trading strategy recursively. Suppose that the random variables $\xi_{k+2}, \ldots, \xi_T$ and

$\eta_{k+1}, \ldots, \eta_T$ (or, equivalently, $V_{k+1}, \ldots, V_T$) have already been prescribed. At time k, we want to choose first ξ_{k+1} and then V_k (respectively $\eta_k = V_k - \xi_k \cdot X_k$) such that the conditional risk

$$(3.6) \qquad R_k := E\left[\left(C_{k+1} - C_k\right)^2 \Big| \mathcal{F}_k \right]$$
$$= E\left[\left(V_{k+1} - V_k - \xi_{k+1} \cdot \Delta X_{k+1}\right)^2 \Big| \mathcal{F}_k \right]$$

is minimized. Note that the expectation in (2.6) has now been replaced by a conditional expectation. In analogy to (2.9), this implies

$$(3.7) \qquad C_k = E\left[C_{k+1} \big| \mathcal{F}_k \right],$$

i.e., the cost process C_k $(k = 0, \ldots, T)$ is a *martingale*. Moreover, the conditional versions of (2.7) and (2.8) yield the recursion formulas

$$(3.8) \qquad \xi_k = \frac{\mathrm{Cov}_{\mathcal{F}_{k-1}}\left(H - \sum\limits_{j=k+1}^{T} \xi_j \cdot \Delta X_j, \; \Delta X_k \right)}{\mathrm{Var}_{\mathcal{F}_{k-1}}\left[\Delta X_k \right]}$$

and

$$(3.9) \qquad \eta_k = E\left[H - \sum_{j=k+1}^{T} \xi_j \cdot \Delta X_j \;\Bigg|\; \mathcal{F}_k \right] - \xi_k \cdot X_k \, .$$

In particular, this recursion leads to the fair price $V_0 = \eta_0$ of our option.

The structure of the optimal strategy becomes much more transparent if P is a *martingale measure*, i.e., if X_k $(k = 0, \ldots, T)$ is a martingale under P:

$$(3.10) \qquad X_k = E\left[X_{k+1} \big| \mathcal{F}_k \right] \qquad (k = 0, \ldots, T-1) \, .$$

By (3.7) and (3.5), it follows that the value process V_k $(k = 0, \ldots, T)$ is also a martingale, hence of the form

$$(3.11) \qquad V_k = E\left[H \big| \mathcal{F}_k \right]$$

due to (3.4). We can now give a more direct construction of the optimal strategy. To begin with, (3.11) determines $\eta_k = V_k - \xi_k \cdot X_k$ as soon as we know ξ_k. In order to compute ξ_k, we use the fact that H can be written as

$$(3.12) \qquad H = V_0 + \sum_{j=1}^{T} \xi_j^H \cdot \Delta X_j + L_T^H$$

where L_k^H $(k = 0, \ldots, T)$ is a martingale which is orthogonal to X_k $(k = 0, \ldots, T)$ in the sense that

$$(3.13) \qquad E\left[\Delta L_k^H \cdot \Delta X_k \,\middle|\, \mathcal{F}_{k-1} \right] \;=\; 0 \,.$$

This allows us to conclude that

$$(3.14) \qquad \xi_k \;=\; \xi_k^H \qquad (k = 1, \ldots, T)$$

is the optimal hedging strategy. In fact, (3.11) and (3.12) imply

$$(3.15) \qquad V_k \;=\; V_0 \,+\, \sum_{j=1}^{k} \xi_j^H \cdot \Delta X_j \,+\, L_k^H \,,$$

hence

$$(3.16) \qquad R_{k-1} \;=\; E\left[\left(\Delta L_k^H \right)^2 \,\middle|\, \mathcal{F}_{k-1} \right] + \left(\xi_k - \xi_k^H \right)^2 \cdot E\left[\left(\Delta X_k \right)^2 \,\middle|\, \mathcal{F}_{k-1} \right]$$

due to (3.13), and this is minimized by the choice of (3.14).

Now consider the very special case where H can actually be generated by X_k $(k = 0, \ldots, T)$ in the sense that

$$(3.17) \qquad H \;=\; V_0 \,+\, \sum_{j=1}^{T} \xi_j^H \cdot \Delta X_j \,,$$

i.e., where $L_T^H = 0$ in (3.12). Then (3.16) implies $R_k = 0$ $(k = 0, \ldots, T)$, hence

$$C_{k+1} \;=\; C_k \;=\; C_0$$

due to (3.6). Thus, the optimal strategy is *self-financing*. By (3.3) and (3.15),

$$\begin{aligned}
\eta_k \;&=\; V_k - \xi_k \cdot X_k \\
&=\; V_{k-1} + \xi_k \cdot (X_k - X_{k-1}) - \xi_k \cdot X_k \\
&=\; V_{k-1} - \xi_k \cdot X_{k-1} \,.
\end{aligned}$$

In this special case, the process η_k $(k = 0, \ldots, T)$ is *predictable*, i.e.

$$(3.18) \quad \eta_k \text{ is } \mathcal{F}_{k-1}\text{-measurable} \qquad (k = 1, \ldots, T) \,.$$

This means that all the ingredients of the trading strategy can already be fixed at the beginning of each period. Note also that the strategy remains the same if we change the measure P to any measure $\widetilde{P}$ which is equivalent to P because this preserves the structure (3.17). Thus, the preceding discussion is valid for any model P which can be obtained by an equivalent change of measure from a martingale model P^* satisfying (3.17). In such a case, the explicit use of P^* permits a direct computation of the fair option price. In fact, (3.17) implies

$$V_0 \; = \; E^*[H] \, ,$$

in analogy to (2.17) and (1.5).

4. Option trading and stochastic calculus

Let us briefly comment on the extension of our previous discussion to a continuous-time setting where X_t $(0 \le t \le T)$ is a square-integrable semimartingale on some probability space $(\Omega, \mathcal{F}, P)$ with a right-continuous filtration $\mathcal{F}_t$ $(0 \le t \le T)$. We can use the same conceptual approach to the pricing and hedging of options as in the discrete-time case, but the technicalities are much more involved.

Here again, the situation is most transparent in the *complete* case where, in analogy to (3.17), our square-integrable contingent claim H can be represented as

$$(4.1) \qquad\qquad H \; = \; V_0 + \int_0^T \xi_u^H \, dX_u$$

P-almost surely, i.e., as a stochastic integral of some predictable process ξ_t^H $(0 \le t \le T)$ with respect to the basic semimartingale X_t $(0 \le t \le T)$. In this case, we take $\xi_t = \xi_t^H$ and determine $\eta_t = V_t - \xi_t \cdot X_t$ in such a way that

$$V_t \; = \; V_0 + \int_0^t \xi_u \, dX_u \qquad (0 \le t \le T) \, .$$

Then we have $V_T = H$, i.e., the strategy produces the contingent claim H at the initial cost $C_0 = V_0$, and the resulting cost process

$$C_t = V_t - \int_0^t \xi_u \, dX_u$$

satisfies

$$C_t = C_0 = V_0 \qquad (0 \le t \le T) \, .$$

Thus, the strategy is self-financing, and V_0 is the fair price of H. Now assume that there exists an equivalent martingale measure P^* which preserves the structure (4.1). Let us switch from P to P^*; this could be done more explicitly by means of a Girsanov transformation. In this new model, the price can now be computed directly as the expected value

(4.2) $$V_0 = E^*[H]$$

of our contingent claim H. In the special situation of (1.3), any square-integrable contingent claim admits a representation (4.1); this is a variant of a fundamental representation theorem of K. Itô [9]. Thus, the problem is completely solved, and for an option of the form $H = (X_T - K)^+$, the pricing formula (4.2) reduces to the Black-Scholes formula (1.5). For a detailed introduction to the general complete case, we refer to Harrison and Pliska [6], [7].

In the *incomplete* case, but under the assumption that P is a *martingale measure*, the optimal trading strategy can be determined as in (3.14). Here the risk can be reduced to the intrinsic risk, but it can no longer be eliminated completely. This case has been worked out in Föllmer/Sondermann [5] and Schweizer [12].

In a general incomplete model where X_t $(0 \le t \le T)$ is a semimartingale but not a martingale, the situation becomes very delicate. Here it is no longer possible to compute the optimal strategy by a simple backwards recursion as in the discrete-time case above. But it is shown in the second author's thesis [13] that one can derive an *optimality equation* for the strategy which is based on the Doob-Meyer decomposition of the semimartingale X into a martingale and a predictable process of finite variation. This nonlinear stochastic optimality equation can be solved by means of a suitable Girsanov transformation. We refer to [13] for further details.

5. The valuation of a stop-loss contract

D. Sondermann [14] has pointed out that the valuation of a stop-loss contract can be viewed as an exact analogue to the pricing of an option. Consider a stochastic cumulative claim process S_t $(0 \le t \le T)$, a deterministic cumulative premium process

$$p(t) = \int_0^t \dot{p}(u)\,du \qquad (0 \le t \le T)$$

with $\dot{p} > 0$, and the associated stochastic process

$$(5.1) \qquad\qquad X_t = S_t - p(t) \qquad (0 \le t \le T)\,.$$

We assume that (S_t) is a Poisson process with fixed jump height and deterministic intensity $\lambda(t)$. This model is *complete* in the sense that any square-integrable H can be represented in the form

$$(5.2) \qquad\qquad H = V_0 + \int_0^T \xi_u^H\,dX_u\,.$$

In particular, this is true for the random variable $H = (S_T - c)^+$ associated to a stop-loss contract. Now we consider an insurance strategy given by ξ_t $(0 \le t \le T)$ and η_t $(0 \le t \le T)$. Here, η_t is the cash reserve held at time t, and ξ_t represents a proportional reinsurance contract inducing the capital flow

$$\int_0^t \xi_u\,(dS_u - \dot{p}(u)\,du) = \int_0^t \xi_u\,dX_u\,.$$

We take $\xi_t = \xi_t^H$, and we determine $\eta_t = V_t - \xi_t \cdot X_t$ by setting

$$(5.3) \qquad\qquad V_t = V_0 + \int_0^t \xi_u^H\,dX_u\,.$$

As in the previous section, we conclude that this strategy is self-financing and produces the random payment H at the terminal time T. The required initial cost V_0 can be computed directly as the expected value

$$V_0 = E^*[H] = E^*\left[(S_T - c)^+\right]$$

if we switch from P to the new model P^* where the process X_t $(0 \le t \le T)$ is a martingale; this is achieved by choosing the rate $\lambda^*(t) = \dot{p}(t)$. Note that, in complete analogy to our discussion of currency options, the fair premium V_0 does not depend on the a priori rate $\lambda(t)$. For a more detailed account see [14].

References

[1] L. Bachelier, "Théorie de la Spéculation", *Ann. Sci. Ec. Norm. Sup. III–17 (1900)*, 21–86

[2] J. Bernoulli, "Ars coniectandi" *(1713)*

[3] F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy 81 (1973)*, 637–659

[4] J. C. Cox and S. A. Ross, "The Valuation of Options for Alternative Stochastic Processes", *Journal of Financial Economics 3 (1976)*, 145–166

[5] H. Föllmer and D. Sondermann, "Hedging of Non-Redundant Contingent Claims", *in: W. Hildenbrand and A. Mas-Colell (eds.), Contributions to Mathematical Economics (1986)*, 205–223

[6] J. M. Harrison and S. R. Pliska, "Martingales and Stochastic Integrals in the Theory of Continuous Trading", *Stochastic Processes and their Applications 11 (1981)*, 215–260

[7] J. M. Harrison and S. R. Pliska, "A Stochastic Calculus Model of Continuous Trading : Complete Markets", *Stochastic Processes and their Applications 15 (1983)*, 313–316

[8] Chr. Huygens, "De ratiociniis in ludo aleae" *(1657)*

[9] K. Itô, "Multiple Wiener Integral", *Journal of the Mathematical Society of Japan 3 (1951)*, 157–169

[10] R. C. Merton, "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science 4 (1973)*, 141–183

[11] P. A. Samuelson, "Rational Theory of Warrant Pricing", *in: P. H. Cootner (ed.), "The Random Character of Stock Market Prices", MIT Press, Cambridge, Massachusetts (1964)*, 506–525

[12] M. Schweizer, "Varianten der Black-Scholes-Formel", *Diplomarbeit ETHZ, Zürich (1984)*

[13] M. Schweizer, "Hedging of Options in a General Semimartingale Model", *Diss. ETHZ no. 8615, Zürich (1988)*

[14] D. Sondermann, "Reinsurance in Arbitrage-Free Markets", *discussion paper no. B–82, University of Bonn (1988)*

[15] N. Wiener, "Differential-Space", *Journal of Mathematics and Physics 2 (1923)*, 131–174 ; *reprinted in: N. Wiener, "Collected Works", Volume I, MIT Press (1976)*

Hans Föllmer
Martin Schweizer
Institut für Angewandte Mathematik
Universität Bonn
Wegelerstr. 6
D – 5300 Bonn