Semi-efficient valuations and put-call parity

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Abstract

We propose an approach to the valuation of payoffs in general semimartingale models of financial markets where prices are nonnegative. Each asset price can hit 0; we only exclude that this ever happens simultaneously for all assets. We start from two simple, economically motivated axioms, namely absence of arbitrage (in the sense of NUPBR) and absence of relative arbitrage among all buy-and-hold strategies (called static efficiency). A valuation process for a payoff is then called semi-efficient consistent if the financial market enlarged by that process still satisfies this combination of properties. It turns out that this approach lies in the middle between the extremes of valuing by risk-neutral expectation and valuing by absence of arbitrage alone. We show that this always yields put-call parity, although put and call values themselves can be nonunique, even for complete markets. We provide general formulas for put and call values in complete markets and show that these are symmetric and that both contain three terms in general. We also show that our approach recovers all the put-call parity respecting valuation formulas in the classic theory as special cases, and we explain when and how the different terms in the put and call valuation formulas disappear or simplify. Along the way, we also define and characterize completeness for general semimartingale financial markets and connect this to the classic theory.

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1 Introduction

Recent years have seen many models for financial markets with less stringent assumptions than the existence of a true martingale measure for discounted asset prices S. This has impacts on the valuation of options. If S is only a strict local martingale, these so-called bubble models create difficulties with put-call parity. A paper by Ruf [37] constructs an arbitrage-free economy with negative call prices and points out that "many no-arbitrage arguments [...] implicitly rely on stronger assumptions than just the existence of an ELMM." In a similar vein, Heston et al. [22] say that "Option textbooks typically present 'universal' properties of option values based on seemingly weak assumptions about underlying spot prices. These properties include the ideas that option values should not exceed the stock price, put-call parity should hold, and other relations between option values and intrinsic values should hold [Merton (1973)]. Such properties were originally motivated by the absence of arbitrage or by dominated asset arguments, but [...] some solutions [*of the valuation PDE might*] satisfy these properties, whereas others do not. In some cases [...], there may be no [...] solutions that simultaneously satisfy all these properties." These are some of the many indications that valuing options by no-arbitrage arguments is still not fully understood. The present paper proposes to address this gap.

Our goal is to present an approach to valuation which starts from simple, economically motivated axioms and rigorously derives conclusions as well as limitations. Our model for asset prices (nonnegative semimartingales whose sum never vanishes) is almost completely general. We need not distinguish whether these assets have "bubbles" or not; the valuation principle works independently of this. We obtain sharper results for complete markets, but the method also works under incompleteness. To the best of our knowledge, such a comprehensive and general approach does not exist in the literature so far.

Our paper makes several contributions:

1) We propose a valuation approach that lies strictly between valuation by risk-neutral expectation and valuation by absence of arbitrage alone. Simple examples (Sections 4.1 and 4.2) show that these two extremes do not yield satisfactory results in general.

2) We treat all underlying assets in a symmetric way, viewing puts and calls as exchange options. We always obtain put-call parity (Theorem 6.1), even if the values of puts and calls separately are not unique. We emphasize that this is a result, not an axiom. We also recover as special cases all classic results and formulas (see Section 7) that also respect put-call parity. So our approach is a genuine extension of the existing literature.

3) Our framework allows us to understand why puts and calls are treated differently in the classic theory. We show (Theorem 6.2) that in general, correction terms to risk-neutral option pricing appear in both put and call values; we argue on economic grounds where the corrections belong; and we explain (Section 7.1) why this symmetry never occurs when discounted prices satisfy (classic) NFLVR and contain an asset (bank account) with constant price 1.

4) We introduce and characterize completeness (Section 3) for general semimartingale models. We do not a priori assume absence of arbitrage even before defining completeness, and we do not need the existence of an asset with a positive price process. Again the classic theory is contained as a special case.

A detailed comparison with the literature is given in Section 7.1 after we have presented our results. Here we just mention two recent papers fairly closely related to our work. Kardaras [30] studies valuation and parities for exchange options in a general model with finitely many assets having nonnegative prices. He fixes a local martingale deflator, defines prices with respect to this, and shows how different terms in the resulting valuation formulas can be given a financial interpretation. However, this is done for a fixed (although general) linear pricing operator. Fisher et al. [15] start with a symmetric market containing N possibly defaulting assets and describe this by the $[0,\infty]^{N\times N}$-valued matrix process of all exchange rate pairs. By reducing things to earlier work of Yan [39], they present versions of the first and second FTAP, and they also introduce and study so-called martingale valuation operators. The corresponding valuation results, for a fixed such operator, are similar to those of [30].

Our paper is structured as follows. Sections 2.1–2.3 present the setup, define a number of arbitrage-related concepts and provide dual characterizations used later in the paper. Overall, Section 2 is mainly a concise summary of those results from [20] and [21] that we need in our developments. Section 3 introduces and studies attainability and completeness in a general semimartingale market. This is needed later to obtain sharper valuation results for complete markets, but is also of independent interest. Section 4 explains the basic idea for our valuation approach and shows that both risk-neutral expectation and absence of arbitrage alone are too extreme to yield good valuation results. Section 5 presents our intermediate approach, combining absence of arbitrage (in the sense of NUPBR) with absence of relative arbitrage among all buy-and-hold strategies (called static efficiency). We characterize the latter concept via martingale properties and use those to construct and characterize all valuations one can obtain with our approach. Due to the generality of our market setup, valuations of general payoffs can be nonunique even under completeness.

Section 6 is devoted to put-call parity and pricing formulas. We first show that our approach always yields put-call parity, even for incomplete markets, without imposing it axiomatically. For complete markets, we then provide general formulas for put and call value processes when we view puts and calls as exchange options between two assets. These values are symmetric and each contains three terms. One is the payoff's risk-neutral expectation; one appears if and only if the corresponding underlying asset has a bubble; and one, which causes possible nonuniqueness, depends on the maximality properties of the two underlying assets. Finally, Section 7 compares our results to the existing literature and illustrates them by a number of examples.

2 Setup and background

Fix a time horizon $T\in(0,\infty)$ and a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0\le t\le T}$ with the usual conditions of right-continuity and completeness. We also assume that $\mathcal{F}_0$ is $\mathbb{P}$-trivial. Unless stated otherwise, σ,τ are always stopping times with values in $[0,T]$ and processes $Y=(Y_t)_{0\le t\le T}$ are indexed by $t\in[0,T]$. We denote by $\mathbf{L}^0_+(\mathcal{F}_\tau)$, $\bar{\mathbf{L}}^0_+(\mathcal{F}_\tau)$, $\mathbf{L}^0_{++}(\mathcal{F}_\tau)$ the sets of all $\mathcal{F}_\tau$-measurable random variables taking $\mathbb{P}$-a.s. values in $[0,\infty)$, $[0,\infty]$, $(0,\infty)$, respectively, and write $\mathbf{L}^0_- = -\mathbf{L}^0_+$. Finally, $e^i=(0,\dots,0,1,0,\dots,0)$ for $i=1,\dots,N$ is the i-th unit vector in $\mathbb{R}^N$ and $\mathbf{1}:=\sum_{i=1}^N e^i=(1,\dots,1)\in\mathbb{R}^N$.

A product-measurable process ξ is *predictable on* $[\![\sigma,T]\!]$ if ξ_σ is $\mathcal{F}_\sigma$-measurable and $\xi\mathbb{1}_{]\!]\sigma,T]\!]}$ is predictable. So if ξ is predictable on $[\![\sigma,T]\!]$ and $A\in\mathcal{F}_\sigma$, also $\xi\mathbb{1}_A$ is predictable on $[\![\sigma,T]\!]$. If Z is an $\mathbb{R}^m$-valued semimartingale, ${}^\sigma L(Z)$ is the set of all $\mathbb{R}^m$-valued processes ζ which are predictable on $[\![\sigma,T]\!]$ and for which the stochastic integral process $\int_\sigma^t \zeta_s \,\mathrm{d}Z_s := \int_{(0,t]} \zeta_s \mathbb{1}_{]\!]\sigma,t]\!]}(s)\,\mathrm{d}Z_s$, $0\le t\le T$, is defined in the sense of m-dimensional stochastic integration (see [27, Section III.6]). We also write $\zeta\bullet Z := \int \zeta\,\mathrm{d}Z$ for $\zeta\in {}^0L(Z)$. Note that normal dots $\cdot$ are reserved for scalar products.

An important extra comment on notation appears in Section 2.4.

2.1 Model and basic concepts

This section lays out basic definitions and concepts. It is similar to [21], where more discussion and interpretation can be found.

Our model is given by an $\mathbb{R}^N$-valued semimartingale $\mathbf{S}$, where $\mathbf{S}_t$ describes the prices at time t, in some unspecified unit, of N basic traded assets. A *numéraire* is a semimartingale D with $\inf_{0\le t\le T} D_t > 0$ $\mathbb{P}$-a.s., and $\mathcal{D}$ is the set of all numéraires. We view $D_t(\omega)$ as a (state- and time-dependent) conversion factor from one unit to another; so $S_t = \mathbf{S}_t/D_t$ gives time-t prices of our assets in the unit corresponding to D.

Remark 2.1. Economic sense says that natural qualitative properties of a financial market should not depend on the chosen unit; hence all concepts and results should apply for the model $\mathbf{S}$ if and only if they are valid for all other models $S=\mathbf{S}/D$ with $D\in\mathcal{D}$. This *numéraire-independent* paradigm is developed in Herdegen [20] and used also in [21]. For easier understanding, we work here with the fixed model $\mathbf{S}$ and only comment in Section 2.4 on the relation to numéraire-independence. However, we shall see that changes of numéraire are indispensable to formulate our results if we want to keep $\mathbf{S}$ general.

To accommodate as many setups as possible, our semimartingale model $\mathbf{S}$ is kept very general. In particular, it extends the *classic setup* of mathematical finance as follows. Suppose we have some reference asset (e.g., a bond) with price process $B>0$ and d other risky assets whose prices are all expressed in units of B. Then the corresponding model has $N=1+d$ and the classic setup uses $\mathbf{S}:=(1,X)$, or sometimes $\tilde{\mathbf{S}}:=B\mathbf{S}=(B,Y)$ with $Y:=BX$, where the $\mathbb{R}^d$-valued semimartingale X describes the (B-)discounted prices. Any result for a general $\mathbf{S}$ then holds in particular also for $\mathbf{S}=(1,X)$; but the converse is only true if the result does not depend (in an open or hidden way) on the particular structure of $(1,X)$ or on B. We return to this important distinction later.

Example 2.2. Suppose we have a bond given by $B_t=\exp(\int_0^t r_s\,\mathrm{d}s)$ and one stock whose price follows the SDE

$$\mathrm{d}Y_t = Y_t(\mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t),$$

with a one-dimensional Brownian motion W and suitably integrable predictable processes r,μ,σ. The filtration $\mathbb{F}$ could be generated by W or include extra randomness, e.g., for stochastic volatility models. This setup includes in particular the following three cases:

a) the classic *Black–Scholes model* $B_t=e^{rt}$, $\mathrm{d}Y_t=\mu Y_t\,\mathrm{d}t+\sigma Y_t\,\mathrm{d}W_t$, with constants r,μ and $\sigma>0$.

b) the classic *CEV* (constant elasticity of variance) *model*, viewed under an equivalent local martingale measure; here $B_t = e^{rt}$, $\mathrm{d}Y_t = rY_t\,\mathrm{d}t + \sigma|Y_t|^\beta\,\mathrm{d}W_t$, with constants r, $\sigma > 0$ and $\beta > 0$.

c) a version of the CEV model with *stochastic volatility*, where $B \equiv 1$, $\mathrm{d}X_t = \sigma_t|X_t|^\beta\,\mathrm{d}W_t$ and $\mathrm{d}\sigma_t = \alpha(\overline{\sigma} - \sigma_t)(\sigma_t - \underline{\sigma})\,\mathrm{d}W'_t$ for a two-dimensional (possibly correlated) Brownian motion (W, W') and constants $\beta > 1$, $\alpha > 0$ and $\overline{\sigma} > \sigma_0 > \underline{\sigma} > 0$.

We use these examples later to illustrate some specific features of our results.

We next discuss dynamic trading; without special mention, σ and τ below are always stopping times $\leq T$. A *self-financing strategy on* $[\![\sigma, T]\!]$, shortly $\vartheta \in {}^\sigma L^{\mathrm{sf}}(\mathbf{S}) =: {}^\sigma L^{\mathrm{sf}}$, is an $\mathbb{R}^N$-valued process ϑ which is predictable on $[\![\sigma, T]\!]$, in ${}^\sigma L(\mathbf{S})$, and such that its *wealth process* (in the unit corresponding to $\mathbf{S}$) satisfies

$$V(\vartheta)[\mathbf{S}] := \mathbf{V}(\vartheta) := \vartheta \cdot \mathbf{S} = \vartheta_\sigma \cdot \mathbf{S}_\sigma + \int_\sigma \vartheta_u\,\mathrm{d}\mathbf{S}_u \qquad \text{on } [\![\sigma, T]\!],\ \mathbb{P}\text{-a.s.} \tag{2.1}$$

We point out that (2.1) simultaneously contains the definition of wealth and (in its last equality) the self-financing property.

In the classic setup $\mathbf{S} = (1, X)$, self-financing strategies on $[\![\sigma, T]\!]$ are neatly and conveniently described by pairs $(v_\sigma, \psi) \in \mathbf{L}^0(\mathcal{F}_\sigma) \times {}^\sigma L(X)$, with their wealth process given by $\tilde{V}(v_\sigma, \psi) := v_\sigma + \int_\sigma \psi\,\mathrm{d}X$ on $[\![\sigma, T]\!]$, $\mathbb{P}$-a.s. So (v_σ, ψ) corresponds to $\vartheta := (\vartheta^1, \psi)$ given by $\vartheta^1 := v_\sigma + \int_\sigma \psi\,\mathrm{d}X - \psi \cdot X$, and then $\tilde{V}(v_\sigma, \psi) = \mathbf{V}(\vartheta)$, both on $[\![\sigma, T]\!]$, $\mathbb{P}$-a.s.

In particular, for any $v_\sigma \in \mathbf{L}^0(\mathcal{F}_\sigma)$, any $\mathbb{R}^d$-valued integrand $\psi \in {}^\sigma L(X)$ can be uniquely augmented to a ($\mathbb{R}^N$-valued) self-financing strategy $\vartheta \mathrel{\widehat{=}} (v_\sigma, \psi)$ on $[\![\sigma, T]\!]$ for $\mathbf{S} = (1, X)$, with $\vartheta^i = \psi^{i-1}$ for $i = 2, \dots, N = d + 1$. The "missing bond component" ϑ^1 is obtained by solving the self-financing condition (2.1) for ϑ^1. In contrast, for general $\mathbf{S}$, not every $\mathbb{R}^N$-valued integrand $\vartheta \in {}^\sigma L(\mathbf{S})$ can be viewed as a self-financing strategy on $[\![\sigma, T]\!]$ for $\mathbf{S}$, because it need not satisfy the last equality in (2.1). As a consequence, properties of self-financing strategies ϑ in a general model should be expressed in terms of their wealth $\mathbf{V}(\vartheta)$ and cannot be written as properties of stochastic integrals $\int \vartheta\,\mathrm{d}\mathbf{S}$ — not for *arbitrary* integrands ϑ for $\mathbf{S}$. (A nice illustration of this point appears in [20, Remark 4.14].)

In our general model $\mathbf{S}$, a self-financing strategy $\vartheta \in {}^\sigma L^{\mathrm{sf}}$ is called *undefaultable on* $[\![\sigma, T]\!]$, written briefly $\vartheta \in {}^\sigma L^{\mathrm{sf}}_+ := {}^\sigma L^{\mathrm{sf}}_+(\mathbf{S})$, if $\mathbf{V}(\vartheta) \geq 0$ on $[\![\sigma, T]\!]$, $\mathbb{P}$-a.s. We could also say that ϑ is *0-admissible for* $\mathbf{S}$ *on* $[\![\sigma, T]\!]$.

In the classic setup $\mathbf{S} = (1, X)$, there is a well-known notion of admissibility. For clarity, we distinguish between self-financing strategies ϑ for $\mathbf{S}$ and integrands ψ for X, and we only discuss $\sigma = 0$; so any v_0 is just a constant. We call $\vartheta \mathrel{\widehat{=}} (v_0, \psi) \in \mathbb{R} \times {}^0 L(X)$ an *a-admissible strategy* if $\tilde{V}(v_0, \psi) = v_0 + \psi \bullet X \geq -a$ on $[\![0, T]\!]$, and an *admissible strategy*, $\vartheta \in {}^0 L^{\mathrm{sf}}_{\mathrm{adm}}(1, X)$ for short, if it is a-admissible for some $a \geq 0$. We call $\psi \in {}^0 L(X)$ an *a-admissible integrand for* X if $\psi \bullet X \geq -a$, and an *admissible integrand for* X, $\psi \in {}^0 L^{\mathrm{sf}}_{\mathrm{adm}}(X)$ for short, if it is a-admissible for some $a \geq 0$. Clearly, a strategy $\vartheta \mathrel{\widehat{=}} (v_0, \psi)$ is a-admissible if and only if both $v_0 \geq -a$ and the integrand ψ is $(a + v_0)$-admissible, and admissible strategies for $(1, X)$ correspond to pairs from $\mathbb{R} \times \{\text{admissible integrands for } X\}$. In particular, $\vartheta \mathrel{\widehat{=}} (v_0, \psi)$ is undefaultable if and only if $v_0 \geq 0$ and ψ is a v_0-admissible integrand for X. While all this may look pedantic, it is important — it shows why we often cannot use results from the classic setup in a straightforward way.

Returning to our general model $\mathbf{S}$, a *numéraire strategy* for $\mathbf{S}$ is a self-financing strategy $\eta \in {}^0L^{\mathrm{sf}}$ such that $\mathbf{V}(\eta)$ is a numéraire, i.e., $\inf_{0\le t\le T}\mathbf{V}_t(\eta) > 0$ $\mathbb{P}$-a.s. (This numéraire is even *tradable.*) We call $\mathbf{S}$ a *numéraire model* if such an η exists. Each numéraire strategy η induces a $\mathbb{P}$-a.s. unique model $\mathbf{S}^{(\eta)}$ with the property that $V(\eta)[\mathbf{S}^{(\eta)}] := \eta \cdot \mathbf{S}^{(\eta)} \equiv 1$. It is given by the $\mathbf{V}(\eta)$-*discounted prices*

$$\mathbf{S}^{(\eta)} := \frac{\mathbf{S}}{\mathbf{V}(\eta)} = \frac{\mathbf{S}}{\eta \cdot \mathbf{S}}. \tag{2.2}$$

In the classic setup $\mathbf{S} = (1, X)$, we always have a numéraire model; one possible numéraire strategy is $\eta \equiv e^1$, the buy-and-hold strategy of the first asset (usually the bond). For that choice, $V(e^1)[(1, X)] = e^1 \cdot (1, X) \equiv 1$, and prices are already $\mathbf{V}(e^1)$-discounted.

Let us now list the conditions we impose later on the model $\mathbf{S}$. We want to have $\mathbf{S} \ge 0$ and that the model is *nondegenerate* in the sense that $\mathbf{S} \cdot \mathbf{1}$ is a numéraire, i.e.,

$$\inf_{0\le t\le T}(\mathbf{S}_t \cdot \mathbf{1}) = \inf_{0\le t\le T}\sum_{i=1}^{N}\mathbf{S}^i_t > 0 \qquad \mathbb{P}\text{-a.s.} \tag{2.3}$$

This means that the *market portfolio* $\eta^{\mathbf{S}} := \mathbf{1}$ of holding one unit of each asset is a numéraire strategy. More generally, we could also work under the weaker assumption that there exists a numéraire strategy, i.e., that $\mathbf{S}$ is a numéraire model. However, having $\mathbf{S} \ge 0$ and (2.3) looks natural and makes the statements of some results less technical.

We emphasize that $\mathbf{S} \ge 0$ and (2.3) are not meant to be standing assumptions. Whenever we use these conditions in a result, they are explicitly listed there.

For a stopping time $\tau \le T$, an *improper payoff at time* τ is just a random variable $\mathbf{f} \in \bar{\mathbf{L}}^0_+(\mathcal{F}_\tau)$. We call $\mathbf{f}$ a *payoff* if it is in $\mathbf{L}^0_+(\mathcal{F}_\tau)$, and *strictly positive* if $\mathbf{f} \in \mathbf{L}^0_{++}(\mathcal{F}_\tau)$. The interpretation is that $\mathbf{f}$ is the amount, in the same units as $\mathbf{S}$, we get at time τ from some financial instrument. This needs some care; see Remark 2.19 below.

We now turn to the valuation of payoffs. The basic idea is that we want to introduce a valuation process $\mathbf{U}^{\mathbf{f}}$ for $\mathbf{f}$ in such a way that the extended model consisting of $\mathbf{S}$ together with $\mathbf{U}^{\mathbf{f}}$, viewed as a proposed price process for the new asset with final payoff $\mathbf{f}$, has the same properties as the original model $\mathbf{S}$. More precisely:

Definition 2.3. For a payoff $\mathbf{f}$ at time T, a *valuation (process)*, in the same units as $\mathbf{S}$, is a nonnegative semimartingale $\mathbf{U}^{\mathbf{f}}$ which satisfies the *terminal condition* $\mathbf{U}^{\mathbf{f}}_T = \mathbf{f}$ $\mathbb{P}$-a.s. For a property $\mathcal{E}$ that a model $\mathbf{S}$ can have, a valuation process $\mathbf{U}^{\mathbf{f}}$ for $\mathbf{f}$ is called $\mathcal{E}$*-consistent* or *consistent for* $\mathcal{E}$ if the extended model $(\mathbf{S}, \mathbf{U}^{\mathbf{f}})$ has the property $\mathcal{E}$.

Remark 2.4. 1) We can extend Definition 2.3 to the case where $\mathbf{f} \in \mathbf{L}^0(\mathcal{F}_T; \mathbb{R}^k_+)$ describes a vector of k payoffs. Then $\mathbf{U}^{\mathbf{f}}$ must be an $\mathbb{R}^k_+$-valued semimartingale, and all concepts and definitions are used componentwise.

2) For payoffs $\mathbf{f} \ge 0$, valuation processes $\mathbf{U}^{\mathbf{f}} \ge 0$ seem natural even if this condition is not derived from an economic postulate; see [37]. If we start with basic assets $\mathbf{S} \ge 0$ and consider an extended model $(\mathbf{S}, \mathbf{U}^{\mathbf{f}})$, having the latter consistent also calls for $\mathbf{U}^{\mathbf{f}} \ge 0$.

To proceed with the basic idea, we now introduce a number of desirable properties $\mathcal{E}$ of a model. To be able to work with them, we then also need to characterize them in terms of suitable dual quantities. This will be done next.

2.2 Arbitrage, maximal strategies and viability

This section lists some properties that a model or strategies can have, and characterizes them by dual (martingale) properties. We begin with arbitrage-related concepts.

In the classic setup $\mathbf{S} = (1, X)$, one usually considers the sets

$$
\begin{aligned}
\mathcal{K}_0^1(1,X) &:= \{\psi \bullet X_T : \psi \in {}^0L(X) \text{ is 1-admissible integrand for } X\} \\
&= \{\tilde{V}_T(0,\psi) : \vartheta \mathrel{\widehat{=}} (0,\psi) \text{ is 1-admissible strategy for } (1,X)\} \\
&= \{\mathbf{V}_T(\vartheta) - 1 : \vartheta \mathrel{\widehat{=}} (1,\psi) \text{ is 0-admissible strategy for } \mathbf{S}\}, \qquad (2.4)\\
\mathcal{K}_0(1,X) &:= \{\psi \bullet X_T : \psi \in {}^0L(X) \text{ is admissible integrand}\} = \bigcup_{a\geq 0} a\mathcal{K}_0^1(1,X), \\
\mathcal{C}(1,X) &:= \Big(\mathcal{K}_0(1,X) - \mathbf{L}_+^0(\mathcal{F}_T)\Big) \cap \mathbf{L}^\infty(\mathcal{F}_T).
\end{aligned}
$$

Then X satisfies

- classic NA if $\mathcal{K}_0(1,X) \cap \mathbf{L}_+^0(\mathcal{F}_T) = \{0\}$,
- classic NUPBR if $\mathcal{K}_0^1(1,X)$ is bounded in $\mathbf{L}^0$,
- classic NFLVR if the norm-closure of $\mathcal{C}(1,X)$ in $\mathbf{L}^\infty$ intersects $\mathbf{L}_+^\infty(\mathcal{F}_T)$ only in 0.

It is a well-known result in the classic setup that classic NFLVR for X is equivalent to the combination of classic NA and classic NUPBR, both for X. There are also important connections to maximal strategies, and we return to this a bit later.

We now extend the classic definitions of NUPBR and NFLVR to our general model $\mathbf{S}$.

Definition 2.5. Define, in analogy to (2.4), the sets

$$
\begin{aligned}
\mathcal{K}_0^1(\mathbf{S}) &:= \{\mathbf{V}_T(\vartheta) - 1 : \vartheta \in {}^0L_+^{\mathrm{sf}}(\mathbf{S}) \text{ has } \mathbf{V}_0(\vartheta) = 1\}, \\
\mathcal{K}_0(\mathbf{S}) &:= \bigcup_{a\geq 0} a\mathcal{K}_0^1(\mathbf{S}), \\
\mathcal{C}(\mathbf{S}) &:= \Big(\mathcal{K}_0(\mathbf{S}) - \mathbf{L}_+^0(\mathcal{F}_T)\Big) \cap \mathbf{L}^\infty(\mathcal{F}_T).
\end{aligned}
$$

Then we say that $\mathbf{S}$ *satisfies NUPBR* if $\mathcal{K}_0^1(\mathbf{S})$ is bounded in $\mathbf{L}^0$, and *NFLVR* if the norm-closure of $\mathcal{C}(\mathbf{S})$ in $\mathbf{L}^\infty$ intersects $\mathbf{L}_+^\infty(\mathcal{F}_T)$ only in 0. (We do not need NA here.)

If we are *in the classic setup* $\mathbf{S} = (1, X)$, then *NUPBR for* $\mathbf{S}$ *is the same as classic NUPBR for* X. But we emphasize that in general,

$$
\mathcal{K}_0^1(\mathbf{S}) \subsetneqq \{\vartheta \bullet \mathbf{S}_T : \vartheta \in {}^0L(\mathbf{S}) \text{ is 1-admissible integrand for } \mathbf{S}\};
$$

so *NUPBR and classic NUPBR for* $\mathbf{S}$ *need not coincide outside of the classic setup.*

Remark 2.6. We can also define the sets $\mathcal{K}_0^1(S), \mathcal{K}_0(S), \mathcal{C}(S)$ and hence NUPBR and NFLVR for general $S = \mathbf{S}/D$, with a numéraire $D \in \mathcal{D}$, instead of $\mathbf{S}$. The only change is that we replace $\mathbf{S}$ everywhere by S and $\mathbf{V}(\vartheta)$ by $V(\vartheta)[S] := \vartheta \cdot S = \vartheta_0 \cdot S_0 + \vartheta \bullet S \geq 0$, for $\vartheta \in {}^0L(S)$ to be in ${}^0L_+^{\mathrm{sf}}(S)$. See also Section 2.4 for a more detailed discussion.

To get concise formulations later, we introduce some terminology. Recall that an *equivalent local martingale measure (ELMM)* for an adapted RCLL process Y is a probability $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ such that Y is a local $\mathbb{Q}$-martingale. We briefly write $\mathbb{Q} \in \mathbb{M}_{\mathrm{loc}}^{\mathrm{e}}(Y)$ or sometimes $Y \in \mathcal{M}_{\mathrm{loc}}(\mathbb{Q})$. Recall that nonnegative processes are local martingales iff they are σ-martingales; so whenever $\mathbf{S} \geq 0$, it is enough to work with ELMMs.

Definition 2.7. A *numéraire/ELMM pair* is a pair $(D, \mathbb{Q})$ with a numéraire $D \in \mathcal{D}$ such that $\mathbb{Q}$ is an ELMM for the model $S = \mathbf{S}/D$. A *tradable numéraire/ELMM pair* is a pair $(\eta, \mathbb{Q})$ where η is a numéraire strategy and $(\mathbf{V}(\eta), \mathbb{Q})$ is a numéraire/ELMM pair.

Remark 2.8. Take a probability $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ and denote by Z its density process with respect to $\mathbb{P}$; so $Z_0 = 1$ as $\mathcal{F}_0$ is trivial. If $(D, \mathbb{Q})$ is a numéraire/ELMM pair for $\mathbf{S}$, then $Y := Z/D$ is a strictly positive semimartingale (i.e., in $\mathcal{D}$), with $Y_0 = 1$ if $D_0 = 1$, such that $Y\mathbf{S}$ is a local $\mathbb{P}$-martingale. Such a Y (with $Y_0 = 1$) is often called a *(local martingale) deflator* for $\mathbf{S}$ under $\mathbb{P}$. (If $\mathbf{S} = (1, X)$, then Y is also a local $\mathbb{P}$-martingale.) Many of our results can also be formulated in terms of deflators; but neither statements nor proofs become shorter, and numéraires and ELMMs make the formulations more intuitive.

We next need a notion of maximality for strategies. The basic idea is simple, but there are some subtleties we have to explain. In general, for a subset $C \subseteq \mathbf{L}^0$, an element $c \in C$ is called maximal in C if $c' \in C$ and $c' \geq c$ $\mathbb{P}$-a.s. always implies $c' = c$ $\mathbb{P}$-a.s.

In the classic setup $\mathbf{S} = (1, X)$, this notion was studied by Delbaen/Schachermayer [7, 8] for the case where $C = \{c = \psi \bullet X_T : \psi \text{ is an admissible integrand for } X\}$ consists of all final wealths from admissible strategies $\vartheta \mathrel{\widehat{=}} (v_0, \psi)$ with $v_0 = 0$. An admissible integrand $\psi \in {}^0L(X)$ is thus a *classically maximal integrand for* X if for any admissible integrand $\psi' \in {}^0L(X)$ with $\psi' \bullet X_T \geq \psi \bullet X_T$ $\mathbb{P}$-a.s., we have $\psi' \bullet X_T = \psi \bullet X_T$ $\mathbb{P}$-a.s.

Still in the classic setup, one easily sees that classic NA for X is equivalent to saying that the zero integrand is maximal in the set of all admissible integrands for X. Moreover, [6, Proposition 3.5] shows that under classic NA, an admissible integrand ψ with $\psi \bullet X_T \geq -a$ $\mathbb{P}$-a.s. for some $a \geq 0$ is automatically a-admissible. So *under classic NA*, an a-admissible integrand $\psi \in {}^0L(X)$ is classically maximal in the set of *all* admissible integrands $\psi' \in {}^0L(X)$ if and only if it is classically maximal in the smaller set of all a-admissible integrands $\psi'' \in {}^0L(X)$. This is already pointed out in [8, before Definition 2.2].

The above definition of classic maximality is fine in principle. But all results obtained by Delbaen and Schachermayer impose a priori the condition that X satisfies classic NFLVR (and hence classic NA). (They also assume that X is locally bounded, but this can be eliminated with some extra effort.) We want to study general models which only satisfy NUPBR, and this has two consequences. First, we often cannot use results from the classic setup and have to provide different arguments. Second, we need a different notion of maximality whose properties do not rely so crucially on the classic NA condition.

Our definition of maximality differs from the above approach in two directions. On the one hand, we use a slightly stronger notion of maximality (the analogue of the above notion is called weak maximality). On the other hand, we work with a smaller class of strategies. A priori, therefore, the classic and our approach are not directly comparable.

Definition 2.9. Fix a stopping time $\sigma \leq T$ and let ${}^{\sigma}\Gamma \subseteq {}^{\sigma}L^{\mathrm{sf}}$ be a class of self-financing strategies on $[\![\sigma, T]\!]$. A strategy $\vartheta \in {}^{\sigma}\Gamma$ is *weakly maximal for* ${}^{\sigma}\Gamma$ if there is no $\vartheta' \in {}^{\sigma}\Gamma$ with $\mathbf{V}_{\sigma}(\vartheta') - \mathbf{V}_{\sigma}(\vartheta) \in \mathbf{L}^0_-$ and $\mathbf{V}_T(\vartheta') - \mathbf{V}_T(\vartheta) \in \mathbf{L}^0_+(\mathcal{F}_T) \setminus \{0\}$. It is *maximal for* ${}^{\sigma}\Gamma$ if there is no $\mathbf{f} \in \mathbf{L}^0_+(\mathcal{F}_T) \setminus \{0\}$ such that for all $\varepsilon > 0$, there exists $\bar{\vartheta} \in {}^{\sigma}\Gamma$ with

$$\mathbf{V}_{\sigma}(\bar{\vartheta}) - \mathbf{V}_{\sigma}(\vartheta) \leq \varepsilon \quad \text{and} \quad \mathbf{V}_T(\bar{\vartheta}) - \mathbf{V}_T(\vartheta) \geq \mathbf{f}, \qquad \mathbb{P}\text{-a.s.}$$

The following concept will be used later. For stopping times $\sigma \leq \tau \leq T$, a class ${}^{\sigma}\Gamma \subseteq {}^{\sigma}L^{\mathrm{sf}}$ of strategies on $[\![\sigma, T]\!]$ and a payoff $\mathbf{f}$ at time τ, the *superreplication price of* $\mathbf{f}$ *at time* σ *for* ${}^{\sigma}\Gamma$ (in the same units as $\mathbf{S}$) is defined by

$$\begin{aligned}\pi_{\sigma}(\mathbf{f} \,|\, {}^{\sigma}\Gamma) := \operatorname{ess\,inf}\{v \in \bar{\mathbf{L}}^0_+(\mathcal{F}_{\sigma}) : \exists \vartheta \in {}^{\sigma}\Gamma \text{ such that } \mathbb{P}\text{-a.s. on } \{v < \infty\}, \\ \mathbf{V}_{\sigma}(\vartheta) \leq v \text{ and } \mathbf{V}_{\tau}(\vartheta) \geq \mathbf{f}\}.\end{aligned} \tag{2.5}$$

To be precise, one also needs for ${}^{\sigma}\Gamma$ a cone structure; but we later use only classes ${}^{\sigma}\Gamma$ which have this property. In formal analogy, one can define the *subreplication price* as

$$\begin{aligned}\pi^{\mathrm{sub}}_{\sigma}(\mathbf{f} \,|\, {}^{\sigma}\Gamma) := \operatorname{ess\,sup}\{v \in \mathbf{L}^0_+(\mathcal{F}_{\sigma}) : \exists \vartheta \in {}^{\sigma}\Gamma \text{ such that } \mathbb{P}\text{-a.s.}, \\ \mathbf{V}_{\sigma}(\vartheta) \geq v \text{ and } \mathbf{V}_{\tau}(\vartheta) \leq \mathbf{f}\}.\end{aligned} \tag{2.6}$$

Remark 2.10. **1)** In terms of superreplication prices, maximality can be written more compactly: $\vartheta \in {}^{\sigma}\Gamma$ is maximal for ${}^{\sigma}\Gamma$ if and only if there is no $\mathbf{f} \in \mathbf{L}^0_+(\mathcal{F}_T) \setminus \{0\}$ with $\pi_{\sigma}(\mathbf{V}_T(\vartheta) + \mathbf{f} \,|\, {}^{\sigma}\Gamma) \leq \mathbf{V}_{\sigma}(\vartheta)$ $\mathbb{P}$-a.s.

2) A discussion of other, related notions of maximality is given in [21, Remark 3.4].

In the classic setup $\mathbf{S} = (1, X)$, there is a martingale characterization of classically maximal integrands. If X satisfies classic NFLVR and is locally bounded, an admissible integrand $\psi \in {}^{0}L(X)$ is by [7, Corollary 14] classically maximal for X if and only if $\psi \bullet X$ is a true $\mathbb{Q}$-martingale for some ELMM $\mathbb{Q}$ for X. Earlier versions of such results are due to Ansel/Stricker [2] and Jacka [26], among others. Delbaen/Schachermayer [9] provide a similar result for the case of a general (not locally bounded) X satisfying classic NFLVR, but we give no details here (they involve so-called feasible weight functions).

In order to have an analogous result without NFLVR, we first need an appropriate weak concept of absence of arbitrage.

Definition 2.11. A model $\mathbf{S}$ is called *dynamically viable* if the zero strategy 0 is maximal for ${}^{\sigma}L^{\mathrm{sf}}_+(\mathbf{S})$, for each stopping time $\sigma \leq T$.

Viability is a no-arbitrage condition which says that doing nothing cannot be improved by trading. Using classic terminology from Fernholz [13] and Fernholz/Karatzas/Kardaras [14], viability is slightly stronger than *no relative arbitrage*, in the relevant class of strategies, with respect to the zero strategy 0. More precisely, no relative arbitrage in ${}^{0}L^{\mathrm{sf}}_+$ with respect to 0 is equivalent to the zero strategy being *weakly* maximal in ${}^{0}L^{\mathrm{sf}}_+$.

The next martingale characterization of dynamic viability extends the classic FTAP.

Theorem 2.12. *If $\mathbf{S} \geq 0$ satisfies* (2.3)*, the following are equivalent:*

1) $\mathbf{S}$ *is dynamically viable, or, equivalently, the zero strategy* 0 *is maximal for* ${}^0L_+^{\mathrm{sf}}(\mathbf{S})$.

2) *There exists a tradable numéraire/ELMM pair* $(\eta, \mathbb{Q})$ *for* $\mathbf{S}$ *(so that* $\mathbb{Q}$ *is an ELMM for* $\mathbf{V}(\eta)$*-discounted prices* $\mathbf{S}^{(\eta)}$*). For each such pair,* η *is maximal for* ${}^0L_+^{\mathrm{sf}}(\mathbf{S})$.

3) *There exists a numéraire/ELMM pair* $(D, \mathbb{Q})$ *for* $\mathbf{S}$*. (Equivalently, there exists a local martingale deflator for* $\mathbf{S}$*.)*

4) *There exists a numéraire* D *such that the model* $S = \mathbf{S}/D$ *satisfies NFLVR.*

5) *There exists a numéraire* D *such that the model* $S = \mathbf{S}/D$ *satisfies NUPBR.*

6) *For every numéraire* $D \in \mathcal{D}$*, the model* $S = \mathbf{S}/D$ *satisfies NUPBR.*

7) $\mathbf{S}$ *satisfies NUPBR.*

We point out two important aspects for general models $\mathbf{S}$. First, we usually cannot obtain any martingale properties for the original $\mathbf{S}$ itself. So if we have NUPBR but not NFLVR, the use of numéraire changes and hence of different (but economically equivalent) models cannot be avoided. Second, neither S in 4) nor $\mathbf{S}^{(\eta)}$ in 2) need be a true $\mathbb{Q}$-martingale. Finally, we recall that NUPBR and classic NUPBR need not be the same.

Proof of Theorem 2.12. This result is a slight extension of [20, Theorem 4.10 and Proposition 3.24] and [21, Theorem 4.4]. More precisely, "1) $\Leftrightarrow$ 3)" and the equivalence of the two statements in 1) are from [21, Theorem 4.4], and "1) $\Leftrightarrow$ 2)" is from [20, Theorem 4.10]. Next, "1) $\Leftrightarrow$ 6)" is argued like [20, Proposition 3.24, (b)], and "3) $\Leftrightarrow$ 4)" is simply the classic FTAP for the model S. Finally, "6) $\Rightarrow$ 7) $\Rightarrow$ 5)" is clear and for "5) $\Rightarrow$ 6)", we use that products and ratios of numéraires are again numéraires, ${}^0L_+^{\mathrm{sf}}$ is a cone, and $0 < D_T < \infty$ $\mathbb{P}$-a.s. for every numéraire D, to get the set equality $1 + \mathcal{K}_0^1(\mathbf{S}/D') = D_0'(1 + \mathcal{K}_0^1(\mathbf{S}))/D_T' = (D_0' D_T)(1 + \mathcal{K}_0^1(\mathbf{S}/D))/(D_T' D_0)$. $\square$

We next characterize maximal strategies via martingale properties.

Theorem 2.13. ([20, Theorem 4.12]) *Suppose* $\mathbf{S} \geq 0$ *satisfies* (2.3) *and is dynamically viable, or, equivalently by Theorem 2.12, satisfies NUPBR. For a strategy* $\vartheta \in {}^0L_+^{\mathrm{sf}}(\mathbf{S})$*, the following are then equivalent:*

1) ϑ *is maximal for* ${}^0L_+^{\mathrm{sf}}(\mathbf{S})$.

2) *There exists a numéraire/ELMM pair* $(D, \mathbb{Q})$ *such that the* D*-discounted wealth process* $\mathbf{V}(\vartheta)/D$ *is a (true)* $\mathbb{Q}$*-martingale.*

3) *For each* $\mathbb{Q} \approx \mathbb{P}$ *on* $\mathcal{F}_T$*, there exists a numéraire* D *such that* $\mathbb{Q}$ *is an ELMM for* $\mathbf{S}/D$ *and* $\mathbf{V}(\vartheta)/D$ *is a (true)* $\mathbb{Q}$*-martingale.*

Theorem 2.13 is an analogue of the result in the classic setup discussed after Remark 2.10. Again, because we have NUPBR but not NFLVR, we can only obtain martingale properties after a change of numéraire.

2.3 Buy-and-hold strategies and efficiency

A dynamically viable model is arbitrage-free in the sense of NUPBR, but as we shall see in Section 4, this is not enough to derive valuations with good properties. We therefore introduce an extra concept. For any stopping time $\sigma \leq T$ and any class ${}^\sigma\Gamma \subseteq {}^\sigma L^{\mathrm{sf}}(\mathbf{S})$ of strategies on $[\![\sigma, T]\!]$, we define

$$\mathbf{h}^\sigma\Gamma := \left\{ \vartheta \in {}^\sigma\Gamma : \vartheta \mathbb{1}_{[\![\sigma,T]\!]} = \vartheta_\sigma \mathbb{1}_{[\![\sigma,T]\!]} \right\}$$

and call $\vartheta \in \mathbf{h}^\sigma\Gamma$ a *buy-and-hold* strategy in ${}^\sigma\Gamma$. Because we consider later different models, we point out that a class of strategies also depends on the model it refers to.

Remark 2.14. Our buy-and-hold strategies only make one trade and then keep the portfolio up to the final time T. Alternatively, one might look at *simple self-financing strategies* which have the form $\vartheta = h_{-1}\mathbb{1}_{[\![0]\!]} + \sum_{i=0}^{m-1} h_i \mathbb{1}_{]\!]\sigma_i,\sigma_{i+1}]\!]} \in \Theta^{\mathrm{simple}}$, where $m \in \mathbb{N}$, $0 = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_m = T$ are stopping times and the h_i are bounded $\mathbb{R}^N$-valued $\mathcal{F}_{\sigma_i}$-measurable random variables with $(h_i - h_{i-1}) \cdot \mathbf{S}_{\sigma_i} = 0$ for $i = 0, \dots, m-1$. Then one could call $\mathbf{S}$ *statically efficient for simple strategies* if every $\vartheta \in \Theta^{\mathrm{simple}}$ is maximal for Θ^{simple}. However, this is not a good idea for two reasons. First, for our purposes, it is already enough if we impose static efficiency only with respect to the smaller class of buy-and-hold strategies. Second and more importantly, static efficiency for simple strategies is too strong. In finite discrete time where $\Theta_+^{\mathrm{simple}} := \Theta^{\mathrm{simple}} \cap {}^0L_+^{\mathrm{sf}}$ equals ${}^0L_+^{\mathrm{sf}}$, it is already equivalent to dynamic efficiency (or also to NA, see Remark 2.18 below), and we conjecture that in general, dynamic viability plus static efficiency for simple strategies already implies dynamic efficiency, which is too restrictive as an assumption on the model; see Section 4.2 below.

Definition 2.15. A model $\mathbf{S}$ is called

- *statically viable* if the zero strategy 0 is maximal for $\mathbf{h}^\sigma L_+^{\mathrm{sf}}(\mathbf{S})$, for each $\sigma \leq T$.
- *statically efficient* if each $\vartheta \in \mathbf{h}^\sigma L_+^{\mathrm{sf}}(\mathbf{S})$ is maximal for $\mathbf{h}^\sigma L_+^{\mathrm{sf}}(\mathbf{S})$, for each $\sigma \leq T$.
- *dynamically efficient* if each $\vartheta \in \mathbf{h}^\sigma L_+^{\mathrm{sf}}(\mathbf{S})$ is maximal for ${}^\sigma L_+^{\mathrm{sf}}(\mathbf{S})$, for each $\sigma \leq T$.

In a viable model, one cannot improve the zero strategy of doing nothing. Efficiency is stronger — no buy-and-hold strategy can be improved, in the relevant class, without risk or extra capital. This yields again absence of relative arbitrage, but now with respect to *all buy-and-hold* strategies.

Remark 2.16. In the economic literature, efficiency is often used in the sense of informational efficiency, which then entails a connection to the filtration describing the evolution of information over time. The above notion does not do that; it could thus also be called *trading efficiency* to make the distinction explicit.

Theorem 2.17. ([21, Theorem 5.2 and Corollary 5.3]) *If* $\mathbf{S} \geq 0$ *satisfies* (2.3), *the following are equivalent:*

1) $\mathbf{S}$ *is dynamically efficient.*

2) *The market portfolio $\eta^{\mathbf{S}} \equiv \mathbf{1}$ is maximal for ${}^0L^{\mathrm{sf}}_+(\mathbf{S})$; in other words, the class of undefaultable strategies contains no relative arbitrage with respect to the market portfolio.*

3) *Each bounded numéraire strategy is maximal for ${}^0L^{\mathrm{sf}}_+(\mathbf{S})$.*

4) *There exists a numéraire/ELMM pair $(D,\mathbb{Q})$ such that the D-discounted price process $S = \mathbf{S}/D$ is a (true) $\mathbb{Q}$-martingale.*

Remark 2.18. 1) The work by Yan [39] contains a result which is similar to parts of Theorem 2.17. It uses the particular numéraire $\mathbf{V}(\eta^{\mathbf{S}}) = \sum_{i=1}^N \mathbf{S}^i$ and shows that $\mathbf{S}/\mathbf{V}(\eta^{\mathbf{S}})$ admits a true EMM if and only if $\mathbf{S}$ satisfies NFLVR for so-called allowable strategies. For a more detailed comparison, we refer to [20, Section 5.2].

2) For finite discrete time, dynamic viability and dynamic efficiency are both equivalent to the classic condition NA; see [21, Proposition 4.3] for details.

3) Even in finite discrete time, static efficiency is strictly weaker than dynamic efficiency; see [19, Example VIII.2.5]. (That example has a two-period model on a finite Ω which allows arbitrage in the first time-step and hence is not dynamically efficient; however, all submodels in the second time-step and also the submodel going directly from date 0 to date 2 are arbitrage-free so that one has static efficiency.) For general models, static efficiency is thus strictly weaker than having absence of arbitrage for all submodels of $\mathbf{S}$ where trading is only allowed at finitely many fixed time points; and for finite discrete time, static efficiency is even strictly weaker than having absence of arbitrage for all one-step submodels of $\mathbf{S}$.

2.4 Change of unit and numéraire invariance

Our basic model is given in some fixed unit by the process $\mathbf{S}$. But most results and many of the proofs can only be formulated in some different unit or after a change of numéraire. One way to handle this would be to write everything in a numéraire-independent setup, as in [20, 21]. This is elegant, but needs getting used to. For easier reading, we have therefore chosen to express as much as possible in the fixed model $\mathbf{S}$. This short section presents those facts and results about numéraire invariance that we need in our proofs.

Every numéraire $D \in \mathcal{D}$ induces a model $S = \mathbf{S}/D$. Prices there are given in a different unit, but of course basic qualitative aspects of the financial market do not change. In [20], all these models, with $D \in \mathcal{D}$, are hence called *economically equivalent*.

In units of $S = \mathbf{S}/D$, the wealth process of a self-financing strategy $\vartheta \in {}^\sigma L^{\mathrm{sf}}$ on $[\![\sigma, T]\!]$ is

$$V(\vartheta)[S] := \vartheta \cdot S = \vartheta_\sigma \cdot S_\sigma + \int_\sigma \vartheta_u \, \mathrm{d}S_u \qquad \text{on } [\![\sigma, T]\!], \ \mathbb{P}\text{-a.s.}$$

In view of (2.1), this gives $V(\vartheta)[\mathbf{S}] = \mathbf{V}(\vartheta)$ and also the *numéraire invariance property*

$$V(\vartheta)[\mathbf{S}/D] = V(\vartheta)[\mathbf{S}]/D = \mathbf{V}(\vartheta)/D \qquad \text{for } \vartheta \in {}^0L^{\mathrm{sf}} \text{ and every numéraire } D \in \mathcal{D}. \quad (2.7)$$

For a payoff $\mathbf{f} \in \bar{\mathbf{L}}^0_+(\mathcal{F}_\tau)$ at time τ in the units of $\mathbf{S}$, the payoff in the units of $S = \mathbf{S}/D$ is in analogy to (2.7) given by

$$F_{\mathbf{f}}[S] = F_{\mathbf{f}}[\mathbf{S}/D] := F_{\mathbf{f}}[\mathbf{S}]/D_\tau = \mathbf{f}/D_\tau \qquad \mathbb{P}\text{-a.s.} \quad (2.8)$$

We call the above mapping $F_{\mathbf{f}} : \{\mathbb{R}^d_+\text{-valued semimartingales}\} \to \bar{\mathbf{L}}^0_+(\mathcal{F}_\tau)$ the *contingent claim map* induced by the payoff $\mathbf{f}$ for $\mathbf{S}$.

Notation. We systematically use boldface symbols $\mathbf{S}, \mathbf{f}, \mathbf{V}$ etc. for all quantities in our basic model $\mathbf{S}$, and normal symbols $S, F_{\mathbf{f}}, V$ etc. in a general model $S = \mathbf{S}/D$. If quantities take a model as argument, we use square brackets $[\,\cdot\,]$ as in (2.7) or (2.8). Other arguments (like for instance a strategy ϑ) are put in round brackets; hence notation like $V(\vartheta)[S]$ appears.

Remark 2.19. A contingent claim map F takes a model $S = \mathbf{S}/D$, corresponding to a choice of unit, and returns a payoff $F[S]$, in that same unit. We point out that the mapping $S \mapsto (S_T - K)^+$ for a fixed $K \geq 0$ does *not* satisfy (2.8) and does *not* correctly describe a call option. If we have a model $(B, Y) = B(1, X) = B\mathbf{S}$ with a bond $B > 0$, say, a call option on asset i (with a strike K in units of $\mathbf{S}$) yields the payoff $\mathbf{f} = (X^i_T - K)^+$ for $\mathbf{S} = (1, X)$, and hence for (2.8) the *exchange option* $F_{\mathbf{f}}[B, Y] = (Y^i_T - KB_T)^+$.

If $\mathbf{f} \in \mathbf{L}^0_+(\mathcal{F}_T)$ is a payoff at time T and $\mathbf{U}^{\mathbf{f}}$ a valuation process for $\mathbf{f}$, this induces in analogy to (2.7) a unique *numéraire-invariant valuation map*

$$U^{\mathbf{f}} : \{\mathbb{R}^d_+\text{-valued semimartingales}\} \to \{\text{semimartingales} \geq 0\}$$

via the numéraire invariance property

$$U^{\mathbf{f}}[\mathbf{S}/D] := \mathbf{U}^{\mathbf{f}}/D \qquad \mathbb{P}\text{-a.s.} \tag{2.9}$$

Clearly, we then have $U^{\mathbf{f}}[\mathbf{S}] = \mathbf{U}^{\mathbf{f}}$ and the terminal condition

$$U^{\mathbf{f}}_T[S] = U^{\mathbf{f}}_T[\mathbf{S}/D] = \mathbf{U}^{\mathbf{f}}_T/D_T = \mathbf{f}/D_T = F_{\mathbf{f}}[\mathbf{S}]/D_T = F_{\mathbf{f}}[S].$$

Finally, the superreplication price π_σ from (2.5), in the units of $\mathbf{S}$, extends to the *superreplication price map*, for every $\mathbf{f} \in \bar{\mathbf{L}}^0_+(\mathcal{F}_\tau)$,

$$\begin{aligned}\Pi_\sigma(F_{\mathbf{f}} \,|\, {}^\sigma\Gamma)[S] := \operatorname{ess\,inf}\{v \in \bar{\mathbf{L}}^0_+(\mathcal{F}_\sigma) : \exists \vartheta \in {}^\sigma\Gamma \text{ such that } \mathbb{P}\text{-a.s. on } \{v < \infty\},\\ V_\sigma(\vartheta)[S] \leq v \text{ and } V_\tau(\vartheta)[S] \geq F_{\mathbf{f}}[S]\}.\end{aligned} \tag{2.10}$$

This implies $\Pi_\sigma(F_{\mathbf{f}} \,|\, {}^\sigma\Gamma)[\mathbf{S}] = \pi_\sigma(\mathbf{f} \,|\, {}^\sigma\Gamma)$ by (2.8) and (2.5), and we note for later use that combining (2.10) with the numéraire invariance (2.7) gives, for $\Gamma = L^{\mathrm{sf}}_+$,

$$\Pi_\sigma(F_{\mathbf{f}} \,|\, {}^\sigma L^{\mathrm{sf}}_+)[\mathbf{S}/D] = \frac{1}{D_\sigma}\Pi_\sigma(F_{\mathbf{f}} \,|\, {}^\sigma L^{\mathrm{sf}}_+)[\mathbf{S}] = \frac{1}{D_\sigma}\pi_\sigma(\mathbf{f} \,|\, {}^\sigma L^{\mathrm{sf}}_+). \tag{2.11}$$

So the superreplication price map (at time $\sigma \leq \tau$) is for fixed $\mathbf{f} \in \bar{\mathbf{L}}^0_+(\mathcal{F}_\tau)$ a mapping

$$\Pi_\sigma(F_{\mathbf{f}} \,|\, {}^\sigma\Gamma)[\,\cdot\,] : \{\mathbb{R}^d_+\text{-valued semimartingales}\} \to \bar{\mathbf{L}}^0_+(\mathcal{F}_\sigma)$$

with the numéraire invariance property (2.11). Analogously, the subreplication price $\pi^{\mathrm{sub}}_\sigma$ from (2.6) extends to a map $\Pi^{\mathrm{sub}}_\sigma$.

3 Attainability and completeness for general $\mathbf{S}$

In this section, we define attainable payoffs and completeness of a general model $\mathbf{S}$, and characterize them via dual (martingale) properties. This is similar in spirit to the classic theory; but the definitions are more general because they do not a priori impose any absence-of-arbitrage condition, and the results are more general because we only assume that $\mathbf{S}$ satisfies NUPBR.

For the classic setup $\mathbf{S} = (1, X)$, following Delbaen/Schachermayer [7], we call a payoff $\mathbf{f} \in \mathbf{L}^0_+(\mathcal{F}_T)$ *classically attainable* if it can be written as

$$\mathbf{f} = \mathbf{V}_T(\vartheta) = \tilde{V}_T(v_0, \psi) = v_0 + \psi \bullet X_T \qquad \mathbb{P}\text{-a.s.} \tag{3.1}$$

with a self-financing admissible strategy $\vartheta \mathrel{\widehat{=}} (v_0, \psi)$ such that $v_0 \in \mathbb{R}$ and ψ is a classically maximal integrand for X. Every classically attainable payoff must have a finite superreplication price; in fact, (3.1) implies $\pi_0(\mathbf{f} \,|\, {}^0L^{\mathrm{sf}}_{\mathrm{adm}}) \le v_0 < \infty$ with ${}^0L^{\mathrm{sf}}_{\mathrm{adm}} = {}^0L^{\mathrm{sf}}_{\mathrm{adm}}(1, X)$. Moreover, if X satisfies classic NFLVR, then $\psi \bullet X$ is a $\mathbb{Q}$-supermartingale for any $\psi \in {}^0L^{\mathrm{sf}}_{\mathrm{adm}}$ and $\mathbb{Q} \in \mathbb{M}^{\mathrm{e}}_{\mathrm{loc}}(X)$ and (3.1) gives $\mathbb{E}_{\mathbb{Q}}[\mathbf{f}] \le v_0$ so that actually $v_0 \ge 0$. The model $\mathbf{S}$ is *classically complete* if every $\mathbf{f} \in \mathbf{L}^0_+(\mathcal{F}_T)$ with $\pi_0(\mathbf{f} \,|\, {}^0L^{\mathrm{sf}}_{\mathrm{adm}}) < \infty$ is classically attainable.

If X satisfies classic NFLVR (and is locally bounded), then by [7, Corollary 14], $\mathbf{f}$ is classically attainable if and only if it can be written as in (3.1), with an integrand $\psi \in {}^0L(X)$ such that $\psi \bullet X$ is a true $\mathbb{Q}$-martingale for some $\mathbb{Q} \in \mathbb{M}^{\mathrm{e}}_{\mathrm{loc}}(X)$. The latter is the old definition of attainability used for instance in [18, 28, 26, 2, 4, 29]; it has the crucial disadvantage that it already needs ELMMs in its formulation. The same assumptions on X imply by [7, Theorem 16] that $\mathbf{f}$ is classically attainable if and only if $\sup\{\mathbb{E}_{\mathbb{Q}}[\mathbf{f}] : \mathbb{Q} \in \mathbb{M}^{\mathrm{e}}_{\mathrm{loc}}(X)\}$ is attained in some ELMM $\mathbb{Q}^*$. For the case where X is not locally bounded, see [9, Theorem 5.16].

Remark 3.1. In finite discrete time (fdt, for short), the classic definition of (fdt-)completeness for an arbitrage-free model $\mathbf{S} = (1, X)$ is that *every* payoff $\mathbf{f} \in \mathbf{L}^0_+(\mathcal{F}_T)$ (no condition on the superreplication price) is (fdt-)attainable in the sense that $\mathbf{f} = \mathbf{V}_T(\vartheta)$ for some $\vartheta \in {}^0L^{\mathrm{sf}}_{\mathrm{adm}}(1, X)$ (no maximality requirement); see [17, Definition 5.36]. This looks different at first sight.

To see that these concepts agree with our above definitions, denote by $\mathbb{M}^{\mathrm{e}}(X)$ the (non-empty, by absence of arbitrage) set of all EMMs $\mathbb{Q}$ for X. In finite discrete time, $V(\vartheta)[(1, X)]$ is for $\vartheta \in {}^0L^{\mathrm{sf}}_{\mathrm{adm}}(1, X)$ a (true) $\mathbb{Q}$-martingale for any $\mathbb{Q} \in \mathbb{M}^{\mathrm{e}}(X)$ [17, Theorem 5.14]; so maximality automatically holds, classic coincides with fdt-attainability whenever $\mathbf{f}$ has a finite superreplication price, and fdt- implies classic completeness. Again in finite discrete time, a payoff $\mathbf{f} \ge 0$ is fdt-attainable if and only if [17, Theorem 5.32] $\mathbb{Q} \mapsto \mathbb{E}_{\mathbb{Q}}[\mathbf{f}]$ is constant over $\mathbb{M}^{\mathrm{e}}(X) \ne \emptyset$; and any $\mathbf{f} \ge 0$ has $\mathbb{E}_{\mathbb{Q}}[\mathbf{f}] < \infty$ for some $\mathbb{Q} \in \mathbb{M}^{\mathrm{e}}(X) \ne \emptyset$ [17, (proof of) Theorem 5.29]. Now suppose we have classic completeness. Any bounded $\mathbf{f} \ge 0$ has $\pi_0(\mathbf{f} \,|\, {}^0L^{\mathrm{sf}}_{\mathrm{adm}}) < \infty$ and is hence fdt-attainable, and taking $\mathbf{f} = \mathbb{1}_A$ with $A \in \mathcal{F}_T$ thus implies by classic completeness that the mapping $\mathbb{Q} \mapsto \mathbb{E}_{\mathbb{Q}}[\mathbf{f}] = \mathbb{Q}[A]$ is constant over $\mathbb{M}^{\mathrm{e}}(X)$ so that there is exactly one EMM $\mathbb{Q}^*$ for X (on $\mathcal{F}_T$, to be precise). So any $\mathbf{f} \ge 0$ has $\sup_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[\mathbf{f}] = \mathbb{E}_{\mathbb{Q}^*}[\mathbf{f}] < \infty$, hence a finite superreplication price, and is thus classically and fdt-attainable. This proves fdt-completeness.

For the classic setup $\mathbf{S} = (1, X)$, it is folklore that if X satisfies classic NFLVR, then $\mathbf{S}$ is classically complete if and only if X admits a unique ELMM. Results in this spirit can be found in [18, 28, 4, 29] under extra assumptions; a good reference for general X seems surprisingly difficult to pinpoint. (For finite discrete time, see [17, Theorem 5.37] or Remark 3.1.) If we only have NUPBR (classic or general), there need not exist any ELMM for X, and so everything must be done differently.

Definition 3.2. For a model $\mathbf{S}$, a payoff $\mathbf{f} \in \mathbf{L}^0_+(\mathcal{F}_\tau)$ at a stopping time $\tau \leq T$ is called *attainable* if it can be written as

$$\mathbf{f} = \mathbf{V}_\tau(\vartheta) \qquad \mathbb{P}\text{-a.s.} \tag{3.2}$$

for a self-financing undefaultable strategy, $\vartheta \in {}^0L^{\mathrm{sf}}_+(\mathbf{S})$, which is maximal for ${}^0L^{\mathrm{sf}}_+(\mathbf{S})$. A model $\mathbf{S}$ is *complete* if each $\mathbf{f} \in \mathbf{L}^0_+(\mathcal{F}_T)$ with $\pi_0(\mathbf{f} \,|\, {}^0L^{\mathrm{sf}}_+) < \infty$ is attainable.

Remark 3.3. Note as above that if the replication property (3.2) holds, we automatically have a finite superreplication price because $\pi_0(\mathbf{f} \,|\, {}^0L^{\mathrm{sf}}_+) = \pi_0(\mathbf{V}_\tau(\vartheta) \,|\, {}^0L^{\mathrm{sf}}_+) \leq \mathbf{V}_0(\vartheta) < \infty$.

We first show that our notion of attainability coincides with the classic one in the classic setup under classic NFLVR.

Proposition 3.4. *Suppose $\mathbf{S} = (1, X)$ with an $\mathbb{R}^d$-valued semimartingale $X \geq 0$ satisfying classic NFLVR. Then $\mathbf{f} \in \mathbf{L}^0_+(\mathcal{F}_T)$ is attainable if and only if it is classically attainable.*

Proof. Clearly, $\mathbf{S} \geq 0$ satisfies (2.3). Because X satisfies classic NFLVR and hence classic NUPBR, $\mathbf{S}$ satisfies NUPBR and is therefore dynamically viable by Theorem 2.13. Due to [20, Proposition 3.18], maximality in our sense is thus the same as weak maximality in the sense of [20]; see also [21, Remark 3.4]. Moreover, a self-financing strategy $\vartheta \mathrel{\widehat{=}} (v_0, \psi)$ is undefaultable if and only if $v_0 \geq 0$ and ψ is a v_0-admissible integrand for X.

Now consider $\vartheta \mathrel{\widehat{=}} (v_0, \psi)$ and $\vartheta' \mathrel{\widehat{=}} (v'_0, \psi')$. Then $v'_0 = \mathbf{V}_0(\vartheta') \leq \mathbf{V}_0(\vartheta) = v_0$ is equivalent to $v'_0 \leq v_0$, and $v'_0 + \psi' \bullet X_T = \mathbf{V}_T(\vartheta') \geq \mathbf{V}_T(\vartheta) = v_0 + \psi \bullet X_T$ is equivalent to $\psi' \bullet X_T \geq v_0 - v'_0 + \psi \bullet X_T$. We recall that thanks to classic NA, any admissible integrand ψ for X with $\psi \bullet X_T \geq -a$ is even a-admissible; see before Definition 2.5.

If $\mathbf{f}$ is classically attainable, then $\mathbf{f} = v_0 + \psi \bullet X_T$ with $v_0 \geq 0$ and an admissible integrand ψ for X which is classically maximal. As $\mathbf{f} \geq 0$, we get $\psi \bullet X_T \geq -v_0$ and so ψ is even a v_0-admissible integrand due to classic NA. Therefore $\vartheta \mathrel{\widehat{=}} (v_0, \psi)$ is undefaultable and $\mathbf{V}_T(\vartheta) = \mathbf{f}$. For any undefaultable $\vartheta' \mathrel{\widehat{=}} (v'_0, \psi')$ with $\mathbf{V}_0(\vartheta') \leq \mathbf{V}_0(\vartheta)$ and $\mathbf{V}_T(\vartheta') \geq \mathbf{V}_T(\vartheta)$, we then get $\psi' \bullet X_T \geq \psi \bullet X_T$ and therefore equality from the classic maximality of ψ. But this means that ϑ is weakly maximal among all $\vartheta' \in {}^0L^{\mathrm{sf}}_+$ and hence also maximal for ${}^0L^{\mathrm{sf}}_+$. So $\mathbf{f}$ is attainable.

Conversely, if $\mathbf{f}$ is attainable, we have $\mathbf{f} = \mathbf{V}_T(\vartheta) = v_0 + \psi \bullet X_T$ with $\vartheta \mathrel{\widehat{=}} (v_0, \psi)$. As ϑ is undefaultable, ψ is v_0-admissible and hence an admissible integrand for X. Any admissible integrand ψ' for X with $\psi' \bullet X_T \geq \psi \bullet X_T \geq -v_0$ is then v_0-admissible due to classic NA, and therefore $\vartheta' \mathrel{\widehat{=}} (v_0, \psi')$ is undefaultable and has $\mathbf{V}_0(\vartheta') = v_0 = \mathbf{V}_0(\vartheta)$. Therefore maximality of ϑ in ${}^0L^{\mathrm{sf}}_+$ implies that $\psi' \bullet X_T = \mathbf{V}_T(\vartheta') - v_0 = \mathbf{V}_T(\vartheta) - v_0 = \psi \bullet X_T$, and so ψ is also classically maximal. Hence $\mathbf{f}$ is classically attainable. $\square$

The next result is the analogue of the pricing characterization of classically attainable payoffs. As one expects, its formulation involves a change of numéraire.

Proposition 3.5. ([20, Lemma 4.21]) *Suppose* $\mathbf{S} \geq 0$ *satisfies* (2.3) *and NUPBR. For a payoff* $\mathbf{f}$ *at a stopping time* $\tau \leq T$ *with* $\pi_0(\mathbf{f} \,|\, {}^0L_+^{\mathrm{sf}}) < \infty$*, the following are then equivalent:*

1) $\mathbf{f}$ *is maximal for* ${}^0L_+^{\mathrm{sf}}$*, i.e.,* $\pi_0(\mathbf{f}+\mathbf{g} \,|\, {}^0L_+^{\mathrm{sf}}) > \pi_0(\mathbf{f} \,|\, {}^0L_+^{\mathrm{sf}})$ *for every* $\mathbf{g} \in \mathbf{L}_+^0(\mathcal{F}_\tau) \setminus \{0\}$*.*

2) $\mathbf{f}$ *is attainable.*

3) *There exists a numéraire/ELMM pair* $(D, \mathbb{Q})$ *with* $D_0 = 1$ *and* $\pi_0(\mathbf{f} \,|\, {}^0L_+^{\mathrm{sf}}) = \mathbb{E}_{\mathbb{Q}}[\mathbf{f}/D_\tau]$*.*

4) *For each* $\mathbb{Q} \approx \mathbb{P}$ *on* $\mathcal{F}_T$*, there exists a numéraire* D *with* $D_0 = 1$ *such that* $\mathbb{Q}$ *is an ELMM for* $S = \mathbf{S}/D$ *and* $\pi_0(\mathbf{f} \,|\, {}^0L_+^{\mathrm{sf}}) = \mathbb{E}_{\mathbb{Q}}[\mathbf{f}/D_\tau]$*.*

For the classic setup $\mathbf{S} = (1, X)$, it is folklore that if X is arbitrage-free, then completeness is equivalent to having a unique ELMM for X; see the discussion after Remark 3.1. In the more general case where we only have NUPBR, the dual objects characterizing absence of arbitrage are numéraire/ELMM pairs $(D, \mathbb{Q})$, and one might think that completeness would be characterized by uniqueness of such a pair (if there is one). This is not entirely accurate, because one can only get uniqueness of either D or $\mathbb{Q}$ if the other part of the pair is fixed; the next result gives a precise statement. Moreover, a numéraire is unique at best up to a constant factor; this explains why we must impose $D_0 = 1$ below.

Theorem 3.6. *If* $\mathbf{S} \geq 0$ *satisfies* (2.3)*, the following are equivalent:*

1) $\mathbf{S}$ *is complete.*

2) *For each stopping time* $\tau \leq T$*, each* $\mathbf{f} \in \mathbf{L}_+^0(\mathcal{F}_\tau)$ *with* $\pi_0(\mathbf{f} \,|\, {}^0L_+^{\mathrm{sf}}) < \infty$ *is attainable.*

3) *There exists a tradable numéraire/ELMM pair* $(\eta, \mathbb{Q})$*, and for that (tradable) numéraire* $D = \mathbf{V}(\eta)$*, the model* $S = \mathbf{S}/D = \mathbf{S}^{(\eta)}$ *has exactly one ELMM (namely* $\mathbb{Q}$*).*

4) *There exists a numéraire/ELMM pair* $(D, \mathbb{Q})$*, and for each fixed numéraire* D'*, the model* $S' = \mathbf{S}/D'$ *has at most one ELMM.*

5) *There exists a numéraire/ELMM pair* $(D, \mathbb{Q})$*, and for each fixed* $\bar{\mathbb{Q}} \approx \mathbb{P}$ *on* $\mathcal{F}_T$*, there is exactly one numéraire* $\bar{D}$ *with* $\bar{D}_0 = 1$ *such that* $\bar{S} = \mathbf{S}/\bar{D}$ *has that* $\bar{\mathbb{Q}}$ *as ELMM.*

Proof. "5) $\Rightarrow$ 4)": Assume by way of contradiction that there are a numéraire D' and ELMMs $\mathbb{Q}^1, \mathbb{Q}^2$ for $S' = \mathbf{S}/D'$ with $\mathbb{Q}^1 \neq \mathbb{Q}^2$. Replacing D' by D'/D'_0 if necessary, we may assume that $D'_0 = 1$. Denote by Z^i the density process of $\mathbb{Q}^i$ with respect to $\mathbb{P}$ and note that $1/Z^i$ is a numéraire. Then $S^{(i)} := Z^i S' = \mathbf{S}/(D'/Z^i)$ are both local $\mathbb{P}$-martingales by Bayes' theorem, and both numéraires $D^i := D'/Z^i$ have $D_0^i = 1$ because $Z_0^i = 1$ (as $\mathcal{F}_0$ is trivial). But $D^1 \not\equiv D^2$ as $\mathbb{Q}^1 \neq \mathbb{Q}^2$, which contradicts 5) with $\bar{\mathbb{Q}} := \mathbb{P}$.

"4) $\Rightarrow$ 3)": This is clear once we recall the implication "3) $\Rightarrow$ 2)" from Theorem 2.12.

"3) $\Rightarrow$ 2)": Fix $\mathbf{f} \in \mathbf{L}_+^0(\mathcal{F}_\tau)$ with $\pi_0(\mathbf{f} \,|\, {}^0L_+^{\mathrm{sf}}) < \infty$ and a tradable numéraire/ELMM pair $(\eta, \mathbb{Q})$. So $D = \mathbf{V}(\eta)$ and by replacing η by $\eta/\mathbf{V}_0(\eta)$, we may assume $D_0 = 1$. The usual supermartingale argument gives $\mathbb{E}_{\mathbb{Q}}[\mathbf{f}/D_\tau] \leq \pi_0(\mathbf{f} \,|\, {}^0L_+^{\mathrm{sf}}) < \infty$ so that $\mathbf{f}/D_\tau$

is in $\mathbf{L}^1_+(\mathcal{F}_\tau, \mathbb{Q})$. Because the classic model $(1, \mathbf{S}/D)$ (of dimension $N+1$) has $\mathbb{Q}$ by assumption as unique ELMM for $\mathbf{S}/D$, we can write the $\mathbb{Q}$-martingale $M \geq 0$ with final value $M_\tau = \mathbf{f}/D_\tau$, as $M = M_0 + \zeta \bullet (\mathbf{S}/D) = \zeta_0 \cdot (\mathbf{S}_0/D_0) + \zeta \bullet (\mathbf{S}/D)$ for some $\zeta \in {}^0L(\mathbf{S}/D)$. But $D = \mathbf{V}(\eta)$ is tradable so that $\mathbf{S}/D = \mathbf{S}/\mathbf{V}(\eta) = \mathbf{S}^{(\eta)}$. From [20, Theorem 2.14], we thus obtain a self-financing strategy ϑ for $\mathbf{S}^{(\eta)}$, $\vartheta \in {}^0L^{\mathrm{sf}}_+(\mathbf{S}/D)$, with

$$M = \zeta_0 \cdot \mathbf{S}^{(\eta)}_0 + \zeta \bullet \mathbf{S}^{(\eta)} = \zeta_0 \cdot \mathbf{S}^{(\eta)}_0 + \vartheta \bullet \mathbf{S}^{(\eta)} = V(\vartheta)[\mathbf{S}^{(\eta)}] = V(\vartheta)[\mathbf{S}/D] = \mathbf{V}(\vartheta)/D$$

by the numéraire invariance (2.7). This shows that $\mathbf{f} = D_\tau M_\tau = \mathbf{V}_\tau(\vartheta)$ is attainable.

"2) $\Rightarrow$ 1)": This is trivial.

"1) $\Rightarrow$ 5)": This is the most difficult implication. First, by completeness, the zero payoff 0 at time T is attainable so that there exists a maximal strategy $\vartheta \in {}^0L^{\mathrm{sf}}_+$ with $\mathbf{V}_T(\vartheta) = 0$. By [20, Proposition 3.14], the zero strategy 0 is then also maximal for ${}^0L^{\mathrm{sf}}_+$, and so there exists a numéraire/ELMM pair $(D, \mathbb{Q})$ by Theorem 2.12.

Next, it suffices to show that for *one* $\bar{\mathbb{Q}} \approx \mathbb{P}$ on $\mathcal{F}_T$, there is exactly one numéraire $\bar{D}$ with $\bar{D}_0 = 1$ such that $\bar{S} = \mathbf{S}/\bar{D}$ is a local $\bar{\mathbb{Q}}$-martingale. Indeed, take any other $\mathbb{Q}' \approx \mathbb{P}$ on $\mathcal{F}_T$ and any numéraire D' with $D'_0 = 1$ such that $S' = \mathbf{S}/D'$ is a local $\mathbb{Q}'$-martingale. If Z' denotes the density process of $\mathbb{Q}'$ with respect to $\bar{\mathbb{Q}}$, then $S'Z' = \mathbf{S}/(D'/Z')$ is by Bayes' theorem a local $\bar{\mathbb{Q}}$-martingale, and because $D'' := D'/Z'$ is a numéraire with $D''_0 = 1$ (as $Z'_0 = 1$) and $\bar{D}$ for $\bar{\mathbb{Q}}$ is unique by assumption, we must have $D'' = \bar{D}$. So $D' = Z'\bar{D}$ is uniquely determined from $\bar{D}$ and $\mathbb{Q}'$.

Finally, we show the existence of one $\bar{\mathbb{Q}} \approx \mathbb{P}$ on $\mathcal{F}_T$ as above. By Theorem 2.12, there exists a tradable numéraire/ELMM pair $(\eta, \mathbb{Q})$, and replacing η by $\eta/\mathbf{V}_0(\eta)$, we may assume that the tradable numéraire $\bar{D} := \mathbf{V}(\eta)$ has $\bar{D}_0 = 1$. We first claim that $\bar{\mathbb{Q}}$ is the unique ELMM for $\bar{S} = \mathbf{S}/\bar{D} = \mathbf{S}^{(\eta)}$. Indeed, let $\mathbb{Q}'$ be any ELMM for $\bar{S}$, take any $A \in \mathcal{F}_T$ and let $\mathbf{f}_A := \mathbb{1}_A \mathbf{V}_T(\eta) \leq \mathbf{V}_T(\eta) = \bar{D}_T$. By completeness of $\mathbf{S}$, $\mathbf{f}_A = \mathbf{V}_T(\vartheta^A)$ is attainable with a maximal strategy $\vartheta^A \in {}^0L^{\mathrm{sf}}_+$. So $\mathbf{V}_T(\vartheta^A)/\bar{D}_T = \mathbf{f}_A/\bar{D}_T = \mathbb{1}_A \leq 1 = \mathbf{V}_T(\eta)/\bar{D}_T$, and by the numéraire invariance (2.7), $\mathbf{V}(\vartheta)/\bar{D} = V(\vartheta)[\mathbf{S}/\bar{D}] = V(\vartheta)[\bar{S}]$ for $\vartheta \in \{\vartheta^A, \eta\}$. As ϑ^A is maximal, [20, Proposition 3.8] gives $0 \leq \mathbf{V}(\vartheta^A)/\bar{D} \leq \mathbf{V}(\eta)/\bar{D} \equiv 1$. For $R \in \{\bar{\mathbb{Q}}, \mathbb{Q}'\}$, $\mathbf{V}(\vartheta^A)/\bar{D} = V(\vartheta^A)(\bar{S})$ is thus a uniformly bounded local, and hence a true, R-martingale. Using that $\mathcal{F}_0$ is trivial, this yields for $R \in \{\bar{\mathbb{Q}}, \mathbb{Q}'\}$ that

$$R[A] = \mathbb{E}_R[\mathbb{1}_A] = \mathbb{E}_R[\mathbf{V}_T(\vartheta^A)/\bar{D}_T] = \mathbf{V}_0(\vartheta^A)/\bar{D}_0 = \mathbf{V}_0(\vartheta^A).$$

Because $A \in \mathcal{F}_T$ was arbitrary, this shows that $\mathbb{Q}' = \bar{\mathbb{Q}}$.

For the final step, we use several changes of numéraire and hence need the notations from Section 2.4. Let D be any numéraire with $D_0 = 1$ such that $S = \mathbf{S}/D \in \mathcal{M}_{\mathrm{loc}}(\bar{\mathbb{Q}})$. To show $S \equiv \bar{S}\,(= \mathbf{S}^{(\eta)})$, let Z be the $\bar{\mathbb{Q}}$-a.s. unique numéraire satisfying $S = Z\bar{S}$. Then we claim that $Z \in \mathcal{M}_{\mathrm{loc}}(\bar{\mathbb{Q}})$. Indeed, using $V(\eta)[\mathbf{S}^{(\eta)}] \equiv 1$ and the numéraire invariance (2.7) gives $Z = ZV(\eta)[\mathbf{S}^{(\eta)}] = ZV(\eta)[\bar{S}] = V(\eta)[S]$, and the latter is in $\mathcal{M}_{\mathrm{loc}}(\bar{\mathbb{Q}})$ like S. If we consider the classic model $(1, \bar{S})$ (of dimension $N+1$), the argument in the preceding paragraph shows that $\bar{\mathbb{Q}}$ is the unique ELMM for $\bar{S}$. The optional decomposition [31, 16, 38] then yields a predictable process $\zeta \in {}^0L(\bar{S})$ such that $Z = Z_0 + \zeta \bullet \bar{S}$, and because we have $\bar{S} = \mathbf{S}/\bar{D}$ and $\bar{D}$ is tradable, the same argument as in "3) $\Rightarrow$ 2)" gives some $\vartheta \in {}^0L^{\mathrm{sf}}_+(\bar{S})$ such that $Z = V(\vartheta)[\bar{S}]$. Again by the numéraire invariance (2.7),

$Z^2 = ZV(\vartheta)[\bar{S}] = V(\vartheta)[S]$; but this is in $\mathcal{M}_{\mathrm{loc}}(\bar{\mathbb{Q}})$ like S, and so $Z \equiv Z_0 = 1$ and hence $S \equiv \bar{S}$. This completes the proof. ☐

Remark 3.7. **1)** Suppose we replace 3) in Theorem 3.6 by the slightly weaker condition

3′) There exists a numéraire/ELMM pair $(D, \mathbb{Q})$, and for that (not necessarily tradable) numéraire D, the model $S = \mathbf{S}/D$ has exactly one ELMM (namely $\mathbb{Q}$).

We do not know if we can then obtain that 3′) implies 2).

2) For finite Ω, completeness is sometimes defined as the property that any $\mathcal{F}_T$-measurable random variable can be written as the final value of a self-financing strategy (which is automatically admissible when Ω is finite). Completeness then becomes a purely algebraic property which can hold or not, independently of absence of arbitrage; indeed, in that sense, the binomial model is always complete even if it allows arbitrage. In contrast, our general definition of completeness automatically entails an absence-of-arbitrage property, as can be seen in the implication "1) $\Rightarrow$ 5)" of Theorem 3.6.

For the classic setup under classic NFLVR, we obtain the following general result.

Corollary 3.8. *Suppose* $\mathbf{S} = (1, X)$*, where* $X \geq 0$ *is an* $\mathbb{R}^d$*-valued semimartingale satisfying classic NFLVR. Then* $\mathbf{S}$ *is complete if and only if it is classically complete, and this is also equivalent to the existence of a unique ELMM for* X*.*

Proof. Because attainability is equivalent to classic attainability by Proposition 3.4, the first statement is clear. Due to classic NFLVR, the classic FTAP gives the existence of an ELMM $\mathbb{Q}$ for $\mathbf{S}$ or X, and so $D \equiv 1$ gives the numéraire/ELMM pair $(1, \mathbb{Q})$. If $\mathbf{S}$ is complete, Theorem 3.6 with $D \equiv 1$ implies that $\mathbf{S}$ has at most one ELMM, and so we get uniqueness. Conversely, if $\mathbb{Q}$ is the only ELMM for $\mathbf{S}$, we have 3) in Theorem 3.6 with $D \equiv 1$ there (which is clearly tradable for $\mathbf{S} = (1, X)$), and so $\mathbf{S}$ is complete. ☐

Remark 3.9. **1)** A similar result like Theorem 3.6 can be found in Hunt/Kennedy [25, Section 7.3.5]. But a comparison needs some care — a numéraire in [25] is always a *tradable* numéraire in our sense (whereas our numéraires $D \in \mathcal{D}$ are called units in [25]). (Moreover, asset prices in [25] are continuous semimartingales; they even follow an SDE driven by a Brownian motion, although that is not needed everywhere in [25].) For this reason, our results are stronger than those in [25]; see also Corollary 3.13 below.

2) For the classic setup $\mathbf{S} = (1, X)$, Stricker/Yan [38, Corollary 2.1] also characterize completeness using only classic NUPBR (actually, they impose the equivalent assumption that there exists a so-called local martingale density for X). However, their definition of attainability still involves the dual side via local martingale densities, and their way of using the optional decomposition theorem crucially exploits that one asset has price 1.

Theorem 2.13 gives a dual characterization of maximal strategies in ${}^0L^{\mathrm{sf}}_+$ in terms of martingale properties of their wealth. For a complete model, this looks particularly nice.

Corollary 3.10. *Suppose* $\mathbf{S} \geq 0$ *satisfies* (2.3)*. If* $\mathbf{S}$ *is complete, the following are equivalent for* $\vartheta \in {}^0L^{\mathrm{sf}}_+(\mathbf{S})$*:*

1) ϑ *is maximal for* ${}^0L^{\mathrm{sf}}_+(\mathbf{S})$*.*

2) *There exists a numéraire/ELMM pair* $(D, \mathbb{Q})$ *such that the* D*-discounted wealth process* $\mathbf{V}(\vartheta)/D$ *is a (true)* $\mathbb{Q}$*-martingale.*

3) *There exists a numéraire/ELMM pair* $(D, \mathbb{Q})$*, and for each numéraire/ELMM pair* $(D', \mathbb{Q}')$*, the* D'*-discounted wealth process* $\mathbf{V}(\vartheta)/D'$ *is a (true)* $\mathbb{Q}'$*-martingale.*

Proof. In all three cases, the zero strategy is maximal in ${}^0L_+^{\mathrm{sf}}$ — for 1) as in the proof of "1) $\Rightarrow$ 5)" for Theorem 3.6 by [20, Proposition 3.14], and for 2) and 3) by Theorem 2.12. So "1) $\Leftrightarrow$ 2)" is due to Theorem 2.13, and "3) $\Rightarrow$ 2)" is clear. To show "2) $\Rightarrow$ 3)", take a numéraire/ELMM pair $(D, \mathbb{Q})$ such that $\mathbf{V}(\vartheta)/D$ is a $\mathbb{Q}$-martingale and any other numéraire/ELMM pair $(D', \mathbb{Q}')$. Replacing D' by D'/D'_0 and D by D/D_0 if necessary, we can assume that $D'_0 = 1 = D_0$. If Z' is the density process of $\mathbb{Q}'$ with respect to $\mathbb{Q}$, then both $\mathbf{S}/D'$ and $Z'(\mathbf{S}/D) = \mathbf{S}/(D/Z')$, by Bayes' theorem, are local $\mathbb{Q}'$-martingales. Because $\mathbf{S}$ is complete and $\bar{D} := D/Z'$ is a numéraire with $\bar{D}_0 = 1$, Theorem 3.6 with $\bar{\mathbb{Q}} = \mathbb{Q}'$ implies that $D' \equiv \bar{D}$ so that $Z'D' = D$. Therefore $\mathbf{V}(\vartheta)/D' = Z'\mathbf{V}(\vartheta)/D$, and because $\mathbf{V}(\vartheta)/D$ is a $\mathbb{Q}$-martingale, $\mathbf{V}(\vartheta)/D'$ is a $\mathbb{Q}'$-martingale by Bayes' theorem. □

Corollary 3.11. *Suppose* $\mathbf{S} \geq 0$ *satisfies* (2.3)*. If* $\mathbf{S}$ *is complete, the following are equivalent:*

1) *There exists a numéraire/ELMM pair* $(D, \mathbb{Q})$ *such that the* D*-discounted price process* $\mathbf{S}/D$ *is a (true)* $\mathbb{Q}$*-martingale.*

2) *There exists a numéraire/ELMM pair, and for each numéraire/ELMM pair* $(D', \mathbb{Q}')$*, the* D'*-discounted price process* $\mathbf{S}/D'$ *is a (true)* $\mathbb{Q}'$*-martingale.*

Proof. By Corollary 3.10, 1) is equivalent to saying that each e^i, $i = 1, \ldots, N$, is maximal for ${}^0L_+^{\mathrm{sf}}$ as $\mathbf{V}(e^i)/D = \mathbf{S}^i/D$. Again by Corollary 3.10, this is then equivalent to 2). □

Remark 3.12. In [21], we have introduced a notion of financial bubbles, and we have shown that $\mathbf{S}$ has a (so-called strong) bubble if and only if for *every* numéraire/ELMM pair $(D, \mathbb{Q})$, the D-discounted price process $\mathbf{S}/D$ is under $\mathbb{Q}$ a strict local martingale. Corollary 3.11 illustrates another facet of the robustness of our approach: If $\mathbf{S}$ is complete and if one (economically equivalent) model $S = \mathbf{S}/D$ with $D \in \mathcal{D}$ is a true martingale for its (unique, by completeness) ELMM, the same is true for *every* model $S' = \mathbf{S}/D'$ with $D' \in \mathcal{D}$.

We can reformulate part of Corollary 3.11 in the classic setup to get the following result. Because the appearing numéraire D need not be tradable, it seems not obvious how to prove this in a simple way with methods from the classic theory. Recall that local martingale deflators have been introduced in Remark 2.8.

Corollary 3.13. *Suppose* $\mathbf{S} = (1, X)$*, where* $X \geq 0$ *is an* $\mathbb{R}^d$*-valued semimartingale. Suppose that* X *has a unique ELMM* $\mathbb{Q}^*$ *and that* X *is a true* $\mathbb{Q}^*$*-martingale. Then:*

1) *Let* D *be any strictly positive semimartingale (meaning* $\inf_{0 \leq t \leq T} D_t > 0$ $\mathbb{P}$*-a.s., i.e.,* D *is a numéraire). If* $\mathbf{S}/D$ *admits an ELMM* $\mathbb{Q}$*, then* X/D *is a true* $\mathbb{Q}$*-martingale (and* $\mathbb{Q}$ *is unique).*

2) *If* Y *(with* $Y_0 = 1$*) is any deflator for* $\mathbf{S}$ *under* $\mathbb{P}$*, then* Y *is unique (and coincides with* $Z^{\mathbb{Q}^*;\mathbb{P}}$*, the density process of* $\mathbb{Q}^*$ *with respect to* $\mathbb{P}$*).*

4 Valuation I: Basic ideas and first results

As explained briefly in Section 2.1, we want to study valuations $\mathbf{U}^{\mathbf{f}}$ for a payoff $\mathbf{f}$ which are consistent with a property of our model in the following sense: If $\mathbf{S}$ has property $\mathcal{E}$, the extended model $(\mathbf{S}, \mathbf{U}^{\mathbf{f}})$ has property $\mathcal{E}$ as well. We discuss in this section the properties "dynamic viability" and "dynamic efficiency" and show that both do not give good results — the first is too weak, the second too strong. The better, intermediate property of being "semi-efficient" is introduced and studied in the next section.

Remark 4.1. 1) Let $\mathcal{E}$ be one of the properties {dynamic viability, dynamic efficiency, static efficiency} and assume that $\mathbf{S} \geq 0$ satisfies (2.3). Then the dual characterizations in Theorems 2.12 and 2.17 and Proposition A.2 below imply that if the extended model $(\mathbf{S}, \mathbf{U}^{\mathbf{f}})$ has property $\mathcal{E}$, the original model $\mathbf{S}$ must have the same property as well.

2) Combining 1) and Remark 2.4 implies that if consistent valuations exist and are unique, it does not matter in which order we add payoffs to a model, nor if we add them one by one or (as a vector) all at once. More generally and precisely, existence for a vector payoff implies existence for stepwise addition and vice versa; and if we also have uniqueness, the resulting valuations are the same for both procedures.

4.1 Dynamically viable consistent valuations, or valuation by absence of arbitrage alone

We first study the case where the property $\mathcal{E}$ is dynamic viability. By Theorem 2.12, this is NUPBR, a numéraire-independent form of absence of arbitrage. Consistent valuation thus means that we want to value by absence of arbitrage, and use nothing more.

Theorem 4.2. *Suppose* $\mathbf{S} \geq 0$ *satisfies* (2.3) *and is dynamically viable (or, equivalently, satisfies NUPBR). Let* $\mathbf{f}$ *be a payoff at time* T *with* $\pi_0(\mathbf{f} \,|\, {}^0L^{\mathrm{sf}}_+) < \infty$. *Then:*

1) *There exists a dynamically viable consistent valuation process* $\mathbf{U}^{\mathbf{f}}$ *for* $\mathbf{f}$. *It is not unique in general, not even in a complete model.*

2) *For any valuation process* $\mathbf{U}^{\mathbf{f}}$ *for* $\mathbf{f}$, *the following are equivalent:*

   a) $\mathbf{U}^{\mathbf{f}}$ *is dynamically viable consistent.*

   b) *There exists a numéraire/ELMM pair* $(D, \mathbb{Q})$ *such that* $\mathbf{U}^{\mathbf{f}}/D$ *is a local* $\mathbb{Q}$*-martingale.*

   If $\mathbf{S}$ *is in addition complete, the above are also equivalent to*

   c) *For each numéraire/ELMM pair* $(D', \mathbb{Q}')$, $\mathbf{U}^{\mathbf{f}}/D'$ *is a local* $\mathbb{Q}'$*-martingale.*

Proof. 1) Because $\mathbf{S}$ is dynamically viable, there exists a numéraire/ELMM pair $(D, \mathbb{Q})$ by Theorem 2.12. The usual supermartingale argument and numéraire invariance via (2.11) then give $\mathbb{E}_{\mathbb{Q}}[\mathbf{f}/D_T] \leq \Pi_0(F_{\mathbf{f}} \,|\, {}^0L^{\mathrm{sf}}_+)[\mathbf{S}/D] = \pi_0(\mathbf{f}/D_T \,|\, {}^0L^{\mathrm{sf}}_+)/D_0 < \infty$ by hypothesis, and we can define $\mathbf{U}^{\mathbf{f}}/D$ as an RCCL version of the $\mathbb{Q}$-martingale $(\mathbb{E}_{\mathbb{Q}}[\mathbf{f}/D_T) \,|\, \mathcal{F}_t])_{0 \leq t \leq T}$. Then $(\mathbf{S}/D, \mathbf{U}^{\mathbf{f}}/D)$ is a local $\mathbb{Q}$-martingale and the extended model $(\mathbf{S}, \mathbf{U}^{\mathbf{f}})$ is dynamically viable by Theorem 2.12 again. Nonuniqueness is discussed in Example 4.3 after the proof.

2) "a) $\Leftrightarrow$ b)" follows immediately from Theorem 2.12 and the definition of a dynamically viable consistent valuation. Next, dynamic viability of $\mathbf{S}$ implies by Theorem 2.12 that there exists a numéraire/ELMM pair $(D, \mathbb{Q})$, and if $\mathbf{U}^{\mathbf{f}}/D$ is a local $\mathbb{Q}$-martingale as in c), we have b). This argument does not need completeness of $\mathbf{S}$.

Finally, assume in addition that $\mathbf{S}$ is complete. Let $(D, \mathbb{Q})$ be as in b) and $(D', \mathbb{Q}')$ any numéraire/ELMM pair. Replacing D' by D'/D'_0 if necessary, we may assume $D'_0 = 1$. If Z is the density process of $\mathbb{Q}'$ with respect to $\mathbb{Q}$, then $Z\mathbf{S}/D' = \mathbf{S}/(D'/Z) \in \mathcal{M}_{\mathrm{loc}}(\mathbb{Q})$ by Bayes' theorem, and as $\mathbf{S}$ is complete and $\bar{D} = D'/Z$ is a numéraire with $\bar{D}_0 = 1$, Theorem 3.6 yields $D = \bar{D}$. So $Z\mathbf{U}^{\mathbf{f}}/D' = \mathbf{U}^{\mathbf{f}}/D \in \mathcal{M}_{\mathrm{loc}}(\mathbb{Q})$ by b), $\mathbf{U}^{\mathbf{f}}/D' \in \mathcal{M}_{\mathrm{loc}}(\mathbb{Q}')$ by Bayes' theorem, and we get c). $\square$

Example 4.3. Even if $\mathbf{S}$ is complete, a dynamically viable consistent valuation $\mathbf{U}^{\mathbf{f}}$ for $\mathbf{f}$ need not be unique. This already comes up in the classic Black–Scholes model when valuing a European call with $F_{\mathbf{f}}[(B, Y)] = (Y_T - KB_T)^+$. If we take for $U^{\mathbf{f}}[B, Y]$ the price process from the Black–Scholes formula plus any strict local martingale L with $L_T = 0$ (hence $L_0 \neq 0$), the resulting valuation mapping satisfies the terminal condition $U^{\mathbf{f}}_T[B, Y] = F_{\mathbf{f}}[(B, Y)]$ and is dynamically viable consistent by Theorem 4.2 — but it does not agree with the valuation given by the Black–Scholes formula!

This example clearly shows that absence of arbitrage alone, in the strict sense of NUPBR or dynamic viability, or even classic NFLVR, is too weak as a requirement to obtain useful valuations. It is flexible enough to include markets with bubbles (as basic asset prices can be strict local martingales); but we pay the price that we get nonunique option values and perhaps even negative call prices in even the most basic models.

(To complete the example, a concrete choice for a strict local martingale $L \geq 0$ with $L_T = 0$ is for instance the process $L = \mathcal{E}(M)$ with $M_t = \int_0^t \frac{1}{\sqrt{T-s}}\, \mathrm{d}W_s$, $0 \leq t < T$. Because $\langle M \rangle_t \nearrow \infty$ as $t \nearrow T$, we have $\mathcal{E}(M)_t \to 0$ as $t \nearrow T$ and hence can set $\mathcal{E}(M)_T = 0$.)

4.2 Dynamically efficient consistent valuations, or valuation by risk-neutral expectation

We next turn to the case where the property $\mathcal{E}$ is dynamic efficiency. By Theorem 2.17, this is a numéraire-independent form of absence of arbitrage with an equivalent true martingale measure. The corresponding consistent valuation approach could therefore be called "risk-neutral valuation" or, more explicitly, valuation by risk-neutral expectation.

Theorem 4.4. *Suppose* $\mathbf{S} \geq 0$ *satisfies* (2.3) *and is dynamically efficient, and* $\mathbf{f}$ *is a payoff at time* T *with* $\pi_0(\mathbf{f} \,|\, {}^0L^{\mathrm{sf}}_+) < \infty$. *Then:*

1) *There exists a dynamically efficient consistent valuation process* $\mathbf{U}^{\mathbf{f}}$ *for* $\mathbf{f}$.

2) *For any valuation process* $\mathbf{U}^{\mathbf{f}}$ *for* $\mathbf{f}$, *the following are equivalent:*

   a) $\mathbf{U}^{\mathbf{f}}$ *is dynamically efficient consistent.*

   b) *There exists a numéraire/ELMM pair* $(D, \mathbb{Q})$ *such that* $(\mathbf{S}/D, \mathbf{U}^{\mathbf{f}}/D)$ *is a (true)* $\mathbb{Q}$*-martingale.*

c) *For each bounded numéraire strategy* $\eta \in {}^0L^{\mathrm{sf}}_+(\mathbf{S})$*, there exists an ELMM* $\mathbb{Q}$ *for* $\mathbf{S}^{(\eta)} = \mathbf{S}/\mathbf{V}(\eta)$ *such that* $(\mathbf{S}^{(\eta)}, \mathbf{U}^{\mathbf{f}}/\mathbf{V}(\eta))$ *is a (true)* $\mathbb{Q}$*-martingale. In particular, this applies to the market portfolio* $\eta := \eta^{\mathbf{S}} \equiv \mathbf{1}$*.*

If $\mathbf{S}$ *is in addition complete, the above are also equivalent to*

d) *For each numéraire/ELMM pair* $(D', \mathbb{Q}')$*,* $\mathbf{U}^{\mathbf{f}}/D'$ *is the (unique, true)* $\mathbb{Q}'$*-martingale with final value* $\mathbf{f}/D'_T$*.*

In particular, in a complete model, a dynamically efficient consistent valuation is unique.

Proof. 1) This argument parallels the proof of part 1) in Theorem 4.2. Instead of Theorem 2.12, we use Theorem 2.17 and from there obtain true instead of local martingales.

2) "a) $\Leftrightarrow$ b)" follows directly from Theorem 2.17, and "b) $\Leftrightarrow$ c)" is due to [20, Corollary 4.15], applied to the extended model $(\mathbf{S}, \mathbf{U}^{\mathbf{f}})$ and with $\hat{\eta} \equiv \mathbf{1}^f := (1, \dots, 1) \in \mathbb{R}^{N+1}$ being the corresponding extended market portfolio.

Next, dynamic efficiency of $\mathbf{S}$ implies by Theorem 2.17 that there is a numéraire/ ELMM pair $(D, \mathbb{Q})$ such that $\mathbf{S}/D$ is a true $\mathbb{Q}$-martingale. For any bounded numéraire strategy $\eta \in {}^0L^{\mathrm{sf}}_+$, the model $\mathbf{S}^{(\eta)}$ then admits an equivalent true martingale measure and η is maximal in ${}^0L^{\mathrm{sf}}_+$ by [20, Corollary 4.15] (with $\hat{\eta} := \eta^{\mathbf{S}} \equiv \mathbf{1} \in \mathbb{R}^N$). Under d), also $\mathbf{U}^{\mathbf{f}}/\mathbf{V}(\eta)$ is a true $\mathbb{Q}$-martingale and we get c). Again, completeness of $\mathbf{S}$ is not needed.

Finally, the argument for "c) $\Rightarrow$ d)" goes like the proof of "b) $\Rightarrow$ c)" for Theorem 4.2; the only difference is that we have true instead of local martingales for $\mathbf{U}^{\mathbf{f}}/D$ and $\mathbf{U}^{\mathbf{f}}/D'$. Uniqueness of $\mathbf{U}^{\mathbf{f}}$ in the complete case is clear from the martingale property in d). $\square$

Remark 4.5. We have added the property 2c) in Theorem 4.4 because this gives an easy way to check the applicability of that result: Just look at $\mathbf{S}^{(\eta^{\mathbf{S}})} = \mathbf{S}/\sum_{i=1}^N \mathbf{S}^i$ and try to find a true EMM $\mathbb{Q}$ for this. If then also $\mathbf{U}^{\mathbf{f}}/\sum_{i=1}^N \mathbf{S}^i$ is a true $\mathbb{Q}$-martingale, we have a dynamically efficient consistent valuation.

In summary, the above result shows that valuation "by risk-neutral expectation" has very nice martingale properties and gives unique values in a complete model. However, it needs the nontrivial condition that $\mathbf{S}$ is dynamically efficient, so that we must have an equivalent *true* martingale measure, at least after discounting with *some* numéraire. Many models satisfy this, but there are also interesting concrete examples where it is asking too much; see Remark 7.3 for an illustration. In particular, for any model which has a strong bubble in the sense of Herdegen/Schweizer [21], no dynamically efficient consistent valuation can exist; this follows from Remark 4.1 and [21, Theorem 3.7].

For the classic setup, we conclude that risk-neutral valuation looks nice and simple, but only makes sense if all underlying assets are *true* martingales under the chosen ELMM.

5 Valuation II: Semi-efficient models

This section studies consistent valuations in models which are not only dynamically viable, but in addition statically efficient. These are minimal requirements for a financial market to behave in a reasonable way. In fact, dynamic viability, or NUPBR for $\mathbf{S}$ by Theorem 2.12, is a weak numéraire-independent absence-of-arbitrage condition. Static efficiency means that every buy-and-hold strategy $\vartheta \in \mathbf{h}^\sigma L^{\mathrm{sf}}_+$ is maximal in $\mathbf{h}^\sigma L^{\mathrm{sf}}_+$ for each stopping time $\sigma \le T$, so that there is no relative arbitrage between any two static σ-to-T positions in $\mathbf{S}$. This condition seems very plausible; examples where it fails to hold occur if

- a static position in one stock is improved by a static position in other stocks, or
- a static position in a stock is worse than a static position in call and riskless asset.

Such situations point to a degeneracy and should not happen in a good model $\mathbf{S}$. We give this fundamental combination of properties a name. Anticipating Section 7.2 below, we mention that all three models from Example 2.2 yield semi-efficient models.

Definition 5.1. We call $\mathbf{S}$ *semi-efficient* if it is dynamically viable and statically efficient. A valuation $\mathbf{U^f}$ is *semi-efficient consistent* if the extended model $(\mathbf{S}, \mathbf{U^f})$ is semi-efficient.

Our goal in this section is to study semi-efficient consistent valuations. The main results are Theorem 5.5 on the characterization of static efficiency, and its use to show in Theorem 5.13 existence and structure of the desired valuations.

5.1 Static efficiency

We first need a dual characterization of static efficiency so that we can work with that concept, and this requires analysing the maximality of buy-and-hold strategies ϑ in $\mathbf{h}^\sigma L^{\mathrm{sf}}_+$. Recall that such a ϑ is constant on $[\![\sigma, T]\!]$. So we only compare the market at times σ and T and hence in effect look at a one-period model. This motivates the next definition.

Definition 5.2. *Let $\sigma \le T$ be a stopping time and $\eta \in \mathbf{h}^\sigma L^{\mathrm{sf}}_+$ a (buy-and-hold) numéraire strategy. A one-step equivalent martingale measure (EMM) for $\mathbf{S}^{(\eta)}$ on $\{\sigma, T\}$ is a probability $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ with $\mathbf{S}^{(\eta)}_T \in L^1(\mathbb{Q})$ and $\mathbb{E}_\mathbb{Q}[\mathbf{S}^{(\eta)}_T \mid \mathcal{F}_\sigma] = \mathbf{S}^{(\eta)}_\sigma$ $\mathbb{Q}$-a.s.*

In other words, $\mathbb{Q}$ is simply an equivalent (true) martingale measure in the classic sense for the one-period model with $\mathcal{G}_0 = \mathcal{F}_\sigma, \mathcal{G}_1 = \mathcal{F}_T$ and $X_0 = \mathbf{S}^{(\eta)}_\sigma, X_1 = \mathbf{S}^{(\eta)}_T$. Note that all these one-period models have the same end date T. We discuss that important point in more detail in Remarks 2.14, 2.18 and 5.8 below.

For the development of our results, the main inputs needed from this section are Theorem 5.5 and Corollary 5.4 below. To streamline the presentation, we have moved the underlying more technical results and proofs to Appendix A.

It is part of our approach to work with undefaultable strategies $\vartheta \in {}^0L^{\mathrm{sf}}_+$, i.e., with nonnegative wealth. But some arguments become much simpler if we can take differences of strategies, and then we want to use ${}^0L^{\mathrm{sf}}$ instead of ${}^0L^{\mathrm{sf}}_+$. In the classic setup, it is well known for finite discrete time (hence in particular for one-period models) that for absence-of-arbitrage and valuation questions, it does not matter if one uses all self-financing strategies or only those with nonnegative wealth; see for instance [11, Section 2.2] or [32, Lemma 1.2.7]. The next result generalizes this property to buy-and-hold strategies in continuous-time models. The key ingredient is static efficiency, which is in one-period (but *not* in multiperiod, see Remark 2.18) models equivalent to dynamic efficiency and hence to absence of arbitrage by [21, Proposition 4.3].

Lemma 5.3. *Suppose* $\mathbf{S} \geq 0$ *satisfies* (2.3) *and is statically efficient. Fix a stopping time* $\sigma \leq T$.

1) *If* $\vartheta \in \mathbf{h}^\sigma L^{\mathrm{sf}}$ *satisfies* $\mathbf{V}_T(\vartheta) \geq 0$ $\mathbb{P}$*-a.s., then* $\vartheta \in \mathbf{h}^\sigma L_+^{\mathrm{sf}}$*: A buy-and-hold strategy with nonnegative final wealth has nonnegative wealth over its entire lifetime* $[\![\sigma, T]\!]$.

2) *Valuation with undefaultable or with arbitrary self-financing buy-and-hold strategies yields the same result: If* $\mathbf{f}$ *is a payoff at time* T*, then*

$$\pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}}) = \pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}) \qquad \mathbb{P}\text{-}a.s. \tag{5.1}$$

If moreover $\pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}}) < \infty$ $\mathbb{P}$*-a.s., there exists a strategy* $\vartheta \in \mathbf{h}^\sigma L_+^{\mathrm{sf}}$ *with*

$$\mathbf{V}_T(\vartheta) \geq \mathbf{f} \quad \textit{and} \quad \mathbf{V}_\sigma(\vartheta) = \pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}}), \qquad \mathbb{P}\text{-}a.s.$$

In other words, if the cost of superreplication is finite, the essential infimum from the definition (2.5) *is attained as a minimum.*

Proof. This proof partly refers to Appendix A. Part 1) is shown in "1) ⇒ 2)" for Proposition A.2. For 2), the inequality "$\geq$" in (5.1) is clear because $\mathbf{h}^\sigma L_+^{\mathrm{sf}} \subseteq \mathbf{h}^\sigma L^{\mathrm{sf}}$. For "$\leq$", arguing as in the proof of [21, Proposition A.2] via positive $\mathcal{F}_\sigma$-homogeneity of superreplication prices, we may assume without loss of generality that the right-hand side is finite $\mathbb{P}$-a.s. From static efficiency, parts 4) of Propositions A.2 and A.1 give the existence of a one-step EMM for $\mathbf{S}^{(\eta)}$ on $\{\sigma, T\}$, for some numéraire strategy $\eta \in \mathbf{h}^\sigma L_+^{\mathrm{sf}} \subseteq \mathbf{h}^\sigma L^{\mathrm{sf}}$, and so Proposition A.1 yields the existence of a $\vartheta \in \mathbf{h}^\sigma L^{\mathrm{sf}}$ with

$$\mathbf{V}_T(\vartheta) \geq \mathbf{f} \geq 0 \quad \text{and} \quad \mathbf{V}_\sigma(\vartheta) = \pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}), \qquad \mathbb{P}\text{-a.s.}$$

But $\vartheta \in \mathbf{h}^\sigma L_+^{\mathrm{sf}}$ by part 1); so $\pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}}) \leq \mathbf{V}_\sigma(\vartheta)$ $\mathbb{P}$-a.s. yields "$\leq$" in (5.1). ☐

The next corollary is simple, but very useful. It extends Remark 3.3.

Corollary 5.4. *Suppose* $\mathbf{S} \geq 0$ *satisfies* (2.3) *and is statically efficient. If* $\mathbf{f}$ *is a payoff at a stopping time* $\tau \leq T$ *with* $\pi_0(\mathbf{f} \,|\, \mathbf{h}^0 L_+^{\mathrm{sf}}) < \infty$*, then also* $\pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}}) < \infty$ $\mathbb{P}$*-a.s. for every stopping time* $\sigma \leq \tau$.

Proof. Part 2) of Lemma 5.3 gives a strategy $\vartheta \in \mathbf{h}^0 L_+^{\mathrm{sf}} \subseteq \mathbf{h}^\sigma L_+^{\mathrm{sf}}$ with $\mathbf{V}_\tau(\vartheta) \geq \mathbf{f}$ $\mathbb{P}$-a.s. So the definition (2.5) of superreplication prices yields $\pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}}) \leq \mathbf{V}_\sigma(\vartheta) < \infty$ $\mathbb{P}$-a.s. ☐

With the results above and in Appendix A, we can now formulate a dual characterization of statically efficient markets in a handy version. For an $\mathbb{R}^N$-valued random vector Y and a sub-σ-field $\mathcal{G}$, we write $\mathcal{L}(Y \,|\, \mathcal{G})$ for a regular conditional distribution of Y given $\mathcal{G}$, and $\operatorname{ri}\operatorname{conv}\operatorname{supp}\mathcal{L}(Y \,|\, \mathcal{G})$ for the relative interior of the convex hull of its (ω-dependent) topological support.

Theorem 5.5. *If* $\mathbf{S} \geq 0$ *satisfies* (2.3) *and is statically viable, the following are equivalent:*

1) $\mathbf{S}$ *is statically efficient.*

2) *For each deterministic* $s \in [0, T]$*, there exists a pair* $(\eta, \mathbb{Q})$ *such that* η *is a numéraire strategy with* $\eta \in \mathbf{h}^s L_+^{\mathrm{sf}}$ *and* $\mathbb{Q}$ *is a one-step EMM for* $\mathbf{S}^{(\eta)} = \mathbf{S}/\mathbf{V}(\eta)$ *on* $\{s, T\}$.

3) *For each deterministic* $s \in [0,T]$*, there exists a numéraire strategy* $\eta \in \mathbf{h}^s L^{\mathrm{sf}}_+$ *such that* $\mathbf{S}^{(\eta)}_s \in \operatorname{ri}\operatorname{conv}\operatorname{supp}\mathcal{L}(\mathbf{S}^{(\eta)}_T \mid \mathcal{F}_s)$ $\mathbb{P}$*-a.s.*

In both 2) *and* 3)*, we can choose for* η *the market portfolio* $\eta^{\mathbf{S}} \equiv \mathbf{1}$.

Proof. Static viability of $\mathbf{S}$ allows us to use 3′) from Proposition A.2 and obtain that 1) is equivalent to weak maximality of 0 for ${}^sL^{\mathrm{sf}}$, for each $s \in [0,T)$. By Proposition A.1, with $\sigma := s$, the latter is equivalent to 2) as well as to 3). Finally, choosing $\eta = \eta^{\mathbf{S}}$ is possible by part 4) of Proposition A.1. □

Remark 5.6. 1) Without static viability, almost the same argument shows that static efficiency is equivalent to 2) or 3) with deterministic s replaced by stopping times $\sigma \le T$.

2) Theorem 5.5 is especially useful for markets with diffusion dynamics because one can then exploit the many known results about transition densities to check 3).

Static viability is in particular satisfied if $\mathbf{S}$ is dynamically viable. Combining Theorems 5.5 and 2.12 thus yields the following characterization of semi-efficient models.

Theorem 5.7. *If* $\mathbf{S} \ge 0$ *satisfies* (2.3)*, the following are equivalent:*

1) $\mathbf{S}$ *is semi-efficient.*

2) $\mathbf{S}$ *satisfies NUPBR, and all one-period submodels of* $\mathbf{S}$ *between a deterministic* $s \le T$ *and the final date* T *are arbitrage-free and hence admit a one-step EMM on* $\{s,T\}$ *after discounting with some numéraire* $\mathbf{V}(\eta)$*. (In fact, we can take* $\eta = \eta^{\mathbf{S}} \equiv \mathbf{1}$*.)*

Note that even if we choose $\eta \equiv \eta^{\mathbf{S}}$, the EMM depends on the initial date s of the one-period model; no single EMM will fit all one-period submodels in general. We also remark that 2) in Theorem 5.7 is equivalent to the same statement with deterministic s replaced by stopping times $\sigma \le T$.

Remark 5.8. In Remark 2.14, we have briefly considered (and rejected) the use of simple instead of buy-and-hold strategies. We believe (but did not formally check) that similar arguments as for Theorem 5.7 can be used to show that $\mathbf{S}$ is dynamically viable and statically efficient for simple strategies if and only if $\mathbf{S}$ satisfies NUPBR and all submodels of $\mathbf{S}$ which allow trading only at finitely many fixed time points are arbitrage-free. But as already discussed in Remarks 2.14 and 2.18, this is much stronger than $\mathbf{S}$ being semi-efficient, and we think that it excludes (too) many models of interest for applications.

5.2 Semi-efficient consistent valuations

We now study the case where the property $\mathcal{E}$ is to be semi-efficient, i.e., dynamically viable and statically efficient. For a consistent valuation $\mathbf{U}^{\mathbf{f}}$, Theorem 5.5 and Remark 5.6 then tell us that for each stopping time $\sigma \le T$, $\mathbf{U}^{\mathbf{f}}/\mathbf{V}(\eta)$ should have a martingale property on $\{\sigma,T\}$ for some η. By analogy to the classic setup and in view of the hedging duality in Herdegen [20, Theorem 4.19], this suggests that super- and subreplication prices with respect to $\mathbf{h}^\sigma L^{\mathrm{sf}}_+$ should provide upper and lower bounds for $\mathbf{U}^{\mathbf{f}}$ at time σ. So we first examine these bounds, given by the extensions in Section 2.4 of (2.5) and (2.6) for ${}^\sigma\Gamma := \mathbf{h}^\sigma L^{\mathrm{sf}}_+$. Because we exploit (sub- and super-)martingale properties in this analysis, we cannot work in the units of $\mathbf{S}$.

Proposition 5.9. *Suppose* $\mathbf{S} \geq 0$ *satisfies* (2.3) *and is semi-efficient, and* $\mathbf{f}$ *is a payoff at time* T *with* $\pi_0(\mathbf{f} \,|\, \mathbf{h}^0 L_+^{\mathrm{sf}}) < \infty$*. Take any numéraire/ELMM pair* $(D, \mathbb{Q})$*, so that* $S := \mathbf{S}/D$ *is a local* $\mathbb{Q}$*-martingale. Then:*

1) *The family* $\{\Pi_\sigma(F_{\mathbf{f}} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}})[S] : \sigma \leq T \text{ stopping time}\}$ *of static superreplication prices for* $\mathbf{f}$ *in units of* S *is a nonnegative* $\mathbb{Q}$*-supermartingale system.*

2) *For any* $\bar{\vartheta} \in \mathbf{h}^0 L_+^{\mathrm{sf}}$ *with* $\mathbf{V}_T(\bar{\vartheta}) \geq f$ $\mathbb{P}$*-a.s. and any stopping time* $\sigma \leq T$*, define*

$$\underline{\Pi}_\sigma(F_{\mathbf{f}} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}})[S] := V_\sigma(\bar{\vartheta})[S] - \Pi_\sigma(V_T(\bar{\vartheta}) - F_{\mathbf{f}} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}})[S]. \tag{5.2}$$

Then the family $\{\underline{\Pi}_\sigma(F_{\mathbf{f}} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}})[S] : \sigma \leq T \text{ stopping time}\}$ *is a nonnegative local* $\mathbb{Q}$*-submartingale system. Moreover,* $\underline{\Pi}_\sigma(F_{\mathbf{f}} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}})[S]$ *does not depend on the choice of* $\bar{\vartheta}$ *and satisfies* $\underline{\Pi}_\sigma(F_{\mathbf{f}} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}})[S] \geq \Pi_\sigma^{\mathrm{sub}}(F_{\mathbf{f}} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}})[S]$ $\mathbb{P}$*-a.s.*

Remark 5.10. By Proposition 5.9, the process $(\Pi_t(F_{\mathbf{f}} \,|\, \mathbf{h}^t L_+^{\mathrm{sf}})[S])_{0 \leq t \leq T}$ is a $\mathbb{Q}$-supermartingale, but we do not claim that it has an RCLL version. (We conjecture that there is no RCLL version in general, because knowing only *static* superreplication prices seems insufficient to derive right-continuity of the expectation function. But we did not look for a counterexample and we do not need any RCLL property later; see Proposition B.1.)

Proof of Proposition 5.9. Fix a numéraire/ELMM pair $(D, \mathbb{Q})$, which exists by Theorem 2.12 as $\mathbf{S}$ is dynamically viable. Write $S = \mathbf{S}/D$ for brevity and fix $\sigma \leq T$.

1) Take any stopping time $\tau \geq \sigma$. As $\pi_0(\mathbf{f} \,|\, \mathbf{h}^0 L_+^{\mathrm{sf}}) < \infty$, Corollary 5.4 implies that $\mathbb{P}$-a.s., $\pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}}) < \infty$ and $\pi_\tau(\mathbf{f} \,|\, \mathbf{h}^\tau L_+^{\mathrm{sf}}) < \infty$; so Lemma 5.3 gives a $\vartheta^{(\sigma)} \in \mathbf{h}^\sigma L_+^{\mathrm{sf}}$ with $\mathbf{V}_T(\vartheta^{(\sigma)}) \geq \mathbf{f}$ and $\mathbf{V}_\sigma(\vartheta^{(\sigma)}) = \pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}})$, $\mathbb{P}$-a.s., and analogously for τ. Thus the definition (2.5) of superreplication prices yields $\mathbf{V}_\tau(\vartheta^{(\tau)}) = \pi_\tau(\mathbf{f} \,|\, \mathbf{h}^\tau L_+^{\mathrm{sf}}) \leq \mathbf{V}_\tau(\vartheta^{(\sigma)})$ $\mathbb{P}$-a.s., and hence also $V_\tau(\vartheta^{(\tau)})[S] \leq V_\tau(\vartheta^{(\sigma)})[S]$ $\mathbb{P}$-a.s. by the numéraire invariance (2.7). Combining this with (2.11) and the fact that $V(\vartheta^{(\sigma)})[S]$ is on $[\![\sigma, T]\!]$ a nonnegative local $\mathbb{Q}$-martingale, hence a $\mathbb{Q}$-supermartingale, yields 1) via

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[\Pi_\tau(F_{\mathbf{f}} \,|\, \mathbf{h}^\tau L_+^{\mathrm{sf}})[S] \,|\, \mathcal{F}_\sigma] &= \mathbb{E}_{\mathbb{Q}}[V_\tau(\vartheta^{(\tau)})[S] \,|\, \mathcal{F}_\sigma] \leq \mathbb{E}_{\mathbb{Q}}[V_\tau(\vartheta^{(\sigma)})[S] \,|\, \mathcal{F}_\sigma] \\ &\leq V_\sigma(\vartheta^{(\sigma)})[S] = \Pi_\sigma(F_{\mathbf{f}} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}})[S] \qquad \mathbb{P}\text{-a.s.}\end{aligned}$$

2) By assumption, $\pi_0(\mathbf{f} \,|\, \mathbf{h}^0 L_+^{\mathrm{sf}}) < \infty$. Take a $\bar{\vartheta} \in \mathbf{h}^0 L_+^{\mathrm{sf}}$ with $\mathbf{V}_T(\bar{\vartheta}) \geq \mathbf{f}$ $\mathbb{P}$-a.s. and set $\bar{\mathbf{f}} := \mathbf{V}_T(\bar{\vartheta}) - \mathbf{f} \geq 0$. Now take $\vartheta \in \mathbf{h}^\sigma L_+^{\mathrm{sf}}$ with $\mathbf{V}_T(\vartheta) \leq \mathbf{f} = \mathbf{V}_T(\bar{\vartheta}) - \bar{\mathbf{f}}$. Then $\bar{\vartheta} - \vartheta \in \mathbf{h}^\sigma L^{\mathrm{sf}}$ with $0 \leq \bar{\mathbf{f}} \leq \mathbf{V}_T(\bar{\vartheta} - \vartheta)$, and so first $\bar{\vartheta} - \vartheta \in \mathbf{h}^\sigma L_+^{\mathrm{sf}}$ by Lemma 5.3, due to static efficiency, and then

$$\pi_\sigma(\bar{\mathbf{f}} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}}) \leq \mathbf{V}_\sigma(\bar{\vartheta} - \vartheta) = \mathbf{V}_\sigma(\bar{\vartheta}) - \mathbf{V}_\sigma(\vartheta).$$

Solving for $\mathbf{V}_\sigma(\vartheta)$ and maximizing over ϑ then yields via (2.6)–(2.8) and (2.11) that

$$\begin{aligned}0 \leq \pi_\sigma^{\mathrm{sub}}(\mathbf{f} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}}) &\leq \mathbf{V}_\sigma(\bar{\vartheta}) - \pi_\sigma(\bar{\mathbf{f}} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}}) \\ &= \mathbf{V}_\sigma(\bar{\vartheta}) - \pi_\sigma(\mathbf{V}_T(\bar{\vartheta}) - \mathbf{f} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}}) = \underline{\Pi}_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}})[\mathbf{S}].\end{aligned}$$

Via the numéraire invariance properties (2.7), (2.8) and (2.11), we obtain in the same way

$$0 \leq \Pi_\sigma^{\mathrm{sub}}(\mathbf{f} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}})[S] \leq V_\sigma(\bar{\vartheta})[S] - \Pi_\sigma(V_T(\bar{\vartheta}) - F_{\mathbf{f}} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}})[S] = \underline{\Pi}_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L_+^{\mathrm{sf}})[S].$$

But the family $\{\Pi_\sigma(V_T(\bar\vartheta) - F_{\mathbf{f}} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}_+)[S] : \sigma \le T \text{ stopping time}\}$ is a $\mathbb{Q}$-supermartingale system by part 1) applied to $F_{\bar{\mathbf{f}}} = V_T(\bar\vartheta) - F_{\mathbf{f}}$, and as $\{V_\sigma(\bar\vartheta)[S] : \sigma \le T \text{ stopping time}\}$ is a local $\mathbb{Q}$-martingale system, the difference is a local $\mathbb{Q}$-submartingale system.

To show $\underline{\Pi}_\sigma(F_{\mathbf{f}} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}_+)[S]$ is independent of the choice of $\bar\vartheta \in \mathbf{h}^0 L^{\mathrm{sf}}_+$ with $\mathbf{V}_T(\bar\vartheta) \ge \mathbf{f}$ $\mathbb{P}$-a.s., take two such strategies $\vartheta^{(\pm 1,0)}$ and $\sigma \le T$. Then $\mathbf{V}_T(\vartheta^{(i,0)}) \ge \mathbf{V}_T(\vartheta^{(i,0)}) - \mathbf{f} \ge 0$ $\mathbb{P}$-a.s., and as $\vartheta^{(i,0)} \in \mathbf{h}^0 L^{\mathrm{sf}}_+ \subseteq \mathbf{h}^\sigma L^{\mathrm{sf}}_+$, this implies $\pi_\sigma(\mathbf{V}_T(\vartheta^{(i,0)}) - \mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}_+) \le \mathbf{V}_\sigma(\vartheta^{(i,0)}) < \infty$ $\mathbb{P}$-a.s. Lemma 5.3 thus yields $\vartheta^{(i,\sigma)} \in \mathbf{h}^\sigma L^{\mathrm{sf}}_+$ with

$$\mathbf{V}_T(\vartheta^{(i,\sigma)}) \ge \mathbf{V}_T(\vartheta^{(i,0)}) - \mathbf{f} \quad \text{and} \quad \mathbf{V}_\sigma(\vartheta^{(i,\sigma)}) = \pi_\sigma(\mathbf{V}_T(\vartheta^{(i,0)}) - \mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}_+), \qquad \mathbb{P}\text{-a.s.}$$

If we set $\bar\vartheta^{(i,\sigma)} := \vartheta^{(i,\sigma)} - \vartheta^{(i,0)} + \vartheta^{(-i,0)} \in \mathbf{h}^\sigma L^{\mathrm{sf}}$, then $\mathbb{P}$-a.s.

$$\mathbf{V}_T(\bar\vartheta^{(i,\sigma)}) \ge \mathbf{V}_T(\vartheta^{(i,0)}) - \mathbf{f} - \mathbf{V}_T(\vartheta^{(i,0)}) + \mathbf{V}_T(\vartheta^{(-i,0)}) = \mathbf{V}_T(\vartheta^{(-i,0)}) - \mathbf{f} \ge 0,$$

so $\bar\vartheta^{(i,\sigma)} \in \mathbf{h}^\sigma L^{\mathrm{sf}}_+$ by Lemma 5.3, and (2.5) gives

$$\mathbf{V}_\sigma(\vartheta^{(-i,\sigma)}) = \pi_\sigma(\mathbf{V}_T(\vartheta^{(-i,0)}) - \mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}_+) \le \mathbf{V}_\sigma(\bar\vartheta^{(i,\sigma)}) \qquad \mathbb{P}\text{-a.s.} \tag{5.3}$$

But $\mathbf{V}_\sigma(\bar\vartheta^{(1,\sigma)}) + \mathbf{V}_\sigma(\bar\vartheta^{(-1,\sigma)}) = \mathbf{V}_\sigma(\vartheta^{(1,\sigma)}) + \mathbf{V}_\sigma(\vartheta^{(-1,\sigma)})$ $\mathbb{P}$-a.s., and adding the cases ± 1 thus shows that the inequality in (5.3) is even an equality. So we obtain $\mathbb{P}$-a.s.

$$\begin{aligned}
\mathbf{V}_\sigma(\vartheta^{(1,0)}) - \pi_\sigma(\mathbf{V}_T(\vartheta^{(1,0)}) - \mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}_+) &= \mathbf{V}_\sigma(\vartheta^{(1,0)}) - \mathbf{V}_\sigma(\vartheta^{(1,\sigma)}) \\
&= \mathbf{V}_\sigma(\vartheta^{(-1,0)}) - \mathbf{V}_\sigma(\bar\vartheta^{(1,\sigma)}) \\
&= \mathbf{V}_\sigma(\vartheta^{(-1,0)}) - \mathbf{V}_\sigma(\vartheta^{(-1,\sigma)}) \\
&= \mathbf{V}_\sigma(\vartheta^{(-1,0)}) - \pi_\sigma(\mathbf{V}_T(\vartheta^{(-1,0)}) - \mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}_+),
\end{aligned}$$

which gives the assertion in view of the definition (5.2) and the numéraire invariance properties (2.7), (2.8) and (2.11). □

Proposition 5.11. *Suppose* $\mathbf{S} \ge 0$ *satisfies* (2.3) *and is semi-efficient, and* $\mathbf{f}$ *is a payoff at time* T *with* $\pi_0(\mathbf{f} \,|\, \mathbf{h}^0 L^{\mathrm{sf}}_+) < \infty$. *Fix a numéraire/ELMM pair* $(D, \mathbb{Q})$, *so that* $S = \mathbf{S}/D$ *is a local* $\mathbb{Q}$*-martingale. Then:*

1) *There exists a unique local* $\mathbb{Q}$*-martingale* L^{super} *which satisfies*

$$L^{\mathrm{super}}_t \le \Pi_t(F_{\mathbf{f}} \,|\, \mathbf{h}^t L^{\mathrm{sf}}_+)[S] = \frac{1}{D_t}\pi_t(\mathbf{f} \,|\, \mathbf{h}^t L^{\mathrm{sf}}_+) \qquad \mathbb{P}\text{-a.s.}, 0 \le t \le T, \tag{5.4}$$

and is maximal among all $L \in \mathcal{M}_{\mathrm{loc}}(\mathbb{Q})$ *with* $L_t \le \Pi_t(F_{\mathbf{f}} \,|\, \mathbf{h}^t L^{\mathrm{sf}}_+)[S]$ $\mathbb{P}$*-a.s.,* $0 \le t \le T$.

2) *There exists a unique local* $\mathbb{Q}$*-martingale* L^{sub} *which satisfies*

$$L^{\mathrm{sub}}_t \ge \underline{\Pi}_t(F_{\mathbf{f}} \,|\, \mathbf{h}^t L^{\mathrm{sf}}_+)[S] \ge \frac{1}{D_t}\pi^{\mathrm{sub}}_t(\mathbf{f} \,|\, \mathbf{h}^t L^{\mathrm{sf}}_+) \qquad \mathbb{P}\text{-a.s.}, 0 \le t \le T, \tag{5.5}$$

and is minimal among all $L \in \mathcal{M}_{\mathrm{loc}}(\mathbb{Q})$ *with* $L_t \ge \underline{\Pi}_t(F_{\mathbf{f}} \,|\, \mathbf{h}^t L^{\mathrm{sf}}_+)[S]$ $\mathbb{P}$*-a.s.,* $0 \le t \le T$.

Moreover, we have $\mathbb{P}$-a.s.

$$L_t^{\text{super}} \geq L_t^{\text{sub}} \geq \mathbb{E}_{\mathbb{Q}}[F_{\mathbf{f}}[S] \,|\, \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[\mathbf{f}/D_T \,|\, \mathcal{F}_t] \qquad \textit{for } 0 \leq t \leq T, \tag{5.6}$$

$$L_T^{\text{super}} = L_T^{\text{sub}} = F_{\mathbf{f}}[S] = \mathbf{f}/D_T. \tag{5.7}$$

(We emphasize that both L^{super} and L^{sub} depend on the pair $(D, \mathbb{Q})$, and we later sometimes write $L^{\text{super}}(D, \mathbb{Q}), L^{\text{sub}}(D, \mathbb{Q})$ to make this explicit.)

Proof. 1) Because $(\Pi_t(F_{\mathbf{f}} \,|\, \mathbf{h}^t L_+^{\text{sf}})[S])_{0 \leq t \leq T}$ is a $\mathbb{Q}$-supermartingale by Proposition 5.9, existence of L^{super} with $L_T^{\text{super}} = \Pi_T(F_{\mathbf{f}} \,|\, \mathbf{h}^T L_+^{\text{sf}})[S] = F_{\mathbf{f}}[S]$ $\mathbb{P}$-a.s. follows directly from Proposition B.1. The equality in (5.4) is due to (2.11). Uniqueness is clear from maximality.

2) Uniqueness of L^{sub} is clear from minimality. To show existence, write as in (5.2) $\underline{\Pi}_t(F_{\mathbf{f}} \,|\, \mathbf{h}^t L_+^{\text{sf}})[S] = V_t(\bar{\vartheta})[S] - \Pi_t(V_T(\bar{\vartheta}) - F_{\mathbf{f}} \,|\, \mathbf{h}^t L_+^{\text{sf}})[S]$ for a $\bar{\vartheta} \in \mathbf{h}^0 L_+^{\text{sf}}$ with $\mathbf{V}_T(\bar{\vartheta}) \geq \mathbf{f}$ $\mathbb{P}$-a.s. and recall from Proposition 5.9 that this representation does not depend on the choice of $\bar{\vartheta}$. Applying part 1) to $\bar{\mathbf{f}} := \mathbf{V}_T(\bar{\vartheta}) - \mathbf{f} \geq 0$ yields a maximal local $\mathbb{Q}$-martingale $L_t^{\text{super}}(\bar{\mathbf{f}}) \leq \Pi_t(F_{\bar{\mathbf{f}}} \,|\, \mathbf{h}^t L_+^{\text{sf}})[S]$ with $L_T^{\text{super}}(\bar{\mathbf{f}}) = F_{\bar{\mathbf{f}}}[S]$ $\mathbb{P}$-a.s., and as $V(\bar{\vartheta})[S]$ is a local $\mathbb{Q}$-martingale, one easily verifies that $L^{\text{sub}} := V(\bar{\vartheta})[S] - L^{\text{super}}(\bar{\mathbf{f}})$ does the job. The last inequality in (5.5) follows from Proposition 5.9 and the numéraire invariance property (2.11) for Π_σ^{sub} and π_σ^{sub}.

For the last part, we have already argued (5.7). Next, L^{sub} is a local $\mathbb{Q}$-martingale and nonnegative due to (5.5) as $\underline{\Pi}_t(F_{\mathbf{f}} \,|\, \mathbf{h}^t L_+^{\text{sf}})[S] \geq 0$ by Proposition 5.9. So L^{sub} is a $\mathbb{Q}$-supermartingale with final value $F_{\mathbf{f}}[S]$ by (5.7), and this yields the second inequality in (5.6). Because $\mathbf{S}$ is statically efficient, we have $V_t(\bar{\vartheta})[S] = \Pi_t(V_T(\bar{\vartheta}) \,|\, \mathbf{h}^t L_+^{\text{sf}})[S]$ from $\bar{\vartheta} \in \mathbf{h}^0 L_+^{\text{sf}}$, and so subadditivity of superreplication prices and $\bar{\mathbf{f}} = \mathbf{V}_T(\bar{\vartheta}) - \mathbf{f}$ yield

$$V_t(\bar{\vartheta})[S] = \Pi_t(V_T(\bar{\vartheta}) \,|\, \mathbf{h}^t L_+^{\text{sf}})[S] \leq \Pi_t(F_{\mathbf{f}} \,|\, \mathbf{h}^t L_+^{\text{sf}})[S] + \Pi_t(F_{\bar{\mathbf{f}}} \,|\, \mathbf{h}^t L_+^{\text{sf}})[S]. \tag{5.8}$$

Now for the right-hand side of (5.8), the maximal local $\mathbb{Q}$-martingales below the first and second terms are L^{super} and $L^{\text{super}}(\bar{\mathbf{f}}) = V(\bar{\vartheta})[S] - L^{\text{sub}}$, respectively. By Corollary B.2, the sum of these is maximal below the sum on the right-hand side and hence at least equal to the local $\mathbb{Q}$-martingale $V(\bar{\vartheta})[S]$. Rearranging this yields the first inequality in (5.6). □

Remark 5.12. Even for complete $\mathbf{S}$, we may have $L_0^{\text{super}} > L_0^{\text{sub}}$ and hence a nontrivial interval for the valuations of $\mathbf{f}$ at time 0. This reflects the fact that without dynamic efficiency, our valuations produce local, but not necessarily true martingales. Section 7.3 below illustrates this point by an example.

Now we can prove the main result about semi-efficient consistent valuations.

Theorem 5.13. *Suppose $\mathbf{S} \geq 0$ satisfies (2.3) and is semi-efficient, and $\mathbf{f}$ is a payoff at time T with $\pi_0(\mathbf{f} \,|\, \mathbf{h}^0 L_+^{\text{sf}}) < \infty$. Then:*

1) *There exists a semi-efficient consistent valuation $\mathbf{U}^{\mathbf{f}}$ for $\mathbf{f}$. It is not unique in general, not even in a complete market.*

2) *For any valuation $\mathbf{U}^{\mathbf{f}}$ for $\mathbf{f}$, the following are equivalent:*

   a) *$\mathbf{U}^{\mathbf{f}}$ is semi-efficient consistent.*

b) *There exists a numéraire/ELMM pair* $(D, \mathbb{Q})$ *such that* $(\mathbf{S}/D, \mathbf{U}^{\mathbf{f}}/D)$ *is a local* $\mathbb{Q}$*-martingale and*

$$L^{\text{sub}}(D, \mathbb{Q}) \le \mathbf{U}^{\mathbf{f}}/D \le L^{\text{super}}(D, \mathbb{Q}) \qquad \mathbb{P}\text{-}a.s. \tag{5.9}$$

If $\mathbf{S}$ *is in addition complete, the above are also equivalent to*

c) *For each numéraire/ELMM pair* $(D', \mathbb{Q}')$*,* $\mathbf{U}^{\mathbf{f}}/D'$ *is a local* $\mathbb{Q}'$*-martingale and*

$$L^{\text{sub}}(D', \mathbb{Q}') \le \mathbf{U}^{\mathbf{f}}/D' \le L^{\text{super}}(D', \mathbb{Q}') \qquad \mathbb{P}\text{-}a.s. \tag{5.10}$$

Proof. 1) Because $\mathbf{S}$ is dynamically viable, there exists by Theorem 2.12 a numéraire/ ELMM pair $(D, \mathbb{Q})$. If we fix such a pair, Proposition 5.11 gives the existence of local $\mathbb{Q}$-martingales $L^{\text{super}}(D, \mathbb{Q}) \ge L^{\text{sub}}(D, \mathbb{Q})$ $\mathbb{P}$-a.s. with final value $F_{\mathbf{f}}[\mathbf{S}/D] = \mathbf{f}/D_T$. So if we define $\mathbf{U}^{\mathbf{f}}/D := L^{\text{super}}(D, \mathbb{Q})$ (or $L^{\text{sub}}(D, \mathbb{Q})$ if we prefer), then 2b) holds and we get 1) from the equivalence of 2b) and 2a). Nonuniqueness is shown in Section 7.3.

"2a) $\Rightarrow$ 2b)": If $\mathbf{U}^{\mathbf{f}}$ is a semi-efficient consistent valuation for $\mathbf{f}$, Theorem 4.2 gives a numéraire/ELMM pair $(D, \mathbb{Q})$ such that $\mathbf{U}^{\mathbf{f}}/D$ is a local $\mathbb{Q}$-martingale. If (5.9) does not hold, (5.4)–(5.7) together with (2.9) and (2.11) yield $\mathbb{P}[\mathbf{U}^{\mathbf{f}}_t > \pi_t(\mathbf{f} \,|\, \mathbf{h}^t L^{\text{sf}}_+)] > 0$ or $\mathbb{P}[\mathbf{U}^{\mathbf{f}}_t < \underline{\Pi}_t(F_{\mathbf{f}} \,|\, \mathbf{h}^0 L^{\text{sf}}_+)[\mathbf{S}]] > 0$, for some $t \in [0, T)$. We argue the first case; the proof for the second is similar. So $\mathbb{P}[\mathbf{U}^{\mathbf{f}}_t > \pi_t(\mathbf{f} \,|\, \mathbf{h}^t L^{\text{sf}}_+)] > 0$, $\pi_0(\mathbf{f} \,|\, \mathbf{h}^0 L^{\text{sf}}_+) < \infty$ implies $\pi_t(\mathbf{f} \,|\, \mathbf{h}^t L^{\text{sf}}_+) < \infty$ $\mathbb{P}$-a.s. by Corollary 5.4, and so Lemma 5.3 yields $\vartheta^{(t)} \in \mathbf{h}^t L^{\text{sf}}_+$ with $\mathbf{V}_T(\vartheta^{(t)}) \ge \mathbf{f}$ and $\mathbf{V}_t(\vartheta^{(t)}) = \pi_t(\mathbf{f} \,|\, \mathbf{h}^t L^{\text{sf}}_+)$, $\mathbb{P}$-a.s. In the extended model $(\mathbf{S}, \mathbf{U}^{\mathbf{f}})$, define the strategy

$$(\bar{\vartheta}, \bar{\vartheta}^f) := (\vartheta^{(t)}, -1)\mathbb{1}_{\{\mathbf{U}^{\mathbf{f}}_t > \mathbf{V}_t(\vartheta^{(t)})\}} \in \mathbf{h}^t L^{\text{sf}}(\mathbf{S}, \mathbf{U}^{\mathbf{f}}).$$

Then $V_t(\bar{\vartheta}, \bar{\vartheta}^f)[\mathbf{S}, \mathbf{U}^{\mathbf{f}}] = (\mathbf{V}_t(\vartheta^{(t)}) - \mathbf{U}^{\mathbf{f}}_t)\mathbb{1}_{\{\mathbf{U}^{\mathbf{f}}_t > \mathbf{V}_t(\vartheta^{(t)})\}} \in \mathbf{L}^0_- \setminus \{0\}$, and $\mathbf{V}_T(\vartheta^{(t)}) \ge \mathbf{f}$ implies that $V_T(\bar{\vartheta}, \bar{\vartheta}^f)[\mathbf{S}, \mathbf{U}^{\mathbf{f}}] = (\mathbf{V}_T(\vartheta^{(t)}) - \mathbf{f})\mathbb{1}_{\{\mathbf{U}^{\mathbf{f}}_t > \mathbf{V}_t(\vartheta^{(t)})\}} \ge 0$ $\mathbb{P}$-a.s. But $(\mathbf{S}, \mathbf{U}^{\mathbf{f}})$ is statically efficient by the assumption in 2a), and using Lemma 5.3 for that extended model yields $(\bar{\vartheta}, \bar{\vartheta}^f) \in \mathbf{h}^t L^{\text{sf}}_+(\mathbf{S}, \mathbf{U}^{\mathbf{f}})$, contradicting $\mathbb{P}[V_t(\bar{\vartheta}, \bar{\vartheta}^f)[\mathbf{S}, \mathbf{U}^{\mathbf{f}}] < 0] > 0$. So (5.9) and 2b) hold.

"2b) $\Rightarrow$ 2a)": Let $(D, \mathbb{Q})$ be a numéraire/ELMM pair such that $\mathbf{U}^{\mathbf{f}}/D$ is a local $\mathbb{Q}$-martingale with (5.9). By Theorem 4.2, the extended market $(\mathbf{S}, \mathbf{U}^{\mathbf{f}})$ is dynamically and hence statically viable. To show it is statically efficient, we argue indirectly via Proposition A.2 and suppose there are $s \in [0, T)$ and $(\vartheta, \vartheta^f) \in \mathbf{h}^s L^{\text{sf}}(\mathbf{S}, \mathbf{U}^{\mathbf{f}})$ with $V_T(\vartheta, \vartheta^f)[\mathbf{S}, \mathbf{U}^{\mathbf{f}}] \ge 0$ $\mathbb{P}$-a.s., but $\mathbb{P}[V_s(\vartheta, \vartheta^f)[\mathbf{S}, \mathbf{U}^{\mathbf{f}}] < 0] > 0$. We split and normalize (ϑ, ϑ^f) by setting

$$\begin{aligned}
(\vartheta^{(0)}, \vartheta^{f,(0)}) &:= (\vartheta, \vartheta^f)\mathbb{1}_{\{\vartheta^f_s = 0\}} &&= (\vartheta, 0)\mathbb{1}_{\{\vartheta^f_s = 0\}}, \\
(\vartheta^{(-)}, \vartheta^{f,(-)}) &:= \tfrac{1}{|\vartheta^f_s|}(\vartheta, \vartheta^f)\mathbb{1}_{\{\vartheta^f_s < 0\}} &&= \Big(-\frac{\mathbb{1}_{\{\vartheta^f_s < 0\}}}{\vartheta^f_s}\vartheta, -1\Big)\mathbb{1}_{\{\vartheta^f_s < 0\}}, \\
(\vartheta^{(+)}, \vartheta^{f,(+)}) &:= \tfrac{1}{|\vartheta^f_s|}(\vartheta, \vartheta^f)\mathbb{1}_{\{\vartheta^f_s > 0\}} &&= \Big(+\frac{\mathbb{1}_{\{\vartheta^f_s > 0\}}}{\vartheta^f_s}\vartheta, +1\Big)\mathbb{1}_{\{\vartheta^f_s > 0\}}.
\end{aligned}$$

Then $(\vartheta^{(i)}, \vartheta^{f,(i)})$ is in $\mathbf{h}^s L^{\text{sf}}(\mathbf{S}, \mathbf{U}^{\mathbf{f}})$ and $V_T(\vartheta^{(i)}, \vartheta^{f,(i)})[\mathbf{S}, \mathbf{U}^{\mathbf{f}}] \ge 0$ $\mathbb{P}$-a.s. for $i \in \{+, 0, -\}$. The form of $\vartheta^{(0)}$, static efficiency of $\mathbf{S}$ and Proposition A.2 imply $V_s(\vartheta^{(0)}, \vartheta^{f,(0)})[\mathbf{S}, \mathbf{U}^{\mathbf{f}}] \ge 0$ $\mathbb{P}$-a.s., and so either $\mathbb{P}[V_s(\vartheta^{(-)}, \vartheta^{f,(-)})[\mathbf{S}, \mathbf{U}^{\mathbf{f}}] < 0] > 0$ or $\mathbb{P}[V_s(\vartheta^{(+)}, \vartheta^{f,(+)})[\mathbf{S}, \mathbf{U}^{\mathbf{f}}] < 0] > 0$. We argue the first case; the proof for the second is analogous. From the form of $\vartheta^{(-)}$, we get $\mathbf{V}_T(\vartheta^{(-)}) \ge \mathbf{f}$ $\mathbb{P}$-a.s. on $\{\vartheta^f_s < 0\}$ and

$$P[\mathbf{V}_s(\vartheta^{(-)}) < \mathbf{U}^{\mathbf{f}}_s, \vartheta^f_s < 0] > 0. \tag{5.11}$$

Using the assumption $\pi_0(\mathbf{f}\,|\,\mathbf{h}^0L^{\mathrm{sf}}_+) < \infty$, Corollary 5.4 and Lemma 5.3, take $\vartheta^{(s)} \in \mathbf{h}^sL^{\mathrm{sf}}_+$ with $\mathbf{V}_T(\vartheta^{(s)}) \geq \mathbf{f}$ $\mathbb{P}$-a.s. and set $\bar{\vartheta} := \vartheta^{(s)}\mathbb{1}_{\{\vartheta^f_s \geq 0\}} + \vartheta^{(-)} \in \mathbf{h}^sL^{\mathrm{sf}}$. As $\vartheta^{(-)} = \vartheta^{(-)}\mathbb{1}_{\{\vartheta^f_s < 0\}}$, it follows that $\mathbf{V}_T(\bar{\vartheta}) \geq \mathbf{f}$ $\mathbb{P}$-a.s., which implies $\mathbf{V}_s(\bar{\vartheta}) \geq \pi_s(\mathbf{f}\,|\,\mathbf{h}^sL^{\mathrm{sf}}) = \pi_s(\mathbf{f}\,|\,\mathbf{h}^sL^{\mathrm{sf}}_+)$ $\mathbb{P}$-a.s. by static efficiency of $\mathbf{S}$ and Lemma 5.3. Combining this with the property (5.4) of $L^{\mathrm{super}}(D,\mathbb{Q})$ and using (5.11) yields

$$\begin{aligned}
\mathbb{P}[\mathbf{U}^{\mathbf{f}}_s/D_s > L^{\mathrm{super}}_s(D,\mathbb{Q})] &\geq \mathbb{P}[\mathbf{U}^{\mathbf{f}}_s > \pi_s(\mathbf{f}\,|\,\mathbf{h}^sL^{\mathrm{sf}}_+)] \\
&\geq \mathbb{P}[\mathbf{U}^{\mathbf{f}}_s > \mathbf{V}_s(\bar{\vartheta})] \\
&\geq \mathbb{P}[\mathbf{U}^{\mathbf{f}}_s > \mathbf{V}_s(\bar{\vartheta}), \vartheta^f_s < 0] \\
&= \mathbb{P}[\mathbf{U}^{\mathbf{f}}_s > \mathbf{V}_s(\vartheta^{(-)}), \vartheta^f_s < 0] > 0,
\end{aligned}$$

in contradiction to (5.9). So $(\mathbf{S}, \mathbf{U}^{\mathbf{f}})$ is statically efficient and we get 2a).

Next, if we have 2c), then (5.10) implies (5.9) and the remaining part of 2b) follows from Theorem 4.2. This does not need completeness of $\mathbf{S}$.

Finally, if $\mathbf{S}$ is in addition complete, the first half of 2b) implies the first half of 2c) as in the proof of "b) $\Rightarrow$ c)" in Theorem 4.2, and the inequality (5.10) for $\mathbf{U}^{\mathbf{f}}/D'$ is proved as in "2a) $\Rightarrow$ 2b)", using that we already have the equivalence of 2a) and 2b). $\square$

6 Put-call parity and pricing formulas

In this section, we study the consistent valuation of call and put options in semi-efficient markets. We know from Section 4.1 that absence of arbitrage alone will not give strong enough results, and because we do not want to exclude a priori markets with bubbles, we also cannot simply use risk-neutral valuation, as seen in Section 4.2.

Consider our $\mathbb{R}^N$-valued model $\mathbf{S}$. In the special case $\mathbf{S} = (1, X)$ with X valued in $\mathbb{R}^d$, $d = N-1$, a call and a put option on the second asset $\mathbf{S}^2 = X^1$ with strike K, in the unit corresponding to $\mathbf{S}$, are given by the payoffs (in the same unit) $(X^1_T - K)^+ = (\mathbf{S}^2_T - K\mathbf{S}^1_T)^+$ and $(K - X^1_T)^+ = (K\mathbf{S}^1_T - \mathbf{S}^2_T)^+$, respectively. In a different unit corresponding to $S = \mathbf{S}/D$ for some numéraire D, the resulting payoffs (in units of S) are by (2.8) given by

$$C[S] = (S^2_T - KS^1_T)^+ \quad \text{and} \quad P[S] = (KS^1_T - S^2_T)^+. \tag{6.1}$$

In particular, as already mentioned in Remark 2.19, we should view calls and puts as *exchange options* if we want to treat all N assets in a symmetric way.

Our first result is completely general and does not need completeness of $\mathbf{S}$.

Theorem 6.1 (Put-call parity)**.** *Suppose* $\mathbf{S} \geq 0$ *satisfies* (2.3) *and is semi-efficient. Let* C *("call option") and* P *("put option") be the contingent claim maps at time* T *given by* (6.1) *with* $K > 0$*, and set* $\mathbf{C} := C[\mathbf{S}]$*,* $\mathbf{P} := P[\mathbf{S}]$*. Let* (U^C, U^P) *be a semi-efficient valuation map for the vector* (C, P) *and set* $(\mathbf{U}^{\mathbf{C}}, \mathbf{U}^{\mathbf{P}}) := (U^C, U^P)[\mathbf{S}]$*. Then we have put-call parity, meaning that* $\mathbf{V}(e^2) + \mathbf{U}^{\mathbf{P}} = K\mathbf{V}(e^1) + \mathbf{U}^{\mathbf{C}}$ *or, written out in units corresponding to* $S = \mathbf{S}/D$ *for any numéraire* D*,*

$$S^2_t + U^P_t[S] = KS^1_t + U^C_t[S], \qquad 0 \leq t \leq T. \tag{6.2}$$

Proof. Note that the valuation $(\mathbf{U^C}, \mathbf{U^P})$ for $(\mathbf{C}, \mathbf{P})$ exists by Remark 4.1 and Theorem 5.13. Call $\mathbf{S}' := (\mathbf{S}, \mathbf{U^C}, \mathbf{U^P})$ the extended model and $e^C = (0,1,0), e^P = (0,0,1) \in \mathbb{R}^{N+2}$ the buy-and-hold strategies of the call and put valuations. For each fixed t, the restrictions of these strategies to the interval $[\![t,T]\!]$ are in $\mathbf{h}^t L^{\mathrm{sf}}_+(\mathbf{S}')$. Now (6.1) gives for the final payoffs $\mathbf{C} = (\mathbf{V}_T(e^2) - K\mathbf{V}_T(e^1))^+ = \mathbf{V}_T(e^2) - K\mathbf{V}_T(e^1) + \mathbf{P}$, so that

$$\mathbf{V}_T(e^2,0,1) = \mathbf{V}_T(e^2) + \mathbf{P} = K\mathbf{V}_T(e^1) + \mathbf{C} = \mathbf{V}_T(Ke^1,1,0).$$

But $(e^2,0,1), (Ke^1,1,0)$ are both in $\mathbf{h}^t L^{\mathrm{sf}}_+(\mathbf{S}')$ for each t and $\mathbf{S}'$ is semi-efficient and hence statically efficient. So the time-t wealths for these strategies must also agree, which gives

$$\mathbf{V}_t(e^2) + \mathbf{U}^{\mathbf{P}}_t = \mathbf{V}_t(e^2,0,1) = \mathbf{V}_t(Ke^1,1,0) = K\mathbf{V}_t(e^1) + \mathbf{U}^{\mathbf{C}}_t.$$

The extension (6.2) to $S = \mathbf{S}/D$ follows from the numéraire invariance in (2.7) and (2.9). $\square$

One key insight from Theorem 6.1 is that static efficiency is the right condition to ensure put-call parity, in general models which only satisfy dynamic viability (or, equivalently by Theorem 2.12, NUPBR). This gives a clear and general answer to the discussion in Andersen [1, Section 2.2] about the validity or not of put-call parity for model prices. Moreover, imposing static efficiency is very intuitive; it axiomatizes the familiar argument of synthesizing a put by a static portfolio in a call and its two underlyings (usually stock and bond, here S^2 and S^1). We repeat that we do not (and cannot) simply use risk-neutral valuation; in particular, we neither impose nor use any assumption of linearity on $\mathbf{U^C}, \mathbf{U^P}$. (Wealth $\mathbf{V}(\vartheta)$ is linear (in ϑ), but valuations need not be.)

While Theorem 6.1 yields put-call parity under natural and weak assumptions in general, incomplete models, it says nothing about pricing formulas; indeed, valuations under incompleteness are not unique in general. So we next study what can be said for *complete* markets. Section 2.2 of [1] has a detailed discussion (with many references to the literature) about possible call option values in a specific complete diffusion model $(1, X)$ with X satisfying a quadratic SDE. (Lewis [33, Chapter 9] has a similar discussion for stochastic volatility models.) Our next result goes beyond the setup of [1] and gives the range of semi-efficient consistent valuations for calls and puts in a general complete market.

Theorem 6.2. *Suppose* $\mathbf{S} \geq 0$ *satisfies* (2.3) *and is semi-efficient and complete. Let* U^C *and* U^P*, respectively, be semi-efficient consistent valuation maps for the call and put option maps* C *and* P *from* (6.1)*. Then for any numéraire/ELMM pair* $(D, \mathbb{Q})$*, we have with* $S = \mathbf{S}/D$ *the explicit formulas*

$$U^C_t[S] = \mathbb{E}_{\mathbb{Q}}[C[S] \,|\, \mathcal{F}_t] + S^2_t - \mathbb{E}_{\mathbb{Q}}[S^2_T \,|\, \mathcal{F}_t] - L_t, \qquad 0 \leq t \leq T, \tag{6.3}$$

$$U^P_t[S] = \mathbb{E}_{\mathbb{Q}}[P[S] \,|\, \mathcal{F}_t] + KS^1_t - K\mathbb{E}_{\mathbb{Q}}[S^1_T \,|\, \mathcal{F}_t] - L_t, \qquad 0 \leq t \leq T, \tag{6.4}$$

where L *is a local* $\mathbb{Q}$*-martingale with*

$$0 \leq L_t \leq \min(S^2_t, KS^1_t) - \mathbb{E}_{\mathbb{Q}}[\min(S^2_T, KS^1_T) \,|\, \mathcal{F}_t], \qquad 0 \leq t \leq T. \tag{6.5}$$

Moreover:

1) *If at least one of* e^1, e^2 *is maximal for* ${}^0L^{\mathrm{sf}}_+$, *then both* U^C *and* U^P *are unique.*

2) *If* e^1 *is maximal for* ${}^0L^{\mathrm{sf}}_+$, *then* U^C *and* U^P *are for any numéraire/ELMM pair* $(D, \mathbb{Q})$ *given, with* $S = \mathbf{S}/D$, *by*

$$U^C_t[S] = \mathbb{E}_{\mathbb{Q}}[C[S] \,|\, \mathcal{F}_t] + S^2_t - \mathbb{E}_{\mathbb{Q}}[S^2_T \,|\, \mathcal{F}_t], \qquad 0 \le t \le T, \tag{6.6}$$

$$U^P_t[S] = \mathbb{E}_{\mathbb{Q}}[P[S] \,|\, \mathcal{F}_t], \qquad 0 \le t \le T. \tag{6.7}$$

3) *If* e^2 *is maximal for* ${}^0L^{\mathrm{sf}}_+$, *then* U^C *and* U^P *are for any numéraire/ELMM pair* $(D, \mathbb{Q})$ *given, with* $S = \mathbf{S}/D$, *by*

$$U^C_t[S] = \mathbb{E}_{\mathbb{Q}}[C[S] \,|\, \mathcal{F}_t], \qquad 0 \le t \le T, \tag{6.8}$$

$$U^P_t[S] = \mathbb{E}_{\mathbb{Q}}[P[S] \,|\, \mathcal{F}_t] + KS^1_t - K\mathbb{E}_{\mathbb{Q}}[S^1_T \,|\, \mathcal{F}_t], \qquad 0 \le t \le T. \tag{6.9}$$

4) *If there is a numéraire/ELMM pair* $(D, \mathbb{Q})$ *such that for* $S = \mathbf{S}/D$, *both* S^1 *and* S^2 *have continuous paths, the upper bound in* (6.5) *can be sharpened to*

$$L_t \le \mathbb{E}_{\mathbb{Q}}\bigg[-\int_t^T (\mathbb{1}_{\{S^2_u \le KS^1_u\}} \,\mathrm{d}S^2_u + K\mathbb{1}_{\{S^2_u > KS^1_u\}} \,\mathrm{d}S^1_u) \,\bigg|\, \mathcal{F}_t \bigg], \qquad 0 \le t \le T. \tag{6.10}$$

Theorem 6.2 gives clear guidance on the question (also raised in [1, Section 2.2]) of which call price one should use in a complete market, if that market is semi-efficient. In general, both call and put prices have for each numéraire/ELMM pair $(D, \mathbb{Q})$ an explicit expression (6.3) or (6.4) consisting of three terms, and there is no uniqueness for either price. The first term is the option's risk-neutral value; the second is a correction if the option's main underlying has a bubble; and the third causes nonuniqueness — there are no rational arguments to decide where it should be assigned. Things change if at least one of the two assets on which the (exchange) option is written is maximal; then the third term (which involves a local $\mathbb{Q}$-martingale L) disappears, both valuations are unique, and one option value at least is given as a risk-neutral expectation under $\mathbb{Q}$. More precisely, pricing by risk-neutrality is used for that option whose underlying asset (S^2 for call, S^1 for put) is maximal; a correction term resulting from a bubble in the market (if there is one) is automatically assigned to the option whose underlying is not maximal. In particular, the choice of call price depends on the model, and our results say precisely which features of the model decide the choice, on the basis of rational economic arguments.

Remark 6.3. Our results do not (and cannot) answer the analogous question from Lewis [33, Chapter 9] about the choice of "the" appropriate option value in a stochastic volatility model, because these models are typically incomplete and hence do not have unique option values. That issue is eliminated in [33] by working with specific risk preferences (linear or logarithmic utility) to fix a market price of volatility risk.

Proof of Theorem 6.2. In view of the put-call parity (6.2) in Theorem 6.1, we only need to do the derivations for one of the two options, and we choose the call. Note that the local martingale L appearing in the formulas is the same for both put and call.

Fix a numéraire/ELMM pair $(D, \mathbb{Q})$ and set $S = \mathbf{S}/D$. According to Theorem 5.13, $U^C[S] = \mathbf{U^C}/D$ is a local $\mathbb{Q}$-martingale with $0 \le L^{\mathrm{sub}}(D, \mathbb{Q}) \le U^C[S] \le L^{\mathrm{super}}(D, \mathbb{Q})$. To get an upper bound for $L^{\mathrm{super}}(D, \mathbb{Q})$, note that Proposition 5.11 and $C \le V_T(e^2)$ yield

$$L_t^{\mathrm{super}}(D, \mathbb{Q}) \le \Pi_t(C \,|\, \mathbf{h}^t L_+^{\mathrm{sf}})[S] \le \Pi_t(V_T(e^2) \,|\, \mathbf{h}^t L_+^{\mathrm{sf}})[S] \le S_t^2, \qquad 0 \le t \le T.$$

So $S^2 - L^{\mathrm{super}}(D, \mathbb{Q}) \ge 0$ is a local $\mathbb{Q}$-martingale and thus a $\mathbb{Q}$-supermartingale. As $L_T^{\mathrm{super}}(D, \mathbb{Q}) = C[S]$ by (5.7), this gives $S_t^2 - L_t^{\mathrm{super}}(D, \mathbb{Q}) \ge \mathbb{E}_{\mathbb{Q}}[S_T^2 - C[S] \,|\, \mathcal{F}_t]$ and hence

$$L_t^{\mathrm{super}}(D, \mathbb{Q}) \le \mathbb{E}_{\mathbb{Q}}[C[S] \,|\, \mathcal{F}_t] + S_t^2 - \mathbb{E}_{\mathbb{Q}}[S_T^2 \,|\, \mathcal{F}_t], \qquad 0 \le t \le T. \tag{6.11}$$

For a lower bound for $L^{\mathrm{sub}}(D, \mathbb{Q})$, set $\bar{C} := V_T(e^2) - C$ so that $\bar{C}[S] = \min(S_T^2, KS_T^1)$ and

$$\Pi_t(\bar{C} \,|\, \mathbf{h}^t L_+^{\mathrm{sf}})[S] \le \min(S_t^2, KS_t^1), \qquad 0 \le t \le T.$$

Combining (5.5) for $F_{\mathbf{f}} = C$ with (5.2) for $\bar{\vartheta} = e^2$ therefore gives

$$L^{\mathrm{sub}}(D, \mathbb{Q}) \ge S^2 - \min(S^2, KS^1).$$

Now $\min(S^2, KS^1)$ as the minimum of two $\mathbb{Q}$-supermartingales is a $\mathbb{Q}$-supermartingale, and so $L^{\mathrm{sub}}(D, \mathbb{Q}) - S^2 + \min(S^2, KS^1)$ is a nonnegative local and hence a true $\mathbb{Q}$-supermartingale. Because $L_T^{\mathrm{sub}}(D, \mathbb{Q}) = C[S]$ by (5.7), this yields

$$L_t^{\mathrm{sub}}(D, \mathbb{Q}) - S_t^2 + \min(S_t^2, KS_t^1) \ge \mathbb{E}_{\mathbb{Q}}[C[S] - S_T^2 + \min(S_T^2, KS_T^1) \,|\, \mathcal{F}_t], \qquad 0 \le t \le T,$$

and rearranging gives

$$\begin{aligned} L_t^{\mathrm{sub}}(D, \mathbb{Q}) \ge{}& \mathbb{E}_{\mathbb{Q}}[C[S] \,|\, \mathcal{F}_t] + S_t^2 - \mathbb{E}_{\mathbb{Q}}[S_T^2 \,|\, \mathcal{F}_t] \\ & - \min(S_t^2, KS_t^1) + \mathbb{E}_{\mathbb{Q}}[\min(S_T^2, KS_T^1) \,|\, \mathcal{F}_t], \qquad 0 \le t \le T. \end{aligned} \tag{6.12}$$

Combining (6.11) and (6.12) with $L^{\mathrm{sub}}(D, \mathbb{Q}) \le U^C[S] \le L^{\mathrm{super}}(D, \mathbb{Q})$ shows that there exists a local $\mathbb{Q}$-martingale L satisfying (6.5) such that the local $\mathbb{Q}$-martingale $U^C[S]$ satisfies (6.3). Formula (6.4) then follows from the put-call parity (6.2).

1) – 3) If e^i is maximal for ${}^0L_+^{\mathrm{sf}}$, $S^i = \mathbf{S}^i/D$ is by Corollary 3.10 a (true) $\mathbb{Q}$-martingale for any numéraire/ELMM pair $(D, \mathbb{Q})$. Thus the local $\mathbb{Q}$-martingale $L \ge 0$ from (6.5) is bounded above by the (true) $\mathbb{Q}$-martingale KS^1 (for $i = 1$) or S^2 (for $i = 2$). So L is a true $\mathbb{Q}$-martingale as well, and then $L \equiv 0$ as $L_T = 0$ by (6.5). This yields uniqueness of U^C, (6.6) and (6.7) follow for $i = 1$, and the $\mathbb{Q}$-martingale property of S^2 gives (6.8) and (6.9) for $i = 2$.

4) Suppose $(D, \mathbb{Q})$ is a numéraire/ELMM pair such that for $S = \mathbf{S}/D$, both S^1 and S^2 have continuous paths. Then Tanaka's formula gives

$$\begin{aligned} \min(S_t^2, KS_t^1) - \min(S_0^2, KS_0^1) &= S_t^2 - S_0^2 - \Big((S_t^2 - KS_t^1)^+ - (S_0^2 - KS_0^1)^+\Big) \\ &= S_t^2 - S_0^2 - \bigg(\int_0^t \mathbb{1}_{\{S_u^2 > KS_u^1\}} \,\mathrm{d}(S_u^2 - KS_u^1) + \frac{1}{2} J_t\bigg) \\ &= \int_0^t (\mathbb{1}_{\{S_u^2 \le KS_u^1\}} \,\mathrm{d}S_u^2 + K \mathbb{1}_{\{S_u^2 > KS_u^1\}} \,\mathrm{d}S_u^1) - \frac{1}{2} J_t, \end{aligned} \tag{6.13}$$

where J is the local time of $S^2 - KS^1$ at 0. Define A by $A_t := \frac{1}{2} J_t$ and $M \in \mathcal{M}_{\mathrm{loc}}(\mathbb{Q})$ by $M_t := \min(S_0^2, KS_0^1) + \int_0^t (\mathbb{1}_{\{S_u^2 \le KS_u^1\}} \,\mathrm{d}S_u^2 + K\mathbb{1}_{\{S_u^2 > KS_u^1\}} \,\mathrm{d}S_u^1)$. Then A is an increasing adapted continuous process and $M - A$ is by (6.13) the Doob–Meyer decomposition of the $\mathbb{Q}$-supermartingale $\min(S^2, KS^1) \ge 0$. It follows that $M \ge 0$ is a local $\mathbb{Q}$-martingale and a $\mathbb{Q}$-supermartingale and (as in the proof of Proposition B.1) that M_T and A_T are $\mathbb{Q}$-integrable. For convenience, define $N \in \mathcal{M}_{\mathrm{loc}}(\mathbb{Q})$ by $N_t := M_t - \mathbb{E}_{\mathbb{Q}}[M_T \,|\, \mathcal{F}_t]$. Then $N \ge 0$ by the $\mathbb{Q}$-supermartingale property of M, and due to (6.5), L satisfies

$$L_t \le M_t - A_t - \mathbb{E}_{\mathbb{Q}}\left[M_T - A_T \,|\, \mathcal{F}_t\right] \le N_t + \mathbb{E}_{\mathbb{Q}}[A_T \,|\, \mathcal{F}_t], \qquad 0 \le t \le T.$$

So $N - L \in \mathcal{M}_{\mathrm{loc}}(\mathbb{Q})$ is bounded below by the $\mathbb{Q}$-martingale with final value A_T and is hence a $\mathbb{Q}$-supermartingale. As $N_T - L_T \ge \mathbb{E}_{\mathbb{Q}}[A_T - A_T \,|\, \mathcal{F}_T] = 0$, we obtain $N - L \ge 0$, and so (6.10) follows because for $0 \le t \le T$,

$$L_t \le N_t = M_t - \mathbb{E}_{\mathbb{Q}}[M_T \,|\, \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}\Big[- \int_t^T (\mathbb{1}_{\{S_u^2 \le KS_u^1\}} \,\mathrm{d}S_u^2 + K\mathbb{1}_{\{S_u^2 > KS_u^1\}} \,\mathrm{d}S_u^1) \,\Big|\, \mathcal{F}_t\Big].$$

□

The formulas (6.3) and (6.4) describe semi-efficient consistent valuations for calls or puts as a sum of three terms, with a local martingale L satisfying (6.5). One can ask conversely if (6.3) and (6.4), with L satisfying (6.5), always *define* consistent valuation maps. This is not true in general, as shown in Section 7.3; but it does hold under a mild additional assumption on $\mathbf{S}$.

Proposition 6.4. *Suppose that $\mathbf{S} = (1, X) = (1, X^1, \dots, X^d) \ge 0$ is semi-efficient and complete. If X satisfies at time T the conditional full support condition*

$$\operatorname{supp} \mathcal{L}(X_T \,|\, \mathcal{F}_t) = (0, \infty)^d, \qquad 0 \le t < T, \tag{6.14}$$

then for any numéraire/ELMM pair $(D, \mathbb{Q})$ and any local $\mathbb{Q}$-martingale L satisfying (6.5)*, the valuation map U^C defined by* (2.9) *and, for $S = \mathbf{S}/D$,*

$$U_t^C[S] := \mathbb{E}_{\mathbb{Q}}[C[S] \,|\, \mathcal{F}_t] + S_t^2 - \mathbb{E}_{\mathbb{Q}}[S_T^2 \,|\, \mathcal{F}_t] - L_t, \qquad 0 \le t \le T, \tag{6.15}$$

is semi-efficient consistent for the call option map C from (6.1)*. Moreover, if S^1 and S^2 are continuous, the upper bound in* (6.5) *can be replaced by* (6.10)*. An analogous result holds for the case of a put option.*

Proof. Set $\bar{C} := V_T(e^2) - C$ so that $\bar{C}[S] = \min(S_T^2, KS_T^1)$. We first show that

$$\Pi_t(C \,|\, \mathbf{h}^t L_+^{\mathrm{sf}})[S] = S_t^2, \qquad 0 \le t < T, \tag{6.16}$$

$$\Pi_t(\bar{C} \,|\, \mathbf{h}^t L_+^{\mathrm{sf}})[S] = \min(S_t^2, KS_t^1), \qquad 0 \le t < T. \tag{6.17}$$

By the numéraire invariance (2.11), it is enough to establish (6.16) and (6.17) for $\mathbf{S}$. Fix $t \in [0, T)$ and take $\vartheta^C, \vartheta^{\bar{C}} \in \mathbf{h}^t L_+^{\mathrm{sf}}$ satisfying $\mathbf{V}_T(\vartheta^C) \ge C[\mathbf{S}] = (X_T^1 - K)^+$ and $\mathbf{V}_T(\vartheta^{\bar{C}}) \ge \bar{C}[\mathbf{S}] = \min(X_T^1, K)$. Then (6.14) implies that for all $x \in (0, \infty)^d$,

$$\vartheta_t^{C,1} + \sum_{i=1}^d \vartheta_t^{C,i+1} x_i \ge (x_1 - K)^+ \qquad \mathbb{P}\text{-a.s.}, \tag{6.18}$$

$$\vartheta_t^{\bar{C},1} + \sum_{i=1}^d \vartheta_t^{\bar{C},i+1} x_i \ge \min(x_1, K) \qquad \mathbb{P}\text{-a.s.} \tag{6.19}$$

In (6.18), letting $|x| \searrow 0$ shows that $\vartheta_t^{C,1} \geq 0$ $\mathbb{P}$-a.s. For $k \in \{1,\dots,d\}$, letting $x_i \searrow 0$ for $i \in \{1,\dots,d\} \setminus \{k\}$, then dividing by x_k and letting $x_k \nearrow \infty$ shows that $\vartheta_t^{C,2} \geq 1$ $\mathbb{P}$-a.s. and $\vartheta_t^{C,i+1} \geq 0$ $\mathbb{P}$-a.s. for $i \in \{2,\dots,d\}$. So $\mathbf{V}_t(\vartheta^C) \geq \mathbf{S}_t^2$ $\mathbb{P}$-a.s., and as $\mathbf{V}_t(e^2) = \mathbf{S}_t^2$, we get (6.16). Moreover, as $\mathbf{S}^1 \equiv 1$, (6.19) directly implies that $\mathbf{V}_t(\vartheta^{\bar{C}}) \geq \min(\mathbf{S}_t^2, K\mathbf{S}_t^1)$, and we have $\mathbf{V}_t(e^2 \mathbb{1}_{\{\mathbf{S}_t^2 \leq K\}} + K e^1 \mathbb{1}_{\{\mathbf{S}_t^2 > K\}}) = \min(\mathbf{S}_t^2, K\mathbf{S}_t^1)$. This yields (6.17).

Now take any numéraire/ELMM pair $(D, \mathbb{Q})$, any $L \in \mathcal{M}_{\mathrm{loc}}(\mathbb{Q})$ satisfying (6.5), set $S := \mathbf{S}/D$ and define $U^C[S]$ by (6.15). Then $U^C[S] \in \mathcal{M}_{\mathrm{loc}}(\mathbb{Q})$. By Theorem 5.13, by (5.2), and in view of the characterization of L^{super} and L^{sub} in Proposition 5.11, it remains to show that for $0 \leq t \leq T$,

$$U_t^C[S] \leq \Pi_t(C \,|\, \mathbf{h}^t L_+^{\mathrm{sf}})[S] = \begin{cases} S_t^2 & \text{if } t \in [0,T), \\ (S_T^2 - K S_T^1)^+ & \text{if } t = T. \end{cases} \tag{6.20}$$

$$U_t^C[S] \geq \bar{\pi}_t(C \,|\, \mathbf{h}^t L_+^{\mathrm{sf}})[S] = V_t(e^2)[S] - \Pi_t(\bar{C} \,|\, \mathbf{h}^t L_+^{\mathrm{sf}})[S] = S_t^2 - \min(S_t^2, K S_t^1). \tag{6.21}$$

But the inequality (6.20) follows from (6.15) due to $L \geq 0$ via

$$U_t^C[S] \leq S_t^2 - \mathbb{E}_{\mathbb{Q}}[S_T^2 - C[S] \,|\, \mathcal{F}_t] \leq \begin{cases} S_t^2 & \text{if } t \in [0,T), \\ (S_T^2 - K S_T^1)^+ & \text{if } t = T, \end{cases}$$

and as $C[S] - S_T^2 = -\min(S_T^2, K S_T^1)$, the inequality (6.21) follows from (6.15) due to (6.5) via $U_t^C[S] \geq S_t^2 - \min(S_t^2, K S_t^1)$. The additional claim with (6.10) follows by arguing as in the proof of 3) in Theorem 6.2. □

In Proposition 6.4, $L \equiv 0$ is always a possible choice. If there exists in addition a local $\mathbb{Q}$-martingale $L \not\equiv 0$ satisfying (6.5) (or (6.10) in the continuous case), then U^C is not unique. Section 7.3 below shows that this situation can indeed occur. By symmetry, one can also construct an example where U^P is not unique.

7 Examples and related work

Our goal in this article is to provide precise conditions which enable us to draw precise conclusions about option valuation, and in particular give economically well-justified arguments for choosing specific formulas for call and put prices in complete models. This topic has a long history, and the literature contains competing (and contradictory) call price formulas in concrete models. Textbooks sometimes use absence-of-arbitrage arguments for deriving the Black–Scholes formula without examining or specifying exact assumptions. In this section, we first give an overview of these issues and explain how our approach resolves them in a satisfactory way, and then illustrate our results and their limitations by examples.

7.1 Connections to the literature

Maybe the earliest major contribution on nonuniqueness of option prices can be found in the book [33] by A. Lewis. He considered stochastic volatility models with a fixed market price of risk (i.e., under one chosen ELMM) and noticed that in some cases, the resulting

PDE for option prices had multiple solutions. A detailed discussion can be found in [33, Chapter 9]; but the arguments for the choice of a particular solution as "the" option price are somewhat ad hoc and not based on a clear economic reasoning.

After the work of Lewis, it was realized that there is a connection between nonunique option prices and strict local martingales. Cox/Hobson [5] consider a model of the form $\mathbf{S} = (1, X)$, where X (called S in [5]) is a continuous semimartingale with a unique ELMM $\mathbb{Q}$ so that $\mathbf{S}$ is classically complete. Option prices are defined as superreplication prices with respect to a rather technically defined ($\mathbb{Q}$-dependent) class of strategies, and [5, Theorem 3.3] shows that this yields as time-t price of H the risk-neutral value $E_{\mathbb{Q}}[H \,|\, \mathcal{F}_t]$. If X is under $\mathbb{Q}$ a strict local martingale, this leads to the failure of put-call parity for model prices, and [5] then also discuss alternative definitions for the prices of European calls (or options of the form $H = h(X_T)$). In conclusion, the paper states that "Great care is needed when pricing options under such a model [*where X is a strict local $\mathbb{Q}$-martingale*] as many intuitively obvious statements turn out to be false [...] It may be that the standard mathematical definition is not the appropriate financial definition [...]."

Essentially the same model as in [5] was considered by Madan/Yor [34] who took for $X \geq 0$ a continuous local $\mathbb{Q}$-martingale and directly discussed pricing (but not hedging) under $\mathbb{Q}$. For a payoff of the form $H = h(X_T)$, they define the time-0 price of H as $\lim_{n\to\infty} E_{\mathbb{Q}}[h(X_{\tau_n \wedge T})]$, where $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence of stopping times for X; so each X^{τ_n} is a true $\mathbb{Q}$-martingale on $[0, T]$, and [34] propose to use the limit of the corresponding risk-neutral prices. If h is bounded (like $h(x) = (K - x)^+$ for the European put), this yields the risk-neutral price $E_{\mathbb{Q}}[H]$. For the European call with $h(x) = (x - K)^+$, [34, Proposition 2 and Theorem 1] show that

$$\begin{aligned}
\lim_{n\to\infty} E_{\mathbb{Q}}[(X_{\tau_n \wedge T} - K)^+] &= E_{\mathbb{Q}}[(X_T - K)^+] + \lim_{n\to\infty} n\mathbb{Q}\Big[\sup_{0\leq s\leq T} X_s \geq n\Big] \\
&= E_{\mathbb{Q}}[(X_T - K)^+] + (X_0 - E_{\mathbb{Q}}[X_T]).
\end{aligned}$$

A consequence of this correction to risk-neutral pricing if X is a strict local $\mathbb{Q}$-martingale is that put-call parity is restored. [34] also show (as can be seen above) that the limits defining call and put prices do not depend on the choice of the localizing sequence (τ_n).

Both [5] and [34] give a lot of credit to Heston, Loewenstein and Willard for a detailed and thorough analysis of the connections between option pricing and bubbles. They cite a preprint version from 2004; the actual paper by Heston et al. [22] appeared in 2007. The model in [22] has one bond B (with stochastic short rate) and one stock Y (given by a diffusion model), with coefficients r, μ, σ all being functions of a factor U given by another diffusion (and driven by a second Brownian motion). [22] enforce completeness by assuming that the market price of risk for U (or an ELMM for $X = Y/B$, if it exists) has been fixed. They then discuss very carefully possible conditions on r, μ, σ and their implications on the properties of the model and of option prices, and they illustrate by three examples in well-known models how nonunique option prices can arise if those conditions do not hold. In modern terminology, their (weakest) Condition 1 is that X satisfies classic NUPBR (see [24, Theorem 7]). Condition 2 in [22] is that X admits an ELMM or, equivalently, satisfies classic NFLVR; and Condition 3 is that X is under the ELMM $\mathbb{Q}$ not only a local, but a true martingale. [22] axiomatically impose put-call parity and on that basis discuss when and how, under the respective conditions, bubbles can appear in the basic assets and/or in option prices. Given that the authors work in a diffusion setup, it is not surprising that many discussions are in terms of PDEs (and the uniqueness or not of their solutions), but the economic message is nevertheless very clear.

The more recent paper by Andersen [1] presents explicit formulas for call prices in one-dimensional diffusion models with quadratic volatility. The latter often lead to asset prices which are strict local martingales under the unique ELMM, and hence to nonunique option prices. [1] does not take a stand on the choice of price; but the paper gives a very balanced and readable discussion on the issue and hence serves as a good overview.

With the exception of [22] in the case where only Condition 1 is imposed, the models discussed so far have one important common feature: There is a bond with price process $B > 0$ in the market, and discounted prices $\mathbf{S} = (1, X) = (1, Y/B)$ satisfy classic NFLVR. Equivalently, $\mathbf{S}$ admits an ELMM $\mathbb{Q}$ and hence is a local $\mathbb{Q}$-martingale. Almost all papers focus on the effects of X being a *strict* local $\mathbb{Q}$-martingale (a "bubble" in the usually chosen terminology) and do not pay much attention to the discounted bond price 1. But this automatically makes the analysis completely asymmetric. Indeed, if $(1, X)$ satisfies classic NFLVR, it is a local $\mathbb{Q}$-martingale under some ELMM $\mathbb{Q}$. Trivially, 1 is then a true $\mathbb{Q}$-martingale, which means by Theorem 2.13 that e^1 (and hence asset 1) is maximal. The discussion about X being a strict local or a true martingale thus only asks whether e^2 (i.e., asset 2) is *als*o maximal or not; the possibility of only the stock, but not the bond being maximal is never considered. Even if it is not pointed out directly, this aspect is behind many works on the benchmark approach to quantitative finance; see for instance Platen/Heath [36, Sections 10.3 and 12.2]. By Theorem 6.2, maximality of asset 1 (i.e., the bond in the above models) implies that both call and put valuations are unique, and that any correction in comparison to risk-neutral values is always assigned to the call. Our analysis shows that in general, the situation is more symmetric and also more interesting. An explicit study in a very particular case can also be found in [23].

Remark 7.1. If neither e^1 nor e^2 is maximal, call and put prices are not unique in general; see Section 7.3 below. This implies in particular that the limits in the Madan/Yor [34] approach will become dependent on the localizing sequence; their uniqueness in the context of [34] crucially rests on the maximality of the bond (asset 1).

More recently, a number of papers have also examined a similar symmetric setting as our paper. Kardaras [30] studies valuation and parities for exchange options in a general model with finitely many assets $S^i \geq 0$ in an economy with a possibly stochastic lifetime ζ. The main assumptions are that $S^i = 0$ on $[\![\zeta, \infty[\![$ and that there is a nonnegative local martingale deflator Z, i.e., Z and each ZS^i is a local $\mathbb{P}$-martingale (and $Z_0 = 1$). By definition, a time-T payoff H_T is then valued at time 0 by $E_{\mathbb{P}}[Z_T H_T I_{\{T<\zeta\}}]$, and the focus of [30] is on deriving expressions for this value, when $H_T = (S^j_T - S^i_T)^+$ is an exchange option, in terms of expectations under measures $\mathbb{Q}^i$ and $\mathbb{Q}^j$ under which S^i respectively S^j have (almost) a local martingale property. There is no deeper economic justification for the above choice of valuation (nor for the choice of Z, if there are several), and there is no discussion of valuations at later times $t > 0$.

Two recent papers by Carr/Fisher/Ruf [3] and Fisher/Pulido/Ruf [15] also use a setting with N assets treated in a symmetric way. The earlier paper [3] has $N = 2$ and studies some of the effects that appear in valuation when one of the assets hits 0, with results that look similar to the ones in [30]. The notion of valuation in [3] is superreplication, but simultaneously with respect to two, possibly not equivalent measures; they are associated to the two assets being used as numéraires. The more recent paper [15] works with general N and presents links between absence-of-arbitrage conditions and the existence and structure of so-called martingale valuation operators. Because [15] is written in terms of exchange rate matrices whose entries are processes valued in $[0, +\infty]$, a translation into the standard framework of N assets is not straightforward, and the connections to other works are difficult to see.

Remark 7.2. Both [30] and [3] impose a technical condition on the underlying probability space to use an extension result [35, Theorem V.4.1] based on projective limits. This assumption is not entirely harmless; it fails for example if one wants to work on the space $C([0,T]; \mathbb{R}^N_{++})$ of strictly positive finite-valued continuous functions on $[0,T]$ with a strict local martingale for the coordinate process. In particular, the put-call parity result as stated in [3, Corollary 3.8] cannot hold in general without this technical condition.

7.2 Examples I: Some semi-efficient models

To have concrete examples, we show here that *all three models a), b), c) from Example 2.2 yield semi-efficient markets*. First, they all are dynamically viable by Theorem 2.12 because each admits a numéraire/ELMM pair $(D, \mathbb{Q})$. Indeed, take $\mathbf{S} = (1, Y/B) =: (1, X)$. For the Black–Scholes model in a), X (and hence $\mathbf{S}$) has an equivalent (true) martingale measure; for the CEV models without (in b)) and with (in c)) stochastic volatility, X is even already a local martingale under $\mathbb{P}$ itself.

Next, to check static efficiency via Theorem 5.5, we verify, using $(1, X) = \mathbf{S}^{(e^1)}$, that

$$X_s \in \operatorname{ri} \operatorname{conv} \operatorname{supp} \mathcal{L}(X_T \,|\, \mathcal{F}_s) \qquad \mathbb{P}\text{-a.s. for each } s \in [0,T]. \tag{7.1}$$

This is clear for the Black–Scholes model in a) because $X_s > 0$ and $X_T = X_s(X_T/X_s)$ has X_T/X_s under $\mathbb{P}$ independent of X_s with a lognormal distribution. For the CEV model in b), [12, Equation (7)] gives the explicit formula for the transition density $f(T, y; s, x)$ of the conditional distribution of X at time T, given we are in x at time s. One can see from that expression that $\mathcal{L}(X_T \,|\, \mathcal{F}_s)$ has $(0, \infty)$ as its support, and this readily yields (7.1).

For the CEV model in c) with stochastic volatility, we have to work a bit more because X alone is there not a Markov process. Recall that

$$\mathrm{d}X_t = \sigma_t |X_t|^\beta \,\mathrm{d}W_t, \tag{7.2}$$

$$\mathrm{d}\sigma_t = \alpha(\sigma_t - \overline{\sigma})(\sigma_t - \underline{\sigma}) \,\mathrm{d}W'_t, \tag{7.3}$$

with a two-dimensional Brownian motion (W, W') under $\mathbb{P}$ and constants $\beta > 1$, $\alpha > 0$ and $\overline{\sigma} > \sigma_0 > \underline{\sigma} > 0$. It has already been shown in [21, Example 6.3] that $\mathbf{S}$ is dynamically viable (and that it fails to be dynamically efficient). For each $s \in [0,T)$, e^1 is a numéraire strategy in $\mathbf{h}^s L^{\mathrm{sf}}_+$ and $\mathbf{S} = \mathbf{S}^{(e^1)}$. By Theorem 5.5, it is thus enough to find some $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ such that $\mathbb{Q}$ is a one-step EMM for $\mathbf{S}$ on $\{s, T\}$, i.e., $\mathbb{E}_\mathbb{Q}[X_T \,|\, \mathcal{F}_s] = X_s$ $\mathbb{Q}$-a.s., and for that, it even suffices to show $\mathbb{P}[X_T > X_s \,|\, \mathcal{F}_s] > 0$ $\mathbb{P}$-a.s. and $\mathbb{P}[X_T < X_s \,|\, \mathcal{F}_s] > 0$

$\mathbb{P}$-a.s. Now the pair (X,σ) is a strong Markov process for the filtration $(\mathcal{F}_t)_{t\in[0,T]}$; so it is enough to show $\mathbb{P}_{x,v}[X_{T-s}>y]>0$ and $\mathbb{P}_{x,v}[X_{T-s}<y]>0$ for all $y>0$ and $v\in(\underline{\sigma},\overline{\sigma})$, where $\mathbb{P}_{x,v}$ is the distribution of the solution (X,σ) of (7.2), (7.3) with initial value (x,v).

To make the presentation more transparent, we work without loss of generality on the canonical path space, denote by (X,σ) the canonical Markov process and by ϑ the shift operator. Fix $x>0$. Then $\mathbb{E}_{x,v}[X_{T-s}]<x$ for all v by [21, Example 6.3]; see the general result just before Equation (6.5) there. This directly gives $\mathbb{P}_{x,v}[X_{T-s}<x]>0$. To derive the other inequality, take $\varepsilon\in(0,T-s)$ small enough that $\mathbb{E}_{2x,v}[X_\varepsilon]>x$ for all $v\in(\underline{\sigma},\overline{\sigma})$; see [21, Equation (6.5)]. Define the stopping times $\tau_x^\uparrow:=\inf\{t\geq 0: X_t\geq x\}$ and $\tau_x^\downarrow:=\inf\{t\geq 0: X_t\leq x\}$. We claim that $\mathbb{P}_{y,v}[\tau_{2x}^\uparrow<\varepsilon]>0$ for all $y\in(0,2x]$ and $v\in(\underline{\sigma},\overline{\sigma})$, and $\mathbb{P}_{2x,v}[\tau_x^\downarrow>\varepsilon]>0$ for all $v\in(\underline{\sigma},\overline{\sigma})$. The first claim follows because X is a strict local $\mathbb{P}_{y,v}$-martingale on $[0,\varepsilon]$ and hence cannot be uniformly bounded. The second claim holds because if we had $\mathbb{P}_{2x,v}[\tau_x^\downarrow\leq\varepsilon]=1$ for some $v\in(\underline{\sigma},\overline{\sigma})$, then the choice of ε and the supermartingale property of X under $\mathbb{P}_{2x,v}$ would yield a contradiction via

$$x<\mathbb{E}_{2x,v}[X_\varepsilon]\leq\mathbb{E}_{2x,v}[X_{\tau_x^\downarrow}]=\mathbb{E}_{2x,v}[x]=x.$$

Note also that $\mathbb{P}_{x,v}[X_{T-s-\varepsilon}\leq x]>0$ by the fact that $\mathbb{E}_{x,v}[X_{T-s-\varepsilon}]<x$. Combining everything then yields

$$\begin{aligned}
\mathbb{P}_{x,v}[X_{T-s}>x]&\geq\mathbb{E}_{x,v}\left[\mathbb{1}_{\{X_\varepsilon\circ\vartheta_{T-s-\varepsilon}>x\}}\mathbb{1}_{\{X_{T-s-\varepsilon}\leq x\}}\right]\\
&=\mathbb{E}_{x,v}\left[\mathbb{E}_{X_{T-s-\varepsilon},\sigma_{T-s-\varepsilon}}\left[\mathbb{1}_{\{X_\varepsilon>x\}}\right]\mathbb{1}_{\{X_{T-s-\varepsilon}\leq x\}}\right]\\
&\geq\mathbb{E}_{x,v}\left[\mathbb{E}_{X_{T-s-\varepsilon},\sigma_{T-s-\varepsilon}}\left[\mathbb{1}_{\{X_\varepsilon>x\}}\mathbb{1}_{\{\tau_{2x}^\uparrow<\varepsilon\}}\right]\mathbb{1}_{\{X_{T-s-\varepsilon}\leq x\}}\right]\\
&\geq\mathbb{E}_{x,v}\left[\mathbb{E}_{X_{T-s-\varepsilon},\sigma_{T-s-\varepsilon}}\left[\mathbb{1}_{\{(\inf_{0\leq u\leq\varepsilon}X_u)\circ\vartheta_{\tau_{2x}^\uparrow}>x\}}\mathbb{1}_{\{\tau_{2x}^\uparrow<\varepsilon\}}\right]\mathbb{1}_{\{X_{T-s-\varepsilon}\leq x\}}\right] \qquad (7.4)\\
&=\mathbb{E}_{x,v}\left[\mathbb{E}_{X_{T-s-\varepsilon},\sigma_{T-s-\varepsilon}}\left[\mathbb{E}_{2x,\sigma_{\tau_{2x}^\uparrow}}\left[\mathbb{1}_{\{\inf_{0\leq u\leq\varepsilon}X_u>x\}}\right]\mathbb{1}_{\{\tau_{2x}^\uparrow<\varepsilon\}}\right]\mathbb{1}_{\{X_{T-s-\varepsilon}\leq x\}}\right]\\
&=\mathbb{E}_{x,v}\left[\mathbb{E}_{X_{T-s-\varepsilon},\sigma_{T-s-\varepsilon}}\left[\mathbb{E}_{2x,\sigma_{\tau_{2x}^\uparrow}}\left[\mathbb{1}_{\{\tau_x^\downarrow>\varepsilon\}}\right]\mathbb{1}_{\{\tau_{2x}^\uparrow<\varepsilon\}}\right]\mathbb{1}_{\{X_{T-s-\varepsilon}\leq x\}}\right]>0,
\end{aligned}$$

where we use at the start a trivial inclusion, the Markov property and another trivial inclusion, and after (7.4) the strong Markov property, the definition of $\tau_x^\downarrow$ and that $\mathbb{P}_{2x,v}[\tau_x^\downarrow>\varepsilon]>0$, $\mathbb{P}_{x,v}[\tau_{2x}^\uparrow<\varepsilon]>0$ and $\mathbb{P}_{x,v}[X_{T-s-\varepsilon}\leq x]>0$. For (7.4), we use that if after time $\tau_{2x}^\uparrow<\varepsilon$, we stay above x for at least ε units of time, then we must be above x at time ε. This proves (7.1) for Example 2.2 c).

Remark 7.3. It is shown in [21, Example 6.3] that the market (there called $\mathcal{S}$) generated by the stochastic volatility CEV model $\mathbf{S}$ from Example 2.2 c) is dynamically viable, but not dynamically efficient. In the language of [21], $\mathcal{S}$ has therefore a *strong bubble*, which means by [21, Theorem 3.7] that for any numéraire/ELMM pair $(D,\mathbb{Q})$, the process $S=\mathbf{S}/D$ is a strict local $\mathbb{Q}$-martingale. This shows that risk-neutral valuation is impossible in this market — *any* choice of ELMM $\mathbb{Q}$ (if there is one) for *any* model $S\in\mathcal{S}$ will misprice at least one basic asset i at some time t, because $\mathbb{E}_{\mathbb{Q}}[S_T^i\,|\,\mathcal{F}_t]\neq S_t^i$. In other words, no dynamically efficient consistent valuation can exist for $\mathbf{S}$; this follows of course also from Remark 4.1 because $\mathbf{S}$ itself is not dynamically efficient.

However, as argued above, the model $\mathbf{S}=(1,X)$ from Example 2.2 c) is statically efficient (it satisfies (2.3) as $X\geq 0$). So semi-efficient valuations exist by Theorem 5.13.

7.3 Examples II: Pitfalls

While Theorems 5.13, 6.1 and 6.2 provide good results about semi-efficient valuations, there are still some pitfalls to be aware of. A semi-efficient model need not be dynamically efficient — it can have a strong bubble in the sense of [21] so that for every numéraire/ELMM pair $(D, \mathbb{Q})$, the process $S = \mathbf{S}/D$ is a strict local $\mathbb{Q}$-martingale. It seems clear that we should then expect some tricky issues with valuations, and the present section illustrates this. In fact, we show that even in a complete and semi-efficient market,

a) call options can have nonunique consistent valuations;

b) put options can at the same time have nonunique consistent valuations;

c) in Proposition 5.11, we can have $L_0^{\mathrm{super}} > L_0^{\mathrm{sub}}$;

d) the formulas (6.3) and (6.4), with L satisfying (6.5), do not always define consistent valuations.

Note that despite a) and b), we still have put-call parity due to Theorem 6.1.

We first explain how suitable abstract assumptions imply the above assertions, and then exhibit a concrete example satisfying these assumptions. In abstract terms, let $\mathbf{S} = (\mathbf{S}^1, \mathbf{S}^2)$, where $\mathbf{S} \geq 0$ satisfies (2.3) and is a continuous local $\mathbb{P}$-martingale. We also assume that for some (or equivalently all) $K > 0$,

$$\min(\mathbf{S}^2, K\mathbf{S}^1) \text{ is not of class (D).} \tag{7.5}$$

Then $\mathbf{S}^1$ and $\mathbf{S}^2$ are both also not of class (D) and therefore strict local $\mathbb{P}$-martingales.

In view of (6.10), we first look at the local $\mathbb{P}$-martingale $\bar{L} \geq 0$ defined by

$$\bar{L}_t := \mathbb{E}\Big[-\int_t^T (\mathbb{1}_{\{\mathbf{S}_u^2 \leq K\mathbf{S}_u^1\}} \, \mathrm{d}\mathbf{S}_u^2 + \mathbb{1}_{\{\mathbf{S}_u^2 > K\mathbf{S}_u^1\}} \, \mathrm{d}\mathbf{S}_u^1) \Big| \mathcal{F}_t \Big], \qquad 0 \leq t \leq T.$$

Recall from the proof of Theorem 6.2 that $\bar{L}_t = \mathbb{E}[M_t - M_T \,|\, \mathcal{F}_t]$, where $M \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P})$ is from the Doob–Meyer decomposition of the $\mathbb{P}$-supermartingale $Z := \min(\mathbf{S}^2, K\mathbf{S}^1)$. So $\bar{L}$ is not identically 0 if and only if M is a strict local $\mathbb{P}$-martingale, and to obtain $\bar{L} \not\equiv 0$, it suffices to show that M is not of class (D), or equivalently that Z is not of class (D). But this is exactly our assumption (7.5).

Now suppose also that $\mathbf{S}^1 > 0$ and $\mathbf{S}^2 > 0$ and that $X := \mathbf{S}^2/\mathbf{S}^1 = V(e^2)[\mathbf{S}^{(e^1)}]$ satisfies the conditional full support condition (6.14) with $d = 1$. Then we have

$$\mathbf{S}_t^{(e^1)} = (1, X_t) \in \operatorname{ri} \operatorname{conv} \operatorname{supp} \mathcal{L}(\mathbf{S}_T^{(e^1)} \,|\, \mathcal{F}_t) \qquad \mathbb{P}\text{-a.s., for each } t \in [0, T].$$

Moreover, $\mathbf{S}$ is dynamically viable by Theorem 2.12 because $(1, \mathbb{P})$ is by assumption a numéraire/ELMM pair. So Theorem 5.5 implies that $\mathbf{S}$ is statically efficient and therefore semi-efficient. If $\mathbf{S}$ is also complete, Proposition 6.4 says that for any local $\mathbb{P}$-martingale L satisfying $0 \leq L \leq \bar{L}$, the valuation $\mathbf{U}^{\mathbf{C}}(L)$ defined by

$$U_t^C[\mathbf{S}; L] := \mathbf{U}_t^{\mathbf{C}}(L) := \mathbb{E}[(\mathbf{S}_T^2 - K\mathbf{S}_T^1)^+ \,|\, \mathcal{F}_t] + \mathbf{S}_t^2 - \mathbb{E}[\mathbf{S}_T^2 \,|\, \mathcal{F}_t] - L_t, \qquad 0 \leq t \leq T, \tag{7.6}$$

is semi-efficient consistent for the call option $\mathbf{C} = (\mathbf{S}^2_T - K\mathbf{S}^1_T)^+$. Choosing $L \equiv 0$ and $L := \bar{L} \not\equiv 0$ thus gives two different consistent valuations for a call, which proves the nonuniqueness assertion in a). Statement b) is then obvious because we always have put-call parity. (Alternatively, one can argue b) in the same way as a), using that the local martingale L appearing in put and call valuations is the same.) Finally, $\bar{L} \geq 0$ is a local $\mathbb{P}$-martingale and hence a $\mathbb{P}$-supermartingale, and we know from (5.9) in Theorem 5.13, for $\mathbf{f} = \mathbf{C}$, that for any $L \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P})$ with $0 \leq L \leq \bar{L}$,

$$L^{\mathrm{sub}}(1, \mathbb{P}) \leq \mathbf{U}^{\mathbf{C}}(L) \leq L^{\mathrm{super}}(1, \mathbb{P}).$$

So if we had $L^{\mathrm{sub}}_0 = L^{\mathrm{super}}_0$, (7.6) would give $\mathbf{U}^{\mathbf{C}}_0(\bar{L}) = \mathbf{U}^{\mathbf{C}}_0(0)$, which would in turn yield $\bar{L}_0 = L_0 = 0$ and hence $\bar{L} \equiv 0$. This is a contradiction, and so we also get statement c).

The example to illustrate d) is slightly different, but leverages what we have already done. We start with $\mathbf{S} = (\mathbf{S}^1, \mathbf{S}^2)$ as above and define a new model $\mathbf{S}' := (\mathbf{S}^1, \mathbf{S}^2, \mathbf{S}^3)$, where

$$\mathbf{S}^3 := U^C[\mathbf{S}; \bar{L}] = \mathbf{U}^{\mathbf{C}}(\bar{L}).$$

Then $\mathbf{S}' \geq 0$ satisfies (2.3) and is complete (like $\mathbf{S}$, in view of our choice of $\mathbf{S}^3$) and semi-efficient by Theorem 6.2 applied to $\mathbf{S}$. However, taking $L \equiv 0$ in (6.3) or in (7.6) now gives a valuation for the call $\mathbf{C}$ which is not consistent with $\mathbf{S}'$ because

$$\mathbf{U}^{\mathbf{C}}_T(L) = (\mathbf{S}^2_T - K\mathbf{S}^1_T)^+ = \mathbf{U}^{\mathbf{C}}_T(\bar{L}) = \mathbf{S}^3_T,$$

but

$$\mathbf{U}^{\mathbf{C}}_0(L) = \mathbf{U}^{\mathbf{C}}_0(\bar{L}) - L_0 + \bar{L}_0 \neq \mathbf{U}^{\mathbf{C}}_0(\bar{L}) = S^3_0$$

because $\bar{L}_0 \neq 0 = L_0$, as seen above. So for the model $\mathbf{S}'$, not all choices from (6.3) lead to consistent valuations, proving d). (This is clear from the construction of $\mathbf{S}'$: we have added as a new third asset one of the several possible valuations of a call, and this automatically excludes all other possible valuations of that same call. Again by construction, $X' := (\mathbf{S}^2/\mathbf{S}^1, \mathbf{S}^3/\mathbf{S}^1) = (X, \mathbf{S}^3/\mathbf{S}^1)$ also does not satisfy the condition (6.14) with $d = 2$.)

To complete the example, it remains to construct some concrete $\mathbf{S} = (\mathbf{S}^1, \mathbf{S}^2)$ with all the above properties. For that, let X^1 and X^2 be the unique strong solutions to the SDEs

$$\begin{aligned} \mathrm{d}X^1_t &= X^1_t \,\mathrm{d}W^1_t, \qquad X^1_0 = 1, \\ \mathrm{d}X^2_t &= \frac{1}{X^2_t}\,\mathrm{d}t + \mathrm{d}W^2_t, \qquad X^2_0 = 1, \end{aligned}$$

where $(W^1_t)_{0\leq t\leq T}$ and $(W^2_t)_{0\leq t\leq T}$ are independent $\mathbb{P}$-Brownian motions, and the underlying filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ is the natural (augmented) filtration of (W^1, W^2). So $X^1 > 0$ is a geometric Brownian motion without drift, a special case of Example 2.2 a), and $X^2 > 0$ is a three-dimensional Bessel process BES^3. For subsequent use, we remark that $U := 1/X^2$ satisfies the SDE

$$\mathrm{d}U_t = -|U_t|^2 \,\mathrm{d}W^2_t, \qquad U_0 = 1,$$

which is a special case ($\beta = 2$) of the SDE for the CEV model in Example 2.2 b).

We now define

$$\mathbf{S}^1 := \frac{1}{X^2} = U, \qquad \mathbf{S}^2 := \frac{X^1}{X^2} = UX^1$$

and claim that this gives us all the desired properties. First of all, we clearly have $\mathbf{S}^1 > 0$, $\mathbf{S}^2 > 0$ and $\mathbf{S} = (\mathbf{S}^1, \mathbf{S}^2)$ is continuous so that (2.3) is satisfied. It is well known that $\mathbf{S}^1 = U$ is a (strict) local $\mathbb{P}$-martingale, and because it is independent from the true $\mathbb{P}$-martingale X^1, it follows that $\mathbf{S}^2 = UX^1$ is a (strict) local $\mathbb{P}$-martingale, too. To verify (7.5), we observe that U is not of class (D) as it is a strict local $\mathbb{P}$-martingale, and $\min(X^1, K)$ is independent from U and satisfies $\inf_{0\le t\le T} \mathbb{E}[\min(X^1_t, K)] > 0$ because each X^1_t has a lognormal distribution. By Proposition B.3, the supermartingale $\min(\mathbf{S}^2, K\mathbf{S}^1) = U\min(X^1, K)$ is therefore also not of class (D). Next, $X := \mathbf{S}^2/\mathbf{S}^1 = X^1$ is a geometric Brownian motion and hence clearly satisfies the conditional full support condition (6.14) with $d = 1$. Finally, it is not difficult to check that $\mathbb{P}$ itself is the only probability equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ under which $\mathbf{S}$ is a local martingale; see Example 3.3 of [21]. So $\mathbf{S}$ is complete by Theorem 3.6, and all the above conditions on $\mathbf{S}$ are satisfied.

Remark 7.4. The key property driving the features of the above example is that we look in a complete market at an exchange option between two assets which are both strict local martingales under their unique ELMM. By Corollary 3.10, this implies that neither of these assets is maximal, and so we cannot use Theorem 6.2 to deduce any uniqueness result for the semi-efficient consistent valuations for calls and puts.

A Buy-and-hold strategies and static efficiency

This section provides a dual characterization of static efficiency in terms of martingale properties. In preparation, we first analyse the maximality of buy-and-hold strategies ϑ in $\mathbf{h}^\sigma L^{\mathrm{sf}}_+$. Recall from Section 5.1 the concept of a one-step EMM for $\mathbf{S}^{(\eta)}$ on $\{\sigma, T\}$.

We begin with a technical result which characterizes the maximality of the zero strategy 0 not in $\mathbf{h}^\sigma L^{\mathrm{sf}}_+$, but in $\mathbf{h}^\sigma L^{\mathrm{sf}}$. The (classic) key ingredients are the Dalang–Morton–Willinger theorem, the one-step superhedging duality or optional decomposition, and the geometric characterization of absence of arbitrage, all with contingent initial data. Recall from Section 5.1 (before Theorem 5.5) the notation $\operatorname{ri}\operatorname{conv}\operatorname{supp}\mathcal{L}(Y \mid \mathcal{G})$ and from Definition 2.9 the notion of a weakly maximal strategy. Note that maximality implies weak maximality; see [20, Proposition 3.12].

Proposition A.1. *Suppose* $\mathbf{S} \ge 0$ *satisfies* (2.3). *For any stopping time* $\sigma \le T$, *the following are equivalent:*

1) 0 *is weakly maximal for* $\mathbf{h}^\sigma L^{\mathrm{sf}}$.

2) 0 *is maximal for* $\mathbf{h}^\sigma L^{\mathrm{sf}}$.

3) *Each* $\vartheta \in \mathbf{h}^\sigma L^{\mathrm{sf}}$ *is maximal for* $\mathbf{h}^\sigma L^{\mathrm{sf}}$.

4) *For each numéraire strategy* $\eta \in \mathbf{h}^\sigma L^{\mathrm{sf}}_+$, *there exists a one-step EMM* $\mathbb{Q}$ *for* $\mathbf{S}^{(\eta)}$ *on* $\{\sigma, T\}$.

4′) *For each numéraire strategy* $\eta \in \mathbf{h}^\sigma L^{\mathrm{sf}}_+$, *we have*

$$\mathbf{S}^{(\eta)}_\sigma \in \operatorname{ri}\operatorname{conv}\operatorname{supp}\mathcal{L}(\mathbf{S}^{(\eta)}_T \mid \mathcal{F}_\sigma) \qquad \mathbb{P}\text{-}a.s.$$

5) *There exist a numéraire strategy* $\eta \in \mathbf{h}^\sigma L_+^{\mathrm{sf}}$ *and a one-step EMM* $\mathbb{Q}$ *for* $\mathbf{S}^{(\eta)}$ *on* $\{\sigma, T\}$.

5′) *There exists a numéraire strategy* $\eta \in \mathbf{h}^\sigma L_+^{\mathrm{sf}}$ *such that*

$$\mathbf{S}_\sigma^{(\eta)} \in \operatorname{ri}\operatorname{conv}\operatorname{supp}\mathcal{L}(\mathbf{S}_T^{(\eta)} \,|\, \mathcal{F}_\sigma) \qquad \mathbb{P}\text{-}a.s.$$

Moreover, if one of the above equivalent conditions is satisfied, then for any payoff $\mathbf{f}$ *at time* T *with* $\pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}) < \infty$ $\mathbb{P}$-*a.s., there exists* $\vartheta \in \mathbf{h}^\sigma L^{\mathrm{sf}}$ *with*

$$\mathbf{V}_T(\vartheta) \geq \mathbf{f} \quad \textit{and} \quad \mathbf{V}_\sigma(\vartheta) = \pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}), \qquad \mathbb{P}\text{-}a.s.$$

Proof. "3) ⇒ 2) ⇒ 1)" is trivial.

"2) ⇒ 3)": If ϑ is not maximal for $\mathbf{h}^\sigma L^{\mathrm{sf}}$, Remark 2.10 yields $\mathbf{f} \in \mathbf{L}_+^0(\mathcal{F}_T) \setminus \{0\}$ with

$$\pi_\sigma(\mathbf{V}_T(\vartheta) + \mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}) \leq \mathbf{V}_\sigma(\vartheta) < \infty \qquad \mathbb{P}\text{-a.s.} \tag{A.1}$$

Fix $\delta > 0$ and $\mathbf{g} \in \mathbf{L}_+^0(\mathcal{F}_\sigma) \setminus \{0\}$. Using (A.1) and applying [21, Lemma A.1] for $C := F_{\mathbf{g}}$ to the contingent claim map $S \mapsto V_T(\vartheta)[S] + F_{\mathbf{f}}[S]$ gives a $\bar{\vartheta} \in {}^\sigma L^{\mathrm{sf}}$ with $\mathbf{V}_T(\bar{\vartheta}) \geq \mathbf{V}_T(\vartheta) + \mathbf{f}$ and $\mathbf{V}_\sigma(\bar{\vartheta}) \leq \pi_\sigma(\mathbf{V}_T(\vartheta) + \mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}) + \delta\mathbf{g} \leq \mathbf{V}_\sigma(\vartheta) + \delta\mathbf{g}$, $\mathbb{P}$-a.s. For $\hat{\vartheta} := \bar{\vartheta} - \vartheta \in \mathbf{h}^\sigma L^{\mathrm{sf}}$, we thus obtain $\mathbf{V}_T(\hat{\vartheta}) \geq \mathbf{f}$ and $\mathbf{V}_\sigma(\hat{\vartheta}) \leq \delta\mathbf{g}$, $\mathbb{P}$-a.s. So $\pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}) \leq \delta\mathbf{g}$, and letting $\delta \searrow 0$ gives $\pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}) = 0$ which contradicts the maximality of 0 for $\mathbf{h}^\sigma L^{\mathrm{sf}}$.

"1) ⇒ 4)": For a numéraire strategy $\eta \in \mathbf{h}^\sigma L_+^{\mathrm{sf}}$, define $\mathcal{G}_0 := \mathcal{F}_\sigma$, $\mathcal{G}_1 := \mathcal{F}_T$, $\bar{X}_0 := \mathbf{S}_\sigma^{(\eta)}$, $\bar{X}_1 := \mathbf{S}_T^{(\eta)}$. Then 1) implies that the classic discounted one-period model $(1, \bar{X})$ (of dimension $N+1$) is arbitrage-free in the classic sense that there is no $\mathcal{G}_0$-measurable $\mathbb{R}^N$-valued random vector $\xi \in \mathbf{L}_+^0 \setminus \{0\}$. Indeed, if such a ξ exists, $\vartheta := \xi \mathbb{1}_{[\![\sigma,T]\!]} - (\xi \cdot \mathbf{S}_\sigma^{(\eta)})\eta$ is like η in $\mathbf{h}^\sigma L^{\mathrm{sf}}$, and as $V(\eta)[\mathbf{S}^{(\eta)}] \equiv 1$ by (2.2), we get

$$\begin{aligned} V_\sigma(\vartheta)[\mathbf{S}^{(\eta)}] &= \xi \cdot \mathbf{S}_\sigma^{(\eta)} - (\xi \cdot \mathbf{S}_\sigma^{(\eta)}) = 0 \qquad \mathbb{P}\text{-a.s.}, \\ V_T(\vartheta)[\mathbf{S}^{(\eta)}] &= \xi \cdot \mathbf{S}_T^{(\eta)} - (\xi \cdot \mathbf{S}_\sigma^{(\eta)}) = \xi \cdot (\bar{X}_1 - \bar{X}_0) \in \mathbf{L}_+^0 \setminus \{0\} \end{aligned}$$

which contradicts 1). So by the Dalang–Morton–Willinger theorem, for instance in the form of [17, Theorem 1.54]), there exists an EMM $\mathbb{Q}$ for the above one-period model, and translating everything back to our setup, we see that $\mathbb{Q}$ is a one-step EMM for $\mathbf{S}^{(\eta)}$ on $\{\sigma, T\}$.

"3) ⇔ 3′)" and "4) ⇔ 4′)" follow directly from Jacod/Shiryaev [27, Theorem 3].

"4) ⇒ 5)" is trivial as the market portfolio $\eta^{\mathbf{S}} \equiv \mathbf{1}$ is a numéraire strategy in $\mathbf{h}^\sigma L_+^{\mathrm{sf}}$.

"5) ⇒ 2)" and additional assertion: Let $\eta, \mathbb{Q}$ be as in 5) and $\mathbf{f}$ a payoff at time T with $\pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}}) < \infty$ $\mathbb{P}$-a.s. Recall from (2.11) with $D := \mathbf{V}(\eta)$ and $\mathbf{S}^{(\eta)} = \mathbf{S}/\mathbf{V}(\eta)$ that $\Pi_\sigma(F_{\mathbf{f}} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}})[\mathbf{S}^{(\eta)}] = \frac{1}{D_\sigma}\pi_\sigma(\mathbf{f} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}})$. If $\mathcal{Q} \neq \emptyset$ denotes the set of all one-step EMMs for $\mathbf{S}^{(\eta)}$ on $\{\sigma, T\}$, then for $\vartheta \in \mathbf{h}^\sigma L^{\mathrm{sf}}$ with $V_T(\vartheta)[\mathbf{S}^{(\eta)}] \geq F_{\mathbf{f}}[\mathbf{S}^{(\eta)}]$ and $\tilde{\mathbb{Q}} \in \mathcal{Q}$, we have $\mathbb{P}$-a.s.

$$V_\sigma(\vartheta)[\mathbf{S}^{(\eta)}] = \vartheta_\sigma \cdot \mathbf{S}_\sigma^{(\eta)} = \mathbb{E}_{\tilde{\mathbb{Q}}}[\vartheta_\sigma \cdot \mathbf{S}_T^{(\eta)} \,|\, \mathcal{F}_\sigma] = \mathbb{E}_{\tilde{\mathbb{Q}}}[V_T(\vartheta)[\mathbf{S}^{(\eta)}] \,|\, \mathcal{F}_\sigma] \geq \mathbb{E}_{\tilde{\mathbb{Q}}}[F_{\mathbf{f}}[\mathbf{S}^{(\eta)}] \,|\, \mathcal{F}_\sigma].$$

Thus, by the definition of superreplication prices,

$$\operatorname*{ess\,sup}_{\tilde{\mathbb{Q}} \in \mathcal{Q}} \mathbb{E}_{\tilde{\mathbb{Q}}}[F_{\mathbf{f}}[\mathbf{S}^{(\eta)}] \,|\, \mathcal{F}_\sigma] \leq \Pi_\sigma(F_{\mathbf{f}} \,|\, \mathbf{h}^\sigma L^{\mathrm{sf}})[\mathbf{S}^{(\eta)}] < \infty \qquad \mathbb{P}\text{-a.s.} \tag{A.2}$$

In particular, $\mathbb{P}[\Pi_\sigma(F_{\mathbf{f}}\,|\,\mathbf{h}^\sigma L^{\mathrm{sf}})[\mathbf{S}^{(\eta)}]>0]>0$ if $\mathbb{P}[F_{\mathbf{f}}[\mathbf{S}^{(\eta)}]>0]>0$, which gives 2) in view of (2.11) and the description of maximality in Remark 2.10.

Now define a one-period market (of dimension $N+1$) as in the proof of "1) ⇒ 4)" and set $\gamma_1 := F_{\mathbf{f}}[\mathbf{S}^{(\eta)}]$ and $\gamma_0 := \operatorname{ess\,sup}_{\tilde{\mathbb{Q}}\in\mathcal{Q}} \mathbb{E}_{\tilde{\mathbb{Q}}}[\gamma_1\,|\,\mathcal{G}_0] < \infty$ $\mathbb{P}$-a.s. Then $\mathcal{Q}$ is the set of all EMMs for $(1,\bar{X})$ and γ_1 is a payoff in this market, all in the classic sense. By the superhedging duality or optional decomposition (see for instance [17, Corollary 7.15]), there exists an $\mathbb{R}^N$-valued $\mathcal{F}_\sigma$-measurable random vector ξ satisfying $\gamma_0+\xi\cdot(\bar{X}_1-\bar{X}_0)\ge\gamma_1$ $\mathbb{P}$-a.s. Thus $\vartheta := \xi\mathbb{1}_{[\![\sigma,T]\!]}+(\gamma_0-\xi\cdot\mathbf{S}^{(\eta)}_\sigma)\eta$ is in $\mathbf{h}^\sigma L^{\mathrm{sf}}$, and as in the proof of "1) ⇒ 4)", we get $\mathbb{P}$-a.s.

$$V_\sigma(\vartheta)[\mathbf{S}^{(\eta)}] = \xi\cdot\mathbf{S}^{(\eta)}_\sigma+(\gamma_0-\xi\cdot\mathbf{S}^{(\eta)}_\sigma) = \gamma_0 = \operatorname*{ess\,sup}_{\tilde{\mathbb{Q}}\in\mathcal{Q}} \mathbb{E}_{\tilde{\mathbb{Q}}}[F_{\mathbf{f}}[\mathbf{S}^{(\eta)}]\,|\,\mathcal{F}_\sigma],$$
$$V_T(\vartheta)[\mathbf{S}^{(\eta)}] = \xi\cdot\mathbf{S}^{(\eta)}_T+(\gamma_0-\xi\cdot\mathbf{S}^{(\eta)}_\sigma) = \gamma_0+\xi\cdot(\bar{X}_1-\bar{X}_0)\ge\gamma_1 = F_{\mathbf{f}}[\mathbf{S}^{(\eta)}].$$

Together with (A.2), this establishes the additional assertion. □

The next technical result provides equivalent primal descriptions of static efficiency.

Proposition A.2. *If $\mathbf{S}\ge 0$ satisfies* (2.3)*, the following are equivalent:*

1) *$\mathbf{S}$ is statically efficient.*
2) *$\mathbf{S}$ is statically viable, and for each stopping time $\sigma\le T$, every $\vartheta\in\mathbf{h}^\sigma L^{\mathrm{sf}}$ which satisfies $\mathbf{V}_T(\vartheta)\ge 0$ $\mathbb{P}$-a.s. is in $\mathbf{h}^\sigma L^{\mathrm{sf}}_+$.*

2′) *$\mathbf{S}$ is statically viable, and for each deterministic $s\in[0,T)$, every $\vartheta\in\mathbf{h}^s L^{\mathrm{sf}}$ which satisfies $\mathbf{V}_T(\vartheta)\ge 0$ $\mathbb{P}$-a.s. is in $\mathbf{h}^s L^{\mathrm{sf}}_+$.*

3) *$\mathbf{S}$ is statically viable, and 0 is weakly maximal for $\mathbf{h}^\sigma L^{\mathrm{sf}}$ for each stopping time $\sigma\le T$.*

3′) *$\mathbf{S}$ is statically viable, and 0 is weakly maximal for $\mathbf{h}^s L^{\mathrm{sf}}$ for each deterministic $s\in[0,T)$.*

4) *The zero strategy 0 is weakly maximal for $\mathbf{h}^\sigma L^{\mathrm{sf}}$, for each stopping time $\sigma\le T$.*
5) *Every strategy $\vartheta\in\mathbf{h}^\sigma L^{\mathrm{sf}}$ is maximal for $\mathbf{h}^\sigma L^{\mathrm{sf}}$, for each stopping time $\sigma\le T$.*

Proof. "1) ⇒ 2)": (Static) Viability follows from efficiency. For $\sigma\le T$, take $\vartheta\in\mathbf{h}^\sigma L^{\mathrm{sf}}$ with $\mathbf{V}_T(\vartheta)\ge 0$ $\mathbb{P}$-a.s. As $\mathbf{S}\ge 0$, both $|\vartheta| := (|\vartheta^1|,\dots,|\vartheta^d|)$ and $|\vartheta|+\vartheta$ are in $\mathbf{h}^\sigma L^{\mathrm{sf}}_+\subseteq\mathbf{h}^\tau L^{\mathrm{sf}}_+$ for any stopping time $\tau\le T$ with $\tau\ge\sigma$, and $\mathbf{V}_T(|\vartheta|+\vartheta)\ge\mathbf{V}_T(|\vartheta|)$ $\mathbb{P}$-a.s. Because $|\vartheta|$ is weakly maximal for $\mathbf{h}^\tau L^{\mathrm{sf}}_+$ by static efficiency, we first get $\mathbf{V}_\tau(\vartheta+|\vartheta|)\ge\mathbf{V}_\tau(|\vartheta|)$ and hence $\mathbf{V}_\tau(\vartheta)\ge 0$, $\mathbb{P}$-a.s. So $\vartheta\in\mathbf{h}^\sigma L^{\mathrm{sf}}_+$ because $\tau\ge\sigma$ was arbitrary.

"2) ⇒ 3)": By way of contradiction, suppose 0 is not weakly maximal for $\mathbf{h}^\sigma L^{\mathrm{sf}}$ for some $\sigma\le T$. Then there is $\vartheta\in\mathbf{h}^\sigma L^{\mathrm{sf}}$ with $\mathbf{V}_\sigma(\vartheta)\le 0$ and $\mathbf{V}_T(\vartheta)\ge 0$, $\mathbb{P}$-a.s., where the second inequality is strict with positive probability. Now 2) gives that $\vartheta\in\mathbf{h}^\sigma L^{\mathrm{sf}}_+$, and so 0 fails to be maximal for $\mathbf{h}^\sigma L^{\mathrm{sf}}_+$, contradicting the static viability of $\mathbf{S}$.

"3) ⇒ 3′)" is trivial.

"3′) ⇒ 2′)": Fix $s\in[0,T)$ and take $\vartheta\in\mathbf{h}^s L^{\mathrm{sf}}$ with $\mathbf{V}_T(\vartheta)\ge 0$ $\mathbb{P}$-a.s. If ϑ is not in $\mathbf{h}^s L^{\mathrm{sf}}_+$, right-continuity of the paths of $\mathbf{V}(\vartheta)$ gives some $r\in(s,T)$ such that we have $\mathbb{P}[\mathbf{V}_r(\vartheta)<0]>0$. Let η be a numéraire strategy in $\mathbf{h}^r L^{\mathrm{sf}}_+$ and define $\tilde{\vartheta}\in\mathbf{h}^r L^{\mathrm{sf}}$ by

$$\tilde{\vartheta} := (\vartheta+|V_r(\vartheta)[\mathbf{S}^{(\eta)}]|\,\eta)\mathbb{1}_{\{V_r(\vartheta)[\mathbf{S}^{(\eta)}]<0\}}\mathbb{1}_{[\![r,T]\!]}.$$

Note that by the numéraire invariance (2.9), $\mathbf{V}(\,\cdot\,) = V(\,\cdot\,)[\mathbf{S}^{(\eta)}]\mathbf{V}(\eta)$. Using this together with $V(\eta)[\mathbf{S}^{(\eta)}] \equiv 1$ and $\mathbf{V}_T(\vartheta) \geq 0$ $\mathbb{P}$-a.s., we get $\mathbb{P}$-a.s.

$$\mathbf{V}_r(\tilde{\vartheta}) = 0 \quad \text{and} \quad \mathbf{V}_T(\tilde{\vartheta}) \geq |V_r(\vartheta)[\mathbf{S}^{(\eta)}]|\mathbf{V}_T(\eta)\mathbb{1}_{\{V_r(\vartheta)[\mathbf{S}^{(\eta)}]<0\}}.$$

So $\mathbf{V}_T(\tilde{\vartheta}) \in \mathbf{L}^0_+ \setminus \{0\}$, contradicting the weak maximality of 0 for $\mathbf{h}^r L^{\mathrm{sf}}$.

"2′) ⇒ 2)": Note that the second statement in 2′) trivially also holds for $s = T$. Fix $\sigma \leq T$ and take $\vartheta \in \mathbf{h}^\sigma L^{\mathrm{sf}}$ with $\mathbf{V}_T(\vartheta) \geq 0$ $\mathbb{P}$-a.s. First, let $\tau \geq \sigma$ be of the form $\tau = \sum_{i=1}^n t_i \mathbb{1}_{A_i}$, where $n \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_n \leq T$, and $(A_i)_{i\in\{1,\dots,n\}}$ is a partition of Ω into pairwise disjoint sets $A_i \in \mathcal{F}_{t_i}$. For $i \in \{1,\dots,n\}$, set $\vartheta^{(i)} := \mathbb{1}_{A_i}\vartheta \in \mathbf{h}^\tau L^{\mathrm{sf}} \cap \mathbf{h}^{t_i} L^{\mathrm{sf}}$ so that $\vartheta = \sum_{i=1}^n \vartheta^{(i)}$. As each $\mathbf{V}_{t_i}(\vartheta^{(i)}) \geq 0$ $\mathbb{P}$-a.s. by 2′), we have $\mathbf{V}_\tau(\vartheta) \geq 0$ $\mathbb{P}$-a.s. For general $\tau \geq \sigma$, there is a decreasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times $\leq T$ each taking only finitely many values and such that $\lim_{n\to\infty} \tau_n = \tau$. As each $\mathbf{V}_{\tau_n}(\vartheta) \geq 0$ $\mathbb{P}$-a.s. by the first part of the argument, right-continuity of the paths of $\mathbf{V}(\vartheta)$ yields $\mathbf{V}_\tau(\vartheta) \geq 0$ $\mathbb{P}$-a.s. So $\vartheta \in \mathbf{h}^\sigma L^{\mathrm{sf}}_+$ because $\tau \geq \sigma$ was arbitrary.

"3) ⇒ 4)" is trivial.

"4) ⇒ 5)" follows from "1) ⇒ 3)" in Proposition A.1.

"5) ⇒ 1)" is clear because $\mathbf{h}^\sigma L^{\mathrm{sf}}_+ \subseteq \mathbf{h}^\sigma L^{\mathrm{sf}}$, for each $\sigma \leq T$. □

Remark A.3. The equivalence of 3) and 4) in Proposition A.2 shows that 3) is in fact equivalent to the same statement *without* static viability of $\mathbf{S}$. This is not true for 3′).

B Results from stochastic analysis

This appendix collects some auxiliary results used in the body of the paper.

Proposition B.1. *Let X be a nonnegative supermartingale on $[0,T]$, not assumed to be right-continuous. Then there exists a unique RCLL local martingale L which is maximal below X in the sense that $L_t \leq X_t$ $\mathbb{P}$-a.s., $0 \leq t \leq T$, and $L \geq \tilde{L}$ $\mathbb{P}$-a.s. for any other RCLL local martingale $\tilde{L}$ with $\tilde{L}_t \leq X_t$ $\mathbb{P}$-a.s., $0 \leq t \leq T$. Moreover, L is nonnegative and satisfies $L_T = X_T$ $\mathbb{P}$-a.s.*

Proof. Uniqueness of L follows directly from maximality. For existence, [10, Theorem VI.2] yields a nonnegative RCLL supermartingale Z on $[0,T]$ with $Z_t = \lim_{q\downarrow\downarrow t} X_q$ $\mathbb{P}$-a.s., $t \in [0,T)$, where the limit can be taken along rational numbers $q > t$, and $Z_T = X_T$ $\mathbb{P}$-a.s. Moreover, right-continuity of $(\mathcal{F}_t)$ and [10, Theorem VI.2] also give $Z_t \leq X_t$ $\mathbb{P}$-a.s., $0 \leq t \leq T$. So any right-continuous process R satisfies $R_t \leq X_t$ $\mathbb{P}$-a.s., $0 \leq t \leq T$, if and only if $R \leq Z$ $\mathbb{P}$-a.s. Let $Z = M - A$ be the Doob–Meyer decomposition of Z into a local martingale M and an increasing locally integrable process A null at 0, both RCLL. As $Z \geq 0$ $\mathbb{P}$-a.s., M is nonnegative and thus a supermartingale. So M_T and hence also $A_T \leq M_T$ are in L^1, and the RCLL process L defined by $L_t := M_t - \mathbb{E}[A_T \,|\, \mathcal{F}_t]$ is a local martingale with $L_T = Z_T$ $\mathbb{P}$-a.s. To show that L is maximal below X, we note first that

$$Z_t - L_t = \mathbb{E}[A_T \,|\, \mathcal{F}_t] - A_t = \mathbb{E}[A_T - A_t \,|\, \mathcal{F}_t] \geq 0 \qquad \mathbb{P}\text{-a.s.}, 0 \leq t \leq T,$$

gives $L \leq Z$ $\mathbb{P}$-a.s. If $\tilde{L}$ is any RCLL local martingale with $\tilde{L} \leq Z$ $\mathbb{P}$-a.s., then

$$L_t - \tilde{L}_t \geq L_t - Z_t = A_t - \mathbb{E}[A_T \,|\, \mathcal{F}_t] \geq -\mathbb{E}[A_T \,|\, \mathcal{F}_t] \qquad \mathbb{P}\text{-a.s.}, 0 \leq t \leq T.$$

So $L - \tilde{L} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P})$ is bounded below by a uniformly integrable martingale and hence a supermartingale. As $L_T - \tilde{L}_T = Z_T - \tilde{L}_T \geq 0$ $\mathbb{P}$-a.s., we get $L - \tilde{L} \geq 0$, so $\tilde{L} \leq L$ $\mathbb{P}$-a.s. □

The following corollary follows immediately from the proof of Proposition B.1 and the uniqueness of the Doob–Meyer decomposition.

Corollary B.2. *For $i = 1, 2$, let X^i be nonnegative supermartingales on $[0, T]$, which are not assumed to be right-continuous, and L^i the corresponding maximal RCLL local martingales below X^i. Then $L^1 + L^2$ is the maximal local martingale below $X^1 + X^2$.*

Proposition B.3. *On $[0, T]$, let X be a nonnegative right-continuous supermartingale that is not of class (D) and Y a nonnegative adapted RCLL process with $\inf_{0 \leq t \leq T} \mathbb{E}[Y_t] > 0$. If X and Y are independent, then the product XY is not of class (D).*

Proof. The stopping times $\tau_n := \inf\{t \geq 0 : X_t \geq n\}$, $n \in \mathbb{N}$, are independent of Y because X is. Next, $\tau_n \nearrow \infty$ $\mathbb{P}$-a.s. gives $\lim_{n\to\infty} X_{\tau_n} Y_{\tau_n} \mathbb{1}_{\{\tau_n < \infty\}} = 0$ $\mathbb{P}$-a.s., and so it suffices to show that $\liminf_{n\to\infty} \mathbb{E}[X_{\tau_n} Y_{\tau_n} \mathbb{1}_{\{\tau_n<\infty\}}] > 0$. Now $\liminf_{n\to\infty} \mathbb{E}[X_{\tau_n} \mathbb{1}_{\{\tau_n<\infty\}}] > 0$ by the Johnson–Helms criterion in [10, Theorem VI.25] because X is not of class (D), and combining this with the independence of Y and (X, τ_n), we get first $\mathbb{E}[Y_{\tau_n} \mid X, \tau_n] = \mathbb{E}[Y_t]|_{t=\tau_n}$ and hence, as desired,

$$\liminf_{n\to\infty} \mathbb{E}[X_{\tau_n} Y_{\tau_n} \mathbb{1}_{\{\tau_n<\infty\}}] \geq \inf_{0 \leq t \leq T} \mathbb{E}[Y_t] \liminf_{n\to\infty} \mathbb{E}[X_{\tau_n} \mathbb{1}_{\{\tau_n<\infty\}}] > 0.$$

□

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