Lannes' T-Functor and the Cohomology of BG

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Introduction.

Let G be a compact, not necessarily connected Lie group and p a fixed prime. We shall show how to reconstruct in a functorial way $H^*(BG;\mathrm{Z}/p)$ out of $H^{*\gg 0}(BG;\mathrm{Z}/p)$. As a consequence, we obtain in particular the following theorem.

THEOREM. *Let $\rho : G \to H$ be a morphism of compact Lie groups (for instance finite groups) such that for some $n \geq 0$*

$$H^j(B\rho;\mathrm{Z}) : H^j(BH;\mathrm{Z}) \to H^j(BG;\mathrm{Z})$$

is an isomorphism for all $j \geq n$. Then ρ is an isomorphism of Lie groups.

Namely, the assumption implies that for every prime p, one has induced isomorphisms $H^{*>n}(BH;\mathrm{Z}/p) \to H^{*>n}(BG;\mathrm{Z}/p)$ and thus, by applying the "reconstruction functor", isomorphisms $H^*(BH;\mathrm{Z}/p) \to H^*(BG;\mathrm{Z}/p)$. Since BG and BH are spaces of finite type, it follows that the induced map $H^*(BH;\mathrm{Z}) \to H^*(BG;\mathrm{Z})$ is an isomorphism too. Thus, by Jackowski [6] and Minami [10], ρ is an isomorphism of Lie groups.

In section one we will recall some basic facts concerning the T-functor and in section two we will describe the "reconstruction functor", following the work of Dwyer and Wilkerson [2]. We will show how this functor can be used to reconstruct certain graded algebras out of their structure in high degrees. In section three we will apply the functor to the cohomology of BG and discuss a few applications.

1. Some basic facts on Lannes' T-functor.

Let $\mathcal{K}$ denote the category of unstable algebras over the mod-p Steenrod algebra $\mathcal{A}_p$, and let $\mathcal{U}$ be the category of unstable $\mathcal{A}_p$-modules. If A denotes a finite elementary abelian p-group, p a fixed prime, and $H^*A = H^*(A;\mathrm{Z}/p) \in \mathcal{K}$ its mod-p cohomology, Lannes defined in [8] functors $T_A : \mathcal{K} \to \mathcal{K}$ and $T_A : \mathcal{U} \to \mathcal{U}$, characterised by the adjointness relations

$$Hom_{\mathcal{K}}(T_A R, S) \cong Hom_{\mathcal{K}}(R, H^*A \otimes S)\,,$$

$$Hom_{\mathcal{U}}(T_A M, N) \cong Hom_{\mathcal{U}}(M, H^*A \otimes N)\,.$$

Here R, S are objects in $\mathcal{K}$, and M, N in $\mathcal{U}$. The underlying $\mathcal{A}_p$-module of the $\mathcal{A}_p$-algebra $T_A R$ agrees with the $\mathcal{A}_p$-module obtained by applying T_A to the underlying $\mathcal{A}_p$-module of R; this justifies the use of the same letter T_A for the two functors on $\mathcal{K}$ and $\mathcal{U}$ respectively.

An object $R \in \mathcal{K}$ is called *connected*, if $1 \in R$ gives rise to an isomorphism $\mathrm{F}_p \to R^0$. The ideal of elements of positive degree of R is denoted by $I(R)$, and the *indecomposable quotient* $Q(R) \in \mathcal{U}$ of R is defined as usual by $Q(R) = I(R)/I(R)^2$. An object $M \in \mathcal{U}$ is called *locally finite*, if each $x \in M$ lies in a finite $\mathcal{A}_p$-submodule of M. Note that a homomorphism $\varphi : B \to A$ of elementary abelian p-groups gives rise to a natural transformation $T_B \to T_A$. Since $B = 0$ yields for T_B the identity functor, one has canonical split injective morphisms $R \to T_A R$ and $M \to T_A M$. For $f \in Hom_{\mathcal{K}}(R, H^*A)$, with adjoint morphism $ad(f) : T_A R \to \mathrm{F}_p$ inducing the ring map $(T_A R)^0 \to \mathrm{F}_p$, one defines following [3] $T_f R$ to be the quotient $T_A R \otimes_{(T_A R)^0} \mathrm{F}_p$, where F_p is considered as a $(T_A R)^0$-module via $ad(f)$. In case R is connected, there is a unique morphism

$$\phi(R) \in Hom_{\mathcal{K}}(R, H^*A)$$

mapping $I(R)$ to 0. We denote by $T_{\phi(R)} R$ the corresponding quotient of $T_A R$. The following two basic results will be used in the next section.

(1.1) LEMMA [9]. *Let $M \in \mathcal{U}$ and $A \neq 0$ a finite elementary abelian p-group. Then the canonical map*

$$M \to T_A M$$

is an isomorphism if and only if M is a locally finite $\mathcal{A}_p$-module.

An object $R \in \mathcal{K}$ is called of *finite type*, if for each degree i the F_p-vector space R^i is finite dimensional. From the first part of Theorem 3.2 in [4] one obtains the following.

(1.2) LEMMA. *Suppose that $R \in \mathcal{K}$ is connected and that $Q(R)$ is locally finite as a module over $\mathcal{A}_p$. Then the canonical map*

$$R \to T_{\phi(R)}R$$

is an isomorphism.

If $R \in \mathcal{K}$ admits only a finite number of $\mathcal{K}$-maps $R \to H^*A$ (for instance, if R is finitely generated as a ring), then one has $T_A R \cong \prod T_f R$, where f ranges over $Hom_{\mathcal{K}}(R, H^*A)$. This follows from the p-boolean algebra structure of $(T_A R)^0$ (cf.[8]), which implies that $(T_A R)^0 \cong \prod \mathrm{F}_p$, a finite product, with projections corresponding to the elements of $Hom_{\mathcal{K}}(R, H^*A) \cong Hom_{\mathcal{K}}(T_A R, \mathrm{F}_p)$; one then has

$$\prod T_f R = \prod (T_A R \otimes_{(T_A R)^0} \mathrm{F}_p) \cong T_A R \otimes_{(T_A R)^0} (\prod \mathrm{F}_p) \cong T_A R\,.$$

2. The reconstruction functor

The basic reference for this section is [2]. For $R \in \mathcal{K}$ one defines the category $\mathcal{A}_R$, with objects the *finite* morphisms $f : R \to H^*A$ (i.e., H^*A is a finitely generated R-module), where A is any finite *non-trivial* elementary abelian p-group (thus, if the ideal $I(R)$ is nilpotent, then $\mathcal{A}_R$ is the empty category); the morphisms $f_1 \to f_2$ in $\mathcal{A}_R$ are group homomorphisms $\varphi : A_1 \to A_2$ such that one has a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{f_1} & H^*A_1 \\ \| & & \uparrow{\scriptstyle \varphi^*} \\ R & \xrightarrow{f_2} & H^*A_2 \end{array} \qquad (1)$$

Note that φ^* will be a finite morphism, since f_1 is finite; therefore, the homomorphism φ is necessarily injective. By adjointness, the diagram (1) gives rise to a commutative diagram

$$\begin{array}{ccc} T_{f_1}R & \longrightarrow & \mathrm{F}_p \\ \downarrow{\scriptstyle \varphi_*} & & \| \\ T_{f_2}R & \longrightarrow & \mathrm{F}_p \end{array} \qquad (2)$$

One obtains thus a functor

$$\alpha_R : \mathcal{A}_R \to \mathcal{K}$$

by mapping the object $(f : R \to H^*A)$ to $T_f R$, and the morphism $\varphi : f_1 \to f_2$ to φ_* as in diagram (2). The inverse limit of the functor α_R, $\lim \alpha_R$, defines then an object of $\mathcal{K}$, which we denote by $\alpha(R)$. This all makes perfect sense as long as the category $\mathcal{A}_R$ is not empty; in case $\mathcal{A}_R$ is empty, we use the following convention.

(2.1) DEFINITION. *For $R \in \mathcal{K}$ we put*

$$\alpha(R) = \begin{cases} \lim \alpha_R, & \textit{if } \mathcal{A}_R \textit{ is not empty;} \\ R^0, & \textit{if } \mathcal{A}_R \textit{ is empty.} \end{cases}$$

The canonical maps $R \to T_f R$, $f \in \mathcal{A}_R$, form a compatible family and, in case $\mathcal{A}_R$ is not empty, give rise to a $\mathcal{K}$-morphism $R \to \alpha(R)$ which we call the *canonical map*. If the category $\mathcal{A}_R$ is empty, we have $\alpha(R) = R^0$ and we define the canonical map $R \to \alpha(R) = R^0$ to be the natural projection.

Next, we want to define $\alpha(?)$ on certain morphisms of $\mathcal{K}$. Recall that (following Quillen) a ring map $\psi : R \to S$ is called an *F-isomorphism*, if *ker*(ψ) consist entirely of nilpotent elements, and every $s \in S$ admits some power s^{p^n} which lies in the image of ψ. If $\psi : R \to S$ is an F-isomorphism in $\mathcal{K}$, then one has, according to [8], an induced bijection

$$\psi^* : Hom_{\mathcal{K}}(S, H^*A) \cong Hom_{\mathcal{K}}(R, H^*A)\,.$$

Furthermore, if $\psi : R \to S$ is a finite morphism in $\mathcal{K}$ (i.e. S is finitely generated as an R-module), then $f : S \to H^*A$ is finite if and only if $f \circ \psi : R \to H^*A$ is finite. Therefore, the following holds.

(2.2) LEMMA. *Let $\psi : R \to S$ be a finite F-isomorphism in $\mathcal{K}$.Then ψ induces an equivalence of categories*

$$\psi^* : \mathcal{A}_S \to \mathcal{A}_R.$$

We will write $\mathcal{K}_{finF}$ for the subcategory of $\mathcal{K}$ which has the same objects as $\mathcal{K}$, but the morphisms $R \to S$ in $\mathcal{K}_{finF}$ are the finite F-isomorphisms in $\mathcal{K}$. Our discussion above shows that we may view α as a functor

$$\alpha : \mathcal{K}_{finF} \to \mathcal{K} .$$

Namely, on morphisms $\psi : R \to S$ we define α using the maps $T_f R \to T_g S$ with $f \in Hom_{\mathcal{K}}(R, H^*A)$ and $g \in Hom_{\mathcal{K}}(S, H^*A)$ each a finite morphism, $A \neq 0$, and f and g related by $f = g \circ \psi$. In case $\mathcal{A}_R$ and $\mathcal{A}_S$ are not empty, these maps fit together to give rise to

$$\alpha(\psi) : \lim \alpha_R = \alpha(R) \to \lim \alpha_S = \alpha(S) .$$

In the trivial case that the categories $\mathcal{A}_R$ and $\mathcal{A}_S$ are empty, we already defined $\alpha(R) = R^0$ and $\alpha(S) = S^0$; we define then $\alpha(\psi) : R^0 \to S^0$ to be the degree zero component of the morphism ψ. One easily checks now that $\alpha : \mathcal{K}_{finF} \to \mathcal{K}$ is a well defined functor.

(2.3) PROPOSITION. *Suppose that $\psi : R \to S$ is a finite morphism in $\mathcal{K}$ and assume that $ker(\psi)$ as well as $coker(\psi)$ are locally finite $\mathcal{A}_p$-modules. Supose furthermore that R and S are connected and finitely generated as rings. Then ψ lies in $\mathcal{K}_{finF}$ and induces an isomorphism*

$$\alpha(\psi) : \alpha(R) \xrightarrow{\cong} \alpha(S) .$$

Proof. Since $ker(\psi)$ and $coker(\psi)$ are locally finite $\mathcal{A}_p$-modules, ψ is an F-isomorphism and lies therefore in $\mathcal{K}_{finF}$. We first consider the special case of an empty index category $\mathcal{A}_R$; then $\mathcal{A}_S$ is empty too as ψ is an F-isomorphism. According to our conventions, the map $\alpha(\psi)$ agrees then with $\psi^0 : R^0 \to S^0$, which is an isomorphism since R and S are both assumed to be connected. Next, we consider the case of non-empty index categories $\mathcal{A}_R$ and $\mathcal{A}_S$. The map ψ gives then rise to a commutative diagram of the following form .

$$\begin{array}{ccccccc}
ker\psi & \longrightarrow & R & \xrightarrow{\psi} & S & \longrightarrow & coker\psi \\
 & & \downarrow & & \downarrow & & \\
 & & T_A R & \xrightarrow{T_A\psi} & T_A S & & \\
 & & \downarrow \sigma & & \downarrow \rho & & \\
 & & \prod T_f R & \xrightarrow{\pi(T_f\psi)} & \prod T_g S & & \\
 & & \downarrow & & \downarrow & & \\
kerT_\phi\psi & \longrightarrow & T_{\phi(R)}R & \xrightarrow{T_\phi\psi} & T_{\phi(S)}S & \longrightarrow & cokerT_\phi\psi
\end{array}$$

We used here the notation *coker* to denote cokernels in the category $\mathcal{U}$. By Lannes [8], $T_A : \mathcal{U} \to \mathcal{U}$ is an exact functor, so that by (1.1), because $ker\psi$ and $coker\psi$ are locally finite $\mathcal{A}_p$-modules,

$$ker\psi \cong T_A ker\psi \cong ker T_A \psi\,,$$

and

$$coker\psi \cong T_A coker\psi \cong coker T_A \psi\,.$$

Furthermore, the vertical arrows σ and ρ are isomorphisms , since R and S are finitely generated as rings ; the products are taken over all $f \in Hom_{\mathcal{K}}(R, H^*A)$, respectively all $g \in Hom_{\mathcal{K}}(S, H^*A)$, and these index sets correspond bijectively via the F-isomorphism ψ. Note that the two vertical composite maps are isomorphisms by Lemma (1.2), since R and S are finitely generated as rings. It follows that the kernel of $\pi(\psi)$ is mapped isomorphically onto the kernel of $T_\phi(\psi)$, and similarly for the cokernels. We infer that the induced map

$$\prod_{f \neq \phi(R)} T_f R \to \prod_{g \neq \phi(S)} T_g S$$

is an isomorphism. In particular, all the quotient maps

$$T_f R \to T_g S\,, \quad \text{with} \quad f = g \circ \psi \neq \phi(R)$$

are isomorphisms. Since, by Lemma (2.2), ψ also induces an equivalence of categories $\mathcal{A}_S \to \mathcal{A}_R$, and since $\phi(R)$ (respectively $\phi(S)$) are not in $\mathcal{A}_R$ (respectively $\mathcal{A}_S$) we conclude that

$$\alpha(\psi) : \lim_{f \in \mathcal{A}_R} T_f R \to \lim_{g \in \mathcal{A}_S} T_g S$$

is an isomorphism, completing the proof of the proposition.

A typical example arises as follows. Define for every $R \in \mathcal{K}$ and integer $n \geq 0$ a subobject $R < n > \in \mathcal{K}$ of R by

$$(R < n >)^j = \begin{cases} R^0, & \text{if } j = 0; \\ 0, & \text{if } 0 < j < n; \\ R^j, & \text{if } j \geq n. \end{cases}$$

We will primarily be interested in the case where R is finitely generated as a ring. Then the inclusion $R < n > \to R$ is a finite F-isomorphism. Note also that if R is finitely generated as a ring then so is $R < n >$; namely, if R is generated by elements of degree $\leq k$ and $N = max(k, n)$, then $R < n >$ will be generated by its elements of degree $\leq nN$, which is a finite set. Proposition (2.3) now implies immediately the following corollary.

(2.4) COROLLARY. *Suppose $R \in \mathcal{K}$ is connected and finitely generated as a ring. Then, for every $n \geq 0$, the inclusion $R < n > \to R$ induces an isomorphism*

$$\alpha(R < n >) \cong \alpha(R)\,.$$

Recall that (cf. [2]) $R \in \mathcal{K}$ is said to have a *non-trivial center*, if there exists a finite $\mathcal{K}$-morphism $f : R \to H^*A$ with $A \neq 0$, such that the induced map $R \to T_f R$ is an isomorphism; to avoid confusion, we will say in that situation that R has a *non-trivial DW-center*. According to [2, Prop. 4.10], the natural map

$$R \to \alpha(R) \tag{3}$$

is an isomorphism, if $Q(R)$ is a locally finite $\mathcal{A}_p$-module and R has a non-trivial DW-center. Using Corollary (2.4) and observing that if R is finitely generated as a ring then $Q(R)$ is locally finite (even finite as an abelian group), we obtain the following theorem concerning the reconstruction of certain objects of $\mathcal{K}$ from their structure in high degrees.

(2.5) PROPOSITION. *Let $R \in \mathcal{K}$ be connected, finitely generated as a ring, and assume that R has a non-trivial DW-center. Then, for every $n \geq 0$, there is a natural $\mathcal{K}$-isomorphism*

$$\alpha(R < n >) \cong R\,.$$

The assumptions on R concerning the DW-center can be weakened along the lines of [2, Thm. 1.2], where it is shown that the map (3) is already an isomorphism, if there is a morphism $R \to S$ in $\mathcal{K}$ satisfying the conditions a), b) and c) of the following theorem.

(2.6) THEOREM. *Suppose that $R \in \mathcal{K}$ is connected and admits a map $i : R \to S$ in $\mathcal{K}$ such that :*

- a) *Both R and S are finitely generated as rings, and i makes S into a finitely generated R-module.*
- b) *The map i has a left inverse in $\mathcal{U}$ which is also a map of R-modules.*
- c) *S has a non-trivial DW-center.*

Then, for every $n \geq 0$, there is a natural $\mathcal{K}$-isomorphism

$$\alpha(R < n >) \cong R\,.$$

3. The case of the cohomology of BG

Let G be a compact, not necessarily connected Lie group and p a fixed prime. Then the cohomology $H^*(BG;\mathrm{Z}/p)$ of the classifying space of G is well-known to be finitely generated as a ring. If G contains a central element of order p, then $H^*(BG;\mathrm{Z}/p) \in \mathcal{K}$ has a non-trivial DW-center (cf. [2]), and one has a natural isomorphism $\alpha(H^*(BG;\mathrm{Z}/p)) \cong H^*(BG;\mathrm{Z}/p)$. Using a tranfer argument it was proved in [2] that for an *arbitrary* compact Lie group the canonical map

$$H^*(BG;\mathrm{Z}/p) \to \alpha(H^*(BG;\mathrm{Z}/p))$$

is an isomorphism. Using basic properties of the T-functor, this isomorphism can also be obtained from the homotopy decompostion theorem for the classifying space of a compact Lie group, proved by Jackowski and McClure in [7].

From Corollary (2.4) we thus obtain the following theorem.

(3.1) THEOREM. *Let G be a compact Lie group and $n \geq 0$ an integer. Then there is a natural isomorphism*

$$\alpha(H^*(BG;\mathrm{Z}/p) < n >) \cong H^*(BG;\mathrm{Z}/p)\,.$$

Theorem (3.1) indicates that elements in $H^*(BG;\mathrm{Z}/p)$ must have "implications" in arbitrary high dimensions. Indeed, the following holds. We shall write $<< x >>$ for the $\mathcal{A}_p$-ideal generated by $x \in R \in \mathcal{K}$, that is, the smallest ideal of R which contains x and which is also an $\mathcal{A}_p$-module; it can also be described as the ideal in R generated by the elements $\{\theta x | \theta \in \mathcal{A}_p\}$.

(3.2) THEOREM. *Let G be a compact Lie group, $n > 0$, and $x \in H^n(BG;\mathrm{Z}/p)$ a non-zero element. Let $<< x >>$ denote the $\mathcal{A}_p$-ideal generated by x. Then $<< x >>$ is not locally finite as an $\mathcal{A}_p$-module and, in particular, $<< x >>^j$, the subgroup of elements of degree j in $<< x >>$, is non-zero for infinitely many values of j.*

Proof. Suppose that $<< x >>$ is locally finite as an $\mathcal{A}_p$-module. We then consider the commutative diagram

$$\begin{array}{ccc} H^*(BG;\mathrm{Z}/p) & \xrightarrow{\pi} & H^*(BG;\mathrm{Z}/p)/ << x >> \\ \downarrow & & \downarrow \\ \alpha(H^*(BG;\mathrm{Z}/p)) & \xrightarrow{\alpha(\pi)} & \alpha(H^*(BG;\mathrm{Z}/p)/ << x >>) \end{array}$$

in which π denotes the natural projection. As observed in the beginning of this section, the left vertical arrow is an isomorphism; since we are assuming $<< x >>$ to be locally finite, $\alpha(\pi)$ is an isomorphism too (cf. Proposition (2.3)), and we conclude that $<< x >>$ must be 0, contradicting our assumption that $x \neq 0$. It follows that $<< x >>$ is not locally finite as an $\mathcal{A}_p$-module. Since $H^*(BG;\mathrm{Z}/p)$ is of finite type, we conclude that $<< x >>^j \neq 0$ for infinitely many values of j.

(3.3) *Remark.* The statement concerning $<< x >>^j$ in the previous Theorem is also an immediate consequence of the fact that if $H^{*>0}(BG;\mathrm{Z}/p) \neq 0$, $H^*(BG;\mathrm{Z}/p)$ contains an element in positive degree, which is not a zero divisor. This is well-known for G a finite group, and can be reduced to that case for an arbitrary compact Lie group as follows. Choose a cyclic subgroup $C \subset G$ of order p, which is central in a maximal p-toral subgroup $N \subset G$ (such a C exists if $H^{*>0}(BG;\mathrm{Z}/p) \neq 0$, see for instance [2; Remark 1.4]). We claim then that any $z \in H^*(BG;\mathrm{Z}/p)$ which restricts to a non-trivial element of even degree in $H^*(BC;\mathrm{Z}/p)$, is not a zero divisor in the ring $H^*(BG;\mathrm{Z}/p)$. This is true for $G = F$, a finite p-group, by Duflot [1, Corollary 1]. For G an arbitrary compact Lie group it is well-known that the finite p-subgroups of G detect the mod-p cohomology of G, and that every finite p-subgroup of G is conjugate to one contained in N. Thus, the finite p-subgroups $F \subset N$ containing C detect the mod-p cohomology of G, proving the statement concerning z above. The existence of a $z \in H^{*>0}(BG;\mathrm{Z}/p)$ of even degree, which restricts non-trivially to $H^*(BC;\mathrm{Z}/p)$, follows from the fact that the restriction map $H^*(BG;\mathrm{Z}/p) \to H^*(BC;\mathrm{Z}/p)$ is a finite map of rings.

A typical application of Theorem (3.1) is the theorem mentioned in the introduction. The following corollary is a variation of it. We recall that for a compact Lie group G, a Sylow p-subgroup G_p of G is, by definition, a maximal p-subgroup; it is unique up to conjugation. For basic properties of Sylow p-subgroups of compact Lie groups the reader is referred to [12], or the classical literature.

(3.4) COROLLARY. *Suppose that $\rho : G \to H$ is a morphism of compact Lie groups such that ρ induces an isomorphism*

$$H^{*\gg 0}(BH; \mathbb{Z}/p) \to H^{*\gg 0}(BG; \mathbb{Z}/p)\,,$$

where p denotes a fixed prime. Then ρ induces an isomorphism

$$H^*(BH, \mathbb{Z}/p) \cong H^*(BG; \mathbb{Z}/p)\,,$$

and ρ maps a Sylow p-subgroup of G isomorphically onto a Sylow p-subgroup of H.

Proof. From Theorem (3.1) we see that ρ induces an isomorphism $H^*(BH; \mathbb{Z}/p) \cong H^*(BG; \mathbb{Z}/p)$. It follows then from [11] in the case of finite groups G and H, and [12, Corollary 2.4] in the general case, that ρ maps, as claimed, a Sylow p-subgroup of G isomorphically onto a Sylow p-subroup of H.

(3.5) *Remark.* Corollary (3.3) can be viewed as a counterpart to Jackowski's result (cf. [6]), where it is proved that if $(B\rho)^*$ is an isomorphism $H^*(BH; \mathbb{Z}/p) \to H^*(BG; \mathbb{Z}/p)$ in *low* dimensions, then $(B\rho)^*$ is an isomorphism in all dimensions.

To conclude, we present as an immediate corollary the following criterion for p-nilpotence (cf. Quillen [13]).

(3.6) COROLLARY. *Let G be a finite group and G_p a Sylow p-subgroup of G. If the restriction map $H^*(BG; \mathbb{Z}/p) \to H^*(BG_p; \mathbb{Z}/p)$ is an isomorphism in large dimensions, then G is p-nilpotent.*

Proof. If $H^*(BG; \mathbb{Z}/p) \to H^*(BG_p; \mathbb{Z}/p)$ is an isomorphism in large dimensions, it is an isomorphism in all dimensions (cf.(3.5)). Thus $BG_p \to BG$ induces a homotopy equivalence of Bousfield-Kan $\mathbb{Z}/p$-completions

$$(\mathbb{Z}/p)_\infty BG_p \to (\mathbb{Z}/p)_\infty BG\,.$$

Since BG_p is $\mathbb{Z}/p$-complete, that is,

$$BG_p \simeq (\mathbb{Z}/p)_\infty BG_p\,,$$

the natural map $BG \to (\mathrm{Z}/p)_\infty BG \simeq BG_p$ provides a left homotopy inverse to the map $BG_p \to BG$. But BG and BG_p are Eilenberg-Mac Lane spaces and we conclude that $G_p \subset G$ admits a retraction $r : G \to G_p$. The kernel of r is then a normal complement for G_p in G which, by definition, means that G is p-nilpotent.

(3.7) *Remark.* In the above proof, we also could have used Tate's criterion (cf.[14]) to conclude, from the isomorphism of the restriction map $H^*(BG; \mathrm{Z}/p) \to H^*(BG_p; \mathrm{Z}/p)$, that G is p-nilpotent. Furthermore, by adapting the definitions of p-nilpotence suitably, it is possible to generalize (3.6) to the case of compact Lie groups. This was done, using techniques from stable homotopy theory, by Henn in [5; Thm.2.5].

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