CHAPTER 13

Wall's finiteness obstruction

Guido Mislin

ETH-Mathematik, Zürich

Contents

HANDBOOK OF ALGEBRAIC TOPOLOGY
Edited by I.M. James

1. Historical background

1.1. Borsuk's question

A space X is called an *absolute neighborhood retract* or ANR, if every map $F \to X$ from a closed subspace $F \subset Y$ of a normal space Y admits an extension to some neighborhood of F in Y. A typical example is a compact topological n-manifold (compact Hausdorff space, locally homeomorphic to $\mathbb{R}^n$). It is easy to see that a compact ANR has the homotopy type of a CW-complex (see section 2). Moreover, if X is a compact ANR and $\phi : X \to P$ a homotopy equivalence with P a CW-complex then, as $\phi(X)$ will be contained in a finite subcomplex Q of P, the restriction to Q of a homotopy inverse to ϕ shows that X is a retract up to homotopy of the *finite* complex Q.

Borsuk asked at the International Congress in Amsterdam in 1954 (cf. [8]) whether a compact *metric* ANR is actually homotopy equivalent to a finite CW-complex and he was able to settle this question positively for a special class of compact metric ANR's (admitting a "brick decomposition"). In the simply connected case one can use the concept of a homology decomposition (due to Eckmann and Hilton [19]) to see that such a space is of the homotopy type of a finite CW-complex. In subsequent years many more special cases were settled, in particular the case of a compact n-manifold (see Kirby and Siebenmann [39]). A positive solution to Borsuk's general question was given by West in [75]. Given a compact metric ANR X he constructed a *cell-like* mapping, thus a homotopy equivalence, from $Q \times I_\infty$ onto X, Q a finite polyhedron and I_∞ the Hilbert cube (a countably infinite product of intervals $[0,1]$), showing that X is homotopy equivalent to Q. Later, Chapman found a different proof of the "Borsuk Conjecture" [14].

1.2. Wall's answer to a more general question

A natural generalization of Borsuk's question is the following. Consider an arbitrary topological space X which admits maps $\phi : X \to P$ and $\psi : P \to X$ with P a finite CW-complex, such that $\psi \circ \phi$ is homotopic to id_X; we call such an X *finitely dominated.* A finitely dominated space X is homotopy equivalent to a CW-complex (see section 2), and the obvious question arises whether such a space is actually homotopy equivalent to a *finite* CW-complex. Indeed, that a finitely dominated space X is not far off from being of the homotopy type of a finite CW-complex can be seen from the following result due to Mather [45].

Theorem 1.1. *Suppose X is a finitely dominated space. Then $X \times S^1$ is of the homotopy type of a finite CW-complex.*

Here is a sketch of the pretty argument. Let $\phi : X \to P$ and $\psi : P \to X$ be maps with $\psi \circ \phi \simeq id_X$, where P is a finite CW-complex. The *mapping torus* T of $\phi \circ \psi : P \to P$ is obtained from $P \times [0,1]$ by identifying the points $(x,0) \in P \times [0,1]$ with those of the form $(\phi(\psi(x)),1)$. The homotopy between id_X and $\psi \circ \phi$ gives rise to a homotopy equivalence between $X \times S^1$ and T. But T is obviously homotopy equivalent to a finite CW-complex and thus $X \times S^1$ is too. (We will later discuss a purely algebraic proof of this theorem, using the product formula for finiteness obstructions.) Note also, that Mather's result implies that a finitely dominated space X is homotopy equivalent to a *finite dimensional* CW-complex. Namely, if $X \times S^1 \simeq K$ with K a finite CW-complex of dimension say n, then X is homotopy equivalent to an infinite cyclic covering space of K, which is a locally finite CW-complex of dimension n.

First examples of finitely dominated spaces which are not homotopy equivalent to finite CW-complexes were discovered by de Lyra (see [42]). About at the same time Wall ([71],[72]) found the correct general setup to reduce the topological question to a purely algebraic one. For a finitely dominated connected space X Wall defined an invariant $\tilde{w}(X)$ in the reduced projective class group of $\mathbb{Z}\pi_1(X)$, such that $\tilde{w}(X) = 0$ if and only if X is homotopy equivalent to a finite CW-complex. Moreover, Wall showed that every element in the reduced projective class group of $\mathbb{Z}\pi$, π an arbitrary finitely presented group, may be realized as $\tilde{w}(X)$ for some connected finitely dominated space X with fundamental group π. (If X is a finitely dominated connected space, its fundamental group is necessarily finitely presented, because a retract of a finitely presented group is finitely presented [71]). To get examples of finitely dominated spaces which are not of the homotopy type of any finite CW-complex one therefore just needs to find a finitely presented group whose reduced projective class group is non-trivial. For a finite abelian group π the projective modules over $\mathbb{Z}\pi$ are closely related to ideals in rings of integers of number fields. In particular if $\pi = \mathbb{Z}/p\mathbb{Z}$ with p a prime, the reduced projective class group of π is isomorphic to the ideal class group of the cyclotomic field $\mathbb{Q}(\exp 2\pi\sqrt{-1}/p)$, a group which is known to be trivial for all primes $p \leqslant 19$ and non-trivial for all other primes.

In the following survey we will give an account of Wall's general theory together with some typical applications. A well written and comprehensive introduction to the topic is Varadarajan's book [70], in which the reader will find complete proofs of the many technical details which underlie the basic definitions, and which are omitted in these notes. It is interesting to investigate the question how topological properties of spaces are reflected in properties of their finiteness obstruction. There remain many open questions in that respect. For instance, it is not known whether a finitely dominated loop space (or H-space) is necessarily of the homotopy type of a finite CW-complex. It is also not known if every finitely dominated Eilenberg–MacLane space $K(G,1)$ is homotopy equivalent to a finite CW-complex. The fundamental group G of such a $K(G,1)$ is necessarily finitely presented and torsion-free; at present there are even no examples known of arbitrary torsion-free groups with non-trivial reduced projective class group! We will also discuss the behavior of the finiteness obstruction in fibration and cofibration sequences, and we will indicate the close relationship with *Whitehead torsion.* For the class of finitely dominated nilpotent spaces the situation is relatively well understood, in the sense that one can give a precise description of the elements in the projective class group which can occur as finiteness obstructions of such spaces. For instance, a finitely dominated nilpotent space with *infinite* fundamental group is always of the homotopy type of a finite CW-complex, and the finiteness obstruction vanishes for finitely dominated nilpotent spaces with fundamental group cyclic of prime order. In the nilpotent situation, it is also natural to analyze the finiteness obstruction via p-local information, an approach which leads to several interesting examples and applications. We also sketch briefly the relevance of the finiteness obstruction in the spherical space form problem.

There are many other applications, which we had to omit to keep this exposition relatively short and reasonably self-contained; this applies in particular to Siebenmann's exciting work on the finiteness obstruction to finding a boundary for an open manifold [59].

2. Finitely dominated spaces

Recall that X is said to be *dominated* by Y if there are maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq id_X$. A space X which is dominated by a CW-complex has the homotopy type of a CW-complex. Indeed, if $|S(X)|$ denotes the geometric realization of the singular complex of an arbitrary space X then the natural weak homotopy equivalence $\omega : |S(X)| \to X$ will induce for an arbitrary CW-complex P a bijection

$$\omega_* : [P, |S(X)|] \to [P, X]$$

of homotopy sets and, in case X is dominated by a CW-complex, one thus obtains an induced bijection

$$\omega_* : [X, |S(X)|] \to [X, X] .$$

An inverse up to homotopy for ω is then given by a representative of $(\omega_*)^{-1}([id_X])$. If the connected topological space X is dominated by a CW-complex P then X is also dominated by a connected subcomplex Q of P, which is a path-connected space, and therefore X is path-connected too because $\pi_0(X)$ is a retract of $\pi_0(Q)$. We can therefore refer to the fundamental group of such a space X in an unambiguous way. Moreover, if a not necessarily connected space X is dominated by a finite CW-complex then X is the topological disjoint union of a finite number of connected (and path-connected) spaces each of which is dominated by a finite connected CW-complex. There is therefore no loss in generality to study finiteness conditions for the case of *connected* finitely dominated spaces and, if convenient, we may even assume our spaces to be actual CW-complexes.

Remark. If X is a compact ANR, one can embed X in a linear space $\prod \mathbb{R}$ which is a normal space. Therefore, there exists an open neighborhood $N(X)$ of $X \subset \prod \mathbb{R}$ which retracts onto X. But such an $N(X)$ is easily seen to be homotopy equivalent to some open subset of a finite dimensional space $\mathbb{R}^n$, which is triangulable. It follows, that such an X is dominated by a (finite) CW-complex.

For an arbitrary connected CW-complex X and π-module M, π the fundamental group of X, we write

$$H_*(X;M) \quad \text{and} \quad H^*(X;M)$$

for the homology (respectively cohomology) of X with *local* coefficients in M. These groups are defined as the homology groups of the chain complex $M \otimes_\pi C_*^{cell}(\tilde{X})$ (respectively the cohomology of the cochain complex $\mathrm{Hom}_\pi(C_*^{cell}(\tilde{X}), M)$), where $\tilde{X}$ denotes the universal cover of X and $C_*^{cell}(\tilde{X})$ the cellular chain complex of $\tilde{X}$, considered as a π-complex. Note that if M is a trivial π-module, then $H_*(X;M)$ and $H^*(X;M)$ are just ordinary homology and cohomology of X with coefficients in M.

For an arbitrary connected CW-complex X we define

$$\mathrm{cd}(X) \leqslant n \quad \Longleftrightarrow \quad H^i(X;M) = 0\,,\ \forall M\,, \forall i > n\,.$$

Here M stands as before for an arbitrary $\mathbb{Z}\pi_1(X)$-module. Clearly, if X happens to be homotopy equivalent to a connected m-dimensional CW-complex, then certainly $\mathrm{cd}(X) \leqslant m$. The converse is essentially true too.

Theorem 2.1. *Suppose X is of the homotopy type of a connected CW-complex and $\mathrm{cd}(X) \leqslant n$. Then X is homotopy equivalent to a CW-complex of dimension $\leqslant \max(3,n)$.*

Proof. Consider a homotopy equivalence $\phi : P \to X$. By restricting ϕ to the $(n-1)$-skeleton of P one obtains an $(n-1)$-connected map $\psi : P^{n-1} \to X$ (thus $\psi_* : \pi_i(P^{n-1}) \to \pi_i(X)$ is bijective for $i < n-1$ and surjective for $i = n-1$). We first consider the case $n > 2$ so that $\pi_1(P^{n-1}) \to \pi_1(X)$ is an isomorphism. One shows then, using the assumption $\mathrm{cd}(X) \leqslant n$, that the homotopy theoretic fiber $fib(\psi)$ has for $\pi_{n-1}(fib(\psi))$ a projective π-module, π the fundamental group of X. By replacing P^{n-1} by $P^{n-1} \vee (\bigvee S^{n-1})$, with $\bigvee S^{n-1}$ a suitable (possibly infinite) wedge of spheres, and extending ψ by a constant map over the wedge of spheres, we may assume that the new homotopy fiber has a *free* π-module as its $(n-1)$'st homotopy group. After attaching n-cells corresponding to a basis of that homotopy module, the map extends to a homotopy equivalence between $(P^{n-1} \vee (\bigvee S^{n-1})) \cup \{n-cells\}$ and X. If $n \leqslant 2$, the same argument shows that X is homotopy equivalent to a complex of dimension $\leqslant 3$, and we are done. $\square$

Remark . It is not known whether a CW-complex X with $\mathrm{cd}(X) \leqslant 2$ is actually homotopy equivalent to a 2-dimensional CW-complex. However, if $\mathrm{cd}(X) \leqslant 1$ it follows, using the fact that groups of cohomological dimension 1 are free, that X is homotopy equivalent to a 1-dimensional CW-complex, and therefore $X \simeq K(\pi_1(X), 1)$.

If X is dominated by a finite CW-complex then obviously all the finitely many connected components of X are of finite cohomological dimension and the following theorem is an immediate consequence of the previous one.

Theorem 2.2. *If X is dominated by a finite complex, then X is homotopy equivalent to a finite dimensional CW-complex.*

Recall that a space of the homotopy type of a CW-complex is said to be of *finite type* if it is homotopy equivalent to a CW-complex with finite skeleta.

Theorem 2.3. *Suppose X is finitely dominated. Then X is of finite type.*

Proof. We will only sketch the proof. One may assume that X is connected. The fundamental group of X is then finitely presented, being a retract of a finitely presented group. Thus we can find a finite 2-dimensional CW-complex $W(2)$ and a 2-connected map $W(2) \to X$. One can show that π_2 of the homotopy fiber of that map is a *finitely generated* module over the fundamental group of X so that one can render the map 3-connected by attaching a finite number of 3-cells to $W(2)$, to obtain a new space $W(3)$. The map from $W(2)$ to X then extends to a 3-connected map from $W(3)$ to X. Continuing in this manner one obtains a coherent family of n-connected maps $W(n) \to X$, $n \geqslant 2$, giving rise to a homotopy equivalence $\bigcup W(n) \to X$. □

The two preceding theorems constitute the basic homotopy theoretical properties of finitely dominated spaces. They have purely algebraic counter parts in the framework of chain complexes, which we will discuss in the next section.

3. Finitely dominated chain complexes

We denote by R an associative ring with 1. Modules over R are *unitary left modules*, and all chain complexes are supposed to be *non-negative.* A chain complex $C = \{C_i, d_i\}$ is said to be *projective* if the modules C_i are all projective. As usual, we write $H^*(C; M)$ for the cohomology of the cochain complex $\mathrm{Hom}_R(C, M)$ and we call C *acyclic* when C is exact. It is a basic fact that a morphism $f : C \to D$ of projective chain complexes is a chain homotopy equivalence if and only if the induced map of cohomology $H^*(\ ; R)$ is an isomorphism. We say that the chain complex $C = \{C_i, d_i\}$ is of type FP if each C_i is finitely generated and projective over R, and $C_i = 0$ for i large; C is said to be of type FF, if it is of type FP with all modules C_i free. For a chain complex $C = \{C_i, d_i\}$ of type FP the *Euler characteristic*

$$\chi(C) = \sum (-1)^i [C_i] \in K_0(R)$$

is defined, where $K_0(R)$ denotes the Grothendieck group of finitely generated projective R-modules, and $[C_i]$ stands for the class of the projective module C_i in

$K_0(R)$. (Basic references for $K_0(R)$, the projective class group of R, are the books by Bass [2], Milnor [50] and Swan [64]). If

$$\ldots \to C_i \xrightarrow{d_i} C_{i-1} \to \ldots \to C_0 \to 0$$

is an acyclic complex of type FP then $\chi(C) = 0$, as one easily verifies by observing that all kernels and images of the homomorphisms d_i are projective in this case. More generally, if $\phi : C \to D$ is a chain homotopy equivalence between chain complexes of type FP then $\chi(C) = \chi(D)$. This follows by observing that the mapping cone M of ϕ is an acyclic chain complex of type FP with $M_i = C_i \oplus D_{i-1}$, thus M satisfies $\chi(M) = 0$, which implies $\chi(C) = \chi(D)$. The following definition is therefore meaningful.

Definition 3.1. *Let C be a chain complex which is chain homotopy equivalent to a chain complex D of type FP. Then the Euler characteristic*

$$\chi(C) \in K_0(R)$$

is defined to be the Euler characteristic of D.

We will also use the following definitions.

Definition 3.2. *Let $C = \{C_i, d_i\}$ be a chain complex. Then*

$$\mathrm{cd}(C) \leqslant n \quad \Longleftrightarrow \quad H^i(C; M) = 0\,,\ \forall M\,, \forall i > n\,.$$

Moreover, C is said to be of type FF_∞ if each module C_i is finitely generated and free over R.

In applications it is very useful to have a criterion for deciding when a chain complex is, up to chain homotopy, of type FF_∞. A homological criterion was first proved by Bieri and Eckmann [7] for the case of resolutions of modules. Later Brown [10] dualized the criterion to a cohomological one and proved the following result for arbitrary chain complexes.

Theorem 3.3. *A chain complex C is chain homotopy equivalent to a chain complex of type FF_∞ if and only if the functors $H^n(C; -)$ on the category of R-modules preserve direct limits for all n.*

A chain complex C is said to be of *finite cohomological dimension* if there is an N such that $\mathrm{cd}(C) \leqslant N$. Note that for X a connected CW-complex and C the $\mathbb{Z}\pi_1(X)$-complex $C_*^{cell}(\tilde{X})$, one has

$$\mathrm{cd}(X) \leqslant n \quad \Longleftrightarrow \quad \mathrm{cd}(C_*^{cell}(\tilde{X})) \leqslant n\,,$$

and

$$X \text{ of finite type } \Longrightarrow C_*^{cell}(\tilde{X}) \text{ is, up to chain homotopy, of type } FF_\infty\,.$$

The converse is true too if one assumes $\pi_1(X)$ to be finitely presented. In particular, the following cohomological characterization for finite domination holds.

Theorem 3.4. *Let X be a connected CW-complex. Then*

$$X \text{ is finitely dominated} \Longleftrightarrow \begin{cases} \pi_1(X) \text{ is finitely presented,} \\ \mathrm{cd}(X) < \infty \text{ and} \\ \oplus_n H^n(X;-) \text{ commutes with direct limits.} \end{cases}$$

Remark . A chain complex $C = \{C_i, d_i\}$ is said to be of type FP_∞ if each C_i is finitely generated projective. It is easy to see that a chain complex is chain homotopy equivalent to a chain complex of type FP_∞ if and only if it is chain homotopy equivalent to one of type FF_∞.

The following simple theorem is basic in what follows. We denote by $\tilde{K}_0(R)$ the *reduced* projective class group of R; it is the factor group of $K_0(R)$ modulo the subgroup generated by $[R]$. Thus, the class $[P] \in K_0(R)$ of the finitely generated projective module P maps to zero in $\tilde{K}_0(R)$ if and only if P is *stably free*, meaning that there exists a finitely generated free module F such that $P \oplus F$ is free. We will denote the image of the Euler characteristic $\chi(C)$ in $\tilde{K}_0(R)$ by $\tilde{\chi}(C)$ and we call it the *reduced* Euler characteristic of C. We will use the notation $\{P\} \in \tilde{K}_0(R)$ for the class of the finitely generated projective module P. One easily checks that for finitely generated projective modules P and Q one has

$$\{P\} = \{Q\} \in \tilde{K}_0(P)$$

if and only there exist finitely generated *free* modules F and G such that $P \oplus F \cong Q \oplus G$; such modules P and Q are said to be *stably isomorphic.* Also, every element in $\tilde{K}_0(R)$ is of the form $\{P\}$ for some finitely generated projective P.

Theorem 3.5. *Suppose that the chain complex C is chain homotopy equivalent to a chain complex D of type FF_∞. Furthermore, assume that C satisfies $\mathrm{cd}(C) \leqslant n$. Then C is chain homotopy equivalent to a chain complex of type FP and*

$$\tilde{\chi}(C) = (-1)^{N+1}\{B_N\} \in \tilde{K}_0(R)$$

where N is any integer $\geqslant n+1$ and $B_N = \mathrm{im}(d_{N+1} : D_{N+1} \to D_N) \subset D_N$, the module of N-boundaries of D.

Proof. Since C is chain homotopy equivalent to D we have $\mathrm{cd}(D) \leqslant n$. The reader verifies easily that, in general, for a projective complex the condition $H^i(D; M) = 0, \forall M$ implies that $H_i(D; R) = 0$ and that $\mathrm{im}(d_i) \subset D_{i-1}$ is a direct summand. It follows that B_N is projective and that the inclusion of the truncated complex

$$0 \to B_N \to D_N \to D_{N-1} \ldots \to D_0 \to 0$$

into the complex D is a homology isomorphism, therefore a chain homotopy equivalence, because both complexes are projective. Clearly, as the D_i's are free, the reduced Euler characteristic of the truncated complex equals $(-1)^{N+1}\{B_N\}$ and the result follows. $\square$

The following finiteness theorem is now plain and serves as a motivation for the corresponding topological result which we will discuss in the next section.

Corollary 3.6. *Let C be a chain complex of finite cohomological dimension, chain homotopy equivalent to a chain complex of type FF_∞. Then C is chain homotopy equivalent to a complex of type FF if and only if the reduced Euler characteristic $\tilde{\chi}(C)$ in $\tilde{K}_0(R)$ is zero.*

Namely, we have only to observe that once we have replaced C by the chain homotopy equivalent complex

$$0 \to B_N \to D_N \to \ldots \to D_0 \to 0$$

as above, with the D_i's free and $\tilde{\chi}(C) = (-1)^{N+1}\{B_N\} = 0$, the module B_N is stably free, and we can find a finitely generated free module F with $B_N \oplus F$ free. Therefore we can replaces the truncated complex above by a complex of the form

$$0 \to B_N \oplus F \to D_N \oplus F \to D_{N-1} \to \ldots \to D_0 \to 0$$

with obvious boundary maps, to obtain a complex of type FF, which is chain homotopy equivalent to the original complex C.

4. The finiteness obstruction

Let X denote a connected, finitely dominated CW-complex. Then, as X is homotopy equivalent to a complex of finite type, the cellular chain complex $C_*^{cell}(\tilde{X})$ of the universal cover of X is chain homotopy equivalent to a chain complex of type FF_∞. Moreover, as X is homotopy equivalent to a finite dimensional CW-complex, $C_*^{cell}(\tilde{X})$ is of finite cohomological dimension. The Euler characteristic

$$\chi(C_*^{cell}(\tilde{X})) =: w(X) \in K_0(\mathbb{Z}\pi_1(X))$$

is therefore well defined, and we call it the *Wall* finiteness obstruction of X; the associated reduced Euler characteristic will be denoted by $\tilde{w}(X)$.

We have omitted the mention of *base points*. Indeed, if $x_0, x_1 \in X$ are two base points in the connected CW-complex X, then there exists a homotopy equivalence $f : X \to X$ taking x_0 to x_1 and thus inducing an isomorphism

$$f_* : K_0(\mathbb{Z}\pi_1(X, x_0)) \to K_0(\mathbb{Z}\pi_1(X, x_1)).$$

The point now is that f_* is independent of the particular choice of f. This can be seen by checking that an inner automorphism of a group π induces the identity on $K_0(\mathbb{Z}\pi)$, and using the well-known fact that the class of the homomorphism

$$f_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

modulo inner automorphisms of $\pi_1(X, x_1)$ is independent of the particular choice of f. As a consequence, we may neglect base points when dealing with Wall obstructions. In particular, if $f : X \to Y$ is any map of connected CW-complexes, one has a well-defined induced map

$$f_* : K_0(\mathbb{Z}\pi_1(X)) \to K_0(\mathbb{Z}\pi_1(Y)) .$$

Note that if f is a homotopy equivalence and the spaces X and Y are finitely dominated, one has

$$f_* w(X) = w(Y) ,$$

as f (more precisely, a cellular approximation to f) induces a chain homotopy equivalence of cellular chain complexes of associated universal covers. The following corollary is useful to notice.

Corollary 4.1. *Let X be a connected, finitely dominated CW-complex and let f denote a self-homotopy equivalence of X. Then*

$$f_* w(X) = w(X) .$$

5. Basic properties of $w(X)$

Let π denote an arbitrary group. The natural map $K_0(\mathbb{Z}\pi) \to \tilde{K}_0(\mathbb{Z}\pi)$ admits a splitting σ, given by $\sigma\{P\} = [P] - \mathrm{rk}(P) \cdot [\mathbb{Z}\pi]$, where the *rank* $\mathrm{rk}(P)$ of the finitely generated projective π-module P is defined to be the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_\pi P$. As a result, we have a canonical decomposition

$$K_0(\mathbb{Z}\pi) = \tilde{K}_0(\mathbb{Z}\pi) \oplus \mathbb{Z} , \qquad [P] = (\{P\}, \mathrm{rk}(P)) ,$$

where we have identified the cokernel of the splitting σ with $\mathbb{Z}$, which is the image of the *rank map*

$$\mathrm{rk} : K_0(\mathbb{Z}\pi) \to \mathbb{Z} .$$

Lemma 5.1. *Let X be a connected, finitely dominated CW-complex and $w(X)$ its Wall obstruction. Then*

$$\mathrm{rk}\, w(X) = \chi(X) ,$$

where $\chi(X) = \sum_i (-1)^i \dim_{\mathbb{Q}} H_i(X;\mathbb{Q})$ *denotes the ordinary Euler characteristic of* X.

Namely, if we choose a chain complex P_* of type FP, chain homotopy equivalent to $C_*^{cell}(\tilde{X})$, then

$$w(X) = (\sum_i (-1)^i \{P_i\}, \mathrm{rk} \sum_i (-1)^i [P_i])$$

and

$$\mathrm{rk} \sum_i (-1)^i [P_i] = \sum_i (-1)^i \dim_{\mathbb{Q}} (\mathbb{Q} \otimes_{\pi_1(X)} P_i) = \sum_i (-1)^i \dim_{\mathbb{Q}} H_i(X;\mathbb{Q}) .$$

The following basic theorem of C. T. C. Wall implies that the *reduced Wall obstruction* is indeed the *finiteness obstruction:*

Theorem 5.2. *Let* X *be a connected, finitely dominated* CW*-complex. Then* X *is homotopy equivalent to a finite* CW*-complex if and only if*

$$\tilde{w}\,(X) = 0 \in \tilde{K}_0(\mathbb{Z}\pi_1(X)) .$$

Moreover, if π *denotes a finitely presented group and* y *an arbitrary element of* $K_0(\mathbb{Z}\pi)$, *then there exists a connected, finitely dominated* CW*-complex* Y *with fundamental group* π *and Wall obstruction* $w(Y) = y$.

For the proof see Wall's original papers [71],[72], or Varadarajan's book [70].

Remark. It was proved by Ferry (cf. [27] and [28]) that if the space X is dominated by a finite CW-complex, then X is homotopy equivalent to a *compact metric* space Y; it is an open question whether Y can be chosen locally simply connected.

5.1. A sum theorem for Wall's obstruction

The following *Sum Theorem* is a very useful computational tool. If $f : G \to H$ denotes a group homomorphism, we write f_* for the induced homomorphism $K_0(\mathbb{Z}G) \to K_0(\mathbb{Z}H)$ (given by $[P] \mapsto [\mathbb{Z}H \otimes_G P]$); since $f_*[\mathbb{Z}G] = [\mathbb{Z}H]$, one also obtains an induced map of *reduced* projective class groups, which we will denote by the same symbol f_*. In case $Y \subset X$ denotes a connected, finitely dominated subcomplex of the connected and finitely dominated CW-complex X, and if ι_Y denotes the inclusion map $Y \to X$, then we obtain an induced map

$$\iota_{Y*} : K_0(\mathbb{Z}\pi_1(Y)) \to K_0(\mathbb{Z}\pi_1(X)) ,$$

One assumes here that a base point in X is chosen which lies in Y; the induced map of fundamental groups then gives rise to a map of projective class groups which is independent — in the obvious sense — of the particular choice of the basepoint (see also the corresponding remark earlier on), justifying our notation.

Theorem 5.3. *Let $X = Y \cup Z$ be a connected, finitely dominated CW-complex, with Y , Z and $Y \cap Z$ connected and finitely dominated subcomplexes. Then*

$$w(X) = \iota_{Y*}(w(Y)) + \iota_{Z*}(w(Z)) - \iota_{Y\cap Z*}(w(Y \cap Z)) \in K_0(\mathbb{Z}\pi_1(X)),$$

where ι_Y , ι_Z and $\iota_{Y\cap Z}$ denote the inclusion maps.

We omit the proof and refer to Siebenmann [59], where also a version of this *Sum Theorem* in the non-connected setting is discussed. Note also that under the hypothesis of the sum theorem one sees from the Mayer-Vietoris sequence for rational homology, that the Euler characteristics of the spaces involved satisfy

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

We could therefore have stated the *Sum Theorem* in exactly the same form with $w(-)$ replaced by the reduced Wall obstructions $\tilde{w}(-)$.

5.2. The fibration theorem

Let $F \to E \to B$ be a fibration sequence of connected CW-complexes. If F and B are finitely dominated then $\pi_1(E)$ is finitely presented (cf. [53]) and, by applying the cohomological characterization of finite domination to the Serre spectral sequence of the fibration $F \to E \to B$ one concludes that the total space E is finitely dominated too. We wish now to describe $w(E)$ in terms of the Wall obstructions of the base and fiber and possibly some other data present in the given fibration (the formula in Lal's paper [41] is incorrect; he neglected the action of the fundamental group of the base on the homology of the fiber). The simplest situation is that of a product fibration. For this, it is useful to consider the following pairing. Let G and H denote arbitrary groups. One defines

$$- \otimes - : K_0(\mathbb{Z}G) \times K_0(\mathbb{Z}H) \to K_0(\mathbb{Z}(G \times H))$$

by putting

$$[P] \otimes [Q] := [P \otimes_{\mathbb{Z}} Q],$$

for P (respectively Q) a finitely generated projective $\mathbb{Z}G$- (respectively $\mathbb{Z}H$-) module. Note that the pairing induces also a pairing on the reduced projective class groups, which we will denote by the same symbol "$- \otimes -$".

The *Product Theorem* due to Gersten [34] and Siebenmann [59] can now be expressed by the following attractive formula.

Theorem 5.4. *Let F and B be connected, finitely dominated CW-complexes. Then*

$$w(F \times B) = w(F) \otimes w(B) \in K_0(\mathbb{Z}(\pi_1(F) \times \pi_1(B)))\,.$$

For applications it is useful to decompose the Wall obstructions in this formula into their reduced and Euler characteristic parts, $w = (\tilde{w}, \chi)$, which yields the formula

$$\tilde{w}(F \times B) = \tilde{w}(F) \otimes \tilde{w}(B) + \chi(F) \cdot \iota_{B*}(\tilde{w}(B)) + \chi(B) \cdot \iota_{F*}(\tilde{w}(F))\,,$$

where ι_F and ι_B denote the obvious inclusion maps (as usual we assume that base-points have been chosen, where necessary).

Corollary 5.5. *Let X be a connected, finitely dominated CW-complex and Y a connected finite CW-complex with $\chi(Y) = 0$. Then $\tilde{w}(X \times Y) = 0$ and therefore $X \times Y$ is homotopy equivalent to a finite CW-complex.*

As a special case, we obtain Mather's result stated in the introduction: *If X is a finitely dominated space then $X \times S^1$ is homotopy equivalent to a finite CW-complex*. The formula stated in the *Product Theorem* is also correct for a certain wider class of fibrations with section $\iota_B : B \to E$ and retraction $E \to F$ (with associated decomposition $\pi_1(E) = \pi_1(F) \times \pi_1(B)$); for a precise statement, see Ehrlich [22]. To deal with the case of a general fibration, Ehrlich introduced in [21] a *geometric transfer homomorphism* (see also Pedersen [56]) denoted by

$$p^* : \tilde{K}_0(\mathbb{Z}\pi_1(B)) \to \tilde{K}_0(\mathbb{Z}\pi_1(E))\,,$$

associated with a fibration $p : E \to B$ of connected, finitely dominated CW-complexes, with finitely dominated fiber F. It is useful here to permit the case of a *disconnected* fiber F, e.g. a *finite covering space* situation (in which $\pi_1(E)$ is a subgroup of finite index of $\pi_1(B)$ and the geometric transfer map p^* is given by the usual "restriction map" of projective class groups). If $F = \cup F_i$, the disjoint union of its connected components, one puts $\tilde{w}(F)$ equal $\oplus_i \tilde{w}(F_i)$ in $\oplus_i \tilde{K}_0(F_i)$, and $\iota_{F*}(\tilde{w}(F)) = \sum_i \iota_{F_i*}(\tilde{w}(F_i))$. The general fibration theorem then reads as follows (for a proof see Ehrlich [21]).

Theorem 5.6. *Given a fibration $p : E \to B$ of finitely dominated, connected CW-complexes with finitely dominated (not necessarily connected) fiber F. Then*

$$\tilde{w}(E) = p^* \tilde{w}(B) + \chi(B) \cdot \iota_{F*}(\tilde{w}(F)) \in \tilde{K}_0(\mathbb{Z}\pi_1(E))\,.$$

It seems to be difficult to compute p^* in general. But the composition $p_* p^*$, which is an endomorphism of $\tilde{K}_0(\mathbb{Z}\pi_1(B))$, can be described explicitly: *it is given by* *the $\bullet$-multiplication with the $\pi_1(B)$-module Euler characteristic $\sum_i(-1)^i H_i(F;\mathbb{Z})$.* This dot-multiplication is defined in the following way. For an arbitrary group π let $G(\mathbb{Z}\pi)$ denote the Grothendieck group of π-modules, which are finitely generated as abelian groups. One then defines a pairing

$$- \bullet - : G(\mathbb{Z}\pi) \times K_0(\mathbb{Z}\pi) \to K_0(\mathbb{Z}\pi)$$

by putting $[M]\bullet[P] = [M \otimes_{\mathbb{Z}} P]$, where M is a π-module which is finitely generated and *torsion-free* as an abelian group, and P an arbitrary finitely generated projective π-module. The definition extends to all of $G(\mathbb{Z}\pi)$ by observing that $G(\mathbb{Z}\pi)$ is generated by $[M]$'s which are torsion-free as abelian groups (one checks that the pairing is indeed well-defined, see the paper by Pedersen and Taylor [57]). The tensor product over $\mathbb{Z}$ turns $G(\mathbb{Z}\pi)$ into a commutative ring with unit (given by the class of the trivial π-module $\mathbb{Z}$), and $K_0(\mathbb{Z}\pi)$ becomes a module over $G(\mathbb{Z}\pi)$. One also obtains an induced action of $G(\mathbb{Z}\pi)$ on $\tilde{K}_0(\mathbb{Z}\pi)$, which we will denote by " $\bullet$ " too.

According to Pedersen and Taylor, the image $p_*p^*\tilde{w}(B)$, which by the fibration theorem is equal to $p_*\tilde{w}(E)$, is given by the following formula (their result is actually more general, compare [57]).

Theorem 5.7. *Let $p: E \to B$ be a fibration of connected, finitely dominated CW-complexes, with (not necessarily connected) finitely dominated fiber. Then*

$$p_*\tilde{w}(E) = (\sum_i (-1)^i H_i(F;\mathbb{Z})) \bullet \tilde{w}(B) \in \tilde{K}_0(\mathbb{Z}\pi_1(B)).$$

Note that in case $\pi_1(B)$ acts trivially on the homology of the fiber F, the sum over the homology groups can be replaced by the ordinary Euler characteristic of F, and the formula reduces to

$$p_*\tilde{w}(E) = \chi(F) \cdot \tilde{w}(B).$$

5.3. Relationship with Whitehead torsion

For an associative ring R with unit one defines

$$K_1(R) = Gl(R)/[Gl(R), Gl(R)], \qquad Gl(R) = \bigcup_n Gl_n(R).$$

For the basic properties of the functor $K_1(-)$ see Milnor [49] and [50]. An isomorphism between two finitely generated free, based R-modules gives rise to a well defined "torsion element" in $K_1(\mathbb{Z}\pi)$, which is just the image of the matrix representing the isomorphism. More generally, if $C = \{C_i, d_i\}$ denotes an acyclic, based chain complex of type FF over R, its *torsion* $\mathrm{tor}(C) \in K_0(R)$ is well defined (see

[49] or [15] for details). In case of $R = \mathbb{Z}\pi$, the group ring of the group π, we can consider

$$\pm\pi \subset Gl_1(\mathbb{Z}\pi) \to Gl(\mathbb{Z}\pi) \to K_1(\mathbb{Z}\pi)$$

and one defines the *Whitehead group* of π by

$$\mathrm{Wh}(\pi) := K_1(\mathbb{Z}\pi)/\mathrm{im}(\pm\pi)\,.$$

One checks that an inner automorphism of π induces the identity on $\mathrm{Wh}(\pi)$ and thus $\mathrm{Wh}(\pi_1(X))$ is functorially defined for a path-connected space X . If $f : X \to Y$ is a homotopy equivalence of connected finite CW-complexes, then the *Whitehead torsion* $\tau(f) \in \mathrm{Wh}(\pi_1(Y))$ of f is defined (see Whitehead [76] in the simplicial setting). Roughly speaking, the definition goes as follows. One replaces f up to homotopy by a cellular map g and considers the relative chain complex $C = C_*^{cell}(\tilde{M}_g, \tilde{X})$ with M_g the mapping cylinder of g, considered as a CW-complex in the obvious way. Since C is acyclic and has — because of the cell decomposition — a preferred $\mathbb{Z}\pi_1(M_f)$ basis (up to multiplication by elements $\pm x \in \pi_1(M_f)$), the torsion $\mathrm{tor}(C)$ is well defined modulo the image of $\pm\pi_1(M_f)$ in $K_1(\mathbb{Z}\pi_1(M_f))$, giving rise to a unique element denoted by $\tau(C) \in \mathrm{Wh}(\pi_1(M_f))$. The natural collapsing homotopy equivalence $p : M_f \to Y$ maps $\tau(C)$ to $\mathrm{Wh}(\pi_1(Y)$, leading to the definition

$$\tau(f) := p_*\tau(C) \in \mathrm{Wh}(\mathbb{Z}\pi_1(Y))\,.$$

Remark . It is possible to give a purely geometric description of the Whitehead group, see Eckmann and Maumary [20] as well as Stöcker [61]. A homotopy equivalence f between connected, finite CW-complexes is called *simple*, if $\tau(f) = 0$; for a geometric interpretation of this notion see Cohen's book [15]. A basic result is the *topological invariance of Whitehead torsion* proved by Chapman in [12] : *If f is a homeomorphism between connected, finite CW-complexes, then* $\tau(f) = 0$. Indeed Chapman proves that a homotopy equivalence $f : X \to Y$ between connected finite CW-complexes is simple if and only if $f \times \mathrm{id} : X \times I_\infty \to Y \times I_\infty$ is homotopic to a homeomorphism (cf. [13]), where as earlier I_∞ stands for the Hilbert cube.

Let X be a connected, finitely dominated space so that one has maps $f : K \to X$ and $g : X \to K$ with K a connected finite CW-complex and $f \circ g$ homotopic to the identity of X. Consider a cellular map $\alpha : K \to K$ homotopic to $g \circ f$. Then the mapping torus $T(\alpha)$ is in a natural way a finite CW-complex. Choose any homotopy equivalence $\Phi : T(\alpha) \to X \times S^1$ and let Ψ be a homotopy inverse of Φ. Define the involution $\lambda : X \times S^1 \to X \times S^1$, $\lambda(x, z) = (x, \bar{z})$, and put

$$\sigma(X) := \Phi_*(\tau(\Psi \circ \lambda \circ \Phi)) \in \mathrm{Wh}(\pi_1(X \times S^1))\,.$$

This yields a well-defined homotopy invariant of X (cf. Ferry [29]) .

Now, according to Bass-Heller-Swan [5], one has for any group π a natural decomposition

$$\mathrm{Wh}(\pi \times \mathbb{Z}) = \mathrm{Wh}(\pi) \oplus \mathrm{Nil}(\mathbb{Z}\pi) \oplus \mathrm{Nil}(\mathbb{Z}\pi) \oplus \tilde{K}_0(\mathbb{Z}\pi)\,,$$

where $\mathrm{Nil}(-)$ denotes a certain functor on associative rings with 1. The relationship with Wall's obstruction is given by the following result due to Kwasik [40].

Theorem 5.8. *Let X be a finitely dominated, connected CW-complex. Then one has $\sigma(X) = (0,0,0,\tilde{w}(X))$ if one identifies $\mathrm{Wh}(\pi_1(X \times S^1))$ with the sum $\mathrm{Wh}(\pi_1(X)) \oplus \mathrm{Nil}(\mathbb{Z}\pi_1(X)) \oplus \mathrm{Nil}(\mathbb{Z}\pi_1(X)) \oplus \tilde{K}_0(\mathbb{Z}\pi_1(X))$.*

5.4. Some results on projective class groups of group rings

In order to be able to give examples of finitely dominated spaces which are not homotopy equivalent to finite CW-complexes, one needs to give examples of groups π with $\tilde{K}_0(\mathbb{Z}\pi) \neq 0$. We list a few basic facts on projective class groups of group rings. The first one is due to Swan [63] :

- π a finite group $\implies$ $\tilde{K}_0(\mathbb{Z}\pi)$ a finite group.

Thus, for a finite group π the reduced projective class group $\tilde{K}_0(\mathbb{Z}\pi)$ is naturally isomorphic to the torsion subgroup of $K_0(\mathbb{Z}\pi)$. Note, however, that for an arbitrary finitely generated abelian group π the group $\tilde{K}_0(\mathbb{Z}\pi)$ is *not* finitely generated in general, see Bass–Murthy [4].

Because projective modules over free groups are free (cf. Bass [3], Gersten [33] and Stallings [60]), the following holds:

- π a free group $\implies$ $\tilde{K}_0(\mathbb{Z}\pi) = 0$.

Actually there is no torsion-free group known with non-trivial reduced projective class group (although there exist non-free projective modules over certain torsion-free groups, see the examples of Dunwoody [18], Berridge–Dunwoody [6] and Artamonov [1]). On the other hand, there are many torsion-free groups known for which the reduced projective class group is trivial. The following are a few examples: finitely generated abelian groups, or more generally, Bieberbach groups (cf. Farrell–Hsiang [24]), poly–$\mathbb{Z}$ groups (cf. Farrell–Hsiang [25]) and — even more generally — fundamental groups of closed Riemannian manifolds all of whose sectional curvature values are non-positive (cf. Farrell–Jones [26]). Actually, for all these groups π the Whitehead group $\mathrm{Wh}(\pi \times \mathbb{Z})$ vanishes. Note that if Γ is a direct limit of groups π_α all of which satisfy $\mathrm{Wh}(\pi_\alpha \times \mathbb{Z}) = 0$ then one has $\tilde{K}_0(\mathbb{Z}\Gamma) = 0$. In particular, because the Whitehead group vanishes for finitely generated free abelian groups, one concludes:

- π a torsion-free abelian group $\implies$ $\tilde{K}_0(\mathbb{Z}\pi) = 0$.

We finish this section by briefly describing examples of finite groups with *non-trivial* reduced projective class groups. If Λ denotes a Dedekind domain, we write

$C(\Lambda)$ for the ideal class group of Λ, and $[I]_C$ for the class of a non-zero ideal $I \subset \Lambda$. Because ideals of Λ are finitely generated projective Λ-modules, and because for non-zero ideals I_1, I_2 of a Dedekind domain one has the well-known relation

$$I_1 \oplus I_2 \cong \Lambda \oplus (I_1 \otimes_\Lambda I_2)\,,$$

there is a natural homomorphism

$$C(\Lambda) \to \tilde{K}_0(\Lambda)\,, \qquad [I]_C \mapsto \{I\}\,,$$

which is actually an isomorphism (see for instance Milnor's book [50]). Moreover, for p a prime number, the ideal class group $C(\mathbb{Z}[\exp(2\pi\sqrt{-1}/p)])$ is known to be trivial for $p < 23$, and non-trivial for all primes $p \geqslant 23$ (see Washington's book [74] for references on ideal class groups of number rings). The connection with projective class groups of group rings is given by the following basic theorem due to Rim [58].

Theorem 5.9. *Let $\pi = < x >$ be a cyclic group of prime order p. Consider the map $\mathbb{Z}\pi \to \mathbb{Z}[\exp(2\pi\sqrt{-1}/p)]$ given by mapping x to $\exp(2\pi\sqrt{-1}/p)$. Then the induced map of reduced projective class groups*

$$\tilde{K}_0(\mathbb{Z}\pi) \to \tilde{K}_0(\mathbb{Z}[\exp(2\pi\sqrt{-1}/p)])$$

is an isomorphism.

Corollary 5.10. *For every prime $p \geqslant 23$ there exists a connected, finitely dominated CW-complex X with fundamental group $\mathbb{Z}/p\mathbb{Z}$, such that X is not homotopy equivalent to any finite CW-complex.*

There is a vast literature on projective class groups of finite groups. The following are a few examples. For cyclic groups of prime power order see Galovich [31], Kervaire-Murthy [38] and Ullom [69], for quaternion and dihedral 2-groups see Martinet [44], Keating [36], Fröhlich-Keating-Wilson [30], for metacyclic groups see Keating [37], Galovich-Reiner-Ullom [32], and for general p-groups see Taylor [65]. Examples of *infinite* groups (with torsion) and non-vanishing projective class group are discussed in Bass-Murthy [4] (abelian groups) and in Bürgisser [11] (arithmetic groups).

6. Nilpotent spaces

Let π be an arbitrary group, M a π-module and $I\mathbb{Z}\pi \subset \mathbb{Z}\pi$ the augmentation ideal. The group π is said to operate *nilpotently* on M, and M is termed a *nilpotent* π-module, if there is an $n > 0$ such that

$$(I\mathbb{Z}\pi)^n \cdot M = 0\,.$$

Recall that a connected CW-complex is called *simple* if $\pi_1(X)$ is abelian and operates trivially on all higher homotopy groups $\pi_i(X)$. The notion of a *nilpotent* space is a generalization as follows.

Definition 6.1. *A space is called nilpotent, if it is of the homotopy type of a connected CW-complex X such that $\pi_1(X)$ is a nilpotent group and all homotopy groups $\pi_i(X)$, $i > 1$, are nilpotent $\pi_1(X)$-modules.*

A space X is called *quasi-finite* if $\oplus_n H_n(X;\mathbb{Z})$ is a finitely generated abelian group. For nilpotent spaces, the following criterion for finite domination holds (cf. [52]).

Lemma 6.2. *A nilpotent space X is finitely dominated if and only if it is quasi-finite.*

The Wall obstruction of a finitely dominated nilpotent space is subjected to stringent restrictions. For the case of an infinite fundamental group, the following *vanishing theorem* holds (cf. [52]).

Theorem 6.3. *Let X be a finitely dominated nilpotent space with infinite fundamental group. Then*

$$w(X) = 0 \in K_0(\mathbb{Z}\pi_1(X)).$$

Therefore, X is homotopy equivalent to a finite CW-complex and $\chi(X) = 0$.

Proof. Since $\pi_1(X)$ is an infinite, finitely generated nilpotent group, it admits a surjective homomorphism $\pi_1(X) \to \mathbb{Z}$. The associated covering space $\overline{X}$ is classified by a map $p : X \to S^1$, which has as homotopy theoretic fiber the covering space $\overline{X}$. One checks that $\overline{X}$ is nilpotent and quasi-finite (cf. [52]), thus finitely dominated. Obviously,

$$\chi(X) = \chi(\overline{X}) \cdot \chi(S^1) = 0,$$

and, from the fibration theorem,

$$\tilde{w}(X) = p^*\tilde{w}(S^1) + \chi(S^1) \cdot \tilde{w}(\overline{X}) = 0,$$

which completes the proof. □

The next theorem is from [51]; we will only sketch its proof.

Theorem 6.4. *Let X be a finitely dominated nilpotent space with finite cyclic fundamental group of prime order. Then X is homotopy equivalent to a finite CW-complex.*

Proof. Let $\pi = \pi_1(X) \cong \mathbb{Z}/p\mathbb{Z}$ and $\Lambda = \mathbb{Z}[\exp(2\pi\sqrt{-1}/p)]$. Consider Λ as a π-module via a surjective homomorphism $\mathbb{Z}\pi \to \Lambda$. By Rim's theorem, the induced map

$$\theta : \tilde{K}_0(\mathbb{Z}\pi) \to \tilde{K}_0(\Lambda)$$

is an isomorphism. It therefore suffices to show that

$$\theta(\tilde{\chi}(C_*^{cell}(\tilde{X}))) = \tilde{\chi}(\Lambda \otimes_\pi C_*^{cell}(\tilde{X})) = 0\,.$$

Because Λ has finite cohomological dimension (actually $\mathrm{cd}\Lambda = 1$), one can define the *reduced Euler characteristic*

$$\tilde{\chi}(M) \in \tilde{K}_0(\Lambda)$$

for *any* finitely generated Λ-module M, by putting

$$\tilde{\chi}(M) = \sum_i (-1)^i \{P_i\}\,,$$

where

$$0 \to P_n \to \cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to M$$

denotes a projective resolution of finite length of M, with all the modules P_i finitely generated. One can now compute the Euler characteristic in question by passing to homology:

$$\tilde{\chi}(\Lambda \otimes_\pi C_*^{cell}(\tilde{X})) = \tilde{\chi}(H_i(X;\Lambda)) \in \tilde{K}_0(\Lambda)\,.$$

Because of the nilpotency of X and using the homology sequence associated with the short exact sequence of π-modules

$$0 \to \mathbb{Z} \to \mathbb{Z}\pi \to \Lambda \to 0\,,$$

one can show that, as abelian groups, all homology groups $H_i(X;\Lambda)$ are finite p-groups. On the other, for the trivial π-module $\mathbb{Z}/p\mathbb{Z}$, considered as a Λ-module by letting $\exp(2\pi\sqrt{-1}/p)$ operate via the identity map, one has

$$\tilde{\chi}(\mathbb{Z}/p\mathbb{Z}) = 0 \in \tilde{K}_0(\Lambda)\,.$$

This is plain in view of the following short exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{1-\exp(2\pi\sqrt{-1}/p)} \Lambda \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0\,.$$

Using induction, it is now easy to prove that for *any* Λ-module M such that the underlying abelian group is a finite p-group, one has $\tilde{\chi}(M) = 0$. The claim of the theorem then follows readily. $\square$

The previous theorem admits the following generalization. Let π denote an arbitrary finite group and write $\overline{\mathbb{Z}\pi}$ for a maximal $\mathbb{Z}$-order containing $\mathbb{Z}\pi$ in the rational group algebra $\mathbb{Q}\pi$. Define

$$D(\mathbb{Z}\pi) := \ker\left(j_* : K_0(\mathbb{Z}\pi) \to K_0(\overline{\mathbb{Z}\pi})\right),$$

where $j : \mathbb{Z}\pi \to \overline{\mathbb{Z}\pi}$ denotes the natural inclusion (one verifies that $D(\mathbb{Z}\pi)$ is independent of the particular choice of the maximal order). In case of $\pi = \mathbb{Z}/p\mathbb{Z}$ with p a prime, one has $\overline{\mathbb{Z}\pi} \subset \Lambda \times \mathbb{Z}$ with $\Lambda = \mathbb{Z}[\exp(2\pi\sqrt{-1}/p)]$, and Rim's theorem asserts that j_* is injective in that case, thus $D(\mathbb{Z}/p\mathbb{Z}) = 0$. The previous theorem then becomes a special case of the following result.

Theorem 6.5. *Let X be a finitely dominated nilpotent space with non-trivial finite fundamental group $\pi_1(X) = \pi$. Then*

(i) $w(X) \in D(\mathbb{Z}\pi)$;
(ii) $|\pi| \cdot w(X) = 0$ *in case π is a p-group.*

For the proof, the reader is referred to [55]. Part *(ii)* of the theorem follows from results concerning the exponent of $D(\mathbb{Z}\pi)$ due to Ullom [67]. It is known that $D(\mathbb{Z}\pi) = 0$ for π a dihedral 2-group (cf. Fröhlich–Keating–Wilson [30]), and accordingly the finiteness obstruction vanishes for a finitely dominated nilpotent space with such fundamental groups. But in general, $D(\mathbb{Z}\pi)$ is rather large (see Taylor [65]). Note also that *(ii)* does not hold in general, if π is not a p-group; in [54] it is shown that there exists a finitely dominated nilpotent space with fundamental group cyclic of order 15 and Wall obstruction of order 2. It is also known that $D(\mathbb{Z}/22\mathbb{Z}) \neq 0$, although for a finitely dominated nilpotent space X with $\pi_1(X) \cong \mathbb{Z}/22\mathbb{Z}$ one has $w(X) = 0$ (cf. [54]). Thus, one is led to consider the subset

$$N(\mathbb{Z}\pi) \subset K_0(\mathbb{Z}\pi)$$

consisting of all those elements which arise as finiteness obstructions of finitely dominated nilpotent spaces with fundamental group π. Ewing-Löffler-Pedersen showed that for π a finite nilpotent group $N(\mathbb{Z}\pi)$ is actually a subgroup of $K_0(\mathbb{Z}\pi)$. For $\pi \neq \{e\}$ a finite nilpotent group, we have

$$N(\mathbb{Z}\pi) \subset D(\mathbb{Z}\pi)$$

by the previous theorem, and in general $N(\mathbb{Z}\pi) \neq D(\mathbb{Z}\pi)$, as the case of $\pi = \mathbb{Z}/22\mathbb{Z}$ illustrates. (Note that for the trivial group π one has $N(\mathbb{Z}\pi) = \mathbb{Z}$ but $D(\mathbb{Z}\pi) = 0$; that's why we have to assume $\pi \neq \{e\}$.) For p-groups, the following result holds (see Ewing-Löffler-Pedersen [23]).

Theorem 6.6. *Let π be a non-trivial finite p-group. Then*

$$N(\mathbb{Z}\pi) = D(\mathbb{Z}\pi).$$

In particular it follows that the finiteness obstruction does not vanish in general for a finitely dominated nilpotent space whose fundamental group is a p-group. Indeed, for π a p-group of order p^n with $p > 2$ and $n \geqslant 5$ one has

$$|D(\mathbb{Z}\pi)| \geqslant p^{n-1},$$

see Taylor [65]; (there is a similar result for the prime 2: if $\pi = \mathbb{Z}/2^n\mathbb{Z}$ with $n \geqslant 5$, one has $|D(\mathbb{Z}\pi)| \geqslant 2^{n-2}$, whereas for a generalized quaternion group Q_n of order $2^n \geqslant 8$ one has $|D(\mathbb{Z}Q_n)| = 2$). It should also be noted that one can even find *simple*, finitely dominated spaces not of the homotopy type of finite CW-complexes (cf. [54]). Of course, H-spaces are simple. However, no finitely dominated H-space with non-vanishing finiteness obstruction is known (some partial vanishing results on the Wall obstruction of H-spaces can be found in [52]).

7. Localization techniques

We will be concerned in this section with the study of the finiteness obstruction of a finitely dominated space X with finite fundamental group acting *trivially* on the rational homology of the universal cover of X (e.g., X a nilpotent space). In such a situation, after choosing a basis for the rational homology of X, the *p-local Reidemeister torsion* $\mathrm{RT}_p(X)$ is defined and is used to define the "p-part" $w_p(X)$ of the finiteness obstruction $w(X)$. It turns out that $w_p(X)$ may be non-zero, even if $w(X) = 0$; this is the reason why the "Zabrodsky mixing" of two finite CW-complexes can fail to be homotopy equivalent to a finite CW-complex. We will write as usual $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at the prime ideal (p), i.e., the subring of $\mathbb{Q}$ consisting of fractions with denominators prime to p, and we write $\mathbb{Z}[1/p]$ for the subring consisting of fractions with denominators involving only powers of p. For a nilpotent space X we will write $X_{(p)}$ for its p-localization, and $X_{(0)}$ for its rationalization (cf. [35]).

7.1. p-local Reidemeister torsion

Let X be an arbitrary CW-complex. We will call X *$\mathbb{Q}$-based*, if a fixed basis for $H_*(X;\mathbb{Q})$ is chosen. If X is a finite, $\mathbb{Q}$-based CW-complex, the chain complex $C_*^{cell}(X;\mathbb{Q})$ with its natural basis, together with the given homology basis defines a torsion invariant in $K_1(\mathbb{Q}) \cong \mathbb{Q}^\times$, the group of units of $\mathbb{Q}$ (see Milnor [49]). Now let X be a connected, finitely dominated $\mathbb{Q}$-based CW-complex with finite fundamental group π *operating trivially* on the rational homology of the universal cover of X. If we write $e \in \mathbb{Q}\pi$ for the idempotent $\frac{1}{|\pi|}\sum_{x\in\pi} x$, we obtain a decomposition

$$\mathbb{Q}\pi = e \cdot \mathbb{Q}\pi \times (1-e) \cdot \mathbb{Q}\pi = \mathbb{Q} \times \mathrm{A}\,, \qquad \mathrm{A} := (1-e) \cdot \mathbb{Q}\pi\,.$$

Let P be a chain complex of type FP, chain homotopy equivalent to the cellular chain complex of the universal cover of X. Since, according to Swan [63], for a finitely generated projective $\mathbb{Z}\pi$ module M the $\mathbb{Z}_{(p)}\pi$-module $M_{(p)} := M \otimes \mathbb{Z}_{(p)}$ is free, we can choose a basis for $P_{(p)} := P \otimes \mathbb{Z}_{(p)}$, and obtain from it an induced basis for $P_{(0)} := P \otimes \mathbb{Q}$. Using the natural splitting of $P_{(0)}$ into a $\mathbb{Q}$-complex and an A-complex,

$$P_{(0)} = e \cdot P_{(0)} \times (1-e) \cdot P_{(0)}\,,$$

one observes that, because π operates trivially on the rational homology of X, the homology of $e \cdot P_{(0)}$ is naturally isomorphic via the projection $P_{(0)} \to P \otimes_\pi \mathbb{Q}$ to the homology of X with $\mathbb{Q}$-coefficients, and $H_*(e \cdot P_{(0)})$ has thus a natural basis because X is supposed to be $\mathbb{Q}$-based. On the other hand $(1-e) \cdot P_{(0)}$ is acyclic. From these two based complexes (with based homologies) one obtains thus a pair of torsion invariants in

$$K_1(\mathbb{Q}\pi) = K_1(\mathbb{Q}) \times K_1(A)\,.$$

Since everything depends on the choice of the basis of $P_{(p)}$, the torsion invariant will only be well-defined if we calculate modulo the image of $K_1(\mathbb{Z}_{(p)}\pi)$ in $K_1(\mathbb{Q}\pi)$. The resulting invariant will be denoted by

$$\mathrm{RT}_p(X) \in K_1(\mathbb{Q}\pi)/\mathrm{im}\,(K_1(\mathbb{Z}_{(p)}\pi))$$

and is called (by Ewing-Löffler-Pedersen [23]) the *p-local Reidemeister torsion* of the $\mathbb{Q}$-based space X.

For p any prime and π any group, the commutative diagram with obvious arrows

$$\begin{array}{ccc} \mathbb{Z}\pi & \longrightarrow & \mathbb{Z}_{(p)}\pi \\ \downarrow & & \downarrow \\ \mathbb{Z}[1/p]\pi & \longrightarrow & \mathbb{Q}\pi \end{array}$$

gives rise to a long exact sequence (see Bass [2]) of K-groups

$$\begin{aligned} &K_1(\mathbb{Z}\pi) \to K_1(\mathbb{Z}_{(p)}\pi) \times K_1(\mathbb{Z}[1/p]\pi) \to K_1(\mathbb{Q}\pi) \xrightarrow{\partial^p} \\ &K_0(\mathbb{Z}\pi) \to K_0(\mathbb{Z}_{(p)}\pi) \times K_0(\mathbb{Z}[1/p]\pi) \to K_0(\mathbb{Q}\pi)\,. \end{aligned}$$

We are interested in the *connecting homomorphism*

$$\partial^p : K_1(\mathbb{Q}\pi) \longrightarrow K_0(\mathbb{Z}\pi)\,,$$

which is used in the following definition. Note that the image of $K_1(\mathbb{Z}_{(p)}\pi)$ in $K_1(\mathbb{Q}\pi)$ lies in the kernel of ∂^p.

Definition 7.1. *Let X be a connected, finitely dominated $\mathbb{Q}$-based CW-complex with finite fundamental group π operating trivially on the rational homology of the universal cover of X. Then for every prime p the* p-part *$w_p(X)$ of the Wall obstruction of X is*

$$\partial^p(\mathrm{RT}_p(X)) =: w_p(X) \in K_0(\mathbb{Z}\pi).$$

In [23] Ewing-Löffler-Pedersen show that the p-parts $w_p(X)$ are zero for almost all primes p and it therefore makes sense to form their sum.

Theorem 7.2. *Let X be a connected, finitely dominated $\mathbb{Q}$-based CW-complex with finite non-trivial fundamental group π operating trivially on the rational homology of the universal cover of X. Then*

$$\sum_p w_p(X) = w(X) \in K_0(\mathbb{Z}\pi_1(X)).$$

In case π is a finite nilpotent group, we write π_p for the Sylow p-subgroup of π, and one has a natural decomposition

$$\pi = \pi_p \times \pi',$$

with π' consisting of all elements of order prime to p. The idempotent

$$e_p := \frac{1}{|\pi'|} \sum_{x \in \pi'} x \in \mathbb{Q}\pi$$

provides a splitting of the group algebra

$$\mathbb{Q}\pi = e_p \cdot \mathbb{Q}\pi \times (1 - e_p) \cdot \mathbb{Q}\pi,$$

and yields a corresponding splitting

$$K_1(\mathbb{Q}\pi) = K_1(e_p \cdot \mathbb{Q}\pi) \times K_1((1 - e_p) \cdot \mathbb{Q}\pi).$$

Note that the projection $\pi \to \pi_p$ induces an isomorphism $e_p \cdot \mathbb{Q}\pi \to \mathbb{Q}\pi_p$ so that

$$K_1(e_p \cdot \mathbb{Q}\pi) \cong K_1(\mathbb{Q}\pi_p).$$

Definition 7.3. *Let π be a finite nilpotent group. Then*

$$N_p(\mathbb{Z}\pi) := \partial^p(K_1(e_p \cdot \mathbb{Q}\pi) \times \{0\}) \subset K_0(\mathbb{Z}\pi).$$

Recall that for a finite nilpotent group π we defined $N(\mathbb{Z}\pi)$ to consists of those elements of the projective class group of π which are realizable as finiteness obstructions of finitely dominated nilpotent spaces with fundamental group π. The connection with the subgroups $N_p(\mathbb{Z}\pi)$ is given by the following theorem [23].

Theorem 7.4. *Let π be a non-trivial finite nilpotent group. Then*

$$\sum_p N_p(\mathbb{Z}\pi) = N(\mathbb{Z}\pi) \subset K_0(\mathbb{Z}\pi)\,.$$

If π is an arbitrary finite group, the *Swan subgroup*

$$\mathrm{T}(\mathbb{Z}\pi) \subset K_0(\mathbb{Z}\pi)$$

is defined as the image of the boundary map ∂^{T} in the Milnor exact sequence

$$\cdots \to K_1(\mathbb{Z}/|\pi|\mathbb{Z}) \xrightarrow{\partial^{\mathrm{T}}} K_0(\mathbb{Z}\pi) \to \cdots$$

associated with the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}\pi & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}\pi/(\Sigma) & \longrightarrow & \mathbb{Z}/|\pi|\mathbb{Z} \end{array}$$

where (Σ) denotes the ideal generated by $\Sigma = \sum_{x\in\pi} x$ and the arrows are the obvious maps. The Swan subgroup is quite computable. Its elements are precisely those which can be written in the form $[(k,\Sigma)] - [\mathbb{Z}\pi]$ where $(k,\Sigma) \subset \mathbb{Z}\pi$ denotes the (projective) ideal generated by k and Σ, where $k \in \mathbb{Z}$ is prime to $|\pi|$. According to Ullom [68], the Swan subgroup is trivial for cyclic groups. In general, its exponent divides the Artin exponent of π, and if π is a p-group, $\mathrm{T}(\mathbb{Z}\pi)$ is cyclic (for π a p-group $\mathrm{T}(\mathbb{Z}\pi)$ is completely known, see Taylor [65]). It is not hard to see that in general

$$\mathrm{T}(\mathbb{Z}\pi) \subset D(\mathbb{Z}\pi)\,,$$

and one can show that (cf. [54]) for a finite nilpotent group π one always has has

$$\mathrm{T}(\mathbb{Z}\pi) \subset \mathrm{N}(\mathbb{Z}\pi)\,.$$

In the abelian case one even has the following result [54].

Theorem 7.5. *Let π be a finite abelian group and $x \in \mathrm{T}(\mathbb{Z}\pi)$. Then there exists a finitely dominated connected,* simple *CW-complex X with fundamental group π and $w(X) = x$.*

Remark . The definition of the p-part $w_p(X)$ of the Wall obstruction of X depended on the choice of a basis for the rational homology of X. It is shown in Ewing-Löffler-Pedersen [23] that a different choice results in a change of $w_p(X)$ by an element in $\mathrm{T}(\mathbb{Z}\pi)$. In particular, in case π is cyclic one has $\mathrm{T}(\mathbb{Z}\pi) = 0$ and thus $w_p(X)$ will be independent of that choice.

7.2. Fiberwise localization and genus

For X a connected CW-complex with fundamental group π one has a natural fibration up to homotopy of the form

$$\tilde{X} \to X \to B\pi$$

which, for any prime p, admits a fiberwise p-localization for which we will use the notation

$$\tilde{X}_{(p)} \to X_{(p-\pi)} \to B\pi\,,$$

where $\tilde{X}_{(p)}$ stands for $(\tilde{X})_{(p)}$. In case the group π is finite, of order prime to p and acting trivially on the p-local homology of $\tilde{X}$, this fibration is fiber homotopy trivial, thus $X_{(p-\pi)} \simeq \tilde{X}_{(p)} \times B\pi$. Moreover, if X is nilpotent with finite fundamental group, one has

$$X_{(p-\pi)} \simeq X_{(p)} \times B\pi'\,,$$

where π' denotes the subgroup of the fundamental group of X consisting of all elements of order prime to p. (For a definition of fiberwise localizations on the level of spaces see Bousfield-Kan [9].)

As observed by Wojtkowiak in [78], the following holds.

Theorem 7.6. *Let X be a connected CW-complex with finite fundamental group π and let Y be the universal cover of $X_{(p-\pi)}$ with its natural π action. Then the singular chain complexes $C_*^{sing}(\tilde{X}) \otimes \mathbb{Z}_{(p)}$ and $C_*^{sing}(Y) \otimes \mathbb{Z}_{(p)}$ are naturally chain homotopy equivalent over $\mathbb{Z}_{(p)}\pi$.*

Indeed, the natural π-map $\tilde{X} \to Y$ induces the equivalence. As a result, the p-local Reidemeister torsion $\mathrm{RT}_p(X)$ depends only on the homotopy type of $X_{(p-\pi)}$ and the choice of a basis for the rational homology of that space. In the nilpotent situation this implies that the finiteness obstruction depends, modulo the image of the Swan homomorphism, only on the homotopy types $X_{(p)}$. More precisely, the following holds. Recall that for a nilpotent space of finite type X the genus set $G(X)$ of X consists of all nilpotent homotopy types of finite type Y such that for every prime p one has $Y_{(p)} \simeq X_{(p)}$ (for basic properties of the genus see [35]).

Theorem 7.7. *Let X be a finitely dominated nilpotent space with finite fundamental group and let $Y \in G(X)$. Then there is an isomorphism $\phi : \pi_1(Y) \to \pi_1(X)$*

such that

$$w(X) - \phi_* w(Y) \in \mathrm{T}(\mathbb{Z}\pi_1(X)) \subset K_0(\mathbb{Z}\pi_1(X))\,.$$

For a proof see [23] or [77]. Note that if Y is in the genus of a *finite*, nilpotent CW-complex with finite non-trivial fundamental group, then we can conclude that $w(Y)$ lies in the Swan subgroup (but it is not necessarily zero, for an example see [52]).

The same methods can be used to compute the Wall obstruction for a "Zabrodsky mixing".

Theorem 7.8. *Suppose $P \cup Q$ is a partition of the set of primes and M, X, Y and Z are finitely dominated nilpotent spaces with finite fundamental groups, such that there is a pull-back diagram*

$$\begin{array}{ccc} M & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

with the horizontal arrows being P-equivalences, and the vertical ones Q-equivalences. Then there is a P-equivalence $\phi_X : \pi_1(X) \to \pi_1(M)$ and a Q-equivalence $\phi_Y : \pi_1(Y) \to \pi_1(M)$ such that

$$w(M) = \sum_{p \in P} {\phi_X}_* w_p(X) + \sum_{q \in Q} {\phi_Y}_* w_q(Y)$$

where the p-parts of the Wall obstructions are supposed to be computed with respect to bases of the rational homology of X and Y which correspond to each other via the induced isomorphisms

$$H_*(X;\mathbb{Q}) \xrightarrow{\cong} H_*(Z;\mathbb{Q}) \xleftarrow{\cong} H_*(Y;\mathbb{Q})\,.$$

Another interesting and closely related question is the following. Suppose X is a finitely dominated nilpotent space and p a prime. Is $X_{(p)}$ necessarily homotopy equivalent to $Y_{(p)}$ for some *finite* nilpotent CW-complex Y? The answer is "no" in general. Indeed, if we choose for X a finitely dominated nilpotent space with $\pi_1(X) \cong \mathbb{Z}/p^n\mathbb{Z}$ such that $w(X) \neq 0$, then one has obviously $w(X) = w_p(X) \neq 0$, but for Y a finite CW-complex with the same fundamental group, one has $w(Y) = w_p(Y) = 0$ (recall that $\mathrm{T}(\mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}]) = 0$ so that there is no ambiguity in the definition of the local Wall obstructions). Thus, as $w_p(X)$ depends only on $X_{(p-\pi)} = X_{(p)}$, and similarly for $w_p(Y)$, we see that necessarily $X_p \not\simeq Y_p$. For a thorough discussion of these matters, see [23].

7.3. The spherical space form problem

A classical question asks for the description of all topological manifolds M with universal cover homeomorphic to S^n. The fundamental group of such a manifold is necessarily a finite group with periodic cohomology and, in case M is orientable, the (minimal) period of $\pi_1(M)$ divides $n+1$. If M a non-orientable, then M is easily seen to be homotopy equivalent to the projective space $\mathrm{P}^n(\mathbb{R})$ with n even; we shall concentrate on the orientable case in the sequel.

A finite group with periodic cohomology is called a $\mathcal{P}$-group. The $\mathcal{P}$-groups are characterized by the fact that all their abelian subgroups are cyclic. If the period of a $\mathcal{P}$-group π is k, then

$$H^k(\pi;\mathbb{Z}) = \mathbb{Z}/|\pi|\mathbb{Z}\,.$$

Conversely, if for a finite group π one has $H^i(\pi;\mathbb{Z}) = \mathbb{Z}/|\pi|\mathbb{Z}$, then π is a $\mathcal{P}$-group of period dividing i.

The following emerge as the natural and basic questions:

- *Which $\mathcal{P}$-groups admit free actions on spheres?*
- *If the $\mathcal{P}$-group π acts freely on S^n, what are the possible values of n? Is the minimal value of n equal to $k-1$, where k is the period of π ?*

It has been known for a long time that not every $\mathcal{P}$-group admits a free action on some sphere. Namely, Milnor proved in [48] that if the finite group π acts freely on a sphere, then all elements of order two in π are central. Thus, for instance the symmetric group $\mathcal{S}_3$, which has period 4, does not act freely on any sphere. But according to Madsen, Thomas and Wall ([66],[43]), Milnor's condition is the only obstruction to finding an action, and the following converse holds.

Theorem 7.9. *If the $\mathcal{P}$-group π has the property that all elements of order 2 are central, then π admits a free action on some sphere.*

This answers our first question completely. If the finite group π acts freely on S^n, the orbit space S^n/π is a compact topological manifold, which is an ANR and therefore homotopy equivalent to a finite CW-complex Y. Thus π acts freely and *cellularly* on the finite CW-complex $\tilde{Y}$ homotopy equivalent to S^n. One can therefore divide the second question up into two separate ones, a purely homotopy theoretical one, and a surgery problem:

- •• *Given a $\mathcal{P}$-group π, find all values of n such that π acts freely and cellularly on a finite CW-complex homotopy equivalent to a sphere S^n.*
- •• *Suppose the $\mathcal{P}$-group π acts freely and cellularly on a finite CW-complex homotopy equivalent to S^n. Does π admit a free action on S^n ?*

We will only sketch how the homotopy problem can be reduced to an investigation concerning finiteness obstructions; a thorough analysis as well as a discussion of the surgery problem, which we won't address here, can be found in the excellent survey by Davis and Milgram [17].

If a finite group π acts freely and cellularly on the finite CW-complex X homotopy equivalent to S^n, preserving the orientation, then one obtains a complex

$$\cdots \to C^{cell}_{n+1}(X) \to C^{cell}_n(X) \to \cdots \to C^{cell}_0(X) \to \mathbb{Z} \to 0$$

with $B^{cell}_n(X) = \mathrm{im}(C^{cell}_{n+1}(X) \to C^{cell}_n(X))$ a direct summand. This direct summand is stably free because it represents the reduced finiteness obstruction for the finite CW-complex X/π. It is then easy to modify the complex to obtain a *periodic resolution*

$$0 \to \mathbb{Z} \to F_n \to F_{n-1} \to C^{cell}_{n-2}(X) \to \cdots \to C^{cell}_0(X) \to \mathbb{Z} \to 0\,,$$

with F_n, F_{n-1} and all the modules $C^{cell}_i(X)$, $0 \leqslant i \leqslant n-2$, finitely generated and free over $\mathbb{Z}\pi$. In particular, π is a $\mathcal{P}$-group of period dividing $n+1$. Conversely, according to Swan [62], if π is a $\mathcal{P}$-group of period k then π admits periodic resolutions

$$0 \to \mathbb{Z} \to P_{lk-1} \to P_{lk-2} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0\,, \qquad l \geqslant 1\,,$$

with each P_i finitely generated projective. Swan proved in [62] that

$$\sum_{i=0}^{lk-1} (-1)^i [P_i] \in K_0(\mathbb{Z}\pi)$$

depends, modulo the Swan subgroup $\mathrm{T}(\mathbb{Z}\pi)$, only on the pair (π, l), and hence gives rise to elements

$$s_{lk}(\pi) \in K_0(\mathbb{Z}\pi)/\mathrm{T}(\mathbb{Z}\pi)\,, \qquad l \geqslant 1,$$

with k standing for the period of π. These elements are called *Swan obstructions*, and $s_k(\pi)$ is called *the* Swan obstruction of π. According to Swan the following holds (cf. [62]).

Theorem 7.10. *Let π be a $\mathcal{P}$-group of period k. Then for every $l \geqslant 1$*

(i) $s_{lk}(\pi) = 0$ *if and only if π admits a free, cellular action on a finite CW-complex homotopy equivalent to S^{lk-1};*

(ii) $s_{lk}(\pi) = l \cdot s_k(\pi)$.

It is clear that the elements $s_{lk}(\pi)$ have finite order, that is, $\mathrm{rk}(s_{lk}(\pi)) = 0$. Indeed, because the period k of a $\mathcal{P}$-group is always *even* we obtain, by computing homology with $\mathbb{Q}$ coefficients,

$$\mathrm{rk}(\sum_{i=0}^{lk-1} (-1)^i [P_i]) = 1 - 1 = 0\,.$$

It follows that π acts on *some* finite CW-complex $X \simeq S^{lk-1}$, by taking for l the order of $s_k(\pi)$. Wall improved this considerably (cf. [73]) by showing that one can always take $l = 2$, that is

$$2 \cdot s_k(\pi) = 0 \in K_0(\mathbb{Z}\pi)/\mathrm{T}(\mathbb{Z}\pi)\,.$$

For almost all families of $\mathcal{P}$-groups of period k one can show that the Swan obstruction $s_k(\pi)$ actually vanishes. Moreover, one can show that if $s_k(\pi) \neq 0$, then π must contain a subgroup of the form $Q(2^n a, b, c)$ with a, b, c coprime odd integers and $n \geqslant 3$ (the notation for these groups is due to Milnor [48]). These groups are defined as semi-direct products of the form

$$1 \to \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z} \times \mathbb{Z}/c\mathbb{Z} \to Q(2^n a, b, c) \to Q_n \to 1$$

with

$$Q_n = \mathrm{gp}(x, y \mid x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, yxy^{-1} = x^{-1})$$

the quaternion group of order 2^n, acting so that x inverts the elements of $\mathbb{Z}/a\mathbb{Z}$ and $\mathbb{Z}/b\mathbb{Z}$ while y inverts those of $\mathbb{Z}/a\mathbb{Z}$ and $\mathbb{Z}/c\mathbb{Z}$. These groups all have period 4 and it turns out that the associated Swan obstruction does not vanish in general, but there is no simple minded pattern. For instance, according to Milgram [46] one has $s_4(Q(24,5,1)) \neq 0$ and $s_4(Q(24,13,1)) = 0$. The smallest group with non-vanishing Swan obstruction is $Q(16,3,1)$ a group of order 48 (cf. Davis [16]). Note that one can conclude that the groups $Q(24,5,1)$ and $Q(16,3,1)$ cannot be fundamental groups of 3-manifolds! The computation of the Swan obstruction relies on interpreting it as the image of some Reidemeister torsion element in the following way. Let $\mathbb{Z}_p$ denote the p-adic integers and $\mathbb{Q}_p$ the field of p-adic numbers. Consider the pull-back square

$$\begin{array}{ccc} \mathbb{Z}\pi & \longrightarrow & \mathbb{Z}[1/|\pi|]\pi \\ \downarrow & & \downarrow \\ \prod_{p\mid|\pi|} \mathbb{Z}_p\pi & \longrightarrow & \prod_{p\mid|\pi|} \mathbb{Q}_p\pi \end{array}$$

with associated Milnor sequence

$$\cdots \longrightarrow \prod_{p\mid|\pi|} K_1(\mathbb{Q}_p\pi) \xrightarrow{\partial^\pi} K_0(\mathbb{Z}\pi) \longrightarrow \cdots\,.$$

Composing with the projection $K_0(\mathbb{Z}\pi) \to K_0(\mathbb{Z}\pi)/\mathrm{T}(\mathbb{Z}\pi)$ yields a homomorphism

$$\overline{\partial}^\pi : \prod_{p\mid|\pi|} K_1(\mathbb{Q}_p\pi) \longrightarrow K_0(\mathbb{Z}\pi)/\mathrm{T}(\mathbb{Z}\pi)\,.$$

It turns out that the Swan obstructions $s_{lk}(\pi)$ lie in the image of $\overline{\partial}^{\pi}$. Indeed, one can compute $s_{lk}(\pi)$ as follows. Given a $\mathcal{P}$-group π of period k and a periodic resolution of length lk,

$$\mathrm{P}(\pi, lk) : 0 \to \mathbb{Z} \to P_{lk-1} \to \cdots \to P_0 \to \mathbb{Z} \,,$$

with each P_i finitely generated and projective over $\mathbb{Z}\pi$. One can think of $\mathrm{P}(\pi, lk)$ as being obtained as a pull–back of the form

$$\begin{array}{ccc} \mathrm{P}(\pi, lk) & \longrightarrow & \mathbb{Z}[1/|\pi|] \otimes \mathrm{P}(\pi, lk) \\ \downarrow & & \downarrow \\ \prod_{p||\pi|} \mathbb{Z}_p \otimes \mathrm{P}(\pi, lk) & \longrightarrow & \prod_{p||\pi|} \mathbb{Q}_p \otimes \mathrm{P}(\pi, lk) \,. \end{array}$$

This pull-back is completely determined by a family of "twisting isomorphisms"

$$\phi_{i,p} : \mathbb{Q}_p \otimes P_i \longrightarrow \mathbb{Q}_p \otimes P_i \,, \qquad p\big||\pi| \,,$$

defining a Reidemeister torsion element

$$\tau(\mathrm{P}(\pi, lk)) := \sum_{i=0}^{lk-1} \sum_{p||\pi|} (-1)^i [\phi_{i,p}] \in \prod_{p||\pi|} K_1(\mathbb{Q}_p\pi) \,,$$

which is uniquely determined modulo the image of

$$(\prod_{p||\pi|} K_1(\mathbb{Z}_p\pi)) \times K_1(\mathbb{Z}[1/|\pi|]\pi) \longrightarrow \prod_{p||\pi|} K_1(\mathbb{Q}_p\pi) \,,$$

and it satisfies

$$\overline{\partial}^{\pi}(\tau(\mathrm{P}(\pi, lk))) = s_{lk}(\pi) \in K_0(\mathbb{Z}\pi)/\mathrm{T}(\mathbb{Z}\pi) \,.$$

It turns out that the computation of $s_{lk}(\pi)$ is closely related to the structure of the ring of integers (and strictly positive integers) in cyclotomic number fields. The interested reader is invited to consult [46], [47], [17] and [16] for more details.

References

[1] B. A. Artamonov, Projective, nonfree modules over group rings of solvable groups, (Russian) Mat. sb. (N.S.) 116 (1981) 232–244.

[2] H. Bass, *Algebraic K-Theory*, Benjamin, New York 1968.

[3] H. Bass , Projective modules over free groups are free, J. of Algebra 4 (1964) 367–373.

[4] H. Bass and M. P. Murthy, Grothendieck groups and Picard groups of abelian group rings, Ann. of Math. 86 (1967) 16–73.

[5] H. Bass, A. Heller, R. Swan, The Whitehead group of a polynomial extension, Publ. IHES Paris 22 (1964) 67–79.

[6] P. H. Berridge and M. J. Dunwoody, Non-free projective modules for torsion-free groups, J. London Math. Soc. (2) 19 (1979) 433–436.

[7] R. Bieri and B. Eckmann, Finiteness properties of duality groups, Comment. Math. Helv. 49 (1974) 74–83.

[8] K. Borsuk, Sur l'élimination de phenomènes paradoxaux en topologie générale, Proc. Internat. Congr. Math., Vol 1, Amsterdam, (1954) 197–208.

[9] A. K. Bousfield and D. M. Kan, *Homotopy limits, Completions and Localizations*, Lecture Notes in Math. 304 (1972), Springer Verlag.

[10] K. S. Brown, Homological criteria for finiteness, Comment. Math. Helv. 50 (1975) 129–136.

[11] B. Bürgisser, On the projective class group of arithmetic groups, Math. Z. 184 (1983) 339–357.

[12] T. A. Chapman, Compact Hilbert cube manifolds and the invariance of Whitehead torsion, Bull. Amer. Math. Soc. 79 (1973) 52–56.

[13] T. A. Chapman, Topological invariance of Whitehead torsion, Amer. J. Math. 93 (1974) 488–497.

[14] T. A. Chapman, Invariance of torsion and the Borsuk conjecture, Canad. J. Math. 32 (1980) 1333–1341.

[15] M. M. Cohen, *A Course in Simple Homotopy Theory*, Graduate Texts in Mathematics Vol. 10, Springer Verlag 1970.

[16] J. F. Davis, Thesis, Stanford University 1982.

[17] J. F. Davis and R. J. Milgram, A survey of the spherical space form problem, Mathematical Reports 2 (1985) 223–283, Harwood Academic Publisher GmbH and OPA Ltd.

[18] M. J. Dunwoody, Relation Modules, Bull. London Math. Soc. (1972) 151–155.

[19] B. Eckmann and P. J. Hilton, Décomposition homologique d'un polyèdre simplement connexe, C. R. Acad. Sci. Paris 248 (1959) 2054–2056.

[20] B. Eckmann and S. Maumary, Le groupe des types simples d'homotopie sur un polyèdre, in *Essays on Topology and Related Topics, Memoires dédiés à Georges de Rham*, Springer–Verlag (1970) 173–187.

[21] K. Ehrlich, Fibrations and transfer map in algebraic K-theory, J. Pure Appl. Algebra 14 (1979) 131–136.

[22] K. Ehrlich, The obstruction to the finiteness of the total space of a fibration, Michigan Math. J. 28 (1981) 19–38.

[23] J. Ewing, P. Löffler and E. K. Pedersen, A local approach to the finiteness obstruction, Quart. J. Math. Oxford Ser.(2) 39 (1988) No.156, 443–461.

[24] F. T. Farrell and W. C. Hsiang, The topological-Euclidean space form problem, Invent. Math 45 (1978) 181–192.

[25] F. T. Farrell and W. C. Hsiang, The Whitehead group for poly-(finite or cyclic) groups, J. London Math. Soc. 24 (1981) 308–324.

[26] F. T. Farrell and L. E. Jones, Rigidity and other topological aspects of compact nonpositively curved manifolds, Bull. Amer. Math. Soc. 22 (1990) 59–64.

[27] S. Ferry, Homotopy, simple homotopy, and compacta, Topology 19 (1980) 101–110.

[28] S. Ferry, Finitely dominated compacta need not have finite type, In proceedings : *Shape Theory and Geometric Topology*, Dubrownik 1981, Springer Lecture Notes in Math. 870 (1981) 1–5.

[29] S. Ferry, A simple-homotopy approach to the finiteness obstruction, In proceedings: *Shape Theory and Geometric Topology*, Dubrownik 1981, Springer Lecture Notes in Math. 870 (1981) 73–81.

[30] A. Fröhlich, M. E. Keating and S. M. J. Wilson, The class group of quaternion and dihedral 2-groups, Mathematika 21 (1974) 64–71.

[31] S. Galovich, The class group of a cyclic p-group, J. of Algebra 30 (1974) 368–387.

[32] S. Galovich, I. Reiner and S. Ullom, Class groups of integral representations of metacyclic groups, Mathematika 19 (1972) 105-111.

[33] S. Gersten, Whitehead groups of free associative algebras, Bull. Amer. Math. Soc. 71 (1965) 157–159.

[34] S. Gersten, A product formula for Wall's obstruction, Amer. J. Math. 88 (1966) 337–346.

[35] P. Hilton, G. Mislin and J. Roitberg, *Localization of Nilpotent Groups and Spaces,* Math. Studies vol. 15, North-Holland, 1975.

[36] M. Keating, On the K-theory of the quaternion group, Mathematika 20 (1973) 59–62.

[37] M. Keating, Class groups of metacyclic groups of order $p^r q$, p a regular prime, Mathematika 21 (1974) 90–95.

[38] M. Kervaire and M. P. Murthy, On the projective class group of cyclic groups of prime power order, Comment. Math. Helv. 52 (1977) 415–452.

[39] R. C. Kirby and L. C. Siebenmann, On the triangulation of manifolds and the Hauptvermutung, Bull. Amer. Math. Soc. 75 (1969) 415–452.

[40] S. Kwasik, Wall's obstruction and Whitehead torsion, Comment. Math. Helv. 58 (1983) 503–508.

[41] V. J. Lal, The Wall obstruction of a fibration, Invent. Math. 6 (1968) 67–77.

[42] C. B. de Lyra, On a conjecture in homotopy theory, An. da acad. Brasileira de ciencias 37 (1965) 167–184.

[43] I. Madsen, C. B. Thomas and C. T. C. Wall, The spherical space form problem II, Topology 15 (1976) 375–382.

[44] J. Martinet, Modules sur l'algèbre du groupe quaternonien, Ann. scient. Éc. Norm. Sup., 4^e série t.4 fasc.3 (1971) 399–408.

[45] M. Mather, Counting homotopy types of manifolds, Topology 4 (1965) 93–94.

[46] R. J. Milgram, The Swan finiteness obstruction for periodic groups, Preprint, Stanford University 1980.

[47] R. J. Milgram, Odd index subgroups of units in cyclotomic fields and applications, Lecture Notes in Math. 854 (1981) 269–298, Springer Verlag.

[48] J. Milnor, Groups which act on S^n without fixed points, Amer. J. Math. 79 (1957) 623–630.

[49] J. Milnor, Whitehead torsion, Bull. Amer. Math. Soc. 72 (1966) 358–426.

[50] J. Milnor, *Introduction to Algebraic K–Theory*, Princeton University Press 1971, Princeton, NJ.

[51] G. Mislin, Wall's obstruction for nilpotent spaces, Topology 14 (1975) 311–317.

[52] G. Mislin, Finitely dominated nilpotent spaces, Ann. of Math. 103 (1976) 547–556.

[53] G. Mislin, Conditions for finite domination for certain complexes, Lecture Notes in Math. 673 (1978) 219–224, Springer Verlag.

[54] G. Mislin, The geometric realization of Wall obstructions by nilpotent and simple spaces, Math. Proc. Camp. Phil. Soc. 87 (1980) 199–206.

[55] G. Mislin, K. Varadarajan, The finiteness obstructions for nilpotent spaces lie in $D(\mathbb{Z}\pi)$, Inventiones Math. 53 (1979) 185–191.

[56] E. K. Pedersen, Universal geometric examples for transfer maps in algebraic K- and L-theory, J. Pure Appl. Algebra 22 (1981) 179–192.

[57] E. K. Pedersen and L. Taylor, The Wall finiteness obstruction for a fibration, Amer. J. Math. 100 (1978), 887–896.

[58] D. S. Rim, Modules over finite groups, Ann. of Math. 69 (1959) 700–712.

[59] L. Siebenmann, The obstruction to finding a boundary for an open manifold of dimension greater than five, Ph.D–Thesis, Princeton University 1965.

[60] J. Stallings, Whitehead torsion of free products, Ann. of Math. 82 (1965) 354–363.

[61] R. Stöcker, Whiteheadgruppe topologischer Räume, Inventiones Math. 9 (1970), 271–278.

[62] R. Swan, Periodic resolutions for finite groups, Ann. of Math. 72 (1960) 267–291.

[63] R. Swan, Induced representations and projective modules, Ann. of Math. 71 (1960) 552–578.

[64] R. G. Swan, *K–Theory of Finite Groups and Orders*, Lecture Notes in Math. 149 (1970), Springer Verlag.

[65] M. J. Taylor, Locally free classgroups of prime power order, J. of Algebra 50 (1978) 462–487.

[66] C. B. Thomas and C. T. C. Wall, The topological spherical space form problem I, Comp. Math. 23 (1971) 101–114.

[67] S. Ullom, The exponent of class groups, J. of Algebra 29 (1974) 24–32.

[68] S. Ullom, Nontrivial lower bounds for class groups of integral group rings, Ill. J. Math. 20 (1976) 361–371.

[69] S. Ullom, Fine structure of class groups of cyclic p-groups, J. of Algebra 49 (1977) 112–124.

[70] K. Varadarajan, *The Finiteness Obstruction of C. T. C. Wall*, John Wiley & Sons, Inc., 1989.

[71] C.T.C. Wall, Finiteness conditions for CW-complexes, Ann. of Math. (2) 81 (1965) 55–69.

[72] C.T.C. Wall, Finiteness conditions for CW-complexes II, Proc. Roy. Soc. London Ser. A 295 (1966) 129–139.

[73] C. T. C. Wall, Periodic projective resolutions, Proc. London. Math. Soc. 39 (1979) 509–553.

[74] L. Washington, *Introduction to Cyclotomic Fields*, Graduate Texts in Mathematics Vol. 83, Springer Verlag (1982).

[75] J. E. West, Mapping Hilbert cube manifolds to ANR's: A solution of a conjecture of Borsuk, Ann. of Math. 106 (1977) 1–18.

[76] J. H. C. Whitehead, Simple homotopy types, Amer. J. Math. 72 (1950) 1–57.

[77] Z. Wojtkowiak, On the finiteness obstruction of nilpotent spaces of the same genus, Math. Z. 166 (1979) 103–109.

[78] Z. Wojtkowiak, Localization, finiteness obstructions and Reidemeister torsion, Quart. J. Oxford (2) 35 (1984) 223–241.