# The Z*-theorem for compact Lie groups


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Glauberman's classical $Z^*$-theorem is a theorem about involutions of finite groups (i.e. elements of order 2). It is one of the important ingredients for the classification of finite simple groups, which in turn allows to prove the corresponding theorem for elements of arbitrary prime order $p$. Let us recall the statement: if $G$ is a finite group with a Sylow $p$-subgroup $P$, and if $x$ is an element of $P$ of order $p$ such that no other $G$-conjugate of $x$ lies in $P$, then the image of $x$ in $G/O_{p'}(G)$ is central, where $O_{p'}(G)$ denotes the maximal normal subgroup of $G$ of order prime to $p$. The symbol $Z^*(G)$ is the classical notation for the inverse image in $G$ of the centre of $G/O_{p'}(G)$ and this explains the name of the theorem.

One can restate the assumption on $x$ in terms of control of fusion. For an arbitrary group $G$ and a prime $p$, we say that a subgroup $H$ of $G$ *controls finite $p$-fusion* in $G$ if the following two conditions are satisfied:

(a) every finite $p$-subgroup of $G$ is conjugate to a subgroup of $H$,

(b) if $A$ is a finite $p$-subgroup of $H$ and if $A^g$ is also a subgroup of $H$ for some $g \in G$, then $g = ch$ for some $h \in H$ and $c$ in the centralizer $C_G(A)$ of $A$ in $G$.

This notion is equivalent to the requirement that the inclusion $H \to G$ induces an equivalence between the categories of finite $p$-subgroups (in a suitable sense, see Section 1). The assumption on $x$ in the $Z^*$-theorem is then equivalent (at least for finite groups and more generally for compact Lie groups) to the condition that the centralizer $C_G(x)$ controls finite $p$-fusion in $G$ (see Proposition 1.8 below). Also the conclusion of the $Z^*$-theorem is readily seen to be equivalent to the equation $G = C_G(x) \cdot O_{p'}(G)$ (see Lemma 2.3 below).

It is easy to see (see for instance [Br, Proposition 4]) that one can also restate the $Z^*$-theorem using the centralizer $C_G(A)$ of an *arbitrary* $p$-subgroup $A$. The theorem becomes stronger because the centralizer of a larger subgroup is smaller. Explicitly the statement is now the following.

$Z^*$-THEOREM FOR FINITE GROUPS. *Let $G$ be a finite group and $p$ a prime. Let $A$ be a $p$-subgroup of $G$ and assume that $C_G(A)$ controls (finite) $p$-fusion in $G$. Then*

$$G = C_G(A) \cdot O_{p'}(G).$$

The purpose of this paper is to show that the $Z^*$-theorem holds for compact Lie groups. Recall that a $p$-toral group is a compact Lie group $A$ whose connected component $A^0$ is a torus and whose component group $A/A^0$ is a (finite) $p$-group. Moreover recall that a (not necessarily finite) $p$-group is a group in which every element has order a power of $p$. Our main result takes the following form.

$Z^*$-THEOREM FOR COMPACT LIE GROUPS. *Let $G$ be a compact Lie group and $p$ a prime. Let $A$ be either a (not necessarily finite) $p$-subgroup or a $p$-toral subgroup of $G$ and assume that $C_G(A)$ controls finite $p$-fusion in $G$. Then*

$$G = C_G(A) \cdot [A, G] = C_G(A) \cdot O_{p'}(G).$$

*Moreover $[A, G]$ is a finite normal $p'$-subgroup of $G$. In particular the connected component $G^0$ of $G$ centralizes $A$.*

Here $[A, G]$ denotes the subgroup generated by the commutators $[a, g] = a^{-1}g^{-1}ag$ for $a \in A$, $g \in G$. This is always a normal subgroup of $G$ because $[a, g]^h = [a, h]^{-1}[a, gh]$.

Note that $C_G(A)$ is clearly a closed subgroup of $G$ but $O_{p'}(G)$ need not be closed (since for instance it is dense in $G$ when $G$ is the circle group). However $O_{p'}(G)$ is totally disconnected (since otherwise it would contain a 1-dimensional Lie group), and therefore its intersection with the connected component $G^0$ of $G$ is necessarily central in $G^0$. Thus if $G^0$ is a semi-simple Lie group, it has finite centre and consequently $O_{p'}(G)$ is finite in that case (hence closed).

Our interest in the questions considered in this paper arose from a recent theorem of the first author [Mi1] giving a cohomological criterion for the control of finite $p$-fusion (see Theorem 1.1 below). However we do not need the full strength of this result here. In fact only the easy implication of the theorem is used, namely (a restatement of) the classical result allowing to compute mod-$p$ cohomology using stable elements.

In view of the fundamental importance of the $Z^*$-theorem for finite groups, we hope that its generalization can shed some new light on compact Lie group theory, in particular on the cohomology of these groups. For instance, using the full strength of the theorem of Mislin mentioned above, we prove that a morphism of compact Lie groups $f : H \to G$ which induces a mod-$p$ cohomology isomorphism induces an isomorphism between Sylow $p$-subgroups of $H$ and $G$, as well as Sylow $p$-subgroups of $Z^*(H)$ and $Z^*(G)$ (but only the second case requires the $Z^*$-theorem). For another application of the theorem, we refer the reader to [Mi2], where it is proved that the Dwyer-Wilkerson center $ZH^*(BG, \mathbb{Z}/p\mathbb{Z})$ of the cohomology ring is isomorphic to the group of elements of order $p$ in $Z(G/O_{p'}(G))$.

In the first section of this paper, we give other definitions of control of fusion using either the category of all $p$-subgroups or the category of all $p$-toral subgroups. We prove that they are all equivalent for compact Lie groups.

For a connected group, or more generally for a group whose component group is a $p$-group, we give a direct proof of the $Z^*$-theorem. For the general result however, we use a reduction to the case of finite groups. Thus we need the $Z^*$-theorem for finite groups, but we did not succeed in finding a suitable reference for this theorem (although the result is well known to finite group theorists). It is quoted explicitly (but without proof) in [Pu, Théorème 1.3]. Of course for $p = 2$ this is Glauberman's theorem (see for instance [CR, §63C]). For odd $p$, there is a reduction to the case of simple groups in [Br] and then the proof essentially consists in a direct inspection of the list given by the classification of finite simple groups.

As long as no direct proof of the $Z^*$-theorem exists for finite groups and odd primes, our result unfortunately depends on the classification of finite simple groups. We regret to be reduced to adopting this approach, but fortunately our fellow countryman Armand Borel, facing a similar situation [Bo], has paved the way for a decent excuse by quoting G.B. Shaw:

"You have a low shopkeeping mind. You think of things that would never come into a gentleman's head."

"That's the Swiss national character, dear lady."

## 1. Frobenius categories.

Following Puig [Pu], we define for an arbitrary group $G$ and a prime $p$ the *Frobenius category* $\mathrm{Frob}_p(G)$ as follows. Its objects are the finite $p$-subgroups of $G$ and its morphisms are the group homomorphisms induced by conjugation by some element of $G$. Thus the set of morphisms from $P$ to $Q$ is equal to

$$\mathrm{Mor}(P,Q) = C_G(P)\backslash T_G(P,Q) \qquad \text{where } \; T_G(P,Q) = \{g \in G \mid P^g \subseteq Q\}.$$

In particular the set of endomorphisms of $P$ is the group $N_G(P)/C_G(P)$ (and every endomorphism is an automorphism). The conjugation by an element $g \in G$ will be written $\mathrm{Inn}(g) : x \mapsto x^g = g^{-1}xg$, and $P^g = g^{-1}Pg$.

Any group homomorphism $f : H \to G$ induces a functor $f_* : \mathrm{Frob}_p(H) \to \mathrm{Frob}_p(G)$. When $f$ is the inclusion of a subgroup $H$ in $G$ (which is the only case we consider in this paper), then $f_*$ is an equivalence of categories if and only if $H$ controls finite $p$-fusion in $G$. Indeed condition (a) in the definition of the introduction means that any object of $\mathrm{Frob}_p(G)$ is isomorphic to an object of $\mathrm{Frob}_p(H)$, and condition (b) states that $f_*$ is full (while it is clearly always faithful).

For a finite group $G$, it follows from the description of the cohomology $H^*(G, \mathrm{Z}/p\mathrm{Z})$ in terms of stable elements in the cohomology of a Sylow $p$-subgroup that the restriction $H^*(G, \mathrm{Z}/p\mathrm{Z}) \to H^*(H, \mathrm{Z}/p\mathrm{Z})$ is an isomorphism if $H$ controls (finite) $p$-fusion in $G$. For compact Lie groups, the same result holds for the cohomology $H^*(BG, \mathrm{Z}/p\mathrm{Z})$ by [FM, Theorem 2.3]. The main result of [Mi1] asserts that the converse also holds. (The Frobenius categories are called Quillen categories in [Mi1].) We quote the full result for completeness, although we shall only use the easy part already mentioned for the proof of the $Z^*$-theorem.

(1.1) THEOREM. *Let $f : H \to G$ be a morphism of compact Lie groups. Then $f$ induces an equivalence $f_* : \mathrm{Frob}_p(H) \to \mathrm{Frob}_p(G)$ if and only if the map of classifying spaces $Bf : BH \to BG$ induces an isomorphism in mod-$p$ cohomology.*

It is convenient to introduce also two other categories of subgroups: the category $\mathcal{S}_p(G)$ of all $p$-subgroups of $G$ and, in case $G$ is a compact Lie group, the category $\mathcal{T}_p(G)$ of all $p$-toral subgroups of $G$. In both cases the morphisms are the group homomorphisms induced by conjugation by an element of $G$. The maximal elements of $\mathcal{S}_p(G)$ (viewed as a poset) are called Sylow $p$-subgroup of $G$ (and they always exist by Zorn's lemma). For a compact Lie group, the advantage of those two categories compared to $\mathrm{Frob}_p(G)$ is that they have maximal elements which are all conjugate (i.e. weak terminal objects).

(1.2) LEMMA. *Let $G$ be a compact Lie group.*

*(a) All Sylow $p$-subgroups of $G$ are conjugate.*

*(b) All maximal $p$-toral subgroups of $G$ are conjugate. The connected component of a maximal $p$-toral subgroup is a maximal torus of $G$.*

*(c) The closure of a $p$-subgroup of $G$ is a $p$-toral subgroup of $G$.*

*(d) A Sylow $p$-subgroup of a $p$-toral subgroup $P$ is dense in $P$.*

*Proof.* (b) is proved in [JMO, Lemma A.1]. If $A$ is a $p$-subgroup of $G$, then by [We, 9.4], $A$ contains an abelian normal subgroup $B$ of finite index (because any compact Lie group is a linear group). Therefore the closure of $B$ is an abelian compact Lie group, thus a direct product of a torus and a finite abelian group (a $p$-group in our case). It follows that the closure $\overline{A}$ of $A$ is $p$-toral, proving (c). By (b), we know that $\overline{A}$ is contained in the normalizer $N$ of a maximal torus. By [Fe, Corollary 1.5], all maximal torsion subgroups of $N$ are conjugate. Thus if $U$ and $V$ are two Sylow $p$-subgroups of $G$, they are conjugate to subgroups of $N_t$ where $N_t$ denotes some fixed maximal torsion subgroup of $N$. But within $N_t$ all Sylow $p$-subgroups are conjugate (cf [We, 9.10]). This completes the proof of (a). From (a) and (b) it is clear that every Sylow $p$-subgroup of a $p$-toral group $P$ contains the $p$-torsion subgroup of the torus $P^0$; but that subgroup is dense in $P^0$ and (d) follows. □

The second fact which will be often used is the following.

(1.3) LEMMA. *Let $A$ be a $p$-subgroup of a compact Lie group $G$ and denote by $\overline{A}$ the closure of $A$. Then there exists a countable increasing sequence of finite $p$-subgroups $A_i$ of $A$ such that $\bigcup_i A_i = A$. Moreover $C_G(A_i) = C_G(A) = C_G(\overline{A})$ for $i$ sufficiently large.*

*Proof.* Since $\overline{A}$ is $p$-toral, the torsion elements of $(\overline{A})^0$ form a countable subgroup; thus $A$ is countable. By a result of Schur, since $A$ is a linear torsion group, it is locally finite (cf [We, 4.9]), and it follows that $A = \bigcup_i A_i$ for a suitable increasing sequence of finite subgroups $\{A_i\}$. Clearly $C_G(A_i) = C_G(A) = C_G(\overline{A})$ for $i$ sufficiently large. □

The concept of control of fusion extends immediately to our new categories $\mathcal{S}_p(G)$ and $\mathcal{T}_p(G)$. We shall say that a subgroup $H$ of $G$ *controls $p$-fusion* (respectively *controls $p$-toral fusion*) in $G$ if the inclusion $\mathcal{S}_p(H) \to \mathcal{S}_p(G)$ (respectively $\mathcal{T}_p(H) \to \mathcal{T}_p(G)$) is an equivalence of categories. The reader can easily rewrite this definition with two conditions (a) and (b) as in the case of the control of finite $p$-fusion. If $\mathcal{C}$ denotes either of the three categories $\mathrm{Frob}_p(G)$, $\mathcal{S}_p(G)$ or $\mathcal{T}_p(G)$, we shall also say that $H$ controls fusion in $\mathcal{C}$ to refer to one of the three types of control of fusion.

(1.4) PROPOSITION. *Let $G$ be a compact Lie group and $H$ a closed subgroup of $G$. The following conditions are equivalent.*

*(a) $H$ controls finite $p$-fusion in $G$.*

*(b) $H$ controls $p$-fusion in $G$.*

*(c) $H$ controls $p$-toral fusion in $G$.*

*Proof.* Since any finite $p$-subgroup is a $p$-toral group, it is clear that (c) implies (a). To see that (a) implies (b), write a $p$-subgroup $P$ as a countable union of finite $p$-subgroups $P = \bigcup_i P_i$ (Lemma 1.3). Since $H$ controls finite $p$-fusion in $G$, there exists $g_i \in G$ such that $P_i^{g_i} \leq H$. Since $G$ is compact, we can pass to a subsequence and assume that $(g_i)$ converges to some $g \in G$. Then any element of $P^g$ can be approximated by an element of $P_i^{g_i}$ and since $H$ is closed, it follows that $P^g \leq H$. Now suppose that $P$ and $P^g$ are both subgroups of $H$, for some $g \in G$. Then since $H$ controls finite $p$-fusion in $G$, there exists $h_i \in H$ and $c_i \in C_G(P_i)$ such that $g = c_i h_i$. By Lemma 1.3, there exists $i$ such that $C_G(P_i) = C_G(P)$. Thus $g = ch$ with $c \in C_G(P)$ and $h \in H$. This completes the proof of (b). Finally we prove that (b) implies (c) by a continuity argument: if $Q$ is a $p$-toral subgroup of $G$, choose a dense $p$-subgroup $P$ in $Q$. Since $H$ controls $p$-fusion in $G$, there exists $g \in G$ such that $P^g \leq H$. Then $Q^g \leq H$ because $H$ is closed. Similarly if both $Q$ and $Q^g$ are subgroups of $H$, for some $g \in G$, then since $H$ controls $p$-fusion in $G$, there exists $h \in H$ and $c \in C_G(P)$ such that $g = ch$. But $C_G(P) = C_G(Q)$ and thus $c \in C_G(Q)$. □

For any category $\mathcal{C}$ of subgroups which is closed under conjugacy (such as $\mathrm{Frob}_p(G)$, $\mathcal{S}_p(G)$ or $\mathcal{T}_p(G)$), we shall say that a subgroup $A \in \mathcal{C}$ is *isolated* in $\mathcal{C}$ if for each object $P \in \mathcal{C}$, there is at most one morphism from $A$ to $P$. Translating this condition, we see that $A$ is isolated in $\mathcal{C}$ if and only if for every $h, g \in G$ such that $< A^h, A^g >$ is contained in a subgroup in $\mathcal{C}$, the element $gh^{-1}$ centralizes $A$. Here $< A^h, A^g >$ denotes the subgroup generated by $A^h$ and $A^g$. Conjugating by $h^{-1}$ and replacing $gh^{-1}$ by $g$, we see that actually $A$ is isolated in $\mathcal{C}$ if and only if whenever $< A, A^g >$ is contained in a subgroup in $\mathcal{C}$, then $g \in C_G(A)$. When $\mathcal{C} = \mathrm{Frob}_p(G)$, if an isolated subgroup $A$ is generated by a single element $x$, finite group theorists often say that $x$ is *weakly closed* in a Sylow $p$-subgroup. This condition corresponds to the assumption of the classical statement of the $Z^*$-theorem. Any central subgroup belonging to $\mathcal{C}$ is isolated in $\mathcal{C}$. Also if an isolated subgroup $A$ is contained in a group $P \in \mathcal{C}$, then the definition immediately implies that $A$ is central in $P$.

It is obvious that if $A \in \mathrm{Frob}_p(G)$ is isolated in $\mathcal{S}_p(G)$, then $A$ is isolated in $\mathrm{Frob}_p(G)$. We now show that for a compact Lie group, the converse holds.

(1.5) LEMMA. *Let $G$ be a compact Lie group and let $A \in \mathrm{Frob}_p(G)$. Then $A$ is isolated in $\mathrm{Frob}_p(G)$ if and only if $A$ is isolated in $\mathcal{S}_p(G)$.*

*Proof.* Assume $A$ is isolated in $\mathrm{Frob}_p(G)$. Let $g \in G$ be such that $P =< A, A^g >$ is a $p$-group. In order to prove that $g$ centralizes $A$, it suffices to show that $P$ is finite and then apply the assumption. As observed earlier, $P$ is locally finite since it is a linear torsion group [We, 4.9]. But $P$ is finitely generated, hence finite. □

Now we come to the link between the definition of isolated subgroups and control of fusion.

(1.6) LEMMA. *Let $G$ be an arbitrary group, let $\mathcal{C}$ be any of* $\mathrm{Frob}_p(G)$, $\mathcal{S}_p(G)$ *or* $\mathcal{T}_p(G)$ *(with $G$ a compact Lie group for the latter case) and let $A \in \mathcal{C}$. If $C_G(A)$ controls fusion in $\mathcal{C}$, then $A$ is isolated in $\mathcal{C}$.*

*Proof.* The argument in the three cases is the same. Suppose $< A, A^g > \subseteq P$ where $P \in \mathcal{C}$. By control of fusion, there exists $x \in G$ such that $P^x \subseteq C_G(A)$. For $a \in A$, we have $a^x, a^{gx} \in C_G(A) \subseteq C_G(a)$ and also $a \in C_G(a)$. Clearly $C_G(a)$ also controls fusion (because $C_G(A) \leq C_G(a)$) and applying this to the morphism $\mathrm{Inn}(x) : < a > \to < a^x >$, we obtain $x \in C_G(a)$. Similarly $gx \in C_G(a)$ and therefore $g \in C_G(a)$. This holds for all $a \in A$, showing that $g \in C_G(A)$. Thus $A$ is isolated in $\mathcal{C}$. □

When all maximal elements of our category are conjugate, the converse of Lemma 1.6 holds. We only give the argument for compact Lie groups.

(1.7) LEMMA. *Let $G$ be a compact Lie group.*

*(a) Let $A \in \mathcal{S}_p(G)$. If $A$ is isolated in $\mathcal{S}_p(G)$, then $C_G(A)$ controls $p$-fusion.*

*(b) Let $A \in \mathcal{T}_p(G)$. If $A$ is isolated in $\mathcal{T}_p(G)$, then $C_G(A)$ controls $p$-toral fusion.*

*Proof.* Let $\mathcal{C}$ be either $\mathcal{S}_p(G)$ or $\mathcal{T}_p(G)$ and let $P \in \mathcal{C}$. Since all maximal elements of $\mathcal{C}$ are conjugate (Lemma 1.2), there exists $g \in G$ such that $A$ and $P^g$ lie in such a maximal element $Q$. Since $A$ is isolated, it follows that $Q$ centralizes $A$. Therefore $P^g \subseteq C_G(A)$, proving the first condition for control of fusion.

Now suppose that $P, P^g \leq C_G(A)$ for some $g \in G$. Thus we have $A, A^{g^{-1}} \subseteq C_G(P)$. But $C_G(P)$ is a compact Lie group, so all its Sylow $p$-subgroups (respectively maximal $p$-toral subgroups) are conjugate. Therefore there exists $c \in C_G(P)$ such that $A$ and $A^{g^{-1}c}$ lie in such a maximal element. Since $A$ is isolated, $g^{-1}c$ centralizes $A$. Therefore $g \in C_G(P) \cdot C_G(A)$, proving the second condition for control of fusion. □

Collecting the results above, we obtain the following proposition.

(1.8) PROPOSITION. *Let $G$ be a compact Lie group and let $A$ be either a $p$-subgroup or a $p$-toral subgroup of $G$. Then the following conditions are equivalent.*

*(a)* $C_G(A)$ *controls finite $p$-fusion.*

*(b)* $C_G(A)$ *controls $p$-fusion.*

*(c)* $C_G(A)$ *controls $p$-toral fusion.*

*If $A$ is finite, then these conditions are also equivalent to the following ones.*

*(d)* $A$ *is isolated in* $\mathrm{Frob}_p(G)$.

*(e)* $A$ *is isolated in* $\mathcal{S}_p(G)$.

*(f)* $A$ *is isolated in* $\mathcal{T}_p(G)$.

In order to be able to use the second set of conditions, recall that by Lemma 1.3 one can always replace $A$ by a finite subgroup without changing its centralizer.

The following corollary will be crucial in the proof of the $Z^*$-theorem.

(1.9) COROLLARY. *Let $H$ be a closed subgroup of a compact Lie group $G$ and let $A \leq H$ be a $p$-subgroup or a $p$-toral subgroup. If $C_G(A)$ controls finite $p$-fusion in $G$, then $C_H(A)$ controls finite $p$-fusion in $H$.*

*Proof.* We first replace $A$ by a finite $p$-subgroup $B$ of $A$ such that $C_G(A) = C_G(B)$ and $C_H(A) = C_H(B)$. If $C_G(B)$ controls finite $p$-fusion in $G$, then $B$ is isolated in $\mathrm{Frob}_p(G)$ and therefore $B$ is also isolated in the subcategory $\mathrm{Frob}_p(H)$. But by Proposition 1.8, this implies that $C_H(B)$ controls finite $p$-fusion in $H$. □

## 2. Proof of the $Z^*$-theorem.

We first treat the following special case.

(2.1) PROPOSITION. *Let $G$ be a compact Lie group and assume that $G/G^0$ is a $p$-group. If $A$ is a $p$-group or a $p$-toral subgroup of $G$ such that $C_G(A)$ controls finite $p$-fusion in $G$, then $C_G(A) = G$, that is, $A$ is central in $G$.*

*Proof.* By Lemma 1.3 we can assume that $A$ is a finite $p$-group. Write $K = C_G(A)$. Since $G/G^0$ is a $p$-group, $K/K^0$ is a (finite) $p$-group too (cf. [Ad, Lemma 7.1] or [JMO, Proposition A.4]). By Theorem 1.1, the induced map $BK \to BG$ is a mod-$p$ cohomology isomorphism. First we claim that $K$ covers the quotient $G/G^0$. Otherwise $K \cdot G^0$ is contained in a maximal subgroup $M$ of $G$ of index $p$. Let $\alpha : G/G^0 \to \mathrm{Z}/p$ be a homomorphism with kernel $M/G^0$. Then $\alpha \in H^1(G/G^0, \mathrm{Z}/p)$ is non-trivial but its restriction to $K/K^0$ is trivial. Inflating this to the cohomology of $BG$ and $BK$ (inflation is injective for $H^1$), we obtain a non-trivial element in the kernel of the restriction from $BG$ to $BK$, against our assumption. Thus $K/K^0 \cong G/G^0$ as claimed.

Now we only have to prove that $K^0 = G^0$. Note that the fibration $G \to EG \to BG$ implies that the group $\pi = \pi_1(BG)$ is isomorphic to $\pi_0(G) = G/G^0$, and this is a $p$-group by assumption. Similarly $\pi_1(BK) \cong \pi_0(K) = K/K^0$ and so by the first part of the proof, the inclusion $K \to G$ induces an isomorphism $\pi_1(BK) \cong \pi_1(BG) = \pi$. If $M$ is any finitely generated $\mathrm{Z}/p[\pi]$-module, then $M$ has a finite filtration by submodules $M_i$ (with $1 \le i \le n$) such that the quotients $M_i/M_{i+1}$ are trivial $\mathrm{Z}/p[\pi]$-modules; indeed the trivial module $\mathrm{Z}/p$ is the only simple $\mathrm{Z}/p[\pi]$-module since $\pi$ is a $p$-group. Now we claim that the map $H^*(BG, M) \to H^*(BK, M)$ of cohomology with local coefficients is an isomorphism; indeed since $BK \to BG$ induces an isomorphism in cohomology with trivial coefficients by Theorem 1.1, the claim follows by induction on the length of the filtration, using the long exact sequence of cohomology associated to the sequence $0 \to M_1 \to M \to M/M_1 \to 0$. Now we wish to apply this to the free module $\mathrm{Z}/p[\pi] = \mathrm{Ind}_1^\pi(\mathrm{Z}/p)$. By Shapiro's lemma,

$$H^*(BG, \mathrm{Z}/p[\pi]) \cong H^*(BG^0, \mathrm{Z}/p)\,, \quad \text{and similarly} \quad H^*(BK, \mathrm{Z}/p[\pi]) \cong H^*(BK^0, \mathrm{Z}/p)\,.$$

It follows that the map $H^*(BG^0, \mathrm{Z}/p) \to H^*(BK^0, \mathrm{Z}/p)$ is an isomorphism. But since $G^0 \simeq \Omega(BG^0)$ and $K^0 \simeq \Omega(BK^0)$ and since the spaces $BG^0$ and $BK^0$ are simply connected, the map $H^*(G^0, \mathrm{Z}/p) \to H^*(K^0, \mathrm{Z}/p)$ is an isomorphism too. Now $K^0$ and $G^0$

are compact orientable manifolds, so one can conclude that they have the same dimension. It follows that $K^0 = G^0$ since they are both connected. Therefore $K = G$ and the proof is complete. □

For the proof of the general case, we will also need the following lemma. Although it is certainly well known, we provide a proof for the convenience of the reader.

(2.2) LEMMA. *Let $G$ be a compact Lie group and $A$ a finite $p$-subgroup of $G$. Then there exists a finite subgroup $F$ containing $A$ which maps onto $G/G^0$.*

*Proof.* As observed in Lemma 1.2, $A$ normalizes a maximal torus $T$ of $G$. It is well known that the normalizer $N = N_G(T)$ maps onto $G/G^0$ and that any maximal torsion subgroup of $N$ is dense in $N$. Hence we can choose a torsion subgroup $N_t$ of $N$ containing $A$ and mapping onto $G/G^0$. Since $N_t$ is locally finite (being a linear torsion group [We, 4.9]), and since $A$ and $G/G^0$ are finite, we can find a finite subgroup $F$ of $N_t$ containing $A$ and mapping onto $G/G^0$. □

We also need the following result which was partially mentioned in the introduction. It shows the equivalence between several forms of the conclusion of the $Z^*$-theorem.

(2.3) LEMMA. *Let $A$ be a $p$-subgroup of a finite group $G$. The following conditions are equivalent.*

*(a) The image of $A$ in $G/O_{p'}(G)$ is central.*

*(b) $G = C_G(A) \cdot O_{p'}(G)$.*

*(c) $G = C_G(A) \cdot [A, G]$ and $[A, G]$ is a $p'$-group.*

*Proof.* It is obvious that (c) implies (b) and that (b) implies (a). Assume now (a). Then clearly $[A, G] \subseteq O_{p'}(G)$ so that $N = [A, G]$ is a $p'$-group. Let $\pi : G \to G/N$. We first show that $G = N_G(A) \cdot N$. Let $g \in G$. Since $\pi(g)$ centralizes $\pi(A)$, we have

$$A^g \subseteq \pi^{-1}(\pi(A)) = N \cdot A\,.$$

Since both $A$ and $A^g$ are Sylow $p$-subgroups of $N \cdot A$, we have $A^g = A^n$ for some $n \in N$ and therefore $gn^{-1} \in N_G(A)$. Now we show that $N_G(A) = C_G(A)$. If $h \in N_G(A)$ then for each $a \in A$, the commutator $[a, h]$ belongs to $A$. But this commutator also belongs to $N$ and since $A \cap N = 1$, it follows that $h$ centralizes $a$. □

*Proof of the $Z^*$-theorem.* By Lemma 1.3, we can choose a finite $p$-subgroup $B$ of $A$ such that $C_G(B) = C_G(A)$. The image of $B$ in $G/G^0$ is a $p$-group, whose inverse image in $G$ is a compact Lie group $K$ with component group a $p$-group. Since $C_G(B)$ controls finite $p$-fusion in $G$ by assumption, $C_K(B)$ controls finite $p$-fusion in $K$ by Corollary 1.9, and by Proposition 2.1 we obtain that $B$ is central in $K$, and in particular $G^0$ centralizes $B$.

Choose now a finite subgroup $F$ mapping onto $G/G^0$ and containing $B$ (Lemma 2.2). By Corollary 1.9 again, $C_F(B)$ controls (finite) $p$-fusion in $F$. By the $Z^*$-theorem for finite groups, it follows that we have $F = C_F(B) \cdot O_{p'}(F)$ and therefore by Lemma 2.3, $F = C_F(B) \cdot [B, F]$ and $[B, F]$ is a $p'$-group. But since $G = G^0 \cdot F = G^0 \cdot C_F(B) \cdot [B, F]$ and since $G^0$ centralizes $B$, we conclude that $G = C_G(B) \cdot [B, F]$. Note that $[B, F] = [B, G]$ because $G = G^0 \cdot F$ and $G^0$ centralizes $B$. Thus $G = C_G(B) \cdot [B, G]$, and a fortiori $G = C_G(A) \cdot [A, G]$ and $G = C_G(A) \cdot O_{p'}(G)$ since $[B, G]$ is contained in both $[A, G]$ and $O_{p'}(G)$. This completes the proof of the main statement of the $Z^*$-theorem.

It remains to show that $[A, G]$ is a finite $p'$-group. Let $\overline{A}$ be the closure of $A$. Since $C_G(A) = C_G(\overline{A})$ we have $G = C_G(\overline{A}) \cdot O_{p'}(G)$ and so $[\overline{A}, G]$ lies in $O_{p'}(G)$. Thus $[\overline{A}, G]$ is totally disconnected and therefore $[\overline{A}^0, G]$ and $[\overline{A}, G^0]$ are trivial groups (because $[\overline{A}^0, g]$ and $[a, G^0]$ are connected, hence trivial, for all $g \in G$ and $a \in \overline{A}$). If $U$ and $V$ are finite subgroups such that $\overline{A} = \overline{A}^0 \cdot U$ and $G = G^0 \cdot V$, then by using standard rules for expanding commutators we obtain $[\overline{A}, G] = [U, V]$. It follows that $[\overline{A}, G]$ is finitely generated. Since it is also a subgroup of the torsion group $O_{p'}(G)$, we apply once again Schur's result [We, 4.9] to deduce that $[\overline{A}, G]$ and its subgroup $[A, G]$ are finite. □

For a compact Lie group $G$, let $Z^*(G)$ be the inverse image in $G$ of $Z(G/O_{p'}(G))$. Since there is a unique maximal torsion subgroup in $Z^*(G)$ (the inverse image of the torsion subgroup of the abelian group $Z(G/O_{p'}(G))$ ) and since all Sylow $p$-subgroups of this torsion subgroup are conjugate [We, 9.10], all Sylow $p$-subgroups of $Z^*(G)$ are conjugate. We denote by $Z^*(G)_p$ an arbitrary Sylow $p$-subgroup of $Z^*(G)$. It is not difficult to show (using arguments similar to those of Lemma 2.3) that any subgroup of $Z^*(G)_p$ is an isolated

$p$-subgroup of $G$ (i.e. isolated in $\mathcal{S}_p(G)$). Since conversely the $Z^*$-theorem asserts that any isolated $p$-subgroup of $G$ is contained in $Z^*(G)$, we see that $Z^*(G)_p$ and its conjugates are precisely the maximal isolated $p$-subgroups of $G$.

Denote by $G_p$ a Sylow $p$-subgroup of $G$. We now combine Theorem 1.1 (this time using its full strength) with the results of the present paper.

(2.4) COROLLARY. *Suppose that $f : H \to G$ is a morphism of compact Lie groups inducing a mod-p cohomology isomorphism. Then $f$ induces isomorphisms $H_p \cong G_p$ and $Z^*(H)_p \cong Z^*(G)_p$.*

*Proof.* By Theorem 1.1, $f$ induces an equivalence of categories $\mathrm{Frob}_p(H) \to \mathrm{Frob}_p(G)$. This easily implies that the restriction of $f$ to $H_p$ is injective and that $f(H_p)$ is a Sylow $p$-subgroup $G_p$ of $G$; therefore $H_p \cong G_p$. Now the inclusion $f(H) \to G$ also induces a mod-$p$ cohomology isomorphism, so by Theorem 1.1, $f(H)$ controls finite $p$-fusion in $G$. By Proposition 1.4, $f(H)$ also controls $p$-fusion in $G$ and the equivalence $\mathcal{S}_p(H) \to \mathcal{S}_p(G)$ has to map maximal isolated objects to maximal isolated objects; this implies that $Z^*(H)_p \cong Z^*(G)_p$. $\square$

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