M^6 — On Minimal Market Models and Minimal Martingale Measures

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This paper is dedicated to Eckhard Platen on the occasion of his 60th birthday.

Abstract The well-known absence-of-arbitrage condition NFLVR from the fundamental theorem of asset pricing splits into two conditions, called NA and NUPBR. We give a literature overview of several equivalent reformulations of NUPBR; these include existence of a growth-optimal portfolio, existence of the numeraire portfolio, and for continuous asset prices the structure condition (SC). As a consequence, the minimal market model of E. Platen is seen to be directly linked to the minimal martingale measure. We then show that reciprocals of stochastic exponentials of continuous local martingales are time changes of a squared Bessel process of dimension 4. This directly gives a very specific probabilistic structure for minimal market models.

1 Introduction

Classical mathematical finance has been built on pillars of absence of arbitrage; this is epitomised by the celebrated fundamental theorem of asset pricing (FTAP), due in its most general form to F. Delbaen and W. Schachermayer. However, several recent directions of research have brought up the question whether one should not also study more general models that do not satisfy all the stringent requirements of the FTAP; see also [21] for an early contribution in that spirit. One such line of research is the recent work of R. Fernholz and I. Karatzas on *diverse markets*, of which an overview is given in [17]. Another is the *benchmark approach* and

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the idea of *minimal market models* proposed and propagated by E. Platen and co-authors in several recent publications; see [24] for a textbook account. Finally, also some approaches to *bubbles* go in a similar direction.

Our goal in this paper is twofold. We first give a neutral overview of several equivalent formulations of an L^0-boundedness property, called NUPBR, that makes up a part, but not all of the conditions for the FTAP. For continuous asset prices, we then show that the *minimal market model* of E. Platen is very directly linked to the *minimal martingale measure* introduced by H. Föllmer and M. Schweizer. As a consequence, we exhibit a very specific probabilistic structure for minimal market models: We show that they are *time changes of a squared Bessel process of dimension 4 (a BESQ*^{4}*)*, under very weak assumptions. This extends earlier work in [22] to the most general case of a continuous (semimartingale) financial market.

The paper is structured as follows. Section 2 considers general semimartingale models, introduces basic notations, and recalls that the well-known condition NFLVR underlying the FTAP consists of two parts: no arbitrage NA, and a certain boundedness condition in L^0, made more prominent through its recent labelling as NUPBR by C. Kardaras and co-authors. We collect from the literature several equivalent formulations of this property, the most important for subsequent purposes being the existence of a *growth-optimal portfolio*. Section 3 continues this overview under the additional assumption that the basic price process S is continuous; the main addition is that NUPBR is then also equivalent to the *structure condition (SC)* introduced by M. Schweizer, and that it entails the existence of the minimal martingale density for S.

Both Sections 2 and 3 contain only known results from the literature; their main contribution is the effort made to present these results in a clear, concise and comprehensive form. The main probabilistic result in Section 4 shows that reciprocals of stochastic exponentials of continuous local martingales are automatically time changes of BESQ^4 processes. Combining this with Section 3 then immediately yields the above announced structural result for minimal market models.

2 General financial market models

This section introduces basic notations and concepts and recalls a number of general known results. Loosely speaking, the main goal is to present an overview of the relations between absence of arbitrage and existence of a log-optimal portfolio strategy, in a frictionless financial market where asset prices can be general semimartingales. The only potential novelty in all of this section is that the presentation is hopefully clear and concise. We deliberately only give references to the literature instead of repeating proofs, in order not to clutter up the presentation.

We start with a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions of right-continuity and P-completeness. To keep notations simple, we assume that the time horizon $T \in (0, \infty)$ is nonrandom and finite. All our basic processes will be defined on $[0, T]$ which frees us from worrying about their behaviour at "infinity" or "the end of time". Results from the literature on processes living on $[0,\infty)$ are used by applying them to the relevant processes stopped at T.

We consider a financial market with $d+1$ assets. One of these is chosen as numeraire or unit of account, labelled with the number 0, and all subsequent quantities are expressed in terms of that. So we have an asset $S^0 \equiv 1$ and d "risky" assets whose price evolution is modelled by an $\mathbb{R}^d$-valued semimartingale $S = (S_t)_{0\le t\le T}$, where S_t^i is the price at time t of asset $i \in \{1,\dots,d\}$, expressed in units of asset 0. To be able to use stochastic integration, we assume that S is a semimartingale.

Trading in our financial market is frictionless and must be done in a self-financing way. Strategies are then described by pairs (x,ϑ), where $x\in\mathbb{R}$ is the initial capital or initial wealth at time 0 and $\vartheta = (\vartheta_t)_{0\le t\le T}$ is an $\mathbb{R}^d$-valued predictable S-integrable process; we write $\vartheta \in L(S)$ for short. The latter means that the (real-valued) stochastic integral process $\vartheta \boldsymbol{\cdot} S := \int \vartheta\, dS$ is well defined and then again a semimartingale. We remark in passing that $\vartheta \boldsymbol{\cdot} S$ must be understood as a vector stochastic integral, which may be different from the sum of the componentwise stochastic integrals; see [14] for the general theory and [4] for an amplification of the latter point. In financial terms, ϑ_t^i is the number of units of asset i that we hold in our dynamically varying portfolio at time t, and the self-financing condition means that our wealth at time t is given by

$$X_t^{x,\vartheta} := x + \vartheta \boldsymbol{\cdot} S_t = x + \int_0^t \vartheta_u\, dS_u, \qquad 0 \le t \le T.$$

Not every $\vartheta \in L(S)$ yields a decent trading strategy. To exclude unpleasant phenomena resulting from doubling-type strategies, one has to impose some lower bound on the trading gains/losses $\vartheta \boldsymbol{\cdot} S$. We call $\vartheta \in L(S)$ *a-admissible* if $\vartheta \boldsymbol{\cdot} S \ge -a$, where $a \ge 0$, and *admissible* if it is a-admissible for some $a \ge 0$. We then introduce for $x > 0$ the sets

$$\mathcal{X}^x := \left\{X^{x,\vartheta} \,\middle|\, \vartheta \in L(S) \text{ and } X^{x,\vartheta} \ge 0\right\} = \{x + \vartheta \boldsymbol{\cdot} S \mid \vartheta \in L(S) \text{ is } x\text{-admissible}\},$$
$$\mathcal{X}^{x,++} := \left\{X^{x,\vartheta} \,\middle|\, \vartheta \in L(S) \text{ and } X^{x,\vartheta} > 0 \text{ as well as } X_-^{x,\vartheta} > 0\right\},$$

and we set $\mathcal{X}_T^x := \left\{X_T^{x,\vartheta} \,\middle|\, X^{x,\vartheta} \in \mathcal{X}^x\right\}$, with $\mathcal{X}_T^{x,++}$ defined analogously. So every $f \in \mathcal{X}_T^x$ represents a terminal wealth position that one can generate out of initial wealth x by self-financing trading while keeping current wealth always nonnegative (and even strictly positive, if f is in $\mathcal{X}_T^{x,++}$). We remark that $X_-^{x,\vartheta} > 0$ does not follow from $X^{x,\vartheta} > 0$ since we only know that $X^{x,\vartheta}$ is a semimartingale; we have no local martingale or supermartingale property at this point. Note that all the ϑ appearing in the definition of $\mathcal{X}_T^x$ have the same uniform lower bound for $\vartheta \boldsymbol{\cdot} S$, namely $-x$. Finally, we need the set

$$\mathcal{C} := \left\{X_T^{0,\vartheta} - B \,\middle|\, \vartheta \in L(S) \text{ is admissible}, B \in L_+^0(\mathcal{F}_T)\right\} \cap L^\infty$$

of all bounded time T positions that one can dominate by self-financing admissible trading even from initial wealth 0.

With the above notations, we can now recall from [9] and [18] the following concepts.

Definition. Let S be a semimartingale. We say that *S satisfies NA, no arbitrage*, if $\mathcal{C} \cap L^\infty_+ = \{0\}$; in other words, $\mathcal{C}$ contains no nonnegative positions except 0. We say that *S satisfies NFLVR, no free lunch with vanishing risk*, if $\overline{\mathcal{C}}^\infty \cap L^\infty_+ = \{0\}$, where $\overline{\mathcal{C}}^\infty$ denotes the closure of $\mathcal{C}$ in the norm topology of L^∞. Finally, we say that *S satisfies NUPBR, no unbounded profit with bounded risk*, if $\mathcal{X}^x_T$ is bounded in L^0 for some $x > 0$ (or, equivalently, for all $x > 0$ or for $x = 1$, because $\mathcal{X}^x_T = x\mathcal{X}^1_T$).

The condition NFLVR is a precise mathematical formulation of the natural economic idea that it should be impossible in a financial market to generate something out of nothing without risk. The meta-theorem that "absence of arbitrage is tantamount to the existence of an equivalent martingale measure" then takes the precise form that S satisfies NFLVR if and only if there exists a probability measure Q equivalent to P such that S is under Q a so-called σ-martingale. This is the celebrated *fundamental theorem of asset pricing (FTAP)* in the form due to F. Delbaen and W. Schachermayer; see [6, 8].

In the sequel, our interest is neither in the FTAP nor in equivalent σ-martingale measures Q as above; hence we do not explain these in more detail. Our focus is on the condition NUPBR and its ramifications. The connection to NFLVR is very simple and direct:

S satisfies NFLVR if and only if it satisfies both NA and NUPBR.

This result can be found either in Section 3 of [6] or more concisely in Lemma 2.2 of [15]. Moreover, neither of the conditions NA and NUPBR implies the other, nor of course NFLVR; see Chapter 1 of [12] for explicit counterexamples.

The next definition introduces strategies with certain optimality properties.

Definition. An element $X^{\mathrm{np}} = X^{1,\vartheta^{\mathrm{np}}}$ of $\mathcal{X}^{1,++}$ is called a *numeraire portfolio* if the ratio $X^{1,\vartheta}/X^{\mathrm{np}}$ is a P-supermartingale for every $X^{1,\vartheta} \in \mathcal{X}^{1,++}$. An element $X^{\mathrm{go}} = X^{1,\vartheta^{\mathrm{go}}}$ of $\mathcal{X}^{1,++}$ is called a *growth-optimal portfolio* or a *relatively log-optimal portfolio* if

$$E\left[\log\left(X^{1,\vartheta}_T / X^{\mathrm{go}}_T\right)\right] \le 0$$

for all $X^{1,\vartheta} \in \mathcal{X}^{1,++}$ such that the above expectation is not $\infty - \infty$. Finally, an element $X^{\mathrm{lo}} = X^{1,\vartheta^{\mathrm{lo}}}$ of $\mathcal{X}^{1,++}$ with $E\left[\log X^{\mathrm{lo}}_T\right] < \infty$ is called a *log-utility-optimal portfolio* if

$$E\left[\log X^{1,\vartheta}_T\right] \le E\left[\log X^{\mathrm{lo}}_T\right]$$

for all $X^{1,\vartheta} \in \mathcal{X}^{1,++}$ such that $E\left[\left(\log X^{1,\vartheta}_T\right)^-\right] < \infty$.

For all the above concepts, we start with initial wealth 1 and look at self-financing strategies whose wealth processes (together with their left limits) must remain strictly positive. In all cases, we also commit a slight abuse of terminology by calling "portfolio" what is actually the wealth process of a self-financing strategy. In words, the above three concepts can then be described as follows:

- The *numeraire portfolio* has the property that, when used for discounting, it turns every wealth process in $\mathcal{X}^{1,++}$ into a supermartingale. Loosely speaking, this means that it has the best “performance” in the class $\mathcal{X}^{1,++}$.
- The *growth-optimal portfolio* has, in relative terms, a higher expected growth rate (measured on a logarithmic scale) than any other wealth process in $\mathcal{X}^{1,++}$.
- The *log-utility-optimal portfolio* maximises the expected logarithmic utility of terminal wealth essentially over all wealth processes in $\mathcal{X}^{1,++}$.

The next result gives the first main connection between the notions introduced so far.

Proposition 2.1. **1)** *X^{np}, X^{go} and X^{lo} are all unique.*
2) *X^{np}, X^{go} and X^{lo} coincide whenever they exist.*
3) *X^{np} exists if and only if X^{go} exists. This is also equivalent to existence of X^{lo} if in addition $\sup\big\{E[\log X_T]\,\big|\,X \in \mathcal{X}^{1,++}$ with $E[(\log X_T)^-] < \infty\big\} < \infty$.*
4) *X^{np} (or equivalently X^{go}) exists if and only if S satisfies NUPBR.*

Proof. This is a collection of well-known results; see [3], Propositions 3.3 and 3.5, [5], Theorem 4.1, and [18], Proposition 3.19 and Theorem 4.12. ☐

Our next definition brings us closer again to equivalent σ-martingale measures for S.

Definition. An *equivalent supermartingale deflator* (for $\mathcal{X}^{1,++}$) is an adapted RCLL process $Y = (Y_t)_{0\le t\le T}$ with $Y_0 = 1$, $Y \ge 0$, $Y_T > 0$ P-a.s. and the property that $YX^{1,\vartheta}$ is a P-supermartingale for all $X^{1,\vartheta} \in \mathcal{X}^{1,++}$. The set of all equivalent supermartingale deflators is denoted by $\mathcal{Y}$.

Because $\mathcal{X}^{1,++}$ contains the constant process 1, we immediately see that each $Y \in \mathcal{Y}$ is itself a supermartingale; and $Y_T > 0$ implies by the minimum principle for supermartingales that then also $Y > 0$ and $Y_- > 0$. To facilitate comparisons, we mention that the class $\mathcal{X}^{1,++}$ is called $\mathcal{N}$ in [3], and $\mathcal{Y}$ is called $\mathcal{SM}$ there.

Definition. A *σ-martingale density* (or *local martingale density*) *for S* is a local P-martingale $Z = (Z_t)_{0\le t\le T}$ with $Z_0 = 1$, $Z \ge 0$ and the property that ZS^i is a P-σ-martingale (or P-local martingale, respectively) for each $i = 1,\dots,d$. If $Z > 0$, we call Z in addition *strictly positive*. For later use, we denote by $\mathcal{D}^{++}_{\mathrm{loc}}(S,P)$ the set of all strictly positive local P-martingale densities Z for S.

From the well-known Ansel–Stricker result (see Corollaire 3.5 of [2]), it is clear that $ZX^{1,\vartheta}$ is a P-supermartingale for all $X^{1,\vartheta} \in \mathcal{X}^{1,++}$ whenever Z is a σ- or local martingale density for S. Hence $\mathcal{Y}$ contains all strictly positive σ- and local martingale densities for S. On the other hand, if Q is an equivalent σ- or local martingale measure for S (as in the FTAP, in the sense that each S^i is a Q-σ-martingale or local Q-martingale, respectively), then the density process Z^Q of Q with respect to P is by the Bayes rule a strictly positive σ- or local martingale density for S, if it has $Z^Q_0 = 1$ (which means that $Q = P$ on $\mathcal{F}_0$). In that sense, supermartingale deflators can be viewed as a generalisation of equivalent σ- or local martingale measures for S. This important idea goes back to [20].

The second main connection between the concepts introduced in this section is provided by

Proposition 2.2. *The $\mathbb{R}^d$-valued semimartingale S satisfies NUPBR if and only if there exists an equivalent supermartingale deflator for $\mathcal{X}^{1,++}$. In short:*

$$NUPBR \Longleftrightarrow \mathcal{Y} \neq \emptyset.$$

Proof. This is part of [18], Theorem 4.12. □

Combining what we have seen so far, we directly obtain the main result of this section.

Theorem 2.3. *For an $\mathbb{R}^d$-valued semimartingale S, the following are equivalent:*

1) *S satisfies NUPBR.*
2) *The numeraire portfolio X^{np} exists.*
3) *The growth-optimal portfolio X^{go} exists.*
4) *There exists an equivalent supermartingale deflator for $\mathcal{X}^{1,++}$, i.e. $\mathcal{Y} \neq \emptyset$.*

In each of these cases, X^{np} and X^{go} are unique, and $X^{\mathrm{np}} = X^{\mathrm{go}}$. If in addition $\sup\big\{E[\log X_T] \,\big|\, X \in \mathcal{X}^{1,++} \text{ with } E[(\log X_T)^-] < \infty\big\} < \infty$, then 1)–4) are also equivalent to

5) *The log-utility-optimal portfolio X^{lo} exists.*

In that case, also X^{lo} is unique, and $X^{\mathrm{lo}} = X^{\mathrm{np}} = X^{\mathrm{go}}$.

Remarks. 1) We emphasise once again that all these results are known. In the above most general form, they are due to [18], but variants and precursors can already be found in [21], [3] and [5]. In particular, Theorem 5.1 in [5] shows that under the assumption NA, the existence of the growth-optimal portfolio X^{go} is equivalent to the existence of a strictly positive σ-martingale density for S.
2) It seems that the key importance of the condition NUPBR, albeit not under that name and in the more specialised setting of a complete Itô process model, has first been recognised in [21], who relate NUPBR to the absence of so-called cheap thrills; see Theorem 2 in [21].
3) If the numeraire portfolio X^{np} exists, then it lies in $\mathcal{X}^{1,++}$ and at the same time, $1/X^{\mathrm{np}}$ lies in $\mathcal{Y}$, by the definitions of X^{np} and $\mathcal{Y}$. So another property equivalent to 1)–4) in Theorem 2.3 would be that $\mathcal{X}^{1,++} \cap (1/\mathcal{Y}) \neq \emptyset$ or $\mathcal{Y} \cap (1/\mathcal{X}^{1,++}) \neq \emptyset$.
4) Since we work on the closed interval $[0,T]$, all our processes so far are defined up to and including T. Hence we need not worry about finiteness of X^{np}_T, which is in contrast to [18].
5) In view of the link to log-utility maximisation, it is no surprise that there are also dual aspects and results for the above connections. This is for instance presented in [3] and [18], but is not our main focus here.
6) For yet another property equivalent to NUPBR, see the recent work in [19].

The third important result in this section would be a more explicit description of the numeraire portfolio X^{np} or, more precisely, its generating strategy ϑ^{np}. Such a description can be found in [18], Theorem 3.15, or more generally in [11], Theorem 3.1 and Corollary 3.2. In both cases, ϑ^{np} can be obtained (in principle) by pointwise maximisation of a function (called g or Λ, respectively, in the above references) that is given explicitly in terms of certain semimartingale characteristics. In [18], this involves the characteristics of the returns process R, where each $S^i = S^i_0\,\mathcal{E}(R^i)$ is assumed to be a stochastic exponential. By contrast, the authors of [11] take a general semimartingale S and allow in addition to trading also consumption with a possibly stochastic clock; they then need the (joint) characteristics of (S,M), where M is a certain process defined via the stochastic clock. In the general case where S can have jumps, neither of these descriptions unfortunately gives very explicit expressions for ϑ^{np} since the above pointwise maximiser is only defined implicitly. For this reason, we do not go into more detail here, and focus in the next section on the much simpler case where S is continuous.

3 Continuous financial market models

In this section, we focus on the special case when S on $(\Omega,\mathcal{F},\mathbb{F},P)$ from Section 2 is *continuous*. We introduce some more concepts and link them to those of Section 2. Again, all the results given here are well known from the literature, and we at most claim credit for a hopefully clear and concise overview.

So let $S=(S_t)_{0\le t\le T}$ be an $\mathbb{R}^d$-valued continuous semimartingale with canonical decomposition $S=S_0+M+A$. The processes $M=(M_t)_{0\le t\le T}$ and $A=(A_t)_{0\le t\le T}$ are both $\mathbb{R}^d$-valued, continuous and null at 0. Moreover, M is a local P-martingale and A is adapted and of finite variation. The bracket process $\langle M\rangle$ of M is the adapted, continuous, $d\times d$-matrix-valued process with components $\langle M\rangle^{ij}=\langle M^i,M^j\rangle$ for $i,j=1,\dots,d$; it exists because M is continuous, hence locally square-integrable.

Definition. We say that S satisfies the *weak structure condition (SC′)* if A is absolutely continuous with respect to $\langle M\rangle$ in the sense that there exists an $\mathbb{R}^d$-valued predictable process $\widehat{\lambda}=(\widehat{\lambda}_t)_{0\le t\le T}$ such that $A=\int d\langle M\rangle\,\widehat{\lambda}$, i.e.

$$A^i_t=\sum_{j=1}^d\int_0^t\widehat{\lambda}^j_u\,d\langle M\rangle^{ij}_u=\sum_{j=1}^d\int_0^t\widehat{\lambda}^j_u\,d\langle M^i,M^j\rangle_u\quad\text{for } i=1,\dots,d \text{ and } 0\le t\le T.$$

We then call $\widehat{\lambda}$ the *(instantaneous) market price of risk* for S and sometimes informally write $\widehat{\lambda}=dA/d\langle M\rangle$.

Definition. If S satisfies the weak structure condition (SC′), we define

$$\widehat{K}_t:=\int_0^t\widehat{\lambda}^{\mathrm{tr}}_u\,d\langle M\rangle\,\widehat{\lambda}_u=\sum_{i,j=1}^d\int_0^t\widehat{\lambda}^i_u\widehat{\lambda}^j_u\,d\langle M^i,M^j\rangle_u,\qquad 0\le t\le T$$

and call $\widehat{K} = (\widehat{K}_t)_{0 \le t \le T}$ the *mean-variance tradeoff process* of S. Because $\langle M \rangle$ is positive semidefinite, the process $\widehat{K}$ is increasing and null at 0; but note that it may take the value $+\infty$ in general. We say that S satisfies the *structure condition (SC)* if S satisfies (SC$'$) and $\widehat{K}_T < \infty$ P-a.s.

Remarks. 1) There is some variability in the literature concerning the structure condition; some authors call (SC) what we label (SC$'$) here. The terminology we have chosen is consistent with [26]. For discussions of the difference between (SC$'$) and (SC), we refer to [7] and [16].

2) The weak structure condition (SC$'$) comes up very naturally via Girsanov's theorem. In fact, suppose S is a local Q-martingale under some Q equivalent to P and $1/Z$ is the density process of P with respect to Q. Then the process $M := S - S_0 - \int Z_- \, d\big\langle S, \frac{1}{Z}\big\rangle$ is by Girsanov's theorem a local P-martingale null at 0 and continuous like S, and $A := \int Z_- \, d\big\langle S, \frac{1}{Z}\big\rangle = \int Z_- \, d\big\langle M, \frac{1}{Z}\big\rangle$ is absolutely continuous with respect to $\langle M \rangle$ by the Kunita–Watanabe inequality. These results are of course well known from stochastic calculus; but their relevance for mathematical finance was only discovered later around the time when the importance of equivalent local martingale measures was highlighted by the FTAP.

If S satisfies (SC), the condition $\widehat{K}_T < \infty$ P-a.s. can equivalently be formulated as $\widehat{\lambda} \in L^2_{\mathrm{loc}}(M)$ (on $[0,T]$, to be accurate). This means that the stochastic integral process $\widehat{\lambda} \cdot M = \int \widehat{\lambda} \, dM$ is well defined and a real-valued continuous local P-martingale null at 0, and we have $\widehat{K} = \big\langle \widehat{\lambda} \cdot M \big\rangle$. The stochastic exponential

$$\widehat{Z}_t := \mathcal{E}(-\widehat{\lambda} \cdot M)_t = \exp\Big(-\widehat{\lambda} \cdot M_t - \frac{1}{2}\widehat{K}_t\Big), \qquad 0 \le t \le T$$

is then also well defined and a strictly positive local P-martingale with $\widehat{Z}_0 = 1$. For reasons that will become clear presently, $\widehat{Z}$ is called the *minimal martingale density* for S.

Proposition 3.1. *Suppose S is an $\mathbb{R}^d$-valued continuous semimartingale. Then S satisfies the structure condition (SC) if and only if there exists a strictly positive local martingale density Z for S. In short:*

$$\textit{(SC)} \Longleftrightarrow \mathcal{D}^{++}_{\mathrm{loc}}(S,P) \neq \emptyset.$$

Proof. If S satisfies (SC), we have seen above that $\widehat{Z} = \mathcal{E}(-\widehat{\lambda} \cdot M)$ is a strictly positive local P-martingale with $\widehat{Z}_0 = 1$. Moreover, using (SC), it is a straightforward computation via the product rule to check that each $\widehat{Z}S^i$ is a local P-martingale. Hence we can take $Z = \widehat{Z}$.

Conversely, suppose that ZS and $Z > 0$ are both local P-martingales. If Z is a true P-martingale on $[0,T]$, it can be viewed as the density process of some Q equivalent to P such that, by the Bayes rule, S is a local Q-martingale. Then the Girsanov argument in Remark 2) above gives (SC$'$). In general, applying the product rule shows that ZS has the finite variation part $\int Z_- \, dA + \langle Z, S \rangle$, which must vanish because

ZS is a local P-martingale. This gives $A = -\int \frac{1}{Z_-}\,d\langle S,Z\rangle = -\int \frac{1}{Z_-}\,d\langle M,Z\rangle$, hence again (SC$'$), and with some more work, one shows that even (SC) is satisfied. For the details, we refer to Theorem 1 of [26]. □

Since S is continuous, Theorem 1 of [26] also shows, by an application of the Kunita–Watanabe decomposition with respect to M to the stochastic logarithm N of $Z = \mathcal{E}(N)$, that every strictly positive local martingale density Z for S can be written as $Z = \widehat{Z}\,\mathcal{E}(L)$ for some local P-martingale L null at 0 which is strongly P-orthogonal to M. From that perspective, $\widehat{Z}$ is minimal in that it is obtained for the simplest choice $L \equiv 0$. One can also exhibit other minimality properties of $\widehat{Z}$, but this is not our main focus here.

Remark. While $\widehat{Z}$ is (for continuous S) always strictly positive, it is in general only a local, but not a true P-martingale. But if $\widehat{Z}$ happens to be a true P-martingale (on $[0,T]$), or equivalently if $E_P[\widehat{Z}_T] = 1$, we can define a probability measure $\widehat{P}$ equivalent to P via $d\widehat{P} := \widehat{Z}_T\,dP$. The local P-martingale property of $\widehat{Z}S$ is by the Bayes rule then equivalent to saying that S is a local $\widehat{P}$-martingale. This $\widehat{P}$, if it exists, is called the *minimal martingale measure*; see [10].

As we have already seen in Section 2, the family $\mathcal{Y}$ of all equivalent supermartingale deflators for $\mathcal{X}^{1,++}$ contains the family $\mathcal{D}^{++}_{\mathrm{loc}}(S,P)$ of all strictly positive local martingale densities for S. Therefore $\mathcal{D}^{++}_{\mathrm{loc}}(S,P) \neq \emptyset$ implies that $\mathcal{Y} \neq \emptyset$, and this already provides a strong first link between the results in this section and those in Section 2. A second link is given by the following connection between the minimal martingale density $\widehat{Z}$ and the numeraire portfolio X^{np}.

Lemma 3.2. *Suppose S is an $\mathbb{R}^d$-valued continuous semimartingale. If S satisfies (SC), then X^{np} exists and is given by $X^{\mathrm{np}} = 1/\widehat{Z}$.*

Proof. Since S satisfies (SC), $\widehat{Z} = \mathcal{E}(-\widehat{\lambda}\cdot M)$ exists and

$$1/\widehat{Z} = \exp\Big(\widehat{\lambda}\cdot M + \frac{1}{2}\widehat{K}\Big) = \exp\Big(\widehat{\lambda}\cdot S - \frac{1}{2}\widehat{K}\Big) = \mathcal{E}(\widehat{\lambda}\cdot S),$$

because $S = S_0 + M + \int d\langle M\rangle\,\widehat{\lambda}$. This shows that $1/\widehat{Z} = X^{1,\widehat{\vartheta}}$ lies in $\mathcal{X}^{1,++}$ with $\widehat{\vartheta} = \widehat{\lambda}/\widehat{Z}$. Moreover, $\widehat{Z}S$ is a local P-martingale by (the proof of) Proposition 3.1. A straightforward application of the product rule then shows that also $\widehat{Z}X^{1,\vartheta}$ is a local P-martingale, for every $X^{1,\vartheta} \in \mathcal{X}^{1,++}$, and so this product is also a P-supermartingale since it is nonnegative. Thus $1/\widehat{Z}$ satisfies the requirements for the numeraire portfolio and therefore agrees with X^{np} by uniqueness. □

Of course, also Lemma 3.2 is not really new; the result can essentially already be found in [3], Corollary 4.10. If one admits that the numeraire portfolio X^{np} coincides with the growth-optimal portfolio X^{go}, one can also quote [5], Corollary 7.4. And finally, one could even use the description of X^{np} in [18], Theorem 3.15, because this becomes explicit when S is continuous.

For a complete and detailed connection between Sections 2 and 3, the next result provides the last link in the chain. In the present formulation, it seems due to [12], Theorem 1.25; the proof we give here is perhaps a little bit more compact.

Proposition 3.3. *Suppose S is an $\mathbb{R}^d$-valued continuous semimartingale. If the numeraire portfolio X^{np} exists, then S satisfies the structure condition (SC).*

Proof. For every $X^{1,\vartheta} \in \mathcal{X}^{1,++}$, we can write $X^{1,\vartheta} = \mathcal{E}(\pi \cdot S)$ with $\pi := \vartheta / X^{1,\vartheta}$. Moreover, both ϑ and π are S-integrable and hence also in $L^2_{\mathrm{loc}}(M)$, since the processes $X^{1,\vartheta} > 0$, S and M are all continuous. Using the explicit expression $\mathcal{E}(\pi \cdot S) = \exp\big(\pi \cdot S - \frac{1}{2}\int \pi^{\mathrm{tr}}\, d\langle M\rangle\, \pi\big)$ for the stochastic exponential then gives for every $X^{1,\vartheta} \in \mathcal{X}^{1,++}$ that

$$
\begin{aligned}
\frac{X^{1,\vartheta}}{X^{\mathrm{np}}} = \frac{\mathcal{E}(\pi \cdot S)}{\mathcal{E}(\pi^{\mathrm{np}} \cdot S)} &= \exp\Big((\pi - \pi^{\mathrm{np}}) \cdot S - \frac{1}{2}\int \pi^{\mathrm{tr}}\, d\langle M\rangle\, \pi + \frac{1}{2}\int (\pi^{\mathrm{np}})^{\mathrm{tr}}\, d\langle M\rangle\, \pi^{\mathrm{np}}\Big) \\
&= \exp\Big((\pi - \pi^{\mathrm{np}}) \cdot M - \frac{1}{2}\int (\pi - \pi^{\mathrm{np}})^{\mathrm{tr}}\, d\langle M\rangle\, (\pi - \pi^{\mathrm{np}})\Big) \\
&\quad \times \exp\Big(\int (\pi - \pi^{\mathrm{np}})^{\mathrm{tr}} \big(dA - d\langle M\rangle\, \pi^{\mathrm{np}}\big)\Big),
\end{aligned}
$$

where the last equality is readily verified by multiplying out and collecting terms. But this means that

$$
\frac{X^{1,\vartheta}}{X^{\mathrm{np}}} = \mathcal{E}(L) \exp(B), \tag{3.1}
$$

where $L := (\pi - \pi^{\mathrm{np}}) \cdot M$ $\big($and hence also $\mathcal{E}(L)\big)$ is a continuous local P-martingale and

$$
B := B(\pi) := \int (\pi - \pi^{\mathrm{np}})^{\mathrm{tr}} \big(dA - d\langle M\rangle\, \pi^{\mathrm{np}}\big)
$$

is a continuous adapted process of finite variation. Because X^{np} is the numeraire portfolio, the left-hand side of (3.1) is a P-supermartingale for every ϑ, and the right-hand side gives a multiplicative decomposition of that P-special semimartingale as the product of a local martingale and a process of finite variation; see Théorème 6.17 in [13]. But now the uniqueness of the multiplicative decomposition and the fact that $\mathcal{E}(L)\exp(B)$ is a P-supermartingale together imply that $B = B(\pi)$ must be a decreasing process, for every π (coming from a ϑ such that $X^{1,\vartheta}$ lies in $\mathcal{X}^{1,++}$). By a standard variational argument, this is only possible if $A = \int d\langle M\rangle\, \pi^{\mathrm{np}}$, and because π^{np} is in $L^2_{\mathrm{loc}}(M)$, we see that (SC) is satisfied with $\widehat{\lambda} = \pi^{\mathrm{np}}$. □

Putting everything together, we now obtain the main result of this section.

Theorem 3.4. *For an $\mathbb{R}^d$-valued continuous semimartingale S, the following are equivalent:*

1) *S satisfies NUPBR.*
2) *The numeraire portfolio X^{np} exists.*
3) *The growth-optimal portfolio X^{go} exists.*
4) *There exists an equivalent supermartingale deflator for $\mathcal{X}^{1,++}$, i.e. $\mathcal{Y} \neq \emptyset$.*

5) *There exists a strictly positive local P-martingale density for S, i.e.* $\mathcal{D}^{++}_{\mathrm{loc}}(S,P) \neq \emptyset$.
6) *S satisfies the structure condition (SC).*
7) *S satisfies the weak structure condition (SC′) and* $\widehat{\lambda} \in L^2_{loc}(M)$.
8) *S satisfies the weak structure condition (SC′) and* $\widehat{K}_T < \infty$ *P-a.s.*
9) *S satisfies the weak structure condition (SC′) and the minimal martingale density* $\widehat{Z}$ *exists in* $\mathcal{D}^{++}_{\mathrm{loc}}(S,P)$.

In each of these cases, we then have $X^{\mathrm{np}} = X^{\mathrm{go}} = 1/\widehat{Z}$.

Proof. The equivalence of 1)–4) is the statement of Theorem 2.3. The equivalence of 5)–9) comes from Proposition 3.1 and directly from the definitions. Lemma 3.2 shows that 6) implies 2), and Proposition 3.3 conversely shows that 2) implies 6). The final statement is due to Theorem 2.3 and Lemma 3.2. □

We emphasise once again that all the individual results in this section are known. However, we have not seen anywhere so far the full list of equivalences compiled in Theorem 3.4, and so we hope that the result may be viewed as useful.

Remarks. 1) Because our main interest here lies on the numeraire or growth-optimal portfolio, we have focussed exclusively on the (equivalent) condition NUPBR. There is in fact a whole zoo of absence-of-arbitrage conditions, and an extensive discussion and comparison of these in the framework of a continuous financial market can be found in Chapter 1 of [12]. That work also contains many more details as well as explicit examples and counterexamples.

2) We already know from Proposition 2.1 that there is a close connection between the log-utility-optimal portfolio and the numeraire portfolio. If S is continuous, this turns out to be rather transparent. Indeed, if S satisfies (SC), then $1/\widehat{Z}$ maximises $E\left[\log X^{1,\vartheta}_T\right]$ over all $X^{1,\vartheta} \in \mathcal{X}^{1,++}$, and the maximal expected utility is

$$E\left[\log(1/\widehat{Z}_T)\right] = \frac{1}{2} E\left[\widehat{K}_T\right] \in [0,\infty].$$

So existence of the log-utility-optimal portfolio with finite maximal expected utility is equivalent to the structure condition (SC) plus the extra requirement that $E\left[\widehat{K}_T\right] < \infty$. For more details, we refer to part 1) of Theorem 3.5 in [1].

3) We have already seen in Section 2 that the condition NFLVR is equivalent to the combination of the conditions NUPBR and NA. The latter can be formulated as saying that whenever ϑ is admissible, i.e. a-admissible for some $a \geq 0$, $X^{0,\vartheta}_T \geq 0$ P-a.s. implies that $X^{0,\vartheta}_T = 0$ P-a.s. A slightly different condition is (NA$_+$) which stipulates that whenever ϑ is 0-admissible, $X^{0,\vartheta}_T \geq 0$ P-a.s. implies that $X^{0,\vartheta} \equiv 0$ P-a.s. We mention this condition because it is just a little weaker than NUPBR. In fact, if S is continuous, Theorem 3.5 of [27] shows that S satisfies (NA$_+$) if and only if S satisfies the weak structure condition (SC′) and the mean-variance tradeoff process $\widehat{K}$ does not jump to $+\infty$, i.e.

$$\inf\left\{ t > 0 \,\middle|\, \int_t^{t+\delta} \widehat{\lambda}^{\mathrm{tr}}_u \, d\langle M\rangle_u \, \widehat{\lambda}_u = +\infty \text{ for all } \delta \in (0, T-t] \right\} = \infty.$$

The second condition follows from, but does not imply, $\widehat{K}_T < \infty$ P-a.s., so that (NA_+) is a little weaker than (SC), or equivalently NUPBR.

4 Minimal market models

The notion of a minimal market model is due to E. Platen and has been introduced in a series of papers with various co-authors; see Chapter 13 of [24] for a recent textbook account. Our goal in this section is to link that concept to the notions introduced in Sections 2 and 3 and to exhibit a fundamental probabilistic structure result for such models. The presentation here is strongly inspired by Chapter 5 of [12], but extends and simplifies the analysis and results given there. The latter Chapter 5 is in turn based on Chapters 10 and 13 from [24], although the presentation is a bit different.

The key idea behind the formulation of a minimal market model is an asymptotic diversification result due to E. Platen. Theorem 3.6 of [23] states that under fairly weak assumptions, a sequence of well-diversified portfolios converges in a suitable sense to the growth-optimal portfolio. It is therefore natural to model a broadly based (hence diversified) *index* by the same structure as the growth-optimal portfolio, and to call such a model for an index a *minimal market model*. To study this, we therefore have to take a closer look at the probabilistic behaviour of the growth-optimal portfolio.

We begin by considering a continuous financial market model almost as in Section 3. More precisely, let $S = (S_t)_{t\geq 0}$ be a continuous $\mathbb{R}^d$-valued semimartingale on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. We assume that S satisfies NUPBR on $[0,T]$ for every $T \in (0,\infty)$ so that we can use Theorem 3.4 for every fixed finite T. We thus obtain $S = S_0 + M + \int d\langle M\rangle\,\widehat{\lambda}$ with $\widehat{\lambda} \in L^2_{\mathrm{loc}}(M)$, the minimal martingale density $\widehat{Z} = \mathcal{E}(-\widehat{\lambda}\cdot M)$ exists, and so does the growth-optimal portfolio X^{go}, which coincides with $1/\widehat{Z}$. All this is true on $[0,\infty)$ since it holds on every interval $[0,T]$ and we can simply paste things together.

The minimal market model for the index $I = (I_t)_{t\geq 0}$ is now defined by

$$I := X^{\mathrm{go}} = 1/\widehat{Z} = 1/\mathcal{E}(-\widehat{\lambda}\cdot M). \tag{4.1}$$

In view of the remark after Proposition 3.1, this shows that the *minimal market model (MMM)* is directly connected to the *minimal martingale measure (MMM)*, or more precisely to the minimal martingale density $\widehat{Z}$. It also explains why we have deliberately avoided the use of the abbreviation MMM and gives a clear hint where the title of this paper comes from.

The next result is the key for understanding the probabilistic structure of the process I in (4.1). Recall the notation $\cdot$ for stochastic integrals.

Proposition 4.1. *Suppose $N = (N_t)_{t\geq 0}$ is a real-valued continuous local martingale null at 0, and $V = (V_t)_{t\geq 0}$ is defined by $V := 1/\mathcal{E}(-N) = \mathcal{E}(N + \langle N\rangle)$. Then:*

1) $\{V \bullet \langle N \rangle_\infty = +\infty\} = \{\langle N \rangle_\infty = +\infty\}$ *P-a.s.*
2) *If* $\langle N \rangle_\infty = +\infty$ *P-a.s., then*

$$V_t = C_{\rho_t}, \qquad t \geq 0,$$

where $C = (C_t)_{t \geq 0}$ *is a squared Bessel process of dimension 4 (a BESQ*^{4}*, for short) and the time change* $t \mapsto \rho_t$ *is explicitly given by the increasing process*

$$\rho_t = \frac{1}{4} \int_0^t \frac{1}{\mathcal{E}(-N)_s} \, d\langle N \rangle_s = \frac{1}{4} \int_0^t V_s \, d\langle N \rangle_s, \qquad t \geq 0.$$

3) *The result in 2) is also valid without the assumption that* $\langle N \rangle_\infty = +\infty$ *P-a.s. if we are allowed to enlarge the underlying probability space in a suitable way.*

Proof. The second expression for V follows directly from the explicit formula for the stochastic exponential $\mathcal{E}(-N)$.

1) This is fairly easy, but for completeness we give details. By Proposition V.1.8 of [25], the sets $\{N_\infty := \lim_{t \to \infty} N_t \text{ exists in } \mathbb{R}\}$ and $\{\langle N \rangle_\infty < \infty\}$ are equal with probability 1. So on $\{\langle N \rangle_\infty < \infty\}$, the process $\mathcal{E}(-N)_s = \exp\big(-N_s - \frac{1}{2}\langle N \rangle_s\big)$ converges to $\exp\big(-N_\infty - \frac{1}{2}\langle N \rangle_\infty\big) > 0$ P-a.s. which implies that $s \mapsto V_s = 1/\mathcal{E}(-N)_s$ remains bounded P-a.s. as $s \to \infty$. Hence

$$(V \bullet \langle N \rangle_\infty)(\omega) \leq \text{const.}\,(\omega) \langle N \rangle_\infty(\omega) < \infty \qquad P\text{-a.s. on } \{\langle N \rangle_\infty < \infty\}.$$

On the other hand, $N \in \mathcal{M}^c_{0,\mathrm{loc}}$ implies by Fatou's lemma that $s \mapsto \mathcal{E}(-N)_s$ is a nonnegative supermartingale and therefore converges P-a.s. to a finite limit (which might be 0) as $s \to \infty$. So $s \mapsto 1/V_s = \mathcal{E}(-N)_s$ is bounded P-a.s. and thus

$$(V \bullet \langle N \rangle_\infty)(\omega) \geq \frac{1}{\text{const.}\,(\omega)} \langle N \rangle_\infty(\omega) = +\infty \qquad P\text{-a.s. on } \{\langle N \rangle_\infty = +\infty\}.$$

This proves the assertion.

2) Because $V = \mathcal{E}(N + \langle N \rangle)$ satisfies $dV = V\,dN + V\,d\langle N \rangle$, defining $L \in \mathcal{M}^c_{0,\mathrm{loc}}$ by $dL = \frac{1}{2}\sqrt{V}\,dN$ yields $d\langle L \rangle = \frac{1}{4} V\,d\langle N \rangle$ and

$$dV = 2\sqrt{V}\,dL + 4\,d\langle L \rangle.$$

By 1), $\langle N \rangle_\infty = +\infty$ P-a.s. implies that $\langle L \rangle_\infty = \frac{1}{4} V \bullet \langle N \rangle_\infty = +\infty$ P-a.s., and so the Dambis–Dubins–Schwarz theorem (see Theorem V.1.6 in [25]) yields the existence of some Brownian motion $B = (B_t)_{t \geq 0}$ such that $L_t = B_{\langle L \rangle_t}$ for $t \geq 0$. Hence

$$dV_t = 2\sqrt{V_t}\,dB_{\langle L \rangle_t} + 4\,d\langle L \rangle_t,$$

and if $t \mapsto \tau_t$ denotes the inverse of $t \mapsto \langle L \rangle_t$, we see that $C_t := V_{\tau_t}$, $t \geq 0$, satisfies

$$dC_t = 2\sqrt{C_t}\,dB_t + 4\,dt,$$

so that C is a BESQ^4 process; see Chapter XI of [25]. Finally, since τ and $\langle L\rangle$ are inverse to each other, $V_t = C_{\langle L\rangle_t}$ and as claimed, the time change $t \mapsto \rho_t$ is given by

$$\rho_t = \langle L\rangle_t = \frac{1}{4} V \boldsymbol{\cdot} \langle N\rangle_t, \qquad t \geq 0.$$

3) If we may enlarge the probability space to guarantee the existence of an independent Brownian motion, we can still use the Dambis–Dubins–Schwarz theorem; see Theorem V.1.7 in [25]. Thus the same argument as for 2) still works. □

In view of Theorem 3.4, applying Proposition 4.1 with $N = \widehat{\lambda} \boldsymbol{\cdot} M$ and noting that $\langle N\rangle = \widehat{K}$ now immediately gives the main result of this section.

Theorem 4.2. *Let $S = (S_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued continuous semimartingale and suppose that S satisfies NUPBR on $[0,T]$ for every $T \in (0,\infty)$ (or equivalently that the growth-optimal portfolio X^{go} exists for every finite time horizon T). Denote by $\widehat{Z} = \mathcal{E}(-\widehat{\lambda} \boldsymbol{\cdot} M)$ the minimal martingale density for S and model the index $I = (I_t)_{t\geq 0}$ by $I := X^{\mathrm{go}} = 1/\widehat{Z}$. Then $I_t = C_{\rho_t}$, $t \geq 0$, is a time change of a squared Bessel process C of dimension 4, with the time change given by*

$$\rho_t = \frac{1}{4} \int_0^t \frac{1}{\widehat{Z}_s}\, d\widehat{K}_s, \qquad t \geq 0. \tag{4.2}$$

Theorem 4.2 is a generalisation of Proposition 5.8 in [12], where the same conclusion is obtained under the more restrictive assumption that S is given by a multidimensional Itô process model which is complete. But most of the key ideas for the above proof can already be seen in that result of [12]. Even considerably earlier, the same result as in [12] can be found in Section 3.1 of [22], although it is not stated as a theorem. The main contribution of Theorem 4.2 is to show that neither completeness nor the Itô process structure are needed.

Example 4.3. To illustrate the theory developed so far, we briefly consider the standard, but *incomplete multidimensional Itô process model* for S. Suppose discounted asset prices are given by the stochastic differential equations

$$\frac{dS^i_t}{S^i_t} = (\mu^i_t - r_t)\, dt + \sum_{k=1}^m \sigma^{ik}_t\, dW^k_t \qquad \text{for } i = 1,\dots,d \text{ and } 0 \leq t \leq T.$$

Here $W = (W^1,\dots,W^m)^{\mathrm{tr}}$ is an $\mathbb{R}^m$-valued standard Brownian motion on $(\Omega,\mathcal{F},P)$ with respect to $\mathbb{F}$; there is no assumption that $\mathbb{F}$ is generated by W, and we only suppose that $m \geq d$ so that we have at least as many sources of uncertainty as risky assets available for trade. The processes $r = (r_t)_{t\geq 0}$ (the *instantaneous short rate*), $\mu^i = (\mu^i_t)_{t\geq 0}$ (the *instantaneous drift rate* of asset i) for $i = 1,\dots,d$ and $\sigma^{ik} = (\sigma^{ik}_t)_{t\geq 0}$ for $i = 1,\dots,d$ and $k = 1,\dots,m$ (the *instantaneous volatilities*) are predictable (or even progressively measurable) and satisfy

$$\int_0^T |r_u|\, du + \sum_{i=1}^d \int_0^T |\mu^i_u|\, du + \sum_{i=1}^d \sum_{k=1}^m \int_0^T (\sigma^{ik}_u)^2\, du < \infty \quad P\text{-a.s. for each } T \in (0,\infty).$$

Moreover, to avoid redundant assets (locally in time), we assume that for each $t \geq 0$,

$$\text{the } d \times m\text{-matrix } \sigma_t \text{ has } P\text{-a.s. full rank } d.$$

Then $\sigma_t \sigma_t^{\mathrm{tr}}$ is P-a.s. invertible and we can define the predictable (or progressively measurable) $\mathbb{R}^m$-valued process $\lambda = (\lambda_t)_{t \geq 0}$ by

$$\lambda_t := \sigma_t^{\mathrm{tr}} (\sigma_t \sigma_t^{\mathrm{tr}})^{-1} (\mu_t - r_t \mathbf{1}), \qquad t \geq 0$$

with $\mathbf{1} = (1, \ldots, 1)^{\mathrm{tr}} \in \mathbb{R}^d$. Our final assumption is that $\lambda \in L^2_{\mathrm{loc}}(W)$ or equivalently that

$$\int_0^T |\lambda_u|^2 \, du < \infty \qquad P\text{-a.s. for each } T \in (0, \infty). \tag{4.3}$$

Sometimes λ (instead of $\widehat{\lambda}$ below) is called the *(instantaneous) market price of risk* for S.

It is straightforward to verify that the canonical decomposition of the continuous semimartingale S^i is given by

$$dM_t^i = S_t^i \sum_{k=1}^m \sigma_t^{ik} \, dW_t^k \qquad \text{and} \qquad dA_t^i = S_t^i (\mu_t^i - r_t) \, dt,$$

so that

$$d\langle M^i, M^j \rangle_t = S_t^i S_t^j \sum_{k=1}^m \sigma_t^{ik} \sigma_t^{jk} \, dt = S_t^i S_t^j (\sigma_t \sigma_t^{\mathrm{tr}})^{ij} \, dt.$$

The weak structure condition (SC$'$) is therefore satisfied with the $\mathbb{R}^d$-valued process $\widehat{\lambda} = (\widehat{\lambda}_t)_{t \geq 0}$ given by

$$\widehat{\lambda}_t^i = \frac{1}{S_t^i} \big((\sigma_t \sigma_t^{\mathrm{tr}})^{-1} (\mu_t - r_t \mathbf{1}) \big)^i \qquad \text{for } i = 1, \ldots, d \text{ and } t \geq 0.$$

This gives $\int \widehat{\lambda} \, dM = \int \lambda \, dW$ and therefore the mean-variance tradeoff process as

$$\widehat{K}_t = \big\langle \widehat{\lambda} \bullet M \big\rangle_t = \langle \lambda \bullet W \rangle_t = \int_0^t |\lambda_u|^2 \, du \qquad \text{for } t \geq 0,$$

and so (4.3) immediately implies that S satisfies (SC) on $[0, T]$ for each $T \in (0, \infty)$. Therefore this model directly falls into the scope of Theorem 3.4 (for each fixed T) and of Theorem 4.2. In particular, we of course recover Proposition 5.8 of [12] or the result from Section 13.1 of [24] as a special case (for $m = d$, even without the stronger assumptions imposed there).

Remark. To obtain a more concrete model just for the index I, Theorem 4.2 makes it very tempting to start with a BESQ^4 process C and choose some time change $t \mapsto \rho_t$ to then define the index by

$$I_t := C_{\rho_t}, \qquad t \geq 0. \tag{4.4}$$

Depending on the choice of ρ, this may provide a good fit to observed data and hence yield a plausible and useful model; see Section 13.2 of [24] on the stylized minimal market model. However, a word of caution seems indicated here. In fact, if we accept (as in this section) the basic modelling of the index I by the growth-optimal portfolio X^{go}, then the approach (4.4) raises the following *inverse problem*:

> Given a time change $t \mapsto \rho_t$, when does there exist an $\mathbb{R}^d$-valued continuous semimartingale $S = (S_t)_{t\geq 0}$ which satisfies the structure condition (SC) and whose growth-optimal portfolio is given by the process I defined by (4.4)?

We do not have an answer to this question, but we suspect that the problem is non-trivial. One first indication for this is the observation that the explicit form (4.2) of the time change in Theorem 4.2 implies that

$$\frac{d\widehat{K}_t}{d\rho_t} = 4\widehat{Z}_t, \qquad t \geq 0. \tag{4.5}$$

Since the right-hand side of (4.5) is a local martingale, the processes $\widehat{K}$ and ρ cannot be chosen with an arbitrarily simple structure — for example they cannot both be deterministic.

Acknowledgements

MS gratefully acknowledges financial support by the National Centre of Competence in Research "Financial Valuation and Risk Management" (NCCR FINRISK), Project D1 (Mathematical Methods in Financial Risk Management). The NCCR FINRISK is a research instrument of the Swiss National Science Foundation. We also thank an anonymous referee for helpful suggestions and useful comments.

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