On the Farrell Cohomology of Mapping Class Groups

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Introduction.

For Γ any group of finite virtual cohomological dimension and a prime p, we say that Γ is *p-periodic*, if there exists a positive integer k such that the *Farrell* cohomology groups $\hat{H}^i(\Gamma; M)$ and $\hat{H}^{i+k}(\Gamma; M)$ have naturally isomorphic p-primary components for all $i \in \mathrm{Z}$ and $\mathrm{Z}\Gamma$-modules M. The p-period of Γ is defined as the least value of k (cf. [B]). For instance, if Γ is p-torsion free, then Γ is p-periodic of period one.

The mapping class group, Γ_g, is defined to be the group of path components of the group of orientation preserving homeomorphisms of the oriented closed surface S_g of genus g. For instance, $\Gamma_1 \cong SL(2, \mathrm{Z})$ and the cohomology is well known and easy to compute in this case. By writing $SL(2, \mathrm{Z})$ as an amalgamated product of $\mathrm{Z}/4$ and $\mathrm{Z}/6$ over $\mathrm{Z}/2$, one finds

$$\hat{H}^*(\Gamma_1; \mathrm{Z}) \cong (\mathrm{Z}/12)[x, x^{-1}]$$

with x of degree two. Thus Γ_1 is 2- and 3-periodic, with periods equal to two.

It is well known that Γ_g is of finite virtual cohomological dimension and, if $g > 1$, $vcd(\Gamma_g) = 4g - 5$ (cf. [H]). In the sequel we will always assume that $g > 1$. Recall from [B] that a group of finite vcd is p-periodic if and only if it does not contain a subgroup isomorphic to $\mathrm{Z}/p \times \mathrm{Z}/p$. Because for $g > 1$ the mapping class group Γ_g contains always a subgroup isomorphic to $\mathrm{Z}/2 \times \mathrm{Z}/2$, Γ_g is never 2-periodic. However, for an odd prime p, Γ_g is p-periodic for almost all values of g. This corresponds to the intuitively obvious fact that it is hard to find two "different" homeomorphisms of order p on S_g, which commute with each other. The third author determined in [X1] all the genera g for which Γ_g is p-periodic. In particular, Γ_g is 3-periodic if and only if $g \not\equiv 1 \bmod(3)$. For an odd prime p

and genus $g \not\equiv 1 \bmod(p)$, Γ_g is always p-periodic. Moreover, there are only finitely many "exceptional values" of g with $g \equiv 1 \bmod(p)$ for which Γ_g is p-periodic.

Recall that for a *finite* p-periodic group G and p an odd prime, the Sylow p-subgroup G_p of G is cyclic and, if $G_p \neq 1$, the p-period of G equals $2|N(G_p)/C(G_p)|$, where $N(G_p)$ (respectively $C(G_p)$) denotes the normalizer (respectively centralizer) of G_p in G; in particular, the p-period of G divides $2(p-1)$. Unlike the case of finite groups , the p-period of a p-periodic *infinite* group, p a fixed prime, may be arbitrarily large. A simple example is given by the group $\mathrm{Z}/p^n \circ \mathrm{Z} =< a, b | a^{p^n} = 1, bab^{-1} = a^{p+1} >$. For an odd prime p, the p-period of $\mathrm{Z}/p^n \circ \mathrm{Z}$ equals $2p^{n-1}$.

In this paper, however, we will show the surprising result that for a p-periodic mapping class group Γ_g, the p-period is bounded by $2(p-1)$. The precise theorem reads as follows.

THEOREM 1. *Let p be an odd prime and assume that Γ_g is p-periodic. Then the p-period of Γ_g is given by*

$$lcm\{2[N(\pi) : C(\pi)] | \pi \in S\}$$

where π ranges over S, a set of representatives of conjugacy classes of subgroups of order p of Γ_g, and $N(\pi)$ (respectively $C(\pi)$) denotes the normalizer (respectively centralizer) of π in Γ_g. In particular, the p-period of Γ_g divides $2(p-1)$.

We use the convention that $lcm\{2[N(\pi) : C(\pi)] | \pi \in S\} = 1$ in case S is empty (the p-period of Γ_g equals one in that case).

In case of the prime 3, one can find suitable subgroups of Γ_g to get the following even simpler result.

THEOREM 2. *Let $g > 1$ and assume that Γ_g is 3-periodic. Then the 3-period of Γ_g equals* 4.

Indeed, as we will see, it is also possible to give a more explicit description of the p-period of the mapping class group in the general case. In particular, for $g \equiv 1 \bmod(p)$ one finds the following.

THEOREM 3. *Let p be an odd prime and $g \equiv 1$ mod(p). If Γ_g is p-periodic, then the p-period of Γ_g is $2(p-1)$.*

The basic idea is to study for each subgroup π of order p of Γ_g the action of the normalizer $N(\pi)$ of π in Γ_g on a *spherical space* of the form $\mathrm{R}^d \times (\mathrm{R}^k - \{0\})$, which comes up as a subspace of the normal bundle of the fixed point set of the action of π on the Teichmüller space of S_g. The numbers d and k in $\mathrm{R}^d \times (\mathrm{R}^k - \{0\})$ turn out to depend only on the orbifold S_g/π. We use then the fact that the p-periods of the subgroups of the form $N(\pi)$ of Γ_g determine the p-period of Γ_g.

The rest of the paper is organized as follows. In Section 1 we provide some background material on groups acting on *spherical spaces*. In Section 2 we will establish an upper bound for the p-period of Γ_g. In Section 3 we finish the proof of Theorem 1 in the case that $g \not\equiv 1$ mod(p), and we also establish Theorem 2. Section 4 is devoted to the case $g \equiv 1$ mod(p), and in the last Section 5 we discuss an explicit formula for the p-period; in particular, we will prove Theorem 3.

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1. Groups acting on spherical spaces

We call a (not necessarily compact) manifold X a *spherical space*, if X is homotopy equivalent to a sphere. A classical result on finite groups G states that if G acts freely on an odd dimensional sphere S^{k-1}, then $\hat{H}^*(G;\mathrm{Z})$ is periodic, of period dividing k. For the convenience of the reader we provide below an adaption of the classical proof to the setting of Farrell cohomology of groups of finite virtual cohomological dimension acting on spherical spaces.

(1.1) LEMMA. *Let Γ denote a group of finite virtual cohomological dimension acting properly discontinuously on a spherical space X homotopy equivalent to an odd dimensional sphere S^{k-1}. Assume that the stabilizer of any point $x \in X$ is a finite p-torsion free group, p a fixed prime. Suppose furthermore that Γ acts trivially on $H^{k-1}(X;\mathrm{Z})$. Then Γ is p-periodic of p-period dividing k.*

Proof. For every finite p-subgroup P of Γ, P acts freely on X and there is an orientable spherical fibration (up to homotopy)

$$S^{k-1} \simeq X \longrightarrow X/P \longrightarrow BP\,.$$

The Gysin sequence of this spherical fibration shows that the Euler class $e(P) \in H^k(P;\mathrm{Z})$ induces, for $n > dimX$, an isomorphism

$$- \cup e(P) : H^n(P;\mathrm{Z}) \longrightarrow H^{n+k}(P;\mathrm{Z})\,.$$

It follows that $\hat{H}^*(P;\mathrm{Z})$ is periodic, and $e(P)$ maps under the canonical map

$$H^*(P;\mathrm{Z}) \longrightarrow \hat{H}^*(P;\mathrm{Z})$$

to a unit $\hat{e}(P) \in \hat{H}^k(P;\mathrm{Z})$. If we write $e(\Gamma)$ for the Euler class of

$$S^{k-1} \simeq X \longrightarrow E\Gamma \times_\Gamma X \longrightarrow B\Gamma$$

we conclude that $\hat{e}(\Gamma) \in \hat{H}^k(\Gamma;\mathrm{Z})$ restricts to the unit $\hat{e}(P)$ for every finite p-subgroup $P < \Gamma$. This implies (cf. [B]) that $\hat{e}(\Gamma)_{(p)} \in \hat{H}^k(\Gamma;\mathrm{Z})_{(p)}$ is a unit. As a result, the p-period of Γ divides k.

2. An upper bound for the p-period of Γ_g

Let S_g be a closed oriented surface of genus $g > 1$. It is classical that the Teichmüller space T_g of S_g is homeomorphic to R^{6g-6}, and T_g admits a complex structure such that Γ_g acts on T_g properly discontinuously by holomorphic automorphisms. If a subgroup $G < \Gamma_g$ stabilizes a point of T_g, then G is necessarily finite and there exists a complex structure θ on S_g such that G lifts to a group of holomorphic automorphisms of (S_g, θ) (cf. [K]). When no confusion can arise, we will in our notation not distinguish between G and the lifted group (isomorphic to G) acting on S_g.

According to [M-H], the fixed point set of a finite group $G < \Gamma_g$ acting on T_g is a submanifold homeomorphic to the Teichmüller space $T(\triangle)$ where $\triangle$ is a Fuchsian group containing $\pi_1(S_g)$ with $G \cong \triangle / \pi_1(S_g)$. If the Fuchsian group $\triangle$ admits a presentation of the form

$$\triangle = < x_1, y_1, \ldots, x_h, y_h; e_1, \ldots, e_r \,|\, {e_i}^{m_i} = 1\,, \prod_{j=1}^{h}[x_j, y_j] \prod_{k=1}^{r} e_k = 1 >\,,$$

where all m_i *'s* are greater than one, then the Teichmüller space $T(\triangle)$ is then homeomorphic to $\mathrm{R}^{6h-6+2r}$.

Applied to the case of $G \cong \mathrm{Z}/p$, we obtain the following.

(2.1) PROPOSITION. *Let p be a prime and $\pi < \Gamma_g$ a subgroup of order p. Then the fixed point set $(T_g)^\pi$ is homeomorphic to $\mathrm{R}^{6h-6+2n}$, where h is the genus of S_g/π and n the number of fixed points of the π-action on S_g.*

Proof. Consider the branched covering $S_g \longrightarrow (S_g/\pi) =: S_h$ and define Δ to be the orbifold fundamental group of S_g/π, that is,

$$\Delta = < x_1, y_1, \ldots, x_h, y_h; e_1, \ldots, e_n | {e_1}^p = \ldots = {e_n}^p = 1, \prod [x_j, y_j] \prod e_k = 1 > \ .$$

Let $p_1, \ldots, p_n \in S_g$ be the fixed points of the π-action on S_g. Then, writing $\bar{p}_1, \ldots, \bar{p}_n$ for the images of the p_i 's in S_h, one obtains a regular p-sheeted covering space

$$S_g - \{p_1, \ldots, p_n\} \longrightarrow S_h - \{\bar{p}_1, \ldots, \bar{p}_n\} \, .$$

The induced map of fundamental groups

$$\varphi : \pi_1(S_g - \{p_1, \ldots, p_n\}) \longrightarrow \pi_1(S_h - \{\bar{p}_1, \ldots, \bar{p}_n\})$$

gives rise to an injective map

$$\bar{\varphi} : \pi_1(S_g) \longrightarrow \Delta$$

with image a normal subgroup of index p. It follows that $T(\Delta) \cong \mathrm{R}^{6h-6+2n}$.

Let S_g, $g > 1$, be a closed oriented surface of genus g and $\pi < \Gamma_g$ a subgroup of order p, p a fixed prime. We will write $n(\pi)$ for the number of fixed points of the π-action on S_g, and $h(\pi)$ for the genus of S_g/π. We will establish the following upper bound for the p-period of Γ_g.

(2.2) THEOREM. *Let $g > 1$ and p a prime. Assume that Γ_g is p-periodic. Then the p-period of Γ_g divides*

$$lcm\{6(g - h(\pi)) - 2n(\pi) | \pi \in S\}$$

where S denotes a set of representatives of conjugacy classes of subgroups of order p of Γ_g.

Proof. It is well-known that there is a differentiable structure on T_g such that Γ_g acts smoothly. Let $\pi < \Gamma_g$ be a subgroup of order p. The normalizer $N(\pi)$ acts on $(T_g)^\pi \cong \mathrm{R}^{6h(\pi)-6+2n(\pi)}$ as well as the normal bundle E of $(T_g)^\pi$ in T_g; of course, E is homeomorphic to $\mathrm{R}^{6h(\pi)-6+2n(\pi)} \times \mathrm{R}^{6(g-h(\pi))-2n(\pi)}$ since $T_g \cong \mathrm{R}^{6g-6}$. We now consider the $N(\pi)$-action on the spherical space $E - E_0$, where E_0 denotes the zero section of the bundle $E \longrightarrow (T_g)^\pi$. First we check that the stablizer $N(\pi)_e < N(\pi)$ of every point $e \in E - E_0$ is a finite group of order prime to p. The projection $E \longrightarrow (T_g)^\pi$ is $N(\pi)$-equivariant so that $N(\pi)_e$ is mapped injectively to $N(\pi)_{\bar{e}}$, which is contained in the finite stabilizer $(\Gamma_g)_{\bar{e}}$ of $\bar{e} \in (T_g)^\pi \subset T_g$, where $\bar{e}$ denotes the image of e in $(T_g)^\pi$. Let $x \in N(\pi)_e$. Assume that $x^p = 1$. Since $x \in N(\pi)$, it normalizes π and, as $x^p = 1$, it follows that x even centralizes π. Thus, the subgroup generated by x and π, $< x, \pi >$, is isomorphic to $\mathrm{Z}/p \times \mathrm{Z}/p$, or it is equal to π. Because $N(\pi)$ is a subgroup of Γ_g, it is p-periodic too, and it follows that $< x, \pi > \not\cong \mathrm{Z}/p \times \mathrm{Z}/p$. Thus $x \in \pi$ and if $x \neq 1$, we have $< x >= \pi$. Therefore the (linear) x-action on $V_{\bar{e}} - \{0\}$ must be free; here, $V_{\bar{e}}$ denotes the fibre over $\bar{e}$ of the bundle projection $E \to (T_g)^\pi$. Since $xe = e$, we conclude thus that $x \neq 1$ implies that $e \in E_0$, the 0-section of the bundle $E \longrightarrow (T_g)^\pi$, and it follows that the action of $N(\pi)$ on $(E - E_0) \cong \mathrm{R}^d \times (\mathrm{R}^k - \{0\})$ has finite stabilizers of order prime to p, with $d = 6h(\pi) - 6 + 2n(\pi)$, $k = 6(g - h(\pi)) - 2n(\pi)$. The action is properly discontinuous on E and therefore on $E - E_0$, since it is properly discontinuous on $(T_g)^\pi \subset T_g$, and the fibers of $E \longrightarrow (T_g)^\pi$ are locally compact. Also, $N(\pi)$ acts trivially on $H^*(E - E_0; \mathrm{Z})$, as one can see as follows. The $N(\pi)$ action on T_g and $(T_g)^\pi$ is by complex automorphisms with respect to some fixed complex structure on T_g. The normal bundle of $(T_g)^\pi$ in T_g is the quotient bundle of $\tau(T_g)|(T_g)^\pi$ by $\tau({T_g}^\pi)$, where $\tau(T_g)$ (respectively $\tau({T_g}^\pi)$) denotes the tangent bundle of T_g (respectively $(T_g)^\pi$). All these bundles have a natural orientation, induced by the complex structure of T_g, and these orientations are preserved by the $N(\pi)$-action. It is then obvious that $N(\pi)$ acts trivially on $H^*(E - E_0; \mathrm{Z})$. We are therefore in the situation of (1.1) and conclude that the p-period of $N(\pi)$ divides $k = 6(g - h(\pi)) - 2n(\pi)$. By a result of Brown [B], the fact that Γ_g is p-periodic implies that $\hat{H}^*(\Gamma_g; \mathrm{Z})_{(p)} \cong \prod_{\pi \in S} \hat{H}^*(N(\pi); \mathrm{Z})_{(p)}$, where S denotes a set of representatives for the conjugacy classes of subgroups of order p of Γ_g. The conclusion of the theorem now follows.

In view of the application in Section 3, it is useful to rewrite our result on the p-periodicity of Γ_g as follows.

(2.3) COROLLARY. *Let $g > 1$ and $\pi < \Gamma_g$ a subgroup of order p, p a fixed prime. Assume that Γ_g is p-periodic . Then the p-period of $N(\pi)$ divides*

$$(3m + k)p - 3(m + k) - 2i$$

where the integers $k \geq 0, m > 0$ and i with $0 \leq i \leq p - 1$ are uniquely determined by the equations $2g - 2 = mp - i$, and $n(\pi) = kp + i$.

Proof. From the proof of (2.2) we infer that the p-period of $N(\pi)$ divides $6(g - h(\pi)) - 2n(\pi)$, where $h(\pi)$ denotes the genus of S_g/π and $n(\pi)$ the number of fixed points of the π-action on S_g. By the Riemann-Hurwitz formula applied to the branched covering space $S_g \longrightarrow S_g/\pi$ one has $2 - 2g = p(2 - 2h(\pi)) - n(\pi)(p - 1)$. Also, if one writes $2g - 2$ in the form $mp - i$ with $m > 0$ and $0 \leq i \leq p - 1$, then, the Riemann-Hurwitz formula shows that $n(\pi) = kp + i$ for some $k \geq 0$. It follows that $2g = mp - i + 2$ and $2h(\pi) = 2 + m - k(p - 1) - i$ so that $6(g - h(\pi)) - 2n(\pi) = (3m + k)p - 3(m + k) - 2i$.

3. The p-period of Γ_g for g $\not\equiv$ 1 mod(p)

We will make repeated use of the following Lemma.

(3.1) LEMMA. *Let p be a prime and N a group of finite virtual cohomological dimension which is p-periodic. Suppose N contains a normal subgroup $\pi < N$ of order p. Then the following holds.*

(a) If $x \in \hat{H}^*(N; \mathrm{Z})$ restricts to a unit in $\hat{H}^*(\pi; \mathrm{Z})$ and $degx \neq 0$, then the p-period of N divides $|degx|$.

(b) The p-period of N has the form $2[N : C(\pi)]p^\alpha$, where $C(\pi) < N$ denotes the centralizer of π in N and $\alpha \geq 0$ an integer.

Proof. We can write $x \in \hat{H}^*(N;\mathrm{Z})$ uniquely as a sum of p-primary elements $x_{(p)} \in \hat{H}^*(N;\mathrm{Z})_{(p)}$. If the reduction $\bar{x} \in \hat{H}^*(N;\mathrm{Z}/p)$ is a unit, then so is $x_{(p)} \in \hat{H}^*(N;\mathrm{Z})_{(p)}$. This follows from the fact (cf. [B; Chapter X, 6.6]) that the reduction map

$$\lambda : \hat{H}^*(N;\mathrm{Z})_{(p)} \longrightarrow \hat{H}^*(N;\mathrm{Z}/p)$$

has the property that $ker(\lambda)$ is nilpotent and for every $u \in \hat{H}^*(N;\mathrm{Z}/p)$ there is an integer k such that $u^{pk} \in Im(\lambda)$. Namely, if y is an inverse for $\bar{x}$ and $y^{pk} = \lambda(z)$, then $(x^{pk}z - 1)_{(p)}$ is nilpotent and thus $x_{(p)}$ is a unit. Thus, if $\bar{x}$ is a unit and $deg\bar{x} \neq 0$, then the period of N divides $|degx|$. Because N is p-periodic and $\pi < N$ is normal, π is the only subgroup of order p of N. Quillen's F-isomorphism theorem then implies (cf. [B]) that the restriction map

$$\varphi : \hat{H}^*(N;\mathrm{Z}/p) \longrightarrow \hat{H}^*(\pi;\mathrm{Z}/p)^{N(\pi)/C(\pi)} =: \mathcal{H}^*$$

has the property that $ker(\varphi)$ is nilpotent and that for every $v \in \mathcal{H}^*$ there is an integer s such that $v^{p^s} \in Im(\varphi)$. As before, we conclude that if $\varphi(\bar{x})$ is a unit, then so is $\bar{x}$. But $\varphi(\bar{x}) \in \mathcal{H}^*$ is invertible if and only if it is invertible as an element in $\hat{H}^*(\pi;\mathrm{Z}/p)$, and the invertible elements of $\hat{H}^*(\pi;\mathrm{Z}) \cong \mathrm{Z}/p[w, w^{-1}]$, $degw = 2$, are precisely those, which map to invertible elements in $\hat{H}^*(\pi;\mathrm{Z}/p)$, proving (a). For (b) we observe that the p-period of N is the smallest positive integer k for which $\hat{H}^k(N;\mathrm{Z})_{(p)}$ contains a unit. By the discussion above, this is, up to a p^{th}-power, the smallest positive degree for which $\hat{H}^*(N;\mathrm{Z}/p)$ contains a unit and, using the map φ, this is up to a p^{th}-power the smallest positive degree for which $\mathcal{H}^*$ contains a unit. But $\mathcal{H}^*$ is periodic with period 1 if $p = 2$, and period $2[N(\pi) : C(\pi)]$ if p is odd. Therefore, the p-period of N has the form $2[N(\pi) : C(\pi)]p^{\alpha}$. We will be interested in the case where $\pi < \Gamma_g$ and $N = N(\pi)$, the normalizer of π in Γ_g. To prove our main theorem stated in the introduction, it suffices to show that $\alpha = 0$ in (3.1, (b)) for $N = N(\pi) < \Gamma_g$. In this section, we settle the case $g \not\equiv 1 \bmod(p)$.

(3.2) PROPOSITION. *Let $g > 1$ and assume that p is an odd prime. If $g \not\equiv 1$ mod(p) and $\pi < \Gamma_g$ a subgroup of order p, then $N(\pi)$ is p-periodic with p-period equal to* $2[N(\pi) : C(\pi)]$.

Proof. As observed in the introduction, if p is an odd prime and $g \not\equiv 1 \bmod(p)$, then Γ_g is p-periodic. Thus $N(\pi)$ is p-periodic and to prove the proposition, we need, because of (3.1, (b)), only show that the p-period of $N(\pi)$ is not divisible by p. There will be two cases to consider. In the first case, the upper bound (2.3) for the p-period is prime to p and we are done. For the other case, we will construct an element $c_{p-1}(\rho) \in H^{2(p-1)}(N(\pi); \mathrm{Z})$ with the property that it restricts to a unit in $\hat{H}^{2(p-1)}(\pi; \mathrm{Z})$; this implies by (3.1, (a)) that the p -period of $N(\pi)$ divides $2(p-1)$, which is prime to p. To this end, consider the natural action of Γ_g on $H_1(S_g; \mathrm{Z})$ and write

$$\rho : \Gamma_g \longrightarrow GL(2g, \mathrm{Q})$$

for the corresponding representation over Q. Since $\pi \cong \mathrm{Z}/p$ admits only two irreducible Q-representations, the trivial one, which we denote by τ, and the reduced regular representation $\delta : \pi \longrightarrow GL(p-1, \mathrm{Q})$, the character of $\mu = \rho|\pi$ satisfies

$$\chi_\mu = a(\pi)\chi_\tau + b(\pi)\chi_\delta \tag{3.3}$$

for some natural numbers $a(\pi)$ and $b(\pi)$. It is easy to check (cf. [G-M]) that the Chern class $c_{(p-1)}(\mu) \in H^{2(p-1)}(\pi; \mathrm{Z})$ is non-zero if and only if $b(\pi)$ is relatively prime to p. The number $b(\pi)$ depends on the number of fixed point of the π action on S_g and can be determined as follows. Let $x \in \pi$ be a generator and $1 \in \pi$ the neutral element. Then, for the reduced regular representation δ one has $\chi_\delta(x) = -1$ and therefore (3.3) yields

$$\chi_\rho(x) = a(\pi) - b(\pi),\ \chi_\rho(1) = 2g = a(\pi) + (p-1)b(\pi) \tag{3.4}$$

On the other hand, the Lefschetz Trace Formula shows that

$$2 - \chi_\rho(x) = n(\pi), \tag{3.5}$$

where $n(\pi)$ denotes the number of fixed points of the x-action on S_g. It is convenient to write $2g-2$ in the form $mp-i$ with $0 \le i \le p-1$. Then, from the Riemann-Hurwitz Formula, one has $2g-2 \equiv -n(\pi) \bmod (p)$ and thus $n(\pi) = kp + i$ for some $k \ge 0$. Solving (3.4) and (3.5) for $b(\pi)$, yields

$$b(\pi) = m + k \tag{3.6}$$

As $2g-2 = mp-i$ and $g \not\equiv 1 \bmod (p)$, we have $i \not\equiv 0 \bmod (p)$. By (2.2) the p-period of $N(\pi)$ divides $(3m+k)p-3(m+k)-2i$. As observed, if p does not divide $(3m+k)p-3(m+k)-2i$, we are done, because then necessarily $\alpha = 0$ in (3.1). On the other hand, if p divides $(3m+k)p-3(m+k)-2i$, then $3(m+k) \equiv -2i \not\equiv 0 \bmod (p)$ since p is odd and $i \not\equiv 0 \bmod (p)$; of course, $p \ge 5$ in that case. It then follows that $m+k \not\equiv 0 \bmod (p)$ and by (3.6), that $b(\pi)$ is not divisible by p. We conclude then that $c_{p-1}(\rho) \in H^{2(p-1)}(\pi; \mathrm{Z})$ is nonzero, and it is clear that then $\bar{c}_{p-1}(\mu) \in \hat{H}^{2(p-1)}(\pi; \mathrm{Z})$ is necessarily a unit , concluding the proof of the proposition.

(3.7) COROLLARY. *The 3-period of every 3-periodic mapping class group Γ_g is 4.*

Proof. According to [X1], Γ_g is 3-periodic if and only if $g \not\equiv 1 \bmod (3)$. Thus, by Proposition (3.2), the 3-period of a 3-periodic Γ_g is 2 or 4. To rule out the value 2, it suffices to find a subgroup of Γ_g whose 3-period is 4. We claim that for $g \not\equiv 1 \bmod (3)$, Γ_g contains the Dihedral group D_6 of order 6. Namely, by [X3], $\Gamma_2, \Gamma_3, \Gamma_5$ and Γ_6 contain D_6, and, as shown in [X1], if Γ_g contains a finite subgroup G of order $|G|$ then so does $\Gamma_{g+|G|}$. Thus, every Γ_g, $g \not\equiv 1 \bmod (3)$, contains D_6.

4. The p-period of Γ_g for $g \equiv 1$ mod(p)

Suppose that Γ_g is p-periodic and $1 < g \equiv 1$ mod(p). As discussed in Section 3, to show that the p-period of Γ_g is given by the formula of Theorem 1 in the introduction amounts to showing that for $\pi < \Gamma_g$ any subgroup of order p, the p-period of $N(\pi)$ is relatively prime to p. This will be done by constructing for each such π a symplectic characterstic class

$$d_{k(\pi)}(\rho) \in H^{2k(\pi)}(\Gamma_g; \mathrm{Z})$$

where $k(\pi)$ satisfies $1 \leq k(\pi) \leq p-1$, such that $d_{k(\pi)}(\rho)$ restricts to a unit in $\hat{H}^*(\pi; \mathrm{Z})$. These characteristic classes arise as follows. We look at the natural representation of $\rho : \Gamma_g \longrightarrow Sp(2g, \mathrm{R})$, by letting Γ_g act on $H^1(S_g; \mathrm{R})$, preserving the symplectic form given by the cup-product. Recall that

$$H^*(BSp(2g, \mathrm{R}); \mathrm{Z}) \cong \mathrm{Z}[d_1, \ldots, d_g]$$

where the d_i's are such that they restrict to the universal Chern classes of a maximal compact subgroup $U(g) < Sp(2g, \mathrm{R})$. The images of the d_i's under the induced map

$$\rho^* : H^*(BSp(2g, \mathrm{R}); \mathrm{Z}) \longrightarrow H^*(\Gamma_g; \mathrm{Z})$$

gives rise to *symplectic* characteristic classes

$$d_i(\rho) := \rho^*(d_i) \in H^{2i}(\Gamma_g; \mathrm{Z}),\ 1 \leq i \leq g\,.$$

For our application, the following algebraic Lemma will be useful.

(4.1) LEMMA. *Let p be a prime and $f(x) \in \mathrm{F}_p[x]$ a polynomial satisfying $f(x) \equiv 1$ mod(x^p) and which factors completely over F_p into linear factors. Then each root of f has multiplicity divisible by p.*

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the roots of f with multiplicities. The assumption on f implies that the first $(p-1)$ elementary symmetric functions in the λ 's vanish. As a result

$$\sum_{i=1}^{n} {\lambda_i}^k = 0,\ 1 \leq k \leq p-1\,.$$

If $n_\mu \geq 0$ denotes the multiplicity of $\mu \in \mathrm{F}_p$ as a root of f, then we infer

$$\sum_{\mu \in \mathbb{F}_p - \{0\}} n_\mu \mu^k = 0,\ 1 \leq k \leq p-1\,.$$

Since the Van der Monde matrix $[\mu^k]$ with $\mu \in \mathrm{F}_p - \{0\}$ and $1 \leq k \leq p-1$ is regular, we conclude that $n_\mu \equiv 0 \bmod(p)$ for all $\mu \in \mathrm{F}_p - \{0\}$.

The following useful Lemma on Chern classes of representations of Z/p is now an easy consequence.

(4.2) LEMMA. *Let $\varphi : \mathrm{Z}/p \longrightarrow U(n)$ be a representation of Z/p, p a fixed prime. Assume that $c_i(\varphi) \in H^{2i}(\mathrm{Z}/p;\mathrm{Z})$ is zero for $1 \leq i \leq p-1$. Then φ is of the form $p\psi \oplus \tau$, with τ a trivial representation.*

Proof. Decompose φ into one-dimensional representations, $\varphi = \oplus_{k=0}^{p-1} n_k \omega^k$, with ω a faithful one-dimensional representation of Z/p. We put $\tau = n_0 \omega^0$ and need to show that each $n_i, 1 \leq i \leq p-1$, is divisible by p. Consider the injective ring homomorphism

$$\mathrm{F}_p[x] \longrightarrow H^*(\mathrm{Z}/p;\mathrm{Z}/p)$$

given by mapping x to $\bar{c}_1(\omega) \in H^2(\mathrm{Z}/p;\mathrm{Z}/p)$, the reduction $\bmod(p)$ of $c_1(\omega)$. Note that the polynomial $f(x) \in \mathrm{F}_p[x]$ defined by $\prod_{k=0}^{p-1}(1+kx)^{n_k}$ is mapped to the total Chern class $\bar{c}(\varphi) = \sum \bar{c}_i(\varphi)$. Since $\bar{c}_i(\varphi) = 0$ for $1 \leq i \leq p-1$, we conclude that $f(x) \equiv 1 \bmod(x^p)$. By the previous Lemma, we conclude that n_k is divisible by p for $1 \leq k \leq p-1$.

We now return to the study of the representation

$$\rho : \Gamma_g \longrightarrow Sp(2g, \mathrm{R})$$

and complete the proof of Theorem 1.

(4.3) PROPOSITION. *Let p be an odd prime and $g \equiv 1$ mod(p). Assume that Γ_g is p-periodic. Then, for every subgroup $\pi < \Gamma_g$ of order p there exists an integer $k(\pi)$ with $1 \leq k(\pi) \leq p-1$ such that $d_{k(\pi)}(\rho|\pi) \in \hat{H}^{2i}(\pi; \mathrm{Z})$ is a unit, where $i = k(\pi)$.*

Proof. The representation $\rho : \Gamma_g \longrightarrow Sp(2g, \mathrm{R})$ induces a representation $\rho|\pi : \pi \longrightarrow Sp(2g, \mathrm{R})$ which factors uniquely (up to a conjugation) through a maximal compact subgroup $U(g) < Sp(2g, \mathrm{R})$. We can therefore think of the classes $d_i(\rho|\pi)$ as Chern classes $c_i(\bar{\rho})$ of the representation $\bar{\rho} : \pi \longrightarrow U(g)$ obtained in this way. We want to show that if we had $d_i(\rho|\pi) = 0$ for $1 \leq i \leq p-1$, then $\Gamma_g$would contain a subgroup of the form $\mathrm{Z}/p \times \mathrm{Z}/p$, contradicting the assumption that Γ_g be p-periodic. The representation $\bar{\rho} : \pi \longrightarrow U(g)$ can be realized by choosing a complex structure and a Hermitian metric on $H^1(S_g; \mathrm{R})$ compatible with the π-action and symplectic structure. This can be done by choosing a complex structure on S_g such that π acts by holomorphic automorphisms on S_g; the induced action on the space W^1 of holomorphic 1-forms of S_g is then a model for the representation $\bar{\rho}$. If we decompose $\bar{\rho}$ as $\oplus_{i=0}^{p-1} n_i \omega^i$, ω a faithful irreducible one-dimensional representation then, assuming $d_i(\rho|\pi) = c_i(\bar{\rho}) = 0$ for $1 \leq i \leq p-1$, we infer $n_i \equiv 0$ mod(p) for $1 \leq i \leq p-1$ (cf. (4.2)). Let $x \in \pi$ be a generator and denote by $(\beta_1, \ldots, \beta_{n(\pi)})$ the fixed point datum of the x-action on S_g; thus $1 \leq \beta_i \leq p-1$, and for some numbering of the fixed points of x and in a suitable local coordinate system about the j'th fixed point, the action of x^{-1} is given by $z \mapsto exp(2\pi\sqrt{-1}\beta_j/p)$. We can think of the numbers n_i in the decomposition of $\bar{\rho}$ as dimensions of eigenspaces of the x-action on W^1, the space of holomorphic 1-forms. According to [F-K; Chapter V, 2.2.3 and 2.5.4] one has then

$$n_0 = h(\pi)$$

where $h(\pi)$ denotes the genus of S_g/π, and for $1 \leq j \leq p-1$,

$$n_j = h(\pi) - 1 + n(\pi) - \sum_{i=1}^{n(\pi)} < \delta(j)\beta_i/p > \tag{4.4}$$

for a suitable permutation δ of $\{1, 2, \ldots, p-1\}$; we use the notation $< q >$ to denote the fractional part of a rational number q. By renumbering the fixed points of π suitably, we may assume that δ is the identity permutation. The equations (4.4) imply then, because $n_j \equiv 0 \bmod(p)$ for $1 \leq j \leq p-1$, that

$$n_{p-1} - n_1 = \sum \beta_i/p - \sum < (p-1)\beta_i/p >= 2\sum \beta_i/p - n(\pi) \equiv 0 \; mod(p)\,, \qquad (4.5)$$

and

$$n_{p-1} + n_1 = 2((h(\pi) - 1 + n(\pi)) - n(\pi) \equiv 0 \; mod(p)\,. \qquad (4.6)$$

Moreover, the Riemann-Hurwitz equation $2 - 2g = p(2 - 2h(\pi)) - n(\pi)(p-1)$ shows that, because $g \equiv 1 \bmod(p)$, one has $n(\pi) \equiv 0 \bmod(p)$ so that (4.5) and (4.6) imply

$$h(\pi) \equiv 1 \; mod(p)\,, \qquad (4.7)$$

and

$$\sum \beta_i \equiv 0 \; mod(p^2)\,. \qquad (4.8)$$

With $g = kp + 1$, $h(\pi) = tp + 1$ and $n(\pi) = sp$, the Riemann-Hurwitz equation can now be written in the form

$$(2 - 2g)/p^2 = (2 - 2(t+1)) - s(1 - 1/p)\,. \qquad (4.9)$$

Because of (4.8) we see that either $n(\pi) = 0$, or else $n(\pi) > p$, so that $n(\pi) = sp$ with $s = 0$ or $s > 1$. We wish to prove that this implies that $\mathrm{Z}/p \times \mathrm{Z}/p < \Gamma_g$. For this, it suffices in view of (4.9) to show that $\mathrm{Z}/p \times \mathrm{Z}/p$ admits a generating set $(a_1, \ldots, a_{t+1}, b_1, \ldots, b_{t+1}, c_1, \ldots, c_s)$ satisfying the relations $\prod[a_i, b_i] \prod c_j = 1$ with each c_j of order p (see for instance [T]). By choosing a generating set $\{a, b\}$ for $\mathrm{Z}/p \times \mathrm{Z}/p$, we can proceed as follows. In case $s = 0$ (that is, there are no c_j 's), we choose $a_i = a$ and $b_i = b$ for $1 \leq i \leq t+1$. In case $s \geq 2$, if $s \equiv 1 \bmod(p)$, choose $a_i = a$ and $b_i = b$ $(1 \leq i \leq t+1)$, and $c_1 = c_2 = \ldots = c_{s-2} = a, c_{s-1} = a^2, c_s = a^{-1}$; if $s \not\equiv 1 \bmod(p)$, choose $a_i = a, b_i = b\,(1 \leq i \leq t+1)$ and $c_1 = c_2 = \ldots = c_{s-1} = a, c_s = a^{1-s}$. We conclude that the assumption that $d_i(\rho|\pi) = 0$ for $1 \leq i \leq p-1$ yields a contradiction. Thus, for some $k(\pi)$ with $1 \leq k(\pi) \leq p-1$, we infer that $d_{k(\pi)}(\rho|\pi) \in H^{2k(\pi)}(\pi; \mathrm{Z})$ is a non-zero element. But then $d_{k(\pi)}(\rho|\pi) \in \hat{H}^*(\pi; \mathrm{Z})$ is necessarily a unit, because

$$\hat{H}^*(\pi; \mathrm{Z}) \cong \mathrm{Z}/p[w, w^{-1}]$$

with $\deg w = 2$, finishing the proof of the Proposition.

5. An explicit formula for the p-period of Γ_g

We first give a proof of Theorem 3 of the introduction. It is clear from covering space theory that if $g \equiv 1 \bmod(p)$, say $g = kp + 1$, then the surface S_h of genus $h = k + 1$ will have S_g as p-fold regular covering space, and therefore there exists a subgroup $\pi < \Gamma_g$ of order p, which acts freely on S_g. By a classical result of Nielsen [N], all fixed point free homeomorphisms of S_g, having the same finite order, are conjugate in Γ_g so that we conclude that $N(\pi)/C(\pi) \cong Z/(p-1)$. Theorem 1 then implies immediately that the p-period of Γ_g equals $2(p-1)$, which proves Theorem 3.

For the general case, we use of the following formula.

(5.1) LEMMA [X2; 3.1]. *Let $g > 1$ and write S for the set of conjugacy classes of subgroups $\pi < \Gamma_g$ of order p. Then*

$$lcm\{[N(\pi) : C(\pi)]\,|\pi \in S\} = lcm\{gcd(p-1, n(\pi))\,|\pi \in S\}$$

where $n(\pi)$ denotes the number of fixed points of the π-action of S_g.

It is therefore possible to compute the p-period of a p-periodic Γ_g, if one knows the possible fixed point numbers $n(\pi)$ for homeomorphisms of order p on S_g. These numbers were determined in [X1] and look as follows. If we write $2g-2$ in the form $mp - i$ with $0 \le i \le p-1$, define for an odd prime p

$$B_{g,p} = \begin{cases} \{i, i+p, \ldots, i + [2g/(p-1) - m]p\}, & \text{if } i \not\equiv 1 \bmod(p); \\ \{1+p, \ldots, 1 + [2g/(p-1) - m]p\}, & \text{if } i \equiv 1 \bmod(p). \end{cases}$$

As usual, we use here the notation $[x]$ to denote the integral part of the rational number x. According to [X1], $B_{g,p}$ consists precisely of the set of all those numbers, which occur as cardinalities of the fixed point set for homeomorphisms of order p on S_g; (if, in case $i \not\equiv 1 \bmod(p)$, one has $2g/(p-1) < m$, then $B_{g,p}$ is empty and Γ_g contains no element of order p, similarly in case $i \equiv 1 \bmod(p)$). Combining Theorem 1 with (5.1), one obtains the following explicit formula for the period of a p-periodic mapping class group.

(5.2) THEOREM. *Suppose p is a prime, $g > 1$ and that Γ_g is p-periodic. Then the p-period of Γ_g is given by*

$$lcm\{gcd(2(p-1), 2n)|n \in B_{g,p}\}\,.$$

For instance, (5.2) implies that for large values of g (e.g., $2g > p^3$), the p-period of a p-periodic Γ_g is precisely $2(p-1)$.

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