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TERM STRUCTURES OF IMPLIED VOLATILITIES: ABSENCE OF ARBITRAGE AND EXISTENCE RESULTS

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This paper studies modeling and existence issues for market models of stochastic implied volatility in a continuous-time framework with one stock, one bank account, and a family of European options for all maturities with a fixed payoff function h. We first characterize absence of arbitrage in terms of drift conditions for the forward implied volatilities corresponding to a general convex h. For the resulting infinite system of SDEs for the stock and all the forward implied volatilities, we then study the question of solvability and provide sufficient conditions for existence and uniqueness of a solution. We do this for two examples of h, namely, calls with a fixed strike and a fixed power of the terminal stock price, and we give explicit examples of volatility coefficients satisfying the required assumptions.

KEY WORDS: implied volatility, market model, drift restrictions, infinite SDE system

1. INTRODUCTION

Today, European vanilla options are traded liquidly on the market, and it is well known that their prices are not consistent with the Black–Scholes model. Call option prices are usually quoted by their Black–Scholes implied volatility, i.e., the unique volatility parameter value for which the Black–Scholes formula yields the observed option price. Under the Black–Scholes model, the surface given by European call implied volatilities as a function of strike and maturity would be a plane at the height of the constant volatility parameter. But observed market prices yield a non-flat surface which in addition varies over time. A vast number of alternative stock price models have been developed to account for this; examples are stochastic volatility models or Lévy models. They can mimic most features of the implied volatility surface at a given instant, but fitting them to observed market prices is often not easy.

The fact that many European vanilla prices are given by the market has inspired the alternative approach of *market models for implied volatility*. In these, standard options are treated as basic securities in addition to the underlying stock and bank account. In an Itô process framework over a Brownian filtration, the stock and the implied volatility processes have the form

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$$\frac{dS_t}{S_t} = \mu_t\,dt + \sigma_t\,dW_t, \tag{1.1}$$

$$d\hat{\sigma}_t(T, K) = u_t(T, K)\,dt + v_t(T, K)\,dW_t, \tag{1.2}$$

where W is a multi-dimensional Brownian motion under the real-world measure P and $\hat{\sigma}_t(T, K)$ denotes the implied volatility at time t of a European call on S with strike K and maturity T. Many empirical papers study the statistical behaviour of the surface $\hat{\sigma}(T, K)$; see for example, Cont and Fonseca (2002) for a list of references in this area. Our treatment for analyzing (1.1), (1.2) is based on a more theoretical point of view. It was first pointed out by Lyons (1997) and Schönbucher (1999) that for such a model to be arbitrage-free, the coefficients $\mu_t, \sigma_t, u_t(T, K), v_t(T, K)$ cannot be arbitrarily specified, but must be linked by certain relations. If one takes these *drift restrictions* into account, the question whether the system (1.1), (1.2) admits a solution turns out to be nontrivial. One example of coefficients for one fixed pair (T, K) such that a solution exists is given in Babbar (2001). Schönbucher (1999) also noted that if the processes $v_t(T, K)$ are specified for one fixed K and *all* $T > 0$, then the volatility σ_t is uniquely determined by the $v_t(T, K)$ if there is no arbitrage in the model. The same result was obtained in similar contexts in Brace et al. (2001) and Ledoit et al. (2002). Again, the question arises whether the corresponding, now infinite, system (1.1), (1.2) admits a solution. This is also an important practical issue because it is not possible to specify a concrete model without an existence result. As far as we know, no examples of coefficients which guarantee solvability have been given yet.

A similar phenomenon arises in the theory of interest rates in the Heath–Jarrow–Morton (HJM) frame work (1992). Since one can observe bond prices in the market, these are taken as underlying price processes instead of being derived in a short rate model. Heath et al. (1992) showed that if one starts with a model

$$df_t(T) = \alpha_t(T)\,dt + \sigma_t(T)\,dW_t$$

of the forward rates $f_t(T)$ for all maturities $T > 0$ under the real-world measure, then the drifts $\alpha_t(T)$ are uniquely determined by the processes $\sigma_t(T)$ and one single market price of risk process. These so-called HJM drift restrictions correspond to the above relations between the coefficients of the stock and of the implied volatilities. Moreover, Heath et al. (1992) gave some sufficient conditions on the drift coefficient $\sigma_t(T)$ that guarantee the existence of an arbitrage-free system of forward rate processes $f_t(T)$.

This paper provides a framework for arbitrage-free modeling of a continuous-time market consisting of one stock, one bank account, and a family of European options for all maturities $T > 0$ with a fixed convex payoff function. Our two main contributions are to precisely characterize absence of arbitrage in terms of drift conditions, and above all to address the issue of solvability for the resulting infinite SDE system.

Section 2 introduces, as counterpart to the forward rates, the *forward implied volatilities*

$$X(t, T) = \frac{\partial}{\partial T}\big((T - t)\hat{\sigma}_t^2(T)\big)$$

for a European option with maturity T and convex payoff function h. We consider a modeling framework

$$\frac{dS_t}{S_t} = \mu_t\,dt + \sigma_t\,dW_t, \tag{1.3}$$

$$dX(t, T) = \alpha(t, T)\,dt + v(t, T)\,dW_t, \tag{1.4}$$

and explicitly calculate the drift restrictions and stock volatility specification which are necessary and sufficient for the existence of a local martingale measure for the stock and all options (for all maturities T). Treating a general convex h is rewarded by additional insights into the dependence of the SDE coefficients on the option Greeks. We then specialize to two examples of h, one being a fixed power of the final stock price, the other being the call option with a fixed strike. The latter recovers (in more explicit form) the results of Schönbucher (1999).

In Section 3, we study the question of solvability for the SDEs arising from the preceding drift restrictions and stock volatility specification. This is an infinite SDE system describing the stock S and the family of forward implied volatilities $X(\cdot, T)$ for all $T > 0$. For the example payoff functions considered in Section 2, we provide fairly general classes of coefficients $v(t, T)$ for which there exists a unique solution to this system. In order to prove existence and uniqueness, we apply the results of Wissel (2007). Finally, Section 4 comments on the relation to standard stochastic volatility models and concludes.

When we had already submitted this paper, we learned that similar results have independently been obtained by Jacod and Protter (2006). A more detailed comparison is given in Section 2.2 and at the beginning of Section 3.

2. DRIFT RESTRICTIONS IN IMPLIED VOLATILITY MODELS

In this section, we consider continuous-time models of the form (1.3), (1.4) for a term structure of options. We generalize the results of Schönbucher (1999) from call options to general convex payoff functions and show in Theorem 2.4 that (under some integrability conditions on the coefficients) a model in this class admits an equivalent local martingale measure if and only if the coefficients satisfy certain relations, commonly referred to as *drift restrictions*. One a priori motivation to consider convex payoffs is that under some regularity conditions, a European type contract with such a payoff function can be viewed as an infinite but static portfolio of cash, stock, and call and put options. A posteriori, it additionally turns out that considering general payoff functions allows us to see directly how the drift restrictions and stock volatility specification depend on the Greeks of the options.

We start in Section 2.1 from a collection of European options with a fixed convex payoff function h for all maturities $T > 0$, and introduce implied and forward implied volatilities of their prices. In Section 2.2, we derive the stock volatility specification and the drift restrictions for the forward implied volatilities of such options. Section 2.3 applies these results to several examples of convex payoffs, one of which is the call option with a fixed strike (Corollary 2.9). This yields again the volatility specification and the drift restrictions from Schönbucher (1999) in a slightly more explicit form. Moreover, we show that the stock volatility specification for smooth payoffs has a much simpler form than in the general case. Throughout the paper, interest rates are zero; hence all price processes below denote discounted prices.

REMARK 2.1. One limitation of our setup is that we consider only one fixed payoff function (i.e., one single strike in the call case) for all maturities. We are currently working on an extension to calls with several strikes and several maturities. However, this is substantially more difficult and requires new ideas already at the modeling level, going well beyond the scope of the present paper.

2.1. Implied Volatilities of Options with a Convex Payoff Function

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $h : (0, \infty) \to \mathbb{R}$ a convex function with

$$E[h(L)] < \infty \text{ for each log-normal random variable } L. \tag{2.1}$$

Then it is well known that in the Black–Scholes model, the price at time t of a European option with payoff h and maturity T can be written as $c(S_t, \Upsilon_t)$ for some function $c : (0, \infty) \times [0, \infty) \to \mathbb{R}$, where S_t denotes the current stock price and $\Upsilon_t = (T - t)\sigma^2$. We list some properties of c which we use in this work; their proofs follow from easy calculations within the Black–Scholes model and are left to the reader.

- c is the unique (classical) solution of the boundary value problem

$$\left.\begin{aligned} \frac{1}{2}S^2 c_{SS}(S, \Upsilon) &= c_\Upsilon(S, \Upsilon) \quad (S, \Upsilon > 0), \\ c(S, 0) &= h(S) \quad (S > 0). \end{aligned}\right\} \tag{2.2}$$

- Let Z be a standard normal variable under P. Then we have

$$c(S, \Upsilon) = E\left[h\left(S\exp\left(\sqrt{\Upsilon}Z - \frac{1}{2}\Upsilon\right)\right)\right]. \tag{2.3}$$

- Let c^K denote the Black–Scholes pricing function for a call option with strike K (see Section 2.3.3 below). Then c and its partials can be expressed in terms of c^K as

$$\left.\begin{aligned} c(S, \Upsilon) &= \int_0^\infty c^K_\Upsilon(S, \Upsilon)\frac{2}{K^2}h(K)\,dK, \\ c_\Upsilon(S, \Upsilon) &= \int_0^\infty c^K_{\Upsilon\Upsilon}(S, \Upsilon)\frac{2}{K^2}h(K)\,dK, \\ c_{S\Upsilon}(S, \Upsilon) &= \int_0^\infty c^K_{S\Upsilon\Upsilon}(S, \Upsilon)\frac{2}{K^2}h(K)\,dK, \\ c_{\Upsilon\Upsilon}(S, \Upsilon) &= \int_0^\infty c^K_{\Upsilon\Upsilon\Upsilon}(S, \Upsilon)\frac{2}{K^2}h(K)\,dK. \end{aligned}\right\} \tag{2.4}$$

- Let h' denote the right derivative of the convex function h. The function c is increasing in Υ, and

$$\lim_{\Upsilon\to\infty} c(S, \Upsilon) = h(0+) + Sh'(\infty). \tag{2.5}$$

In Section 2.3, we consider example payoffs with closed form expressions for c, including c^K obtained from (2.3) for $h(S) = (S - K)^+$ with some $K > 0$. To establish the existence of the implied volatility and the positivity of the forward implied volatility for general h, we use the following well-known result.

PROPOSITION 2.1. *(a) Let $(S_t)_{t\geq 0}$ be a process modeling a stock price and let $(C_t^T)_{0\leq t\leq T}$ for each $T > 0$ be a process modeling the price of a European option paying $h(S_T)$ at time T, where $h : (0, \infty) \to \mathbb{R}$ is convex. If there is no elementary arbitrage opportunity in this market, we must have*

$$C_t^{T_1} \leq C_t^{T_2} \quad \forall t \leq T_1 < T_2, \tag{2.6}$$

$$h(S_t) \leq C_t^T \leq h(0+) + S_t h'(\infty) \quad \forall t \leq T. \tag{2.7}$$

(b) If in addition h is non-affine and $P[S_{t_2}/S_{t_1} > a] > 0$, $P[S_{t_2}/S_{t_1} < a] > 0$ *for all* $t_2 > t_1$ *and* $a > 0$, *the above inequalities are strict.*

Now let h be non-affine and denote by S_t $(t \geq 0)$ the price at time t of a stock, and by $C_t^T (0 \leq t \leq T)$ the prices at time t of European options with maturities T and payoff $h(S_T)$. If these prices do not admit elementary arbitrage, Proposition 2.1 yields $c(S_t, 0) \leq C_t^T \leq \lim_{\Upsilon \to \infty} c(S_t, \Upsilon)$. Since $c(S, \Upsilon)$ is strictly increasing in Υ, we can introduce

DEFINITION 2.1. The *implied volatility* of the price C_t^T is the unique parameter $\hat{\sigma}(t, T) \geq 0$ satisfying

$$c\big(S_t, (T-t)\hat{\sigma}^2(t, T)\big) = C_t^T.$$

If $T \mapsto C_t^T$ is differentiable in T, we define the *forward implied volatility* for the maturity T by

$$X(t, T) := \frac{\partial}{\partial T}((T-t)\hat{\sigma}^2(t, T)).$$

REMARKS 2.1.

(1) To be precise, $X(t, T)$ is the *square* of the forward implied volatility; see Section 4.2 and (4.17) in Schönbucher (1999). But since we use here $X(t, T)$ as basic quantity throughout, we refer to $X(t, T)$ as forward implied volatility.
(2) The forward implied volatility is formally analogous to the forward rate in interest rate modeling. However, Proposition 2.1 implies that $X(t, T) \geq 0$ in an arbitrage-free framework. A crucial point in the construction of arbitrage-free models of forward implied volatilities is therefore to ensure positivity of the processes $X(\cdot, T)$. This stands in contrast to interest rate modeling where positivity of the forward rates is a desirable feature, but not necessary for absence of arbitrage.
(3) If h is concave, then $-h$ is convex. Hence if $-h$ satisfies (2.1), implied and forward implied volatilities can be defined as in Definition 2.1.

Proof of Proposition 2.1.

(a) For (2.6), suppose that (on $A \in \mathcal{F}_t$ with $P[A] > 0$, to be accurate) we have $p := C_t^{T_1} - C_t^{T_2} > 0$ for some $t \leq T_1 < T_2$. Then we construct an elementary arbitrage opportunity as follows. On A at time t, set up at zero cost the portfolio consisting of $+1$ unit of the option with maturity T_2, -1 unit of the option with maturity T_1, and p units of cash. At time T_1, its value is given by $V_{T_1} = p + C_{T_1}^{T_2} - h(S_{T_1})$. Since h is convex, there exist $\mathcal{F}_{T_1}$-measurable a, b such that

$$h(S_{T_1}) = aS_{T_1} + b,$$

$$h(x) \geq ax + b \quad \forall x \in \mathbb{R}.$$

Hence we can rearrange the portfolio at time T_1 at zero cost by exchanging the option with maturity T_1 for a units of stock and b units of cash. At time T_2, the value of this new portfolio is given by $V_{T_2} = p + h(S_{T_2}) - (aS_{T_2} + b) \geq p$ a.s. This gives our arbitrage.

For (2.7), we may assume that $h(0+) < \infty$ and $h'(\infty) < \infty$. Suppose $p := C_t^T - h(0+) - S_t h'(\infty) > 0$ for some $t \leq T$. Then we set up at time t at zero

cost a portfolio consisting of $p + h(0+)$ units of cash, $h'(\infty)$ units of stock, and -1 unit of the option. At time T, this has a value of $p + h(0+) + S_T h'(\infty) - h(S_T) \geq p$ by convexity of h, yielding an arbitrage. Finally, if $p := h(S_t) - C_t^T > 0$, we have already seen in the proof of (2.6) how this leads to an elementary arbitrage opportunity.

(b) Whenever $p \geq 0$ in part (a), the above portfolios yield a terminal value ≥ 0. Under the additional assumptions, this is even positive with positive probability, and so we again obtain an elementary arbitrage opportunity if $p = 0$. □

2.2. Drift Restrictions for the Forward Implied Volatilities

Let W be an m-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the P-augmented filtration generated by W, and $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$. We use the notation $x = (x^1, \ldots, x^m)$ for elements of $\mathbb{R}^m$ and denote for $d \in \mathbb{N}$ and $p \geq 1$ by $L^p_{\text{loc}}(\mathbb{R}^d)$ the space of all $\mathbb{R}^d$-valued, progressively measurable, locally p-integrable (in t, P-a.s.) processes. Let $h : (0, \infty) \to \mathbb{R}$ be a non-affine convex function satisfying (2.1). We model a stock price process $(S_t)_{t\geq 0}$ and a family of price processes $(C_t^T)_{0\leq t\leq T}$ $(T > 0)$ of contracts paying $h(S_T)$ at time T by

$$C_t^T = c\left(S_t, \int_t^T X(t, s)\, ds\right) \tag{2.8}$$

with dynamics

$$dS_t = \mu_t S_t\, dt + \sigma_t S_t\, dW_t^1 \quad (t \geq 0), \qquad S_0 = s_0, \tag{2.9}$$

$$dX(t, T) = \alpha(t, T)\, dt + v(t, T)\, dW_t \quad (0 \leq t \leq T), \qquad X(0, T) = X_0(T). \tag{2.10}$$

Here c is the solution of (2.2), and μ, $\alpha(\cdot, T)$ are in $L^1_{\text{loc}}(\mathbb{R})$, $\sigma \in L^2_{\text{loc}}(\mathbb{R})$ is positive-valued and $v(\cdot, T)$ is in $L^2_{\text{loc}}(\mathbb{R}^m)$. Each $X(\cdot, T)$ is a nonnegative process modeling the forward implied volatility for the maturity T, and we assume that

$$\Upsilon_t(T) := \int_t^T X(t, s)\, ds > 0 \tag{2.11}$$

for all $T > t > 0$. By Proposition 2.1, $X(t, T) \geq 0$ and (2.11) are necessary for the model to be arbitrage-free. We also assume that X, α and v are (product-)measurable as functions of (t, T). In the next section, we show a construction of the processes $X(t, T)$ on a suitable space such that this automatically holds.

To simplify notation, we use in the sequel subscripts to denote partial derivatives of c, and we suppress all their arguments $(S_t, \Upsilon_t(T))$.

PROPOSITION 2.2. *Under the measure P, the t-dynamics of C_t^T for each fixed T are given by*

$$\begin{aligned} dC_t^T = &\left(c_S \mu_t S_t + c_\Upsilon \left[\sigma_t^2 - X(t, t) + \int_t^T \alpha(t, s)\, ds\right]\right. \\ &\left. + \frac{1}{2} c_{\Upsilon\Upsilon} \left|\int_t^T v(t, s)\, ds\right|^2 + c_{S\Upsilon} S_t \sigma_t \int_t^T v^1(t, s)\, ds\right) dt \\ &+ c_S \sigma_t S_t\, dW_t^1 + c_\Upsilon \left[\int_t^T v(t, s)\, ds\right] dW_t. \end{aligned}$$

Proof. First, note that (2.10) implies via Fubini as in [Heath et al. 1992, equation (8)] that

$$d\Upsilon_t(T) = \left[\int_t^T \alpha(t,s)\,ds\right] dt + \left[\int_t^T v(t,s)\,ds\right] dW_t - X(t,t)\,dt.$$

Then the claim follows by applying Itô's lemma to (2.8) and using (2.2). □

Our aim is now to show that the existence of a common equivalent local martingale measure for S and C^T for a.e. $T > 0$ is essentially equivalent to the *stock volatility specification*

$$\sigma_t^2 + \sigma_t \lim_{T \searrow t}\left(\frac{S_t c_{S\Upsilon}}{c_\Upsilon}\int_t^T v^1(t,s)\,ds\right) + \frac{1}{2}\lim_{T\searrow t}\left(\frac{c_{\Upsilon\Upsilon}}{c_\Upsilon}\left|\int_t^T v(t,s)\,ds\right|^2\right) - X(t,t) = 0 \tag{2.12}$$

and the *drift restrictions*

$$\mu_t = -\sigma_t b_t^1, \tag{2.13}$$

$$\begin{aligned}\alpha(t,T) = {} & -b_t \cdot v(t,T) - \frac{c_{\Upsilon\Upsilon}}{c_\Upsilon} v(t,T)\cdot\int_t^T v(t,s)\,ds \\ & - \frac{1}{2}\partial_\Upsilon\left(\frac{c_{\Upsilon\Upsilon}}{c_\Upsilon}\right) X(t,T)\left|\int_t^T v(t,s)\,ds\right|^2 \\ & - S_t\frac{c_{S\Upsilon}}{c_\Upsilon}\sigma_t v^1(t,T) - S_t\partial_\Upsilon\left(\frac{c_{S\Upsilon}}{c_\Upsilon}\right) X(t,T)\sigma_t\int_t^T v^1(t,s)\,ds\end{aligned} \tag{2.14}$$

for a *market price of risk* process $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$. (Note that σ_t, μ_t, and $\alpha(t,T)$ depend on S_t and X via the arguments in the derivatives of c.) More precisely, we have the following general result on drift restrictions.

THEOREM 2.1.

(a) If there exists a common equivalent local martingale measure Q for S and C^T for a.e. $T > 0$, then for a.e. t and P-a.s., σ_t is a solution of the quadratic equation (2.12), and there exists a market price of risk process $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$ satisfying (2.13) and (2.14) (a.e. $T > 0$) for a.e. t, P-a.s.

(b) Conversely, suppose that the coefficients $\mu_\cdot, \sigma_\cdot, \alpha(\cdot, T)$ and $v(\cdot, T)$ satisfy, as functions of any positive processes $(S_t)_{t\geq 0}$ and $X(t,T)_{0\leq t\leq T}$ $(T > 0)$, the relations (2.12)–(2.14) (a.e. $T > 0$) for a.e. t, P-a.s., for some bounded (uniformly in t, ω) process $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$. Also suppose that there exists a family of positive continuous adapted processes $(S_t)_{t\geq 0}$, $X(t,T)_{0\leq t\leq T}$ $(T > 0)$ satisfying (2.9) and (2.10) for a.e. $T > 0$. Then for each finite time horizon T^*, there exists a common equivalent local martingale measure Q^{T^*} on $\mathcal{F}_{T^*}$ for $(S_t)_{0\leq t\leq T^*}$ and $(C_t^T)_{0\leq t\leq T}$ from (2.8) for a.e. $T \in [0, T^*]$. One such measure is given by

$$\frac{dQ^{T^*}}{dP} := \mathcal{E}\left(\int b\,dW\right)_{T^*}.$$

Moreover, if $\sigma_\cdot$ is bounded and $h(0+) < \infty, h'(\infty) < \infty$, then S and C^T (for a.e. $T \in [0, T^*]$) are martingales under Q^{T^*}.

(c) In the situation of (a) or (b), the t-dynamics of C^T under P are given by

$$dC_t^T = \left(c_S \mu_t S_t - c_\Upsilon b_t \cdot \int_t^T v(t,s)\,ds \right) dt + c_S \sigma_t S_t\, dW_t^1 + c_\Upsilon \left[\int_t^T v(t,s)\,ds \right] dW_t. \tag{2.15}$$

(d) In the situation of (b), suppose there exist maturities $T^* < T_2 < \cdots < T_m$ such that the matrix

$$\begin{pmatrix} \int_t^{T_2} v^2(t,s)\,ds & \cdots & \int_t^{T_2} v^m(t,s)\,ds \\ \vdots & \ddots & \vdots \\ \int_t^{T_m} v^2(t,s)\,ds & \cdots & \int_t^{T_m} v^m(t,s)\,ds \end{pmatrix} \tag{2.16}$$

is nonsingular P-a.s. for a.e. $t \in [0, T^*]$. Then Q^{T^*} is the only equivalent probability measure on $\mathcal{F}_{T^*}$ under which $(S_t)_{0 \le t \le T^*}$ and $(C_t^{T_j})_{0 \le t \le T^*}$ for $j = 2, \ldots, m$ are local martingales.

The equations (2.12)–(2.14) are the analogues for the forward implied volatility setting of the HJM drift restrictions from interest rate modeling. The free input parameter is the family of processes $v(\cdot, T)$ for all $T > 0$, i.e., the *term structure of the volatilities of X*; they determine σ, μ and $\alpha(\cdot, T)$ via (2.12)–(2.14). If we choose $v(t, T) = 0$ for all t, T, then $\alpha(t, T) = 0$ by (2.14), and (2.10) reads $dX(t, T) = 0$ for all $t \le T$; hence $X(t, T) = X_0(T)$ for all $t \le T$, and (2.12) yields $\sigma_t^2 = X_0(t)$ for all t. This is simply the Black–Scholes model with deterministic time-dependent volatility.

REMARK 2.2.

(1) We have expressed the drift restrictions under the initial (objective) measure P. Like in the HJM framework, one could also write these restrictions under some pricing/martingale measure Q, and the effect would simply be to set $b \equiv 0$; see Björk (2004), Chapter 23. The stock volatility specification (2.12) would not change. But the essential difficulties remain the same under P or under Q: The stock volatility σ_t depends on $X(t, t)$, which couples the evolution of S to that of X, and the drift α of X is quadratic in the volatility v of X.
(2) The equations (2.12)–(2.14) depend on the payoff function h via the price function c given by (2.4). It would be interesting to see if the ratios $\frac{c_{S\Upsilon}}{c_\Upsilon}$ and $\frac{c_{\Upsilon\Upsilon}}{c_\Upsilon}$ appearing in (2.12) and (2.14) have some practical financial interpretation. We do not know yet.
(3) The first main result (theorems 3.4 and 3.7) by Jacod and Protter (2006) is very similar to parts (a) and (b) of Theorem 2.1. They assume that the payoff function is in C^2, use a different parametrization of option prices and work more generally with a filtration generated by a countable family of Brownian motions and a Poisson random measure. Theorem 4.1 in Jacod and Protter (2006) is then an analogue of part (d) of our Theorem 2.4.

Proof of Theorem 2.1.

(a) Since $\mathbb{F}$ is generated by W, Itô's representation theorem gives $E[\frac{dQ}{dP} \mid \mathcal{F}_t] = \mathcal{E}(\int b\, dW)_t$ for some process $b \in L^2_{\text{loc}}(\mathbb{R}^m)$, and

$$\tilde{W} := W - \int b_t\, dt \tag{2.17}$$

is a Q-Brownian motion by Girsanov's theorem. Using this and (2.9), (2.13) follows since S is a local Q-martingale. Combining (2.13) and (2.17) with Proposition 2.2 yields

$$dC_t^T = c_S S_t \sigma_t\, d\tilde{W}_t^1 + c_\Upsilon \left(\mu_t^T\, dt + \left[\int_t^T v(t,s)\, ds \right] d\tilde{W}_t \right), \tag{2.18}$$

where

$$\mu_t^T := \sigma_t^2 - X(t,t) + \int_t^T \alpha(t,s)\, ds + \frac{1}{2} \frac{c_{\Upsilon\Upsilon}}{c_\Upsilon} \left| \int_t^T v(t,s)\, ds \right|^2$$
$$+ \frac{c_{S\Upsilon}}{c_\Upsilon} S_t \sigma_t \int_t^T v^1(t,s)\, ds + b_t \cdot \int_t^T v(t,s)\, ds.$$

Since C^T are local Q-martingales for a.e. T, we must have P-a.s. for a.e. T that $\mu_t^T = 0$ for a.e. t and hence by Fubini's theorem P-a.s. for a.e. t

$$\mu_t^T = 0 \quad \text{for a.e. } T. \tag{2.19}$$

Letting $T \searrow t$ in (2.19), we obtain (2.12). Finally, (2.14) follows after lengthy but straightforward calculations if we differentiate (2.19) with respect to T; note for this that also the arguments $\Upsilon_t(T)$ depend on T.

(c) Under (a) the assertion follows from (2.18) together with (2.19), (2.17), and (2.13). The assertion under (b) is proved together with (b) below.

(b) Another lengthy but straightforward calculation shows that (2.14) and (2.12) imply

$$\int_t^T \alpha(t, s, S_t, X)\, ds = -\frac{1}{2} \frac{c_{\Upsilon\Upsilon}}{c_\Upsilon} \left| \int_t^T v(t,s)\, ds \right|^2 - \frac{c_{S\Upsilon}}{c_\Upsilon} S_t \sigma_t \int_t^T v^1(t,s)\, ds$$
$$- b_t \cdot \left(\int_t^T v(t,s)\, ds \right) + X(t,t) - \sigma_t^2,$$

and plugging this into Proposition (2.2), we obtain (c) under (b) as well. If we define $\frac{dQ^{T^*}}{dP} := \mathcal{E}(\int b\, dW)_{T^*}$ on $\mathcal{F}_{T^*}$, then $W - \int b_t\, dt$ is a Q^{T^*}-Brownian motion on $[0, T^*]$ by Girsanov's theorem. It now follows easily from (c) that S and C^T for all $T > 0$ are Q^{T^*}-local martingales on $[0, T^*]$. If $\sigma_\cdot$ is bounded, then S is a Q^{T^*}-martingale, and if $h'(\infty) < \infty$, so are all the C^T due to (2.5).

(d) Let $\bar{Q}$ be another equivalent local martingale measure for S and all the C^{T_j} on $[0, T^*]$. By Itô's representation theorem, $\frac{d\bar{Q}}{dP} := \mathcal{E}(\int \bar{b}\, dW)_{T^*}$ for some process $\bar{b} \in L^2_{\text{loc}}(\mathbb{R}^m)$, and so it suffices to show that $\bar{b}_t = b_t$ for a.e. t. By Girsanov's theorem, $\bar{W} := W - \int \bar{b}_t\, dt$ is a $\bar{Q}$-Brownian motion. Because S is

a local $\bar{Q}$-martingale, we get $\bar{b}_t^1 = -\mu_t/\sigma_t = b_t^1$, and rewriting (2.15) under $\bar{Q}$ yields

$$dC_t^T = \left(c_\Upsilon(\bar{b}_t - b_t) \cdot \int_t^T v(t,s)\,ds \right) dt + c_S \sigma_t S_t\, d\bar{W}_t^1 + c_\Upsilon \left[\int_t^T v(t,s)\,ds \right] d\bar{W}_t.$$

Since the C^{T_j} are local martingales under $\bar{Q}$, we get $(\bar{b}_t - b_t) \cdot \int_t^{T_j} v(t,s)\,ds = 0$, and if (2.16) is nonsingular, we obtain $\bar{b}_t - b_t = 0$. □

Let us conclude this section with a few comments on market completeness and hedging in forward implied volatility models. Suppose that, as a typical application, we should like to hedge some exotic contingent claim by using a bank account, stock S, and a collection of European options C^T, $T = T_1, T_2, \ldots$, as hedge instruments. A slight complication arises from the fact that as time goes on, some of the options will expire and thus no longer be available as hedge instruments. On the other hand, a total restriction to instruments with higher maturities than the contingent claim to be hedged is both unsatisfactory from a theoretical point of view and might be unfeasible in practice. We therefore consider the following concept.

DEFINITION 2.2. Suppose we are in the situation of Theorem 2.1 (b). We call the model (2.9), (2.10) *complete on* $[T_0, T_1]$ if we have maturities $T_m > \cdots > T_2 > T_1$ such that for each $\mathcal{F}_{T_1}$-measurable $H \geq 0$ with $E^{Q^{T_1}}[H \mid \mathcal{F}_{T_0}] < \infty$, there exists a progressively measurable $\mathbb{R}^m$-valued process $(\eta^1, \ldots, \eta^m)$ on $[T_0, T_1]$ for which

$$V_t := E^{Q^{T_1}}[H \mid \mathcal{F}_{T_0}] + \int_{T_0}^t \eta_u^1\, dS_u + \sum_{j=2}^m \int_{T_0}^t \eta_u^j\, dC_u^{T_j}, \quad T_0 \leq t \leq T_1,$$

is a well-defined continuous semimartingale under P, bounded from below uniformly in t by an $\mathcal{F}_{T_0}$-measurable random variable, and $V_{T_1} = H$ a.s.

The following sufficient condition for completeness can be proved like the standard result, e.g., theorem 1.6.6 in Karatzas and Shreve (1998).

PROPOSITION 2.3. *The model (2.9), (2.10) is complete on* $[T_0, T_1]$ *if there exist maturities* $T_m > \cdots > T_2 > T_1$ *such that the matrix in (2.16) is non-singular P-a.s. for a.e.* $t \in [T_0, T_1]$.

Suppose now that we want to hedge a contingent claim $H \geq 0$ with maturity $T^* > 0$. We assume that the model (2.9), (2.10) is complete on each interval of length $\Delta > 0$, where Δ may be small. If $T^* > \Delta$, the following hedging methodology might be appropriate. Let $n := \lfloor \frac{T^*}{\Delta} \rfloor$ and define $H_j := E^{Q^{T^*}}[H \mid \mathcal{F}_{j\Delta}]$, $j = 0, \ldots, n+1$. Since the model is complete on $[(j-1)\Delta, j\Delta]$, we can hedge the claim H_j on this time interval with initial capital H_{j-1} by using the bank account, the stock S and European options $C^{T_2}, \ldots, C^{T_m}$ with maturities $T_m > \cdots > T_2 > j\Delta$, $j \in \{1, \ldots, n\}$, according to Definition 2.2. So at each time point $j\Delta$, $j = 1, \ldots, n$, the hedge instruments with short time to maturity are replaced by new instruments with later maturity dates. For large T^*, this allows to use as hedge instruments the most liquid options with short to medium time to maturity.

An analogous situation arises in interest rate models if one uses a term structure of bonds as hedge instruments. Dahl (2005) studies this problem in a bond modeling setup which includes incomplete markets.

2.3. Examples of Convex Payoff Functions

2.3.1. Smooth Payoff Functions. We first consider the case where the payoff function h is sufficiently nice. The following result shows how (2.12) then becomes much simpler.

PROPOSITION 2.4. *Suppose that h is in C^4, sufficiently well-behaved, and $h''(S) > 0$ for all $S > 0$. Then the stock volatility specification (2.12) simplifies to*

$$\sigma_t^2 = X(t, t). \tag{2.20}$$

Proof. Let Z be a standard normal variable and define $L := S\exp(\sqrt{\Upsilon}Z - \frac{1}{2}\Upsilon)$. From (2.3) we have

$$c(S, \Upsilon) = E[h(L)]. \tag{2.21}$$

Note that the partial derivatives of L are given by $L_Z = L\sqrt{\Upsilon}$, $L_\Upsilon = \frac{1}{2}L(\frac{Z}{\sqrt{\Upsilon}} - 1)$, $L_S = \frac{1}{S}L$. We use that for a continuously differentiable f with $\lim_{x\to\pm\infty} e^{-x^2/2} f(x) = 0$, we have by partial integration

$$E[f(Z)Z] = E[f'(Z)]. \tag{2.22}$$

Now using that h is *sufficiently well-behaved*, we can differentiate under the expectation in (2.21), plug in L_Υ from above, and then apply (2.22) to obtain

$$\begin{aligned} c_\Upsilon(S, \Upsilon) = E[h'(L)L_\Upsilon] &= \frac{1}{2}E\left[h'(L)L\frac{Z}{\sqrt{\Upsilon}}\right] - \frac{1}{2}E[h'(L)L] \\ &= \frac{1}{2}E\left[(h''(L)L + h'(L))L_Z\frac{1}{\sqrt{\Upsilon}}\right] - \frac{1}{2}E[h'(L)L] = \frac{1}{2}E[h''(L)L^2]. \end{aligned} \tag{2.23}$$

Applying the same procedure now to (2.23), we obtain

$$\begin{aligned} c_{\Upsilon\Upsilon}(S, \Upsilon) &= \frac{1}{2}E[(h'''(L)L^2 + h''(L)2L)L_\Upsilon] \\ &= \frac{1}{4}E\left[\left(h'''(L)L^3 + 2h''(L)L^2\right)\frac{Z}{\sqrt{\Upsilon}}\right] - \frac{1}{4}E\left[h'''(L)L^3 + 2h''(L)L^2\right] \\ &= \frac{1}{4}E\left[\left(h''''(L)L^3 + h'''(L)3L^2 + 2h'''(L)L^2 + 4h''(L)L\right)L_Z\frac{1}{\sqrt{\Upsilon}}\right] \\ &\quad - \frac{1}{4}E\left[h'''(L)L^3 + 2h''(L)L^2\right] \\ &= \frac{1}{4}E\left[h''''(L)L^4 + 4h'''(L)L^3 + 2h''(L)L^2\right], \end{aligned} \tag{2.24}$$

$$c_{S\Upsilon}(S, \Upsilon) = \frac{1}{2}E[(h'''(L)L^2 + h''(L)2L)L_S] = \frac{1}{2S}E[h'''(L)L^3 + 2h''(L)L^2]. \tag{2.25}$$

We now let $\Upsilon \searrow 0$ in (2.23)–(2.25). Since $L \to S$ for $\Upsilon \searrow 0$, h is *sufficiently well-behaved*, and $L \in [S\exp(-\sqrt{\Upsilon_0}|Z| - \frac{1}{2}\Upsilon_0), S\exp(\sqrt{\Upsilon_0}|Z|)]$ for $\Upsilon \in [0, \Upsilon_0]$, the dominated convergence theorem implies that the limits of $c_\Upsilon, c_{S\Upsilon}$ and $c_{\Upsilon\Upsilon}$ for $\Upsilon \searrow 0$ exist, and $\lim_{\Upsilon \searrow 0} c_\Upsilon(S, \Upsilon) = \frac{1}{2}h''(S)S^2 > 0$. Together with (2.12) this gives the assertion. □

REMARK 2.4. The proof of Proposition 2.4 makes explicit where the vague assumption of "sufficiently well-behaved h" is used. One precise sufficient condition for this is for instance that

$$|h''''(S)| \le \gamma(S^p + S^{-p} + 1)$$

for positive constants γ and p, because all functions appearing in the proof are then of polynomial growth in $\exp(|Z|)$ and therefore have finite expectation.

If (2.20) holds, the model (2.9), (2.10) is less sensitive to the volatility coefficients $v(t, T)$ near maturity (for $t \nearrow T$) than in the general case (2.12). This is desirable since it is well known that implied volatilities show a very irregular behaviour close to maturity, and so the coefficients $v(t, T)$ near maturity may be difficult to estimate in practice. The example of call options below shows that (2.20) will not hold in general for payoffs which are not differentiable.

2.3.2. Power Payoff Options. For a specific example of a smooth payoff function, fix $\lambda \in \mathbb{R}\backslash\{0, 1\}$ and consider a contract on the stock paying S_T^λ at time T. This is of interest for fund managers because up to a constant factor, this is the solution to the Merton problem of maximizing expected utility from terminal wealth in the Black–Scholes model; the exponent λ depends on the risk aversion of the power utility $\frac{1}{\gamma}x^\gamma$ used by the investor. The payoff function $h(S) = S^\lambda$ is convex for $\lambda \in \mathbb{R}\backslash[0, 1]$ and concave for $\lambda \in (0, 1)$, and so the results of Sections 2.1 and 2.2 apply. One easily calculates that in the Black–Scholes model, the price at time t of such a contract is given by $C_t^{\lambda,T} = c^\lambda(S_t, (T - t)\sigma^2)$ with

$$c^\lambda(S, \Upsilon) = S^\lambda \exp\left(\frac{1}{2}\lambda(\lambda - 1)\Upsilon\right).$$

One motivation for studying this smooth payoff function is that due to its special form, the SDEs for the forward implied volatilities do not explicitly depend on the stock price S; see Corollary 2.1 below. Hence we obtain results (here and in Section 3) under less restrictive conditions on the SDE coefficients than in the call option case. We call this the *power payoff* example.

To apply Theorem 2.4, we calculate the partial derivatives of c^λ and find

$$\frac{c^\lambda_{\Upsilon\Upsilon}}{c^\lambda_\Upsilon} = \frac{1}{2}\lambda(\lambda - 1), \quad \frac{c^\lambda_{S\Upsilon}}{c^\lambda_\Upsilon} = \lambda\frac{1}{S}.$$

Now suppose we are in the setting of Section 2.2, i.e., for one fixed $\lambda \in \mathbb{R}\backslash\{0, 1\}$ we have processes $(S_t)_{t\ge 0}$, $(C_t^{\lambda,T})_{0\le t\le T}$ $(T > 0)$ satisfying (2.8)–(2.10) with $c = c^\lambda$ in (2.8). Then (2.12)–(2.14) become

$$\sigma_t^2 = X(t, t), \tag{2.26}$$

$$\mu_t = -\sigma_t b_t^1, \tag{2.27}$$

$$\alpha(t, T) = -\frac{1}{2}\lambda(\lambda - 1)v(t, T) \cdot \int_t^T v(t, s)\,ds - \lambda\sigma_t v^1(t, T) - b_t \cdot v(t, T) \tag{2.28}$$

for a suitable process $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$. More precisely, Theorem 2.1 yields

COROLLARY 2.1.

(a) If there exists a common equivalent local martingale measure Q for S and $C^{\lambda,T}$ for a.e. $T > 0$, then for a.e. t and P-a.s., σ_t is a solution of the quadratic equation (2.26), and there exists a market price of risk process $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$ satisfying (2.27) and (2.28) (a.e. $T > 0$) for a.e. t, P-a.s.

(b) Conversely, suppose that the coefficients $\mu_\cdot, \sigma_\cdot, \alpha(\cdot, T)$ and $\nu(\cdot, T)$ satisfy, as functions of any positive processes $(S_t)_{t\geq 0}$ and $X(t,T)_{0\leq t\leq T}$ $(T > 0)$, the relations (2.26)–(2.28) (a.e. $T > 0$) for a.e. t, P-a.s., for some bounded (uniformly in t, ω) process $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$. Also suppose that there exists a family of positive continuous adapted processes $(S_t)_{t\geq 0}$, $X(t,T)_{0\leq t\leq T}$ $(T > 0)$ satisfying (2.9) and (2.10) for a.e. $T > 0$. Then for each finite time horizon T^*, there exists a common equivalent local martingale measure Q^{T^*} on $\mathcal{F}_{T^*}$ for $(S_t)_{0\leq t\leq T^*}$ and $(C_t^{\lambda,T})_{0\leq t\leq T}$ from (2.8) for a.e. $T \in [0, T^*]$. One such measure is given by

$$\frac{dQ^{T^*}}{dP} := \mathcal{E}\left(\int b\, dW\right)_{T^*}.$$

Moreover, if $\sigma_\cdot$ is bounded, then S is a martingale under Q^{T^*}.

2.3.3. Call Options. As a second example, we treat European call options. This is of obvious practical importance, allows the use of explicit formulas, and provides an example of a non-smooth payoff function. Let $h(S) = (S - K)^+$ for a fixed strike $K > 0$ and recall that the option prices are given by $C_t^{K,T} = c^K(S_t, \int_t^T X(t,s)\, ds)$, where c^K is now the Black–Scholes function

$$c^K(S, \Upsilon) = SN\left(\frac{\log(S/K) + \frac{1}{2}\Upsilon}{\Upsilon^{\frac{1}{2}}}\right) - KN\left(\frac{\log(S/K) - \frac{1}{2}\Upsilon}{\Upsilon^{\frac{1}{2}}}\right) \quad (\Upsilon > 0),$$

$$c^K(S, 0) = (S - K)^+,$$

and $N(\cdot)$ denotes the standard normal distribution function. The partial derivatives of c^K are easily computed and with $\log^2 x = (\log x)^2$ we find

$$\frac{c^K_{\Upsilon\Upsilon}}{c^K_\Upsilon} = \frac{1}{2}\left(\frac{\log^2(S/K)}{\Upsilon^2} - \frac{1}{\Upsilon} - \frac{1}{4}\right), \quad \frac{c^K_{S\Upsilon}}{c^K_\Upsilon} = -\frac{1}{S}\left(\frac{\log(S/K)}{\Upsilon} - \frac{1}{2}\right).$$

Now suppose we are in the setting of Section 2.2, i.e., for one fixed strike $K > 0$ we have processes $(S_t)_{t\geq 0}$, $(C_t^{K,T})_{0\leq t\leq T}$ $(T > 0)$ satisfying (2.8)–(2.10) with $c = c^K$ in (2.8). Then (2.12)–(2.14) become

$$\left(\sigma_t - \frac{1}{2}\log(S_t/K)\lim_{T\searrow t}\frac{\int_t^T v^1(t,s)\, ds}{\Upsilon_t(T)}\right)^2 \tag{2.29}$$

$$= X(t,t) - \lim_{T\searrow t}\frac{\sum_{j=2}^m\left(\int_t^T v^j(t,s)\, ds\right)^2}{\Upsilon_t(T)^2}\frac{1}{4}\log^2(S_t/K),$$

$$\mu_t = -\sigma_t b_t^1, \tag{2.30}$$

(2.31)

$$\begin{aligned}\alpha(t,T) = {} & -b_t \cdot v(t,T) \\ & -\frac{1}{2}\left(\frac{\log^2(S_t/K)}{\left(\int_t^T X(t,s)\,ds\right)^2} - \frac{1}{\int_t^T X(t,s)\,ds} - \frac{1}{4}\right) v(t,T) \cdot \int_t^T v(t,s)\,ds \\ & +\frac{1}{2}\left(\frac{\log^2(S_t/K)}{\left(\int_t^T X(t,s)\,ds\right)^3} - \frac{1}{2}\frac{1}{\left(\int_t^T X(t,s)\,ds\right)^2}\right) X(t,T)\left|\int_t^T v(t,s)\,ds\right|^2 \\ & +\left(\frac{\log(S_t/K)}{\int_t^T X(t,s)\,ds} - \frac{1}{2}\right)\sigma_t v^1(t,T) - \frac{\log(S_t/K)}{\left(\int_t^T X(t,s)\,ds\right)^2} X(t,T)\sigma_t \int_t^T v^1(t,s)\,ds\end{aligned}$$

for a suitable process $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$; see the appendix for more details on the derivation of (2.29). More precisely, Theorem 2.1 yields

COROLLARY 2.2.

(a) If there exists a common equivalent local martingale measure Q for S and $C^{K,T}$ for a.e. $T > 0$, then for a.e. t and P-a.s., σ_t is a solution of the quadratic equation (2.29), and there exists a market price of risk process $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$ satisfying (2.30) and (2.31) (a.e. $T > 0$) for a.e. t, P-a.s.

(b) Conversely, suppose that the coefficients $\mu_\cdot, \sigma_\cdot, \alpha(\cdot, T)$ and $v(\cdot, T)$ satisfy, as functions of any positive processes $(S_t)_{t\geq 0}$ and $X(t,T)_{0\leq t\leq T}$ $(T > 0)$, the relations (2.29)–(2.31) (a.e. $T > 0$) for a.e. t, P-a.s., for some bounded (uniformly in t, ω) process $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$. Also suppose that there exists a family of positive continuous adapted processes $(S_t)_{t\geq 0}$, $X(t,T)_{0\leq t\leq T}$ $(T > 0)$ satisfying (2.9) and (2.10) for a.e. $T > 0$. Then for each finite time horizon T^*, there exists a common equivalent local martingale measure Q^{T^*} on $\mathcal{F}_{T^*}$ for $(S_t)_{0\leq t\leq T^*}$ and $(C_t^{K,T})_{0\leq t\leq T}$ from (2.8) for a.e. $T \in [0, T^*]$. One such measure is given by

$$\frac{dQ^{T^*}}{dP} := \mathcal{E}\left(\int b\,dW\right)_{T^*}.$$

Moreover, if $\sigma_\cdot$ is bounded, then S and $C^{K,T}$ (for a.e. $T \in [0, T^*]$) are martingales under Q^{T^*}.

We remark that (2.29)–(2.31) provide an explicit version of the drift restrictions obtained in Schönbucher (1999); see there (4.19), (4.23), and (4.4).

2.3.4. Logarithmic Payoff. Let us finally consider a contract paying the (possibly negative) amount $-\log S_T$ at time T. The practical importance of such a *log contract* comes from its relation with variance swaps. A *variance swap* for the trading period $[0, T]$ on the stock S pays at time T the amount $\langle \log S \rangle_T$; in practice, the payoff is some discrete approximation of the quadratic variation. For details on modeling and pricing such contracts, we refer to Bühler (2006) and Carr et al. (2005). Itô's lemma gives for a positive continuous semimartingale S

$$\langle \log S \rangle_T = 2 \int_0^T \frac{1}{S_t}\, dS_t + 2 \log S_0 - 2 \log S_T.$$

Hence in an arbitrage-free market (with sufficient integrability of the stock price process S), the price of a variance swap for the period $[0, T]$ is a deterministic affine function of the log contract's price.

A nice feature of the log contract is that the drift restrictions for its forward implied volatilities become very simple. For $h(S) = -\log S$, the Black–Scholes price function takes the form

$$c(S, \Upsilon) = -\log S + \frac{1}{2}\Upsilon$$

and hence we have $c_\Upsilon(S, \Upsilon) = \frac{1}{2}$ and $c_{S\Upsilon}(S, \Upsilon) = c_{\Upsilon\Upsilon}(S, \Upsilon) = 0$. Then (2.12)–(2.14) become

$$\sigma_t^2 = X(t, t), \tag{2.32}$$

$$\mu_t = -\sigma_t b_t^1,$$

$$\alpha(t, T) = -b_t \cdot v(t, T).$$

Since the drift α is here linear in v, the existence issue for X is not problematic.

3. ARBITRAGE-FREE IMPLIED VOLATILITY MODELS

In this section, we apply the existence and uniqueness results of Wissel (2007) to the infinite system (2.9), (2.10) of SDEs arising in Section 2. This is not entirely straightforward; while the results in Wissel (2007) impose assumptions on the drift and the volatility coefficients of the SDEs, we are here only allowed to choose the volatility coefficients v in our system (2.9), (2.10). Moreover, (2.14) implies that the drift α is typically quadratic in the volatility v which makes the SDE system rather delicate. The main problem is therefore to find conditions on the processes $v(\cdot, T)$ appearing in Corollaries 2.1(b) and 2.2(b) that allow the application of the results from Wissel (2007). In addition, these conditions must ensure positivity of the solution X to guarantee absence of arbitrage in the model. This latter requirement excludes easy choices of v like constant coefficients.

We first briefly review in Section 3.1 the framework for infinite systems of SDEs developed in Wissel (2007). Then we study power payoff contracts in Section 3.2 (Theorems 3.1, 3.2, and 3.3) and call options in Section 3.3 (Theorem 3.4). Theorem 3.1 is a first-best result that illustrates the basic approach but still has two rather restrictive assumptions. Each of these is (with some extra work) subsequently relaxed, which leads to Theorems 3.2 and 3.3.

In principle, the same technique ought to work for general convex payoffs h; it would most likely require bounds on several higher order derivatives of the Black–Scholes pricing

function (2.2) associated to h. Since this would lead us too far here, we restrict our attention to the above two example payoffs. For the logarithmic payoff, finding coefficients v which guarantee a solution is straightforward due to the linear dependence in (2.32); see section 2 of Bühler (2006).

REMARK 3.1. Section 5 of Jacod and Protter (2006) also contains results on existence of models as well as on their completeness, and even in a more general filtration (allowing jump processes) than we consider here. However, the intention in Jacod and Protter (2006) is different. The authors start with a fairly general exogenous process for the stock S, assume the existence of an equivalent local martingale measure Q for S and define option prices as Q-conditional expectations of discounted payoffs. By using the weak predictable representation property of the filtration, they can write the dynamics of these option prices in terms of predictable processes (corresponding to our α and v), and their main concern is then to study precisely when these processes have certain additional regularity properties. In particular, these processes exist, but cannot really be modeled; they depend in a complicated way on the chosen Q and the process for S. In contrast, we want to build models where one can specify a functional form for the term structure of volatilities $v(\cdot, T)$, depending on the process X itself. We think that for possible applications, this is more natural, and in any case, it leads to a different kind of problem than in Jacod and Protter (2006).

3.1. Construction of the Solution Space

In this section, we define the spaces in which we construct the SDE solutions in Sections 3.2 and 3.3 below. To keep the exposition self-contained, we repeat here some notation from Wissel (2007, section 5.1) which is needed to formulate our results. Some further (rather technical) concepts from Wissel (2007) are presented in the appendix; these are only used in the proofs of Sections 3.2 and 3.3 and can therefore be skipped by those readers who are mainly interested in the results rather than the details of the proofs.

So let $(\Omega, \mathcal{F}, P)$ be a probability space, $T^* \geq T_0 > 0$, $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T_0}$ a filtration on this space satisfying the usual conditions, and W an m-dimensional Brownian motion with respect to P and $\mathbb{F}$. Let $U_{[0,T^*]}$ denote the uniform distribution on $[0, T^*]$ and define

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{G}}, \tilde{P}) := \left([0, T^*] \times \Omega, (\{\emptyset, [0, T^*]\} \otimes \mathcal{F}) \vee \tilde{\mathcal{N}}, \mathcal{B}[0, T^*] \otimes \mathcal{F}, U_{[0,T^*]} \otimes P\right),$$

where $\tilde{\mathcal{N}}$ is the family of $(U_{[0,T^*]} \otimes P)$-zero sets in $\mathcal{B}[0, T^*] \otimes \mathcal{F}$. Also define

$$\tilde{\mathbb{G}} = (\tilde{\mathcal{G}}_t)_{t \in [0,T_0]} \quad \text{with } \tilde{\mathcal{G}}_t := (\mathcal{B}[0, T^*] \otimes \mathcal{F}_t) \vee \tilde{\mathcal{N}}, \quad t \in [0, T_0],$$

$$\tilde{W} = (\tilde{W}_t)_{t \in [0,T_0]} \quad \text{with } \tilde{W}_t(T, \omega) := W_t(\omega) \quad \forall t \in [0, T_0], (T, \omega) \in \tilde{\Omega}.$$

It is straightforward to check that $\tilde{W}$ is a $(\tilde{\mathbb{G}}, \tilde{P})$-Brownian motion on $\tilde{\Omega}$.

We can now introduce the spaces in which we construct our SDE solutions. The following definition coincides with the corresponding one in Wissel (2007).

DEFINITION 3.1. For $p \geq 1$ and $d \in \mathbb{N}$, $\mathcal{S}_c^{p,d}$ or $\mathcal{S}_c^p$ is the space of all $\mathbb{R}^d$-valued, $\tilde{\mathbb{G}}$-adapted, $\tilde{P}$-a.s. continuous processes $X = (X(t))_{0 \leq t \leq T_0}$ on $\tilde{\Omega}$ which satisfy

$$\|X\|^p := E^{\tilde{P}}\left[\sup_{0 \leq t \leq T_0} |X(t)|^p\right] = \frac{1}{T^*} \int_0^{T^*} E\left[\sup_{0 \leq t \leq T_0} |X(t, T)|^p\right] dT < \infty;$$

we identify X and X' in $\mathcal{S}_c^p$ if $\|X - X'\| = 0$.

The following simple result says that stochastic integrals with respect to $\tilde{W}$ can be interpreted as stochastic integrals with respect to W in the natural way.

PROPOSITION 3.1 (Wissel 2007, proposition 5.1). *Let h be a $\tilde{\mathbb{G}}$-progressively measurable process on $\tilde{\Omega}$ such that $\int_0^{T_0} |h_u|^2\, du < \infty$ $\tilde{P}$-a.s. Then we have $\int_0^{T_0} |h_u(T)|^2\, du < \infty$ P-a.s., for a.e. $T \in [0, T^*]$, and the stochastic integral $\int h\, d\tilde{W}$ satisfies*

$$\left(\int_0^t h_u\, d\tilde{W}_u\right)(T) = \left(\int_0^t h_u(T)\, dW_u\right) \quad \forall t \quad P\text{-a.s., for a.e. } T \in [0, T^*].$$

From now on, we identify $\mathbb{F}$-progressively measurable (or $\mathbb{F}$-adapted) processes h on Ω with $\tilde{\mathbb{G}}$-progressively measurable (or $\tilde{\mathbb{G}}$-adapted) processes $\tilde{h}$ on $\tilde{\Omega}$ by setting $\tilde{h}(t, T, \omega) := h(t, \omega)$, and similarly $\mathbb{F}$-stopping times τ on Ω with $\tilde{\mathbb{G}}$-stopping times $\tilde{\tau}$ on $\tilde{\Omega}$ by setting $\tilde{\tau}(T, \omega) := \tau(\omega)$. With a slight abuse of notation, we write τ for $\tilde{\tau}$ and h for $\tilde{h}$, in particular W for $\tilde{W}$.

In Sections 3.2 and 3.3 below, we consider two-dimensional processes (X, Y) on the space $\tilde{\Omega}$ such that $X(t, T, \omega)$ represents the T-forward implied volatility and $Y(t, T, \omega)$ is independent of T and represents the log-price of the underlying stock at time t when the market is in state $\omega \in \Omega$. Proposition 3.1 then implies that for a.e. T, $X(\cdot, T)$ can be interpreted as an Itô process on Ω.

3.2. The Case of a Power Payoff Contract

We first consider a power payoff contract with exponent $\lambda \in \mathbb{R}\backslash\{0, 1\}$; see Section 2.3.2. Let $(\mu_t, b_t^2, \ldots, b_t^m)$ be an $\mathbb{R}^m$-valued $\mathbb{F}$-progressively measurable process on Ω with $|\mu_t|, |b_t^j| \le C\, \forall t$. We take the volatility coefficient $v(t, T) = (v^1(t, T), \ldots, v^m(t, T))$ of the forward implied volatility of the form

$$v^j(t, T, X, Y) = f_j(t, T, X(t, T)) \cdot V_j\left(t, T, \int_t^T X(t, s)\, ds, X(t, t), Y(t, t)\right), \tag{3.1}$$

$j = 1, \ldots, m$, for measurable functions $f_j : [0, T^*]^2 \times [0, \infty) \to \mathbb{R}$ and $V_j : [0, T^*]^2 \times (0, \infty)^2 \times \mathbb{R} \to \mathbb{R}$. We also suppose that

$$f_j(t, T, x) = 0 \quad \forall t > T;$$

this ensures that option prices are constant after maturity. To motivate the dependence of v on the six variables time t, maturity T, forward implied volatility $X(t, T)$, implied volatility to maturity $\int_t^T X(t, s)\, ds$, instantaneous forward implied volatility $X(t, t)$ and log-stock price $Y(t, t)$, note that the drift coefficient $\alpha(t, T)$ in (2.14) for general convex payoff functions depends explicitly on these six quantities (on $X(t, t)$ via the specification (2.12) of σ). Hence it is natural to allow this also for $v(t, T)$. Observe that the only dependence on Y is via the argument $Y(t, t)$ in V_j.

REMARK 3.2. Note that the models we consider here and in Section 3.3 are actually of Markovian type; the dynamics depend only on the current state of the infinite-dimensional process $(X(\cdot), Y(\cdot))$. Hence recalibration is in principle possible. But unlike classical stochastic volatility models like Heston (1993) or Hull and White (1987), our models are

in general *not* Markovian in the finite-dimensional process $(\sigma(\cdot), Y(\cdot))$. This makes them computationally more complex.

Throughout this section, let $Y_0 \in \mathbb{R}$ and X_0 be bounded on $[0, T^*]$ and satisfy $X_0(T) \geq \epsilon$ for a.e. T and some $\epsilon > 0$. As in Corollary 2.1 and (2.26)–(2.28), define

$$\begin{aligned}(3.2)\qquad \alpha(t, T, X, Y) := &-\frac{1}{2}\lambda(\lambda-1)v(t, T, X, Y)\cdot\int_t^T v(t, s, X, Y)\,ds \\ &-\lambda\sqrt{X(t,t)}v^1(t, T, X, Y) + \frac{\mu_t}{\sqrt{X(t,t)}}v^1(t, T, X, Y) \\ &-\sum_{j=2}^m b_t^j v^j(t, T, X, Y),\end{aligned}$$

and note that α becomes zero whenever v does. We take $d = 2$ and consider in $\mathcal{S}_c^{p,2}$ the SDE system

$$\left.\begin{aligned} dX(t, T) &= \alpha(t, T, X, Y)\,dt + v(t, T, X, Y)\,dW_t, \quad X(0, T) = X_0(T), \\ dY(t, T) &= \left(\mu_t - \frac{1}{2}X(t, t)\right)dt + \sqrt{X(t, t)}\,dW_t, \quad Y(0, T) = Y_0. \end{aligned}\right\} \tag{3.3}$$

If (X, Y) solves (3.3), then Y does not depend on T, and $S(t) := e^{Y(t,t)}$ satisfies

$$dS(t) = \mu_t S(t)\,dt + \sqrt{X(t, t)}S(t)\,dW_t.$$

Hence (S, X) is then a solution to (2.9), (2.10).

In order to obtain a unique solution for (3.3), we have to impose some sort of Lipschitz condition on the coefficients. To that end, let $U \subseteq \mathbb{R}^n$ be open and Θ a (possibly empty) set. A function $f : \Theta \times U \to \mathbb{R}$ is called *polynomial Lipschitz on* U if $f(\cdot, x)$ is bounded for fixed $x \in U$ and

$$|f(\theta, x) - f(\theta, x')| \leq |x - x'|\gamma(x, x') \quad \forall x, x' \in U, \theta \in \Theta$$

for a function $\gamma : U \times U \to (0, \infty)$ of at most polynomial growth. One easily checks that if f, g are polynomial Lipschitz on U and $h : f(U) \to \mathbb{R}$ is polynomial Lipschitz on $f(U)$, then $f + g, fg$ and $h \circ f$ are polynomial Lipschitz on U.

As an existence result for (3.3), our first theorem could be viewed as a first-best attempt since it has two rather restrictive assumptions in (3.4) and (3.6). We shall see below how this can be improved.

THEOREM 3.1. *Let $p > 2$ be sufficiently large and $\epsilon > 0$. Suppose that v is of the form (3.1), where V_j is polynomial Lipschitz on $(0, \infty)^2 \times \mathbb{R}$, f_j satisfies*

$$f_j(t, T, x) = 0 \quad \forall t, T, \text{ and for } x \leq \epsilon, \tag{3.4}$$

$$|f_j(t, T, x) - f_j(t, T, x')| \leq C|x - x'| \quad \forall t, T, x, x', \tag{3.5}$$

and v^j satisfies

$$|v^j(t, T, X, Y)| \leq C\left(\sqrt{X(t, T)} + \sqrt{\int_t^T X(t, s)\,ds} + \sqrt{X(t, t)} + \sqrt{Y(t, t)}\right) \tag{3.6}$$

for constants C. Then (3.3) has a unique solution $(X, Y) \in \mathcal{S}_c^{p,2}$. *Y does not depend on T, and* $X \geq \epsilon$.

Proof. Define functions on $\mathbb{R}$ by $\varphi_\epsilon(x) := \sqrt{\max(x, \epsilon)}$, $\psi_\epsilon(x) := 1/\varphi_\epsilon(x)$ and note that these are Lipschitz-continuous for some Lipschitz constant K. Define

$$\begin{aligned}\tilde{\alpha}(t, T, X, Y) := &-\frac{1}{2}\lambda(\lambda - 1)v(t, T, X, Y) \cdot \int_t^T v(t, s, X, Y)\, ds \\ &- \lambda\varphi_\epsilon(X(t, t))v^1(t, T, X, Y) + \mu_t\psi_\epsilon(X(t, t))v^1(t, T, X, Y) \\ &- \sum_{j=2}^m b_t^j v^j(t, T, X, Y)\end{aligned} \tag{3.7}$$

and consider the system

$$\left.\begin{aligned} dX(t, T) &= \tilde{\alpha}(t, T, X, Y) + v(t, T, X, Y)\, dW_t, \\ dY(t, T) &= \left(\mu_t - \frac{1}{2}X(t, t)\right) dt + \varphi_\epsilon(X(t, t))\, dW_t \end{aligned}\right\} \tag{3.8}$$

with initial condition $X(0, T) = X_0(T)$, $Y(0, T) = Y_0$. It suffices to show that the system (3.8) has a unique solution $(X, Y) \in \mathcal{S}_c^{p,2}$, because (3.4) and Proposition A.6 then imply that $X \geq \epsilon$ so that (X, Y) is also a solution of (3.3), and uniqueness follows since one sees in the same way that any solution of (3.3) is also a solution of (3.8).

We now want to apply Proposition A.3 to (3.8), so we check its prerequisites. It is easy to see that the coefficients in (3.8) are strongly progressively measurable in the sense of Definition A.1, due to the specification (3.1) of the function v. To show that the coefficients in (3.8) are locally Lipschitz in the sense of Definition A.2, we use Proposition A.5. By (3.5), the functions

$$g_{1j}(t, T, X, Y) := f_j(t, T, X(t, T)), \quad j = 1, \ldots, m$$

satisfy (A.4), and we claim that (A.5) holds for

$$\begin{aligned} g_2(t, T, X, Y) &:= \varphi_\epsilon(X(t, t)), \\ g_3(t, T, X, Y) &:= \psi_\epsilon(X(t, t)), \\ g_{4j}(t, T, X, Y) &:= V_j\left(t, T, \int_t^T X(t, s)\, ds, X(t, t), Y(t, t)\right), \\ g_{5j}(t, T, X, Y) &:= \int_t^T v^j(t, s, X, Y)\, ds. \end{aligned}$$

This is clear for g_2, g_3 since φ_ϵ, ψ_ϵ are Lipschitz on $\mathbb{R}$, and also for g_4 since V_j is polynomial Lipschitz. For g_5 it is proved in Lemma 3.1 below. Now

$$\tilde{\alpha} = -\frac{1}{2}\lambda(\lambda - 1)\sum_{j=1}^m g_{1j} \cdot g_{4j}g_{5j} - \lambda g_2 g_{11} \cdot g_{41} - \mu g_3 g_{11} \cdot g_{41} - \sum_{j=2}^m b^j g_{1j} \cdot g_{4j},$$

and so Proposition A.5 yields that $\tilde{\alpha}$ is locally Lipschitz. Similarly one can see that the other coefficients of (3.8) are locally Lipschitz. For the growth condition (A.1), note that $v^j = g_{1j} \cdot g_{4j}$ implies

$$|\tilde{\alpha}| \le C_1 \left(\sum_{j=1}^{m} \left(|v^j|^2 + |g_{5j}|^2 \right) + |g_2|^2 + |v^1|^2 + |g_3|^2 + |v^1|^2 + \sum_{j=2}^{m} (1 + |v^j|^2) \right)$$

and therefore by (3.6)

$$|\tilde{\alpha}| \le C_2 \left(|X(t,T)| + \int_0^{T^*} |X(t,s)|\, ds + |X(t,t)| + |Y(t,t)| \right)$$

for some constants C_1, C_2. Using now (A.7) and (A.8) gives for a constant C_3

$$E^{\tilde{P}} \left[\int_0^t |\tilde{\alpha}(u, T, X, Y)|^p \, du \right] \le C_3 (1 + \|(X, Y)\|^p)$$

and hence (A.1). In a similar way, one checks this condition for the other coefficients in (3.8). Finally, the coefficients f of (3.8) satisfy the boundedness condition $|f(u, T, \cdot, (X_0, Y_0))| \le C$ due to (3.6) and because $(\mu, b^2, \ldots, b^m)$ is bounded. This allows to apply Proposition A.3 and ends the proof. □

To complete the proof of Theorem 3.1, we need to show

LEMMA 3.1. *The function* $g_{5j}(t, T, X, Y) := \int_t^T v^j(t, s, X, Y)\, ds$ *satisfies (A.5).*

Proof. Recall that $v^j(t, s, X, Y) = f_j(t, s, X(t, s)) \cdot g_{4j}(t, s, X, Y)$. If we write $g_{4j}(s) := g_{4j}(t, s, X, Y)$ and $g'_{4j}(s) := g_{4j}(t, s, X', Y')$, (3.4) and (3.5) give

$$\begin{aligned}
&|g_{5j}(t, T, X, Y) - g_{5j}(t, T, X', Y')| \\
&\le \int_t^T |f_j(t, s, X(t, s)) - f_j(t, s, X'(t, s))||g_{4j}(s)|\, ds \\
&\quad + \int_t^T |g_{4j}(s) - g'_{4j}(s)||f_j(t, s, X'(t, s))|\, ds \\
&\le C \sup_{s \in [0, T^*]} |g_{4j}(s)| \int_0^{T^*} |X(t, s) - X'(t, s)|\, ds + \sup_{s \in [0, T^*]} |g_{4j}(s) - g'_{4j}(s)| C \int_0^{T^*} |X'(t, s)|\, ds.
\end{aligned}$$

Hence the assertion follows since g_{4j} satisfies (A.5). □

EXAMPLE 3.1. The function v satisfies the conditions of Theorem 3.1 if for instance

$$v(t, T, X, Y) = B(t, T)((X(t, T) - \epsilon)^+)^r$$

for some bounded function $B(\cdot, \cdot)$ and $r \in (0, \frac{1}{2}]$. Using Proposition A.6, one sees as in the proof of Theorem 3.1 that this v gives the same solution for (3.3) as

$$\tilde{v}(t, T, X, Y) = B(t, T)|X(t, T) - \epsilon|^r.$$

Note that condition (3.4) excludes the simplest choice for v one might think of, namely $v(t, T) = B(t, T)$ for some bounded process $B(t, T)$ which does not depend on X. This is natural since for such a v, the forward implied volatilities $X(t, T)$ can become negative, leading to arbitrage opportunities in the model. However, (3.4) is unduly restrictive and we shall relax this below in Theorem 3.3. But let us examine the square-root bound (3.6) on v first. This cannot be relaxed to a linear growth condition in general because if v grows linearly in (X, Y), then α is quadratic in (X, Y) and the solution will typically exist only up to a (possibly finite) explosion time.

Nevertheless, we can abandon (3.6) if we manage to control the quadratic terms from α in a different way. Let v and α still be given by (3.1) and (3.2), but assume now that $V_j \equiv 1$ for all j. This means that v and α are only functions of (t, T, X) but do not depend on Y, and it has the consequence that we can apply the comparison result in Proposition A.4. For that purpose, we want the quadratic terms in α to have negative signs, and this leads us to impose below the sign condition (3.10) on the f_j. Since the coefficients do not depend on Y, we now take $d = 1$ and consider in $\mathcal{S}_c^{p,1}$ the SDE

$$dX(t, T) = \alpha(t, T, X)\,dt + v(t, T, X)\,dW_t, \quad X(0, T) = X_0(T). \tag{3.9}$$

A (unique) solution X to (3.9) gives a (unique) solution (X, Y) for (3.3) via

$$Y(t, T) := Y_0 + \int_0^t \left(\mu_u - \frac{1}{2}X(u, u)\right) du + \int_0^t \sqrt{X(u, u)}\,dW_u,$$

and setting $S(t) := e^{Y(t,t)}$ again yields a (unique) solution (S, X) to (2.9), (2.10).

Our second existence result is now without a square-root growth condition on v; the main price to pay for this is that the SDE for the forward implied volatilities must not be affected by the stock price.

THEOREM 3.2. *Let $p > 2$ be sufficiently large and $\epsilon > 0$. Let $\lambda \in \mathbb{R}\backslash[0, 1]$. Suppose that v is of the form (3.1) with $V_j \equiv 1$ and the f_j satisfy (3.4) and (3.5) as well as*

$$\left.\begin{array}{lll} \lambda f_1(t, T, x) \geq 0 & \forall t, T, x, & \\ \text{for each } j = 2, \ldots, m, \text{ either} & f_j(t, T, x) \geq 0 & \forall t, T, x \\ \text{or} & f_j(t, T, x) \leq 0 & \forall t, T, x. \end{array}\right\} \tag{3.10}$$

Then (3.9) has a unique solution $X \in \mathcal{S}_c^{p,1}$. We have $X \geq \epsilon$.

Proof. As in the proof of Theorem 3.1, define $\tilde{\alpha}$ by (3.1) and consider the SDE

$$dX(t, T) = \tilde{\alpha}(t, T, X) + v(t, T, X)\,dW_t. \tag{3.11}$$

Proposition A.6 and (3.4) again imply that a solution of (3.11) is a solution to (3.9) and vice versa.

To solve (3.11), we want to apply Proposition A.4. As in the proof of Theorem 3.1, $\tilde{\alpha}$ and v are strongly progressively measurable, and by (3.4) and (3.1), we have (A.3) for v and then by (3.1) also for $\tilde{\alpha}$. By (3.5) and (3.1) with $V_j \equiv 1$, v also satisfies the second half of (A.2). Next, note that

$$|v(t, T, X)| \leq C|X(t, T)| \quad \forall t, T, X \tag{3.12}$$

due to (3.4) and (3.5); this gives the first half of (A.2) for v. If we define

$$\beta^*(t, T, X) := \frac{C^2}{\sqrt{\epsilon}}|X(t, T)| + (m-1)C^2|X(t, T)|,$$

then β^* clearly satisfies (A.2). From (3.1), $\lambda \in \mathbb{R}\backslash[0, 1]$, (3.10) and (3.12) we obtain for all X, $X^* \in \mathcal{S}_c^{p,1}$ satisfying $\frac{\epsilon}{2} \leq X \leq X^*$ that

$$\tilde{\alpha}(t, T, X) \leq \mu_t \psi_\epsilon(X(t, t))v^1(t, T, X) - \sum_{j=2}^m b_t^j v^j(t, T, X) \leq \beta^*(t, T, X) \leq \beta^*(t, T, X^*).$$

Finally, the same arguments as in the proof of Theorem 3.1 also show here that $\tilde{\alpha}$ is locally Lipschitz on the set $\{X \in \mathcal{S}_c^{p,1} \mid X \geq \frac{\epsilon}{2}\}$. So Proposition A.4 can be applied and yields the assertion. □

EXAMPLE 3.2. Let $\lambda \in \mathbb{R}\backslash[0, 1]$ and $B(\cdot, \cdot)$ be a bounded function satisfying $\lambda B^1(\cdot, \cdot) \geq 0$ and for each $j = 2, \ldots, m$ either $B^j(\cdot, \cdot) \geq 0$ or $B^j(\cdot, \cdot) \leq 0$. Then one possible choice of v in Theorem 3.2 is

$$v(t, T, X) = B(t, T)((X(t, T) - \epsilon)^+)^r$$

for some $r \in (0, 1]$. As in Example 3.1, this yields the same solution as for

$$\tilde{v}(t, T, X) = B(t, T)|X(t, T) - \epsilon|^r.$$

Of the assumptions of Theorem 3.1, the truly restrictive one is (3.4). It imposes that the volatility v (and by (3.2) also the drift α) of the forward implied volatility X vanishes as soon as X hits $\epsilon > 0$. This ensures that $X(t, T)$ remains $\geq \epsilon$ and as a pleasant side-effect eliminates the problem that the function $\sqrt{\cdot}$ loses its Lipschitz property at zero; but as a condition, (3.4) is rather unnatural and technical. A more natural way to guarantee positive forward implied volatilities is to assume

$$f_j(t, T, 0) = 0 \quad \forall t, T. \tag{3.13}$$

The question whether (3.4) can be weakened to (3.13) is not only of interest here, but will become crucial when we treat call options in the next section. It turns out that we can achieve an existence result under (3.13) by the use of a suitable transformation. But in contrast to Theorems 3.1 and 3.2, we have not yet been able to find explicit examples where the coefficient $v(t, T)$ is only a function of the current forward implied volatility $X(t, T)$.

The basic setting is like for Theorem 3.1. We start from (3.1) and (3.2) for v and α, again allowing V_j to depend on $Y(t, t)$; hence we take $d = 2$ and work in $\mathcal{S}_c^{p,2}$. To avoid the singularity of α at $X(t, t) = 0$, we impose below in (3.15) a particular form for the stock drift μ. Our third existence result, which avoids the restrictive assumption (3.4), is then

THEOREM 3.3. *Let $p > 2$ be sufficiently large. Suppose that v is of the form (3.1), where V_j is polynomial Lipschitz on $(0, \infty)^2 \times \mathbb{R}$, f_j satisfies (3.13) and (3.5), and V_j satisfies*

$$|V_j(t, T, w, x, y)| \leq C\frac{1}{1 + w + \sqrt{x}} \quad \forall t, T. \tag{3.14}$$

Take an $\mathbb{R}$-valued $\mathbb{F}$-progressively measurable process (b_t^1) on Ω with $|b_t^1| \leq C \forall t$ and let μ_t be given by

$$\mu_t = \mu(t, X, Y) := -b_t^1\sqrt{X(t, t)}. \tag{3.15}$$

Then (3.3) has a unique solution $(X, Y) \in \mathcal{S}_c^{p,2}$. Y does not depend on T, and $X > 0$.

The proof of Theorem 3.3 uses a transformation of (3.3) to deal with the fact that the function $\sqrt{\cdot}$ is not polynomial Lipschitz at 0. We introduce the transformation function and derive some of its properties in the following result.

LEMMA 3.2.

(a) Let $a, b \geq 1$ be sufficiently large. Then there exists a convex smooth strictly increasing function $\psi : \mathbb{R} \to (0, \infty)$ such that

$$\psi(z) = \begin{cases} -\dfrac{1}{z} & \text{for } z \leq -a \\ z & \text{for } z \geq a \end{cases} \tag{3.16}$$

and $|\psi''(z)|,\ |\psi'''(z)| \leq b$ $(z \in \mathbb{R})$. We have $|\psi'(z)| \leq 1$ for all $z \in \mathbb{R}$.

(b) If $\varphi : (0, \infty) \to \mathbb{R}$ is the inverse of ψ, we have for $z \in \mathbb{R},\ x = \psi(z)$

$$\varphi'(x) = \frac{1}{\psi'(z)} =: \psi_1(z),$$

$$\varphi''(x) = -\frac{\psi''(z)}{\psi'(z)^3} =: \psi_2(z),$$

and also

$$\psi_1(z) = \begin{cases} z^2 & \text{for } z \leq -a \\ 1 & \text{for } z \geq a, \end{cases} \tag{3.17}$$

$$\psi_2(z) = \begin{cases} 2z^3 & \text{for } z \leq -a \\ 0 & \text{for } z \geq a, \end{cases} \tag{3.18}$$

and the functions ψ_1, ψ_2 are polynomial Lipschitz on $\mathbb{R}$.

(c) Let $f : [0, \infty) \to \mathbb{R}^n$ be globally Lipschitz on $\mathbb{R}$ and satisfy $f(0) = 0$. Then the functions $f \circ \psi$, $(f \circ \psi)\psi_1$ and $(f \circ \psi)^2\psi_2$ are also globally Lipschitz on $\mathbb{R}$.

Proof. Parts (a) and (b) are easily verified, and so is the assertion about $f \circ \psi$ in (c). To prove the assertion for $(f \circ \psi)\psi_1$ and $(f \circ \psi)^2\psi_2$, first note that it suffices to show that the functions are Lipschitz on each of the intervals $(-\infty, -a], [-a, a]$ and $[a, \infty)$, separately. This clearly holds on $[-a, a]$ since f, ψ, ψ_1 and ψ_2 are Lipschitz and bounded on $[-a, a]$.

Next note that $|f(x)| \leq C|x|$ for $x \geq 0$ with the Lipschitz constant C of f. Hence convexity of ψ gives for $z \geq z'$

$$\begin{aligned} &|f(\psi(z))\psi_1(z) - f(\psi(z'))\psi_1(z')| \\ &\quad \leq |f(\psi(z)) - f(\psi(z'))|\,|\psi_1(z)| + |\psi_1(z) - \psi_1(z')|\,|f(\psi(z'))| \\ &\quad \leq C\psi'(z)|z - z'|\frac{1}{\psi'(z)} + |\psi_1(z) - \psi_1(z')|C\psi(z'). \end{aligned}$$

On $[a, \infty)$ we have $|\psi_1(z) - \psi_1(z')| = 0$, and on $(-\infty, -a]$ we have

$$|\psi_1(z) - \psi_1(z')|\psi(z') = \frac{|z^2 - z'^2|}{|z'|} \leq 2|z - z'|.$$

Hence we obtain Lipschitz-continuity for $(f \circ \psi)\psi_1$ on $[a, \infty)$ and $(-\infty, -a]$.

We have $(f \circ \psi)^2 \psi_2 = 0$ on $[a, \infty)$. On $(-\infty, -a]$, we have for $z \geq z'$

$$
\begin{aligned}
&|f(\psi(z))^2 \psi_2(z) - f(\psi(z'))^2 \psi_2(z')| \\
&\leq \left| f\left(-\frac{1}{z}\right)^2 - f\left(-\frac{1}{z'}\right)^2 \right| |2z^3| + |2z^3 - 2z'^3| \left| f\left(-\frac{1}{z'}\right)^2 \right| \\
&\leq \left| f\left(-\frac{1}{z}\right) - f\left(-\frac{1}{z'}\right) \right| \left| f\left(-\frac{1}{z}\right) + f\left(-\frac{1}{z'}\right) \right| |2z^3| + |2z^3 - 2z'^3| \, C^2 \frac{1}{|z'|^2} \\
&\leq C \frac{|z - z'|}{|zz'|} \, C \left(\frac{1}{|z|} + \frac{1}{|z'|} \right) |2z^3| + 2|z - z'| \, |z^2 + zz' + z'^2| \, C^2 \frac{1}{|z'|^2}.
\end{aligned}
$$

This gives Lipschitz-continuity for $(f \circ \psi)^2 \psi_2$ on $[a, \infty)$ and $(-\infty, -a]$. □

We now come to the

Proof of Theorem 3.3. We want to use the transformation $Z = \varphi(X)$. We consider in $\mathcal{S}_c^{p,2}$ the SDE

$$
\left.
\begin{aligned}
dZ(t, T) &= \bar{\alpha}(t, T, Y, Z) \, dt + v(t, T, \psi(Z), Y) \psi_1(Z(t, T)) \, dW_t, \\
dY(t, T) &= \left[\mu(t, \psi(Z), Y) - \frac{1}{2} \psi(Z(t, t)) \right] dt + \sqrt{\psi(Z(t, t))} \, dW_t
\end{aligned}
\right\} \tag{3.19}
$$

with initial condition $Z(0, T) = \varphi(X_0(T))$, $Y(0, T) = Y_0$, where

$$
\bar{\alpha}(t, T, Y, Z) := \alpha(t, T, \psi(Z), Y) \psi_1(Z(t, T)) + \frac{1}{2} |v(t, T, \psi(Z), Y)|^2 \psi_2(Z(t, T)).
$$

If we have a unique solution (Y, Z) to (3.19), then Itô's lemma yields that $(X, Y) = (\psi(Z), Y)$ is the unique solution to (3.3).

We now want to apply Proposition A.3 to (3.19). It is easy to see that the coefficients in (3.19) are strongly progressively measurable due to (3.1). To check that they are locally Lipschitz on $\mathcal{S}_c^p$, we use Proposition A.5. Since f_j satisfies (3.13) and (3.5), Lemma 3.2 (c) implies that the functions

$$
\begin{aligned}
g_{1j}(t, T, Y, Z) &:= f_j(t, T, \psi(Z(t, T))) \psi_1(Z(t, T)), \\
h_{1j}(t, T, Y, Z) &:= f_j(t, T, \psi(Z(t, T)))^2 \psi_2(Z(t, T))
\end{aligned}
$$

$(j = 1, \ldots, m)$ satisfy (A.4). Next we claim that the functions

$$
g_2(t, T, Y, Z) := \sqrt{\psi(Z(t, t))},
$$

$$
g_{4j}(t, T, Y, Z) := V_j \left(t, T, \int_t^T \psi(Z(t, s)) \, ds, \psi(Z(t, t)), Y(t, t) \right), \tag{3.20}
$$

$$
g_{5j}(t, T, Y, Z) := \int_t^T v^j(t, s, \psi(Z), Y) \, ds \tag{3.21}
$$

satisfy (A.5). This is clear for g_2 since $\sqrt{\psi(\cdot)}$ is Lipschitz on $\mathbb{R}$, and it is proved in Lemma 3.3 below for g_{4j} and g_{5j}. Now we have

$$\bar{\alpha} = -\frac{1}{2}\lambda(\lambda - 1)\sum_{j=1}^{m} g_{1j} \cdot g_{4j} g_{5j} - \lambda g_2 g_{11} \cdot g_{41} - \sum_{j=1}^{m} b^j g_{1j} \cdot g_{4j} + \frac{1}{2} h_{1j} |g_{4j}|^2$$

and so Proposition A.5 yields that $\bar{\alpha}$ is locally Lipschitz. Similarly one can see that the other coefficients in (3.19) are locally Lipschitz. To check condition (A.1), we first note that (3.13) and (3.5) give

$$|f_j(t, T, x)| \leq C|x|$$

for $x \geq 0$, and using (3.16)–(3.18) yields for some constants C_1, C_2

$$|g_{1j}| \leq C|\psi(Z(t, T))|\,|\psi_1(Z(t, T))| \leq C_1(1 + |Z(t, T)|),$$

$$|h_{1j}| \leq C^2|\psi(Z(t, T))|^2\,|\psi_2(Z(t, T))| \leq C_2(1 + |Z(t, T)|).$$

Moreover, by (3.14) we have

$$|g_{4j}| \leq C\left(1 + \int_t^T \psi(Z(t, s))\, ds\right)^{-1} \leq C \tag{3.22}$$

and hence by (3.1) and (3.20)

$$|g_{5j}| \leq \int_t^T |f_j(t, s, \psi(Z(t, s)))|\,|g_{4j}(t, s, Y, Z)|\, ds \leq C^2 \int_t^T \psi(Z(t, s))\, ds$$

and then

$$|g_{4j} g_{5j}| \leq C^3.$$

From (3.14) we also obtain

$$|g_{41} g_2| \leq C.$$

These bounds together imply that for some constant C^*

$$|\bar{\alpha}| \leq C^*(1 + |Z(t, T)|),$$

so that we have (A.1) for $\bar{\alpha}$. In a similar way, one can check (A.1) for the other coefficients in (3.19). Finally, the boundedness at $(Z(0, T), Y_0)$ of all coefficients follows by using (3.1), (3.5), (3.13), (3.14) as well as continuity of ψ, ψ_1, ψ_2 and uniform boundedness of $Z(0, T)$, which holds since $\epsilon \leq X(0, T) \leq C$. Hence we may apply Proposition A.3, and this ends the proof. □

It remains to establish

LEMMA 3.3. *The functions g_{4j} and g_{5j} from (3.20) and (3.21) satisfy (A.5).*

Proof. For g_{4j}, note that $|\psi(z) - \psi(z')| \leq |z - z'|$ and $|\psi(z)| \leq a + |z|$ yield

$$|\psi(Z(t, t)) - \psi(Z'(t, t))| \leq |Z(t, t) - Z'(t, t)|,$$

$$\left|\int_t^T \psi(Z(t, s))\, ds - \int_t^T \psi(Z'(t, s))\, ds\right| \leq \int_0^{T^*} |Z(t, s) - Z'(t, s)|\, ds$$

and

$$|\psi(Z(t,t))| \le a + |Z(t,t)|,$$

$$\left|\int_t^T \psi(Z(t,s))\,ds\right| \le T^*a + \int_0^{T^*} |Z(t,s)|\,ds.$$

Hence the claim for g_{4j} follows because V_j is polynomial Lipschitz. For g_{5j}, recall that $v^j(t,s,\psi(Z),Y) = f_j(t,s,\psi(Z(t,s)))g_{4j}(t,s,Y,Z)$. Set $g_{4j}(s) := g_{4j}(t,s,Y,Z)$, $g'_{4j}(s) := g_{4j}(t,s,Y',Z')$ and use (3.13) and (3.5) to obtain

$$\begin{aligned}
&|g_{5j}(t,T,Y,Z) - g_{5j}(t,T,Y',Z')| \\
&\le \int_t^T |f_j(t,s,\psi(Z(t,s))) - f_j(t,s,\psi(Z'(t,s)))|\,|g_{4j}(s)|\,ds \\
&\quad + \int_t^T |g_{4j}(s) - g'_{4j}(s)|\,|f_j(t,s,\psi(Z'(t,s)))|\,ds \\
&\le C^2 \int_0^{T^*} |Z(t,s) - Z'(t,s)|\,ds + C \sup_{S\in[0,T^*]} |g_{4j}(s)^- g'_{4j}(s)| \left(\int_0^{T^*} |Z'(t,s)|\,ds + T^*a\right)
\end{aligned}$$

due to (3.22). Hence (A.5) holds for g_{5j} as well. □

EXAMPLE 3.3. An example satisfying the conditions of Theorem 3.3 is

$$v(t,T,X,Y) = B(t,T)X(t,T)V\left(\int_t^T X(t,s)\,ds,\ X(t,t),\ Y(t,t)\right)$$

for a bounded function $B(\cdot,\cdot)$ and a polynomial Lipschitz function V with

$$|V_j(x,w,y)| \le C\frac{1}{1+w+\sqrt{x}}.$$

3.3. The Case of a European Call

In this section, we do a similar analysis for a European call instead of a power payoff contract. Fix $K > 0$ and let $(b_t^1,\dots,b_t^m)$ be an $\mathbb{R}^m$-valued $\mathbb{F}$-progressively measurable process on Ω with $|b_t^j| \le C\forall t$. Similarly as in (3.1), we take the volatility coefficient $v(t,T) = (v^1(t,T),\dots,v^m(t,T))$ of the forward implied volatility of the form

$$v^j(t,T,X,Y) = X(t,T)\,V_j\left(t,T,\int_t^T X(t,s)\,ds,\ X(t,t),\ Y(t,t)\right), \tag{3.23}$$

$j = 1,\dots,m$, for measurable functions $V_j : [0,T^*]^2 \times (0,\infty)^2 \times \mathbb{R} \to \mathbb{R}$. Assume

$$V_j(t,T,w,x,y) = 0 \quad \forall t > T$$

to ensure constant option prices after maturity. To have existence of the limits in (2.29), we impose the technical condition that for some small constant $\epsilon > 0$

$$V_j(t,T,w,x,y) = V_j(t,t,\epsilon,x,y) \quad \forall w \le \epsilon, \quad \forall T. \tag{3.24}$$

Under this condition, (2.29) with $Y(t,t) = \log S_t$ yields the specification

$$\sigma(t,X,Y) := \frac{1}{2}(Y(t,t) - \log K)V_1(t,t,0,X(t,t),Y(t,t)) + \sqrt{X(t,t) - \frac{1}{4}(Y(t,t) - \log K)^2 \sum_{j=2}^{m} V_j(t,t,0,X(t,t),Y(t,t))^2} \tag{3.25}$$

for the stock volatility. With this function we define as in Corollary 2.9

$$\mu(t,X,Y) := -b_t^1 \sigma(t,X,Y), \tag{3.26}$$

$$\begin{aligned}
\alpha(t,T,X,Y) := {} & -\frac{1}{2}\left(\frac{(Y(t,t) - \log K)^2}{\left(\int_t^T X(t,s)\,ds\right)^2} - \frac{1}{\int_t^T X(t,s)\,ds} - \frac{1}{4} \right) \\
& \times v(t,T,X,Y) \cdot \int_t^T v(t,s,X,Y)\,ds \\
& + \frac{1}{2}\left(\frac{(Y(t,t) - \log K)^2}{\left(\int_t^T X(t,s)\,ds\right)^3} - \frac{1}{2}\frac{1}{\left(\int_t^T X(t,s)\,ds\right)^2} \right) X(t,T) \left| \int_t^T v(t,s,X,Y)\,ds \right|^2 \\
& + \left(\frac{Y(t,t) - \log K}{\int_t^T X(t,s)\,ds} - \frac{1}{2} \right) \sigma(t,X,Y) v^1(t,T,X,Y) \\
& - \frac{Y(t,t) - \log K}{\left(\int_t^T X(t,s)\,ds\right)^2} X(t,T) \sigma(t,X,Y) \int_t^T v^1(t,s,X,Y)\,ds \\
& - \sum_{j=1}^{m} b_t^j v^j(t,T,X,Y).
\end{aligned}$$

Let $Y_0 \in \mathbb{R}$ and X_0 be bounded on $[0, T^*]$ with $X_0 \geq \epsilon$. We take $d = 2$ and consider in $\mathcal{S}_c^{p,2}$ the SDE

$$\left.\begin{aligned}
dX(t,T) &= \alpha(t,T,X,Y)\,dt + v(t,T,X,Y)\,dW_t, \\
dY(t,T) &= \left(\mu(t,X,Y) - \frac{1}{2}(\sigma(t,X,Y))^2 \right) dt + \sigma(t,X,Y)\,dW_t
\end{aligned}\right\} \tag{3.27}$$

with initial condition $X(0,T) = X_0(T)$, $Y(0,T) = Y_0$. If we have a (unique) solution (X, Y) to (3.27), then as in the previous section Y does not depend on T and $S(t) := e^{Y(t,t)}$ gives a (unique) solution (S, X) to (2.9), (2.10).

We can now give sufficient conditions for (3.27) to have a unique solution. Recall from Section 3.2 the definition of a polynomial Lipschitz function.

THEOREM 3.4. *Let $p > 2$ be sufficiently large. Suppose that v is of the form (3.23), where V_j is polynomial Lipschitz on $(0, \infty)^2 \times \mathbb{R}$ and satisfies (3.24) and*

$$x - \frac{1}{4}(y - \log K)^2 \sum_{j=2}^{m} V_j(t, t, 0, x, y)^2 \geq x\epsilon \tag{3.28}$$

for all $t \in [0, T_0]$, $x > 0$, $y \in \mathbb{R}$, as well as

$$|V_j(t, T, w, x, y)| \leq C \frac{1}{w + (1 + \sqrt{x})(1 + |y|)} \quad \forall t, T. \tag{3.29}$$

Then (3.27) has a unique solution $(X, Y) \in \mathcal{S}_c^{p,2}$. Y does not depend on T, and $X > 0$.

REMARKS 3.4.

(1) Despite its appearance, (3.28) is a natural condition. In fact, its slightly weaker form with $\epsilon = 0$ is under (3.23) actually necessary for the model to be arbitrage-free, since we must have (2.29) by Corollary 2.9 (a). Note also that the technical condition (3.24) is trivially satisfied for $\epsilon = 0$.
(2) In comparison to the work of Schönbucher (1999), who also studied the case of call options, a major achievement here is that we give an existence result for the infinite system of SDEs derived from the drift restrictions. To the best of our knowledge, this has not been available so far.

Proof of Theorem 3.4. We use the same transformation $Z = \varphi(X)$ as in the proof of Theorem 3.3. Hence we want to apply Proposition A.3 to the SDE

$$\left.\begin{aligned} dZ(t, T) &= \bar{\alpha}(t, T, Y, Z)\, dt + v(t, T, \psi(Z), Y)\psi_1(Z(t, T))\, dW_t, \\ dY(t, T) &= \left[\mu(t, \psi(Z), Y) - \frac{1}{2}(\sigma(t, \psi(Z), Y))^2\right] dt + \sigma(t, \psi(Z), Y)\, dW_t \end{aligned}\right\} \tag{3.30}$$

with initial condition $Z(0, T) = \varphi(X_0(T))$, $Y(0, T) = Y_0$, where as above

$$\bar{\alpha}(t, T, Y, Z) := \alpha(t, T, \psi(Z), Y)\psi_1(Z(t, T)) + \frac{1}{2}|v(t, T, \psi(Z), Y)|^2 \psi_2(Z(t, T)).$$

Strong progressive measurability follows from (3.23). To check the local Lipschitz property, we again use Proposition A.5. By Lemma 3.2 (c) the functions

$$g_1(t, T, Y, Z) := \psi(Z(t, T))\psi_1(Z(t, T)),$$

$$h_1(t, T, Y, Z) := \psi(Z(t, T)))^2 \psi_2(Z(t, T)),$$

satisfy (A.4). Let $\theta(w) := (0 \vee \frac{2}{\epsilon}(w - \frac{\epsilon}{2})) \wedge 1$ and define the functions

$$\begin{aligned} g_2(t, T, Y, Z) &:= Y(t, t) - \log K, \\ g_3(t, T, Y, Z) &:= \sigma(t, \psi(Z), Y), \\ g_{4j}(t, T, Y, Z) &:= V_j\left(t, T, \int_t^T \psi(Z(t, s))\, ds, \psi(Z(t, t)), Y(t, t)\right), \\ g_{5j}(t, T, Y, Z) &:= \int_t^T v^j(t, s, \psi(Z), Y)\, ds, \end{aligned} \tag{3.31}$$

$$g_6(t, T, Y, Z) := \left(\int_t^T \psi(Z(t,s))\, ds \right)^{-1}, \tag{3.32}$$

$$g_7(t, T, Y, Z) := \theta \left(\int_t^T \psi(Z(t,s))\, ds \right). \tag{3.33}$$

Then we claim that g_2, g_3, g_{4j}, g_{5j} and $g_7,\ g_6 g_7,\ g_6^2 g_7,\ g_6^3 g_7$ satisfy (A.5). This is clear for g_2, it has been proved for g_{4j} and g_{5j} in Lemma 3.3, and it is proved in Lemma 3.4 below for g_3 and $g_7,\ g_6 g_7,\ g_6^2 g_7,\ g_6^3 g_7$. Using the definitions of $\bar{\alpha}$ and α in (3.26) and all the above functions yields

$$\bar{\alpha} = \beta - \sum_{j=1}^m b^j g_1 g_{4j} + \frac{1}{2} h_1 \sum_{j=1}^m g_{4j}^2,$$

where

$$\beta := -\frac{1}{2} \left(g_2^2 g_6^2 - g_6 - \frac{1}{4} \right) \sum_{j=1}^m g_1 g_{4j} g_{5j} + \frac{1}{2} \left(g_2^2 g_6^3 - \frac{1}{2} g_6^2 \right) g_1 \sum_{j=1}^m g_{5j}^2$$

$$+ \left(g_2 g_6 - \frac{1}{2} \right) g_3 g_1 g_{41} - g_2 g_6^2 g_3 g_1 g_{51}.$$

Now if $g_7 < 1$, then $w := \int_t^T \psi(Z(t,s))\, ds \le \epsilon$, and hence (3.23) and (3.24) imply

$$\begin{aligned} g_{5j} &= \int_t^T \psi(Z(t,s)) V_j \left(t, s, \int_t^s \psi(Z(t,r))\, dr, \psi(Z(t,t)),\ Y(t,t) \right) ds \\ &= \int_t^T \psi(Z(t,s)) V_j(t, t, \epsilon, \psi(Z(t,t)),\ Y(t,t))\, ds \\ &= \int_t^T \psi(Z(t,s))\, ds V_j \left(t, T, \int_t^T \psi(Z(t,r))\, dr, \psi(Z(t,t)),\ Y(t,t) \right) \\ &= g_6^{-1} g_{4j}, \end{aligned}$$

and so for $g_7 < 1$ we obtain

$$\beta = \beta_1 := \frac{1}{4} \sum_{j=1}^m g_1 g_{4j}^2 + \frac{1}{8} \sum_{j=1}^m g_1 g_{4j} g_{5j} - \frac{1}{2} g_3 g_1 g_{41}.$$

Therefore we can write $\bar{\alpha}$ as

$$\bar{\alpha} = \beta g_7 + \beta_1 (1 - g_7) - \sum_{j=1}^m b^j g_1 g_{4j} + \frac{1}{2} h_1 \sum_{j=1}^m g_{4j}^2,$$

and Proposition A.5 implies that $\bar{\alpha}$ is locally Lipschitz. Similarly one can see that the other coefficients of (3.30) are locally Lipschitz. To check (A.1), first note that (3.17) and (3.18) yield like for Theorem 3.3

$$|g_1| + |h_1| \le C_1 (1 + |Z(t,T)|)$$

for some constant C_1. Moreover, by (3.29) we have

$$|g_{4j}| \le C \left(\int_t^T \psi(Z(t,s))\, ds + \left(1 + \sqrt{\psi(Z(t,t))} \right) (1 + |Y(t,t)|) \right)^{-1} \le C$$

and hence by (3.23)

$$|g_{5j}| \le \int_t^T \psi(Z(t,s))|g_{4j}(t,s,Y,Z)|\,ds$$

$$\le C\frac{1}{\left(1+\sqrt{\psi(Z(t,t))}\right)(1+|Y(t,t)|)}\int_t^T \psi(Z(t,s))\,ds$$

and by (3.25)

$$|g_3| \le (|Y(t,t)|+|\log K|)|g_{41}(t,t,Y,Z)| + \sqrt{\psi(Z(t,t))} \le |\log K| + 1 + \sqrt{\psi(Z(t,t))},$$

which implies for some constant C_2 that

$$|g_2 g_3| \le C_2\left(1+\sqrt{\psi(Z(t,t))}\right)(1+|Y(t,t)|).$$

Putting all these bounds together implies that for some constant C^* we have

$$|\bar{\alpha}| \le C^*(1+|Z(t,T)|),$$

which is (A.1) for $\bar{\alpha}$. In a similar way, one can check (A.1) for the other coefficients in (3.30). Boundedness at $(Z(0,T), Y_0)$ of all coefficients is checked by straightforward but lengthy verification, and so we have verified the conditions we need to apply Proposition A.3. □

LEMMA 3.4. *For g_3, g_6, and g_7 defined by (3.31)–(3.33), the functions g_3, g_7, g_6g_7, $g_6^2g_7$, $g_6^3g_7$ satisfy (A.5).*

Proof. The functions $\sqrt{\psi(\cdot)}$ and $\psi(\cdot)^{-1}$ are Lipschitz on $\mathbb{R}$ and

$$g_3(t,T,Y,Z) = \frac{1}{2}(Y(t,t)-\log K)g_{41}(t,t,Y,Z)$$

$$+\sqrt{\psi(Z(t,t))}\sqrt{1-\frac{1}{4}\psi(Z(t,t))^{-1}(Y(t,t)-\log K)^2\sum_{j=2}^m g_{4j}(t,t,Y,Z)^2}.$$

We have already proved that g_{4j} satisfies (A.5); hence the expression under the big square root above is polynomial Lipschitz in $(Y(t,t), Z(t,t))$ and $\int_0^{T^*} Z(t,s)\,ds$, and it has values in $[\epsilon,\infty)$ due to (3.28). Since $\sqrt{\cdot}$ is Lipschitz on $[\epsilon,\infty)$, the claim for g_3 follows. For g_7, g_6g_7, $g_6^2g_7$, $g_6^3g_7$, note that the functions $\theta(w), w^{-1}\theta(w), w^{-2}\theta(w), w^{-3}\theta(w)$ are Lipschitz on $[0,\infty)$. Hence the assertion follows from

$$\left|\int_t^T \psi(Z(t,s))\,ds - \int_t^T \psi(Z'(t,s))\,ds\right| \le \int_0^{T^*} |Z(t,s)-Z'(t,s)|\,ds. \qquad \square$$

EXAMPLE 3.4. As in the proof of Theorem 3.4, let $\theta(w) := (0 \vee \frac{2}{\epsilon}(w-\frac{\epsilon}{2}))\wedge 1$. For $j=1,\dots,m$, let C_j be a bounded polynomial Lipschitz function on $[0,T^*]^2\times(0,\infty)$ and U_j a bounded polynomial Lipschitz function on $(0,\infty)^2\times\mathbb{R}$ satisfying

$$|C_j(t,0,x)| \le \sqrt{\frac{1-\epsilon}{m-1}}\sqrt{x} \qquad \forall j=2,\dots,m,$$

$$|U_j(w,x,y)| \le \frac{1}{w+(1+\sqrt{x})(1+|y-\log K|)} \qquad \forall j=1,\dots,m.$$

Then the conditions of Theorem 3.4 are satisfied by the coefficient v given by

$$v^j(t, T, X, Y) = X(t, T)C_j\left(t, (T-t)\theta\left(\int_t^T X(t,s)\,ds\right), X(t,t)\right)$$
$$\times U_j\left(\int_t^T X(t,s)\,ds, X(t,t), Y(t,t)\right).$$

Let us conclude with some comments on the function $v(t, T, X, Y)$ chosen in (3.23). The linear dependence of $v(t, T, X, Y)$ on the forward implied volatility $X(t, T)$ is quite restrictive, and in fact (3.23) could be replaced by (3.1) for some Lipschitz-continuous function f satisfying $f(t, T, 0) = 0$ up to a short time before maturity, i.e., for $T - t \geq \epsilon$. However, we need the existence of the limits in (2.29), and it seems very difficult to avoid the linear dependence form (3.23) close to maturity, i.e., for $T - t \searrow 0$. So for simplicity, we have restricted ourselves to (3.23) here. As a second remark, note that, in contrast to (3.2) for the power payoff contracts, $\alpha(t, T, X, Y)$ in (3.26) is not proportional to $v(t, T, X, Y)$; hence we cannot ensure $X(t, T) \geq \epsilon$ by imposing a condition like (3.4) for v.

4. COMMENTS AND CONCLUSION

Our basic goal in this paper is to study models for a stock S and implied volatilities X. Hence we start from the premise that one wants to specify a model for S and X by prescribing a term structure of volatilities v for X in functional form, and we ask for conditions on v that guarantee the existence and uniqueness of a solution for S, X which does not allow arbitrage. We think it is important to have such results since the drift restrictions in Theorem 2.1 are easy to derive, but not so easy to satisfy.

One possible approach to obtain an arbitrage-free model for implied volatilities is to start from a standard stochastic or local volatility model and examine option prices there. More precisely, consider for deterministic functions $\mu(\cdot,\cdot), \beta(\cdot,\cdot), \gamma(\cdot,\cdot)$ a *stochastic volatility model* of the form

$$dS(t) = \mu(t, \sigma(t))S(t)\,dt + \sigma(t)S(t)\,dW_t^1, \tag{4.1}$$

$$d\sigma(t) = \beta(t, \sigma(t))\,dt + \gamma^1(t, \sigma(t))\,dW_t^1 + \gamma^2(t, \sigma(t))\,dW_t^2 \tag{4.2}$$

for a two-dimensional Brownian motion (W^1, W^2). We assume that the process $\sigma(\cdot)$ is a.s. positive, and that the market price of risk process $b = (b^1, b^2) = (-\frac{\mu}{\sigma}, b^2)$ corresponding to the pricing measure $\frac{dQ^{T^*}}{dP} = \mathcal{E}(\int b\,dW)_{T^*}$ is a deterministic function of $(t, \sigma(t))$. This framework includes classical stochastic volatility models such as Heston (1993) or Hull and White (1987). Under some regularity assumptions on the coefficients μ, β, γ, b, one can then show that the price $C_t^{K,T}$ of a European call with strike K and maturity T can be written as $U(T, t, S(t), \sigma(t))$, where U is a smooth function which is strictly increasing in its last argument σ for all $t < T$; see theorem 3.1 and proposition 4.2 of Romano and Touzi (1997). We can therefore define $\Sigma(T, t, S(t), \cdot)$ to be the inverse of $U(T, t, S(t), \cdot)$ for $t < T$. Thus we have

$$\Sigma\left(T, t, S(t), c^K\left(S(t), \int_t^T X(t,s)\,ds\right)\right) = \sigma(t). \tag{4.3}$$

From (2.18), (2.19) we now obtain with $\tilde{W} := W - \int b\,dt$ that

$$\begin{aligned} dC_t^{K,T} &= c_S^K \sigma(t) S(t) d\tilde{W}_t^1 + c_\Upsilon^K \left(\int_t^T v(t,s)\,ds \right) d\tilde{W}_t \\ &= U_S \sigma(t) S(t)\, d\tilde{W}_t^1 + U_\sigma \gamma(t, \sigma(t))\, d\tilde{W}_t \end{aligned}$$

and therefore

$$\begin{aligned} (4.4) \qquad & \int_t^T v^1(t,s)\,ds \\ &= \left[U_\sigma(T, t, S(t), \sigma(t)) \gamma^1(t, \sigma(t)) \right. \\ &\qquad \left. + \left(U_S(T, t, S(t), \sigma(t)) - c_S^K \left(S(t), \int_t^T X(t,s)\,ds \right) \right) \sigma(t) S(t) \right] \\ &\quad \times \frac{1}{c_\Upsilon^K \left(S(t), \int_t^T X(t,s)\,ds \right)}, \end{aligned}$$

$$(5.5) \qquad \int_t^T v^2(t,s)\,ds = [U_\sigma(T, t, S(t), \sigma(t)) \gamma^2(t, \sigma(t))] \frac{1}{c_\Upsilon^K \left(S(t), \int_t^T X(t,s)\,ds \right)}.$$

If we now plug in $\sigma(t)$ from (4.3) and differentiate (4.4), (4.5) with respect to T, we obtain $v(t, T)$ as a deterministic function of t, T, $X(t, T)$, $\int_t^T X(t,s)\,ds$ and $S(t)$. This specification of $v(t, T)$ together with (4.3), (2.30) and (2.31) yields the forward implied volatility model (2.9), (2.10) corresponding to the stochastic volatility model (4.1), (4.2). Of course, more explicit computations are only possible if we have more explicit results on the call price function U in the given model.

Note that the resulting forward implied volatility model for S, X is Markovian in (S, X), but not in (S, σ); see the remark at the beginning of Section 3.2. Observe also that we cannot and do not expect that $v(t, T)$ from (4.4), (4.5) satisfy the technical conditions of our existence theorems; these are sufficient, but certainly not necessary.

Similarly as above we can treat a *local volatility model* of the form

$$(6.6) \qquad dS(t) = \mu(t, S(t)) S(t)\,dt + \sigma(t, S(t)) S(t)\,dW_t$$

for a one-dimensional Brownian motion W and deterministic functions $\mu(\cdot, \cdot), \sigma(\cdot, \cdot)$. Here we have a complete market, and under some regularity conditions on μ, σ, the price $C_t^{K,T}$ can be written as $U(T, t, S(t))$. With $b := -\frac{\mu}{\sigma}$ and $\tilde{W} := W - \int b\,dt$ we obtain as above from (2.18), (2.19) that

$$dC_t^{K,T} = c_S^K \sigma(t, S(t)) S(t)\, d\tilde{W}_t + c_\Upsilon^K \left(\int_t^T v(t,s)\,ds \right) d\tilde{W}_t = U_S \sigma(t, S(t)) S(t)\, d\tilde{W}_t$$

and therefore

$$\int_t^T v(t,s)\,ds = \left(U_S(T,t,S(t),\sigma(t)) - c_S^K\left(S(t), \int_t^T X(t,s)\,ds\right)\right)\sigma(t,S(t))S(t) \times \frac{1}{c_\Upsilon^K\left(S(t), \int_t^T X(t,s)\,ds\right)}.$$

As above, differentiating with respect to T yields $v(t,T)$ as a deterministic function of t, T, $X(t,T)$, $\int_t^T X(t,s)\,ds$ and $S(t)$ and gives us the forward implied volatility model (2.9), (2.10) corresponding to the local volatility model (4.6).

With the above approach, existence of the model for (S,X) is by construction not an issue; this is similar to the approach in Jacod and Protter (2006), with the same drawback that v is somehow given (as an exogenous process) and cannot be specified in functional form.

REMARK 4.1. As mentioned in the remark before Section 2.1, an obvious open question is how to extend the present approach to the case of several payoff functions (e.g., calls with both different strikes and maturities). This is current work in progress. Another open issue is how our approach works for a concrete model, including calibration. This is beyond the scope of the present paper and left for future research.

APPENDIX

In this Appendix, we first present some further definitions and results from Wissel (2007) which are used in the proofs of the existence results in Section 3. Recall the definitions of $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathcal{G}}, \tilde{\mathbb{G}}, \tilde P, \tilde W)$ and $\mathcal{S}_c^p$ from Section 3.1. Note that tilde quantities like $\tilde\Omega$, etc. correspond to the analogous quantities (like Ω) without tilde in Wissel (2007), and quantities here without tilde (like Ω, W etc.) correspond to the analogous quantities (like Ω^1, W^1, etc.) with superscript^1 in Wissel (2007). In particular, $\tilde\omega = (T,\omega)$ here corresponds to ω in Wissel (2007).

We repeat some further definitions from Wissel (2006). The coefficients of our SDEs lie in the following class.

DEFINITION A.1. Let $n \in \mathbb{N}$. A map $f : [0,T_0] \times \tilde\Omega \times \{X \mid X\,\tilde{\mathbb{G}}\text{-adapted process}\} \to \mathbb{R}^n$ is called *strongly* ($\mathcal{S}_c^p$-)*progressively measurable* if for each $X \in \mathcal{S}_c^p$ the map

$$(t,T,\omega) \mapsto f(t,T,\omega,X)$$

is progressively measurable and satisfies for all $X \in \mathcal{S}_c^p$ and for each $\tilde{\mathcal{F}}$-measurable stopping time τ

$$f(t,\cdot,X)I_{\{t\le\tau(\cdot)\}} = f(t,\cdot,X^\tau)I_{\{t\le\tau(\cdot)\}} \quad \forall t \quad \tilde P\text{-a.s.}$$

Note that a quantity on $\tilde\Omega$ does not depend on T iff it is $\tilde{\mathcal{F}}$-measurable. For a process $X \in \mathcal{S}_c^p$ define the process $q(X)$ by

$$q(X)(t) := \left(\frac{1}{T^*}\int_0^{T^*} \sup_{0\le u\le t} |X(u,T,\cdot)|^p\,dT\right)^{\frac{1}{p}}, \quad t \in [0,T_0].$$

It is easy to check that $q(X)$ is $\tilde{\mathcal{F}}$-measurable and $\tilde{\mathbb{G}}$-adapted, and dominated convergence yields that it is $\tilde{P}$-a.s. continuous in t since $X \in \mathcal{S}_c^p$. Define for each $X \in \mathcal{S}_c^p$ a sequence of $[0, T_0] \cup \{\infty\}$-valued stopping times $\tau_N(X)$, $N \in \mathbb{N}$, by

$$\tau_N(X) := \inf\{t \in [0, T_0] \,|\, q(X)(t) \geq N\}$$

with $\inf \emptyset = \infty$. Note that as a random variable, $\tau_N(X)$ is $\tilde{\mathcal{F}}$-measurable.

DEFINITION A.2. A strongly progressively measurable function f is called *locally Lipschitz (on $\mathcal{S}_c^p$)* if there exist functions C_N with $C_N(t) \overset{t \to 0}{\longrightarrow} 0$ such that for all $t \in [0, T_0]$ and $X, X' \in \mathcal{S}_c^p$ we have

$$\frac{1}{T^*} \int_0^{T^*} \left(\int_0^{t \wedge \tau_N(X) \wedge \tau_N(X')} |f(u, T, \cdot, X) - f(u, T, \cdot, X')|^2 \, du \right)^{\frac{p}{2}} dT$$

$$\leq C_N(t) \big(q(X - X') \big(t \wedge \tau_N(X) \wedge \tau_N(X') \big) \big)^p.$$

We use the following results; they follow in each case from the quoted results in Wissel (2007) by checking the latter's assumptions.

PROPOSITION A.1 (Wissel 2007, theorem 3.1). *Let $d = 2$, $p > 2$ and (X_0, Y_0) be uniformly bounded functions on $[0, T^*]$. Suppose that β and ν are strongly progressively measurable and locally Lipschitz on $\mathcal{S}_c^{p,2}$. Suppose that $f \in \{\beta, \nu\}$ satisfy $|f(u, T, \cdot, (X_0, Y_0))| \leq C$ as well as the growth condition*

$$E^{\tilde{P}} \left[\int_0^{T_0} |f(u, T, \cdot, (X, Y))|^p \, du \right] \leq C(1 + \|(X, Y)\|^p) \tag{A.1}$$

for $(X, Y) \in \mathcal{S}_c^{p,2}$. Then the SDE system

$$d(X, Y)(t, T, \cdot) = \beta(t, T, \cdot, (X, Y)) \, dt + \nu(t, T, \cdot, (X, Y)) \, dW_t$$

with $(X, Y)(0, T, \cdot) = (X_0(T), Y_0(T))$ has a unique solution $(X, Y) \in \mathcal{S}_c^{p,2}$.

PROPOSITION A.2 (Wissel 2007, theorem 4.2). *Let $d = 1$, $p > 2$, and $X_0 \in L^p[0, T^*]$ satisfy $X_0 \geq \epsilon$ for some $\epsilon > 0$. Suppose $\beta^* \geq 0$, β and ν are strongly progressively measurable functions with β locally Lipschitz on the set $\{X \in \mathcal{S}_c^{p,1} \mid X \geq \frac{\epsilon}{2}\}$. Also suppose that for $f \in \{\beta^*, \nu\}$ and $X, X' \in \mathcal{S}_c^{p,1}$ we have*

$$\left.\begin{aligned} f(t, T, \cdot, \epsilon) &\leq C \quad \forall t, T, \\ |f(t, T, \cdot, X) - f(t, T, \cdot, X')| &\leq C |X(t, T) - X'(t, T)| \quad \forall t, T, \end{aligned}\right\} \tag{A.2}$$

that for $X, X^* \in \mathcal{S}_c^{p,1}$ we have the implication

$$\frac{\epsilon}{2} \leq X \leq X^* \Longrightarrow \beta(\cdot, \cdot, \cdot, X) \leq \beta^*(\cdot, \cdot, \cdot, X^*),$$

and that for $f \in \{\beta, \nu\}$ and $X \in \mathcal{S}_c^{p,1}$ we have

$$\text{for all } t, T, \omega, \quad X(t, T, \omega) \leq \epsilon \Longrightarrow f(t, T, \omega, X) = 0. \tag{A.3}$$

Then the SDE

$$dX(t, T, \cdot) = \beta(t, T, \cdot, X) \, dt + \nu(t, T, \cdot, X) \, dW_t$$

with $X(0, T, \cdot) = X_0$ has a unique solution $X \in \mathcal{S}_c^{p,1}$. It satisfies $X \geq \epsilon$.

The local Lipschitz condition in Definition A.2 is often difficult to verify. We therefore, give a useful criterion for the local Lipschitz property which we apply in Section 3.

PROPOSITION A.3. *Suppose that $g_1, \dots, g_n$ are strongly progressively measurable functions satisfying $|g_j(t, T, 0)| \le C$ for $j = 1, \dots, n$ and for all $X, X' \in \mathcal{S}_c^p$*

$$|g_1(t, T, X) - g_1(t, T, X')| \le C|X(t, T) - X'(t, T)|, \tag{A.4}$$

$$|g_j(t, T, X) - g_j(t, T, X')| \le C_j \left(X(t, t), X'(t, t), \int_0^{T^*} |X(t, s)|\, ds, \int_0^{T^*} |X'(t, s)|\, ds \right)$$

$$\times \left(|X(t, t) - X'(t, t)| + \int_0^{T^*} |X(t, s) - X'(t, s)|\, ds \right) \tag{A.5}$$

for $j = 2, \dots, n$, where C_j is a function of at most polynomial growth. If $p \ge 1$ exceeds a constant which only depends on n and the functions C_j, then the product $g_1 \cdots g_n$ is locally Lipschitz (on $\mathcal{S}_c^p$).

Proof. We use proposition 3.3 in Wissel (2007). Since C_j has at most polynomial growth, Jensen's inequality yields constants $c, k > 1$ such that

$$B(u) := C_j \left(X(u, u), X'(u, u), \int_0^{T^*} |X(u, s)|\, ds, \int_0^{T^*} |X'(u, s)|\, ds \right) \tag{A.6}$$

$$\le c \left(1 + |X(u, u)|^k + |X'(u, u)|^k + \int_0^{T^*} |X(u, s)|^k\, ds + \int_0^{T^*} |X'(u, s)|^k\, ds \right).$$

Let $p := 2p'k$ for some $p' > 2n$. Note that

$$\int_0^t |X(u, u)|^p\, du \le \int_0^t \sup_{0 \le u \le t} |X(u, s)|^p\, ds \le T^*(q(X)(t))^p, \tag{A.7}$$

$$\int_0^t \int_0^{T^*} |X(u, s)|^p\, ds\, du \le T_0 \int_0^{T^*} \sup_{0 \le u \le t} |X(u, s)|^p ds \le (T^*)^2 (q(X)(t))^p, \tag{A.8}$$

and similarly by using Jensen's inequality for the power k

$$\int_0^t |X(u, u) - X'(u, u)|^{2p'}\, du \le ((T^*)^{k-1} T^*(q(X - X')(t))^p)^{\frac{1}{k}} = T^*(q(X - X')(t))^{2p'}, \tag{A.9}$$

$$\int_0^t \int_0^{T^*} |X(u, s) - X'(u, s)|^{2p'}\, ds\, du \le (T^*)^2 (q(X - X')(t))^{2p'}. \tag{A.10}$$

Set $\tau := t \wedge \tau_N(X) \wedge \tau_N(X')$ for $t \in [0, T_0]$ and use (A.5) to get for $j = 2, \dots, n$

$$|g_j(u, T, X) - g_j(u, T, X')|^{p'}$$

$$\le B(u)^{p'} \left(|X(u, u) - X'(u, u)| + \int_0^{T^*} |X(u, s) - X'(u, s)|\, ds \right)^{p'}.$$

Combining the Cauchy–Schwarz inequality with (A.6) and (A.7)–(A.10) gives

$$\begin{aligned}
&\int_0^\tau |g_j(u,T,X) - g_j(u,T,X')|^{p'}\,du \\
&\le \left(\int_0^\tau \left(|X(u,u) - X'(u,u)| + \int_0^{T^*} |X(u,s) - X'(u,s)|\,ds\right)^{2p'} du\right)^{\frac{1}{2}} \left(\int_0^\tau B(u)^{2p'}\,du\right)^{\frac{1}{2}} \\
&\le \Bigg[2^{2p'-1}\left(\int_0^\tau |X(u,u) - X'(u,u)|^{2p'}\,du + (T^*)^{2p'-1}\int_0^\tau\int_0^{T^*} |X(u,s) - X'(u,s)|^{2p'}\,ds\,du\right) \\
&\quad \times c^{2p'}5^{2p'-1}\left(T_0 + \int_0^\tau |X(u,u)|^{2p'k}\,du + \int_0^\tau |X'(u,u)|^{2p'k}\,du\right. \\
&\qquad \left.+ (T^*)^{2p'-1}\int_0^\tau\int_0^{T^*} |X(u,s)|^{2p'k}ds\,du + (T^*)^{2p'-1}\int_0^\tau\int_0^{T^*} |X'(u,s)|^{2p'k}ds\,du\right)\Bigg]^{\frac{1}{2}} \\
&\le (2^{2p'-1}(T^* + (T^*)^{2p'+1}))^{\frac{1}{2}}(q(X - X')(\tau))^{p'}(c^{2p'}5^{2p'-1})^{\frac{1}{2}}(T^* + 2T^*N^p + 2(T^*)^{2p'+1}N^p)^{\frac{1}{2}};
\end{aligned}$$

the last step uses the definition of τ. Finally, from (A.4) we have

$$\frac{1}{T^*}\int_0^{T^*}\int_0^\tau |g_1(u,T,X) - g_1(u,T,X')|^p\,du\,dT \le T^*C(q(X - X')(\tau)^p.$$

Since $\frac{1}{p} + (n-1)\frac{1}{p'} \le n\frac{1}{p'} < \frac{1}{2}$, it follows from proposition 3.3 in Wissel (2007) that $g_1 \cdots g_n$ is locally Lipschitz.

We conclude the first part of the appendix with a result that is used to establish positivity of the forward implied volatilities. A similar result is given in Miltersen (1994, theorem 3.5) to obtain positivity of forward rates in an HJM interest rate framework.

PROPOSITION A.4 (Wissel 2007, proposition 2.4). *Let $\Gamma \subseteq \mathbb{R}^d$ be a closed set, $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $v \in L^2_{\mathrm{loc}}(\mathbb{R}^{d\times n})$, and $X(0) \in \Gamma$. Let X be given by*

$$X(t) = X(0) + \int_0^t u(s)\,ds + \int_0^t v(s)\,dW_s \quad (0 \le t < \infty).$$

If X and u, v satisfy $\tilde{P}$-a.s.

$$\text{for all } t,\ X(t) \in \mathbb{R}^d\backslash\Gamma \Longrightarrow u(t) = 0 \text{ and } v(t) = 0,$$

then $X(t) \in \Gamma$ for all $t \ge 0$ $\tilde{P}$-a.s.

Finally, we explain in some more detail how to obtain (2.29) from (2.12). By using that $\sigma_t > 0$ and $S_t > 0$ for a.e. t, P-a.s., one first shows that for fixed $K > 0$, we have $S_t \ne K$ for a.e. $t \ge 0$, Q-a.s. and then P-a.s. With the equations for the partials before (2.29) and setting $q_j(T) := \int_t^T v^j(t,s)\,ds \big/ \Upsilon_t(T)$, we thus obtain from (2.12)

$$\begin{aligned}
\text{(A.11)} \qquad \sigma_t^2 - X(t,t) &= \sigma_t \log\left(\frac{S_t}{K}\right)\lim_{T\searrow t} q_1(T) \\
&\quad - \frac{1}{4}\log^2\left(\frac{S_t}{K}\right)\lim_{T\searrow t}\left[\left(1 - \frac{\Upsilon_t(T)}{\log^2(S_t/K)}\right)\sum_{j=1}^m q_j(T)^2\right]
\end{aligned}$$

for a.e. t, P-a.s. Now fix ω outside a P-nullset as above. By assumption, $(\sigma_t^2)_{t\geq 0} \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $(X(t,t))_{t\geq 0} \in L^1_{\mathrm{loc}}(\mathbb{R})$ (this is implicitly assumed in Section 2.2; it is actually used earlier in Proposition 2.2). Thus for a.e. t, the LHS of (A.11) is finite, and therefore so is the RHS. If we had $\limsup_{T\searrow t} \sum_{j=1}^m q_j(T)^2 = \infty$, then $\lim_{T\searrow t}(1 - \frac{\Upsilon_t(T)}{\log^2(S_t/K)}) = 1$ and $|q_1(T)| \leq \sqrt{\sum_{j=1}^m q_j(T)^2}$ would imply that the term with $\sum_{j=1}^m q_j(T)^2$ dominates the $q_1(T)$-term for $T \searrow t$, and then the RHS of (A.11) would be $-\infty$, a contradiction. Hence we must have

$$\limsup_{T\searrow t} \sum_{j=1}^m q_j(T)^2 < \infty,$$

so the term $\lim_{T\searrow t}[\frac{\Upsilon_t(T)}{\log^2(S_t/K)} \sum_{j=1}^m q_j(T)^2]$ vanishes, and the RHS of (A.11) becomes

$$\sigma_t \log\left(\frac{S_t}{K}\right) \lim_{T\searrow t} \frac{\int_t^T v^1(t,s)\,ds}{\Upsilon_t(T)} - \frac{1}{4}\log^2\left(\frac{S_t}{K}\right) \lim_{T\searrow t} \sum_{j=1}^m \left(\frac{\int_t^T v^j(t,s)\,ds}{\Upsilon_t(T)}\right)^2.$$

This yields (2.29) after rearranging terms.

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