On the Cohomology of Finite Groups of Lie Type

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Introduction

Let $\mathbb{F}_q$ denote the finite field with q elements and let $G = X_{\mathbb{F}_q}$ be a connected (not necessarily split) reductive group scheme over $spec(\mathbb{F}_q)$. We will be interested in the cohomology of the finite groups $G(\mathbb{F}_{q^n})$ of $\mathbb{F}_{q^n}$–rational points of G, with coefficients in $\mathbb{Z}/\ell$ where ℓ is a prime different from $p = char(\mathbb{F}_q)$. These groups are closely related and present special cases of the groups referred to in the title. A finite group of Lie type is, by definition, a central quotient of a group of the form $G(\mathbb{F}_{q^n})$. For instance, all finite Chevalley groups (as defined in [Gor] or [Car]) are of this kind. For simplicity, we will formulate our results for the groups $G(\mathbb{F}_{q^n})$ rather than for these more general central quotients; it should be clear to the reader how to apply the results, mutatis mutandis, to the general finite groups of Lie type. The basic references for reductive group schemes are [DeGr] or [Dem] (see also [Jan] for an account of the basic results).

It is well known that if a homomorphism $\phi : \tau \to \pi$ of finite groups induces an isomorphism in $H^*(\ ;\mathbb{Z}/\ell)$, ℓ a prime, then the kernel of ϕ has order prime to ℓ and the image of ϕ has an index prime to ℓ in π (cf. [Jac]); in particular, it will follow that τ and π will have isomorphic ℓ–Sylow subgroups. Obviously, the converse statement is in general false, as one can see by looking at the inclusion map of an ℓ–Sylow subgroup. Our main theorem shows, however, that for the natural inclusions of groups of Lie type a converse statement holds. The precise statement is as follows.

Theorem. Let G be a connected reductive $\mathbb{F}_q$–group scheme and let ℓ be a prime different from $char(\mathbb{F}_q) = p$. Then the following are equivalent:

(i) the inclusion $G(\mathbb{F}_q) \to G(\mathbb{F}_{q^n})$ induces an $H^*(\ ;\mathbb{Z}/\ell)$–isomorphism

(ii) the groups $G(\mathbb{F}_q)$ and $G(\mathbb{F}_{q^n})$ have isomorphic ℓ–Sylow subgroups.

In Section 1 we will discuss trace formulas and prove the Theorem. We will also point out the relationship of the Theorem with conjugacy questions concerning the ℓ–subgroups of $G(\mathbb{F}_q)$ and $G(\mathbb{F}_{q^n})$. In Section 2 we show how to adapt the proof of the Theorem to cover the Suzuki and Ree groups (as defined in [Gor]).

The notational conventions of the Introduction will be kept throughout this paper.

1. Trace Formulas

As in the Introduction, $G = X_{\mathbb{F}_q}$ denotes always a connected reductive $\mathbb{F}_q$–group scheme. Let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$ and put

$$\overline{G} = G \times_{spec(\mathbb{F}_q)} spec(\overline{\mathbb{F}}_q)\,,$$

the reductive $\overline{\mathbb{F}}_q$–group scheme obtained by base change from G. We will write $\phi : \overline{G} \to \overline{G}$ for the Frobenius endomorphism associated with the $\mathbb{F}_q$–form G of $\overline{G}$. Similarly, ϕ^n will denote the Frobenius of the $\mathbb{F}_{q^n}$–form $G \times_{spec(\mathbb{F}_q)} spec(\mathbb{F}_{q^n})$ of $\overline{G}$. Thus, if $G = spec(A)$ and $\overline{G} = spec(A \otimes \overline{\mathbb{F}}_q)$ then ϕ is given by the $\mathbb{F}_q$-homomorphism which maps $x \in A \otimes \overline{\mathbb{F}}_q$ to x^q. One has therefore a natural diagram of Lang maps

$$\begin{array}{ccccccc} G(\mathbb{F}_q) & \longrightarrow & \overline{G} & \xrightarrow{1/\phi} & \overline{G} & & \\ \downarrow & & \| & & \downarrow & \psi_n & , \\ G(\mathbb{F}_{q^n}) & \longrightarrow & \overline{G} & \xrightarrow[1/\phi^n]{} & \overline{G} & & \end{array} \qquad (1)$$

where ψ_n is given on $\overline{\mathbb{F}}_q$–rational points by $x \mapsto x \cdot \phi(x) \cdot \phi^2(x) \cdot \ldots \cdot \phi^{n-1}(x)$. We will write $H^*_{et}(\overline{G}; \mathbb{Z}/\ell^j)$ for the etale cohomology of $\overline{G}$ with coefficients in the constant sheaf with stalks $\mathbb{Z}/\ell^j$, ℓ a prime different from $p = char(\mathbb{F}_q)$. As usual, the ℓ–adic cohomology $H^*_{et}(\overline{G}; \mathbb{Q}_\ell)$ is defined as

$$(\lim_{\substack{\leftarrow \\ j}} H^*_{et}(\overline{G}; \mathbb{Z}/\ell^j)) \otimes \mathbb{Q}.$$

For the convenience of the reader, we recall some basic facts on $H^*_{et}(\overline{G}; \mathbb{Q}_\ell)$. By the classification of connected reductive group schemes over an algebraically closed field, there exists a unique reductive algebraic group L over $\mathbb{C}$ with the same *Root Data* as $\overline{G}$. The associated Lie group of complex points $L(\mathbb{C})^{top}$, with the strong topology, satisfies (cf. [FrPa])

$$H^*_{sing}(L(\mathbb{C})^{top}; \mathbb{Q}_\ell) \cong H^*_{et}(\overline{G}; \mathbb{Q}_\ell).$$

Moreover, the Lie group $L(\mathbb{C})^{top}$ has complex dimension $N = dim(\overline{G})$, and $L(\mathbb{C})^{top}$ is homeomorphic to $K \times \mathbb{R}^N$, where K denotes a maximal compact subgroup of $L(\mathbb{C})^{top}$ and, of course,

$$H^*_{sing}(L(\mathbb{C})^{top};\mathbb{Q}_\ell) \cong H^*_{sing}(K;\mathbb{Q}_\ell).$$

It is well known that $H^*_{sing}(K;\mathbb{Q}_\ell)$ is an exterior algebra over $\mathbb{Q}_\ell$ on odd dimensional generators. Thus

$$H^*_{et}(\overline{G};\mathbb{Q}_\ell) \cong \bigwedge W$$

with W a graded vector space over $\mathbb{Q}_\ell$. In particular,

$$H^N_{et}(\overline{G};\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell \ ,$$

where $N = dim(\overline{G})$, and therefore a morphism $f : \overline{G} \to \overline{G}$ of schemes over $spec(\overline{\mathbb{F}_q})$ has a well-defined degree

$$d(f) \in \mathbb{Z}_\ell \ ,$$

$\mathbb{Z}_\ell \subset \mathbb{Q}_\ell$ the ℓ–adic integers, satisfying

$$f^*(x) = d(f) \cdot x$$

for all $x \in H^N_{et}(\overline{G};\mathbb{Q}_\ell)$. Because the algebra $H^*_{et}(\overline{G};\mathbb{Z}/\ell)$ satisfies Poincaré duality it is clear that

$$H^*_{et}(f;\mathbb{Z}/\ell) : H^*_{et}(\overline{G};\mathbb{Z}/\ell) \to H^*_{et}(\overline{G};\mathbb{Z}/\ell)$$

is an isomorphism if and only if $d(f) \in \mathbb{Z}_\ell$ is a unit. It is also useful to observe that $H^*_{et}(\overline{G};\mathbb{Q}_\ell)$ has the structure of a Hopf-algebra, with coalgebra structure induced by the multiplication in $\overline{G}$. If we put

$$V = PH^*_{et}(\overline{G};\mathbb{Q}_\ell) \ ,$$

the subspace of primitive elements, then we have a natural isomorphism

$$\bigwedge V = H^*_{et}(\overline{G};\mathbb{Q}_\ell) \ .$$

It follows that if $f : \overline{G} \to \overline{G}$ is a morphism of schemes over $\overline{\mathbb{F}}_q$, such that the induced map $f^* : H^*_{et}(\overline{G};\mathbb{Q}_\ell) \to H^*_{et}(\overline{G};\mathbb{Q}_\ell)$ maps $V = PH^*_{et}(\overline{G};\mathbb{Q}_\ell)$ to itself (e.g., if f is a morphism of group schemes), then

$$d(f) = det(f^* \mid V) \ ,$$

because $H^N_{et}(\overline{G};\mathbb{Q}_\ell) \cong \bigwedge^{max} V$, the largest non-vanishing exterior power of V. Moreover, if we put $f^* = 1 - g$, then the linear map g maps V into itself and satisfies

$$det(f^* \mid V) = det(1 - g \mid V) = grTr(g) \ ,$$

where we mean by $grTr(g)$ the *graded trace*

$$\sum(-1)^i Tr(g : H^i_{et}(\overline{G};\mathbb{Q}_\ell) \to H^i_{et}(\overline{G};\mathbb{Q}_\ell)) \, .$$

This is a consequence of the fact that V has a bases consisting of elements of odd degrees, so that $grTr(g)$, as defined above, is also equal to

$$\sum(-1)^j Tr(\wedge^j Pg : \wedge^j V \to \wedge^j V) \ ,$$

where Pg denotes the restriction of g to V. Our next goal is to analyse the degree of the morphism ψ_n in the diagram (1). An easy spectral sequence argument, applied to the diagram (1), shows the following (see [FrMi] for the case of an $\mathbb{F}_q$–split G) .

Proposition 1.1. Let $G = X_{\mathbb{F}_q}$ and $\psi_n : \overline{G} \to \overline{G}$ be as above. Then for every prime ℓ different from $p = char(\mathbb{F}_q)$ the following are equivalent:

(i) $G(\mathbb{F}_q) \to G(\mathbb{F}_{q^n})$ induces an $H_*(\ ;\mathbb{Z}/\ell)$–isomorphism,

(ii) the degree $d(\psi_n) \in \mathbb{Z}_\ell$ is an ℓ–adic unit.

To prove the theorem of the Introduction, we need to express the degree $d(\psi_n)$ in terms of the orders of the groups $G(\mathbb{F}_q)$ and $G(\mathbb{F}_{q^n})$. This will be done by interpreting $d(\psi_n)$ in terms of graded traces and by relating these to the orders of the groups $G(\mathbb{F}_q)$ and $G(\mathbb{F}_{q^n})$ using the Lefschetz trace formula applied to the Frobenius endomorphism of $\overline{G}$.

Lemma 1.2. Let $\overline{G}$ be a connected reductive $\overline{\mathbb{F}}_q$–group scheme and $\phi : \overline{G} \to \overline{G}$ the Frobenius endomorphism associated with an $\mathbb{F}_q$–form of $\overline{G}$. Then the degree $d(1/\phi)$ of the Lang map $1/\phi : \overline{G} \to \overline{G}$ is given by

$$d(1/\phi) = \sum(-1)^i Tr(\phi^* : H^i_{et}(\overline{G};\mathbb{Q}_\ell) \to H^i_{et}(\overline{G};\mathbb{Q}_\ell)) \ .$$

Proof: As observed above, the Hopf–algebra $H^*_{et}(\overline{G};\mathbb{Q}_\ell)$ is an exterior algebra $\wedge V$ on $V = PH^*_{et}(\overline{G};\mathbb{Q}_\ell)$, the subspace of primitive elements. If we denote by $gr\,Tr(\phi^*)$ the graded trace $\sum(-1)^i Tr(\phi^* : H^i_{et}(\overline{G};\mathbb{Q}_\ell) \to H^i_{et}(\overline{G};\mathbb{Q}_\ell))$ then, because all homogeneous elements of V have odd cohomological degree, we have

$$gr\,Tr(\phi^*) = \sum(-1)^j Tr(\wedge^j P\phi^* : \wedge^j V \to \wedge^j V)\,, \tag{2}$$

where $P\phi^* : PH^*_{et}(\overline{G};\mathbb{Q}_\ell) \to PH^*_{et}(\overline{G};\mathbb{Q}_\ell)$ is the map induced from the morphism of group schemes $\phi : \overline{G} \to \overline{G}$; as observed earlier, the right hand side of (2) is therefore just $\det(1 - P\phi^*)$. On the other hand, $1/\phi : \overline{G} \to \overline{G}$ maps $x \in PH^*_{et}(\overline{G};\mathbb{Q}_\ell)$ to $x - \phi^* x$, which is equal to $(1 - P\phi^*)x$. Thus $\det(1 - P\phi^*) = d(1/\phi)$ which shows that $d(1/\phi) = grTr(\phi^*)$ as claimed.

From the diagram (1) above we see that $\psi_n \circ (1/\phi) = 1/\phi^n$. The following corollary is then immediate.

Corollary 1.3. The degree of the map ψ_n in the diagram (1) is given by

$$d(\psi_n) = \frac{gr\,Tr(\phi^n)^*}{gr\,Tr(\phi^*)} \in \mathbb{Z}_\ell\,.$$

The Lefschetz trace formula in H^*_c, etale cohomology with compact supports, permits to relate the number of $\mathbb{F}_{q^n}$– rational points of certain schemes to a graded trace. The following proposition is well known (see [Mil], and also [DeLu] for a general discussion of the Lefschetz trace formula; the formula we use here is 1.9.4, page 174, of [Del]).

Proposition 1.4. Let $X_{\mathbb{F}_q}$ be a quasi-projective $\mathbb{F}_q$–scheme, $\overline{X} = X_{\mathbb{F}_q} \times_{spec(\mathbb{F}_q)} spec(\overline{\mathbb{F}}_q)$, and $\phi : \overline{X} \to \overline{X}$ the associated Frobenius endomorphism. Then

$$|\overline{X}(\mathbb{F}_q)| = \sum(-1)^i Tr(\phi^*_c : H^i_c(\overline{X};\mathbb{Q}_\ell) \to H^i_c(\overline{X};\mathbb{Q}_\ell))\,. \tag{3}$$

In accordance with the notation used above we will write $gr\,Tr(\phi^*_c)$ for the right hand side of the formula (3). It remains to relate $gr\,Tr(\phi^*)$ to $gr\,Tr(\phi^*_c)$ in case $X_{\mathbb{F}_q} = G$. Since $\overline{G}$ is a smooth quasi-projective variety of dimension N over $\overline{\mathbb{F}}_q$, there is a natural Poincaré– Duality pairing (cf. [Mil])

$$<\,,\,>: H^j_{et}(\overline{G};\mathbb{Q}_\ell) \times H^{2N-j}_c(\overline{G};\mathbb{Q}_\ell) \longrightarrow \mathbb{Q}_\ell(-N)$$

satisfying $< \phi^* x, \phi_c^* y >= q^N < x, y >$. Therefore $< x, \phi_c^* y >=< q^N (\phi^*)^{-1}(x), y >$ which, since the pairing $<,>$ is non-degenerate, implies that

$$Tr(\phi_c^* \mid H_c^{2N-j}(\overline{G};\mathbb{Q}_\ell)) = q^N Tr((\phi^*)^{-1} \mid H_{et}^j(\overline{G};\mathbb{Q}_\ell)) .$$

Thus, taking graded traces, we infer

$$gr\,Tr(\phi_c^*) = q^N \cdot gr\,Tr((\phi^*)^{-1}) \ . \tag{4}$$

From the computations in the proof of the Lemma 1.2 we see that

$$\begin{aligned} gr\,Tr((\phi^*)^{-1}) &= \det(1-(P\phi^*)^{-1}) = (-1)^r \det(P\phi^*)^{-1} \det(1-P\phi^*) \\ &= (-1)^r \det(P\phi^*)^{-1} \cdot gr\,Tr(\phi^*) \end{aligned}$$

where $r = \dim PH_{et}^*(\overline{G};\mathbb{Q}_\ell)$. Note that $\det P\phi^* = u \in \mathbb{Z}_\ell$ is a unit, because ϕ induces an isomorphism in $H_{et}^*(\overline{G};\mathbb{Z}/\ell)$ for any ℓ prime to p. Applying the same reasoning to ϕ^n, we obtain

$$gr\,Tr(((\phi^n)^*)^{-1}) = (-1)^r u^{-n} \cdot gr\,Tr(\phi^n)^* .$$

Therefore, combining this with Corollary 1.3, Proposition 1.4 and formula (4), we obtain

$$\begin{aligned} d(\psi_n) &= \frac{u^n \cdot gr\,Tr((\phi^n)^*)^{-1}}{u \cdot gr\,Tr(\phi^*)^{-1}} = \frac{q^{-nN} \cdot u^n \cdot gr\,Tr(\phi_c^*)^n}{q^{-N} \cdot u \cdot gr\,Tr(\phi_c^*)} \\ &= \left(\frac{u}{q^N}\right)^{n-1} \cdot \frac{|G(\mathbb{F}_{q^n})|}{|G(\mathbb{F}_q)|} \ , \quad u \in \mathbb{Z}_\ell^* \ . \end{aligned}$$

It is now plain that $d(\psi_n)$ is an ℓ–adic unit if and only if $G(\mathbb{F}_q)$ and $G(\mathbb{F}_{q^n})$ have isomorphic ℓ–Sylow subgroups and, in view of Proposition 1.1 the proof of the Theorem is therefore completed.

The following example illustrates the general result. Take $G = S\ell_2$. It is easy to check the Theorem in this case directly because the cohomology rings $H^*(S\ell_2(\mathbb{F}_q);\mathbb{Z}/\ell)$, ℓ a prime different from the characteristic of $\mathbb{F}_q$, are completely known (cf. [FiPr]). The cohomology is periodic of periode 4 ,and for ℓ an odd (prime prime to q) one has

$$H^*(S\ell_2(\mathbb{F}_q);\mathbb{Z}/\ell) \cong \begin{cases} E(u)\otimes P(v), u \in H^3 \text{and } v \in H^4 \ (\text{ if } \ell \text{ divides } q^2-1) \\ \mathbb{Z}/\ell \quad (\text{if } \ell \text{ does not divide } q^2-1) \end{cases} .$$

Here, $E(u)$ denotes an exterior algebra on u, and $P(v)$ a polynomial algebra on v (over $\mathbb{Z}/\ell$). Since the order of $S\ell_2(\mathbb{F}_q)$ is $q(q^2-1)$ we see that the ℓ-Sylow subgroup of $S\ell_2(\mathbb{F}_q)$ is non-trivial if and only if ℓ divides q^2-1 and thus, by the formula above, one has abstract isomorphisms of rings

$$H^*(S\ell_2(\mathbb{F}_q);\mathbb{Z}/\ell) \cong H^*(SL_2(\mathbb{F}_{q^n});\mathbb{Z}/\ell) \tag{5}$$

if $\ell \mid q^2-1$.But $S\ell_2(\mathbb{F}_q)$ and $S\ell_2(\mathbb{F}_{q^n})$ have isomorphic ℓ-Sylow subgroups if and only if q^2-1 and $q^{2n}-1$ contain the same power of ℓ, that is (assuming that ℓ divides q^2-1), if and only if n is relatively prime to ℓ. Thus, in most cases, the isomorphism (5) is not induced by the restriction map

$$res : H^*(S\ell_2(\mathbb{F}_{q^n});\mathbb{Z}/\ell) \to H^*(S\ell_2(\mathbb{F}_q);\mathbb{Z}/\ell) \,.$$

Our general setting will take the following form in case $G = S\ell_2$. One has

$$H^*_{et}(\overline{S\ell}_2;\mathbb{Q}_\ell) = E(w)\,,$$

an exterior algebra over $\mathbb{Q}_\ell$ in $w \in H^3_{et}$. (Recall that $H^*_{et}(\overline{S\ell}_2;\mathbb{Q}_\ell) \cong H^*_{top}(S\ell_2(\mathbb{C});\mathbb{Q}_\ell)$, and $S\ell_2(\mathbb{C})$ is homotopy equivalent to a three-dimensional sphere).

Clearly, $PH^*_{et} = H^3_{et} \cong \mathbb{Q}_\ell$ and one checks easily that $\phi^* w = q^2 w$ so that

$$d(1/\phi) = \det(1 - P\phi^*) = 1 - q^2$$

and thus

$$d(\psi_n) = \frac{q^{2n}-1}{q^2-1} = \frac{1}{q^{n-1}} \cdot \frac{|S\ell_2(\mathbb{F}_{q^n})|}{|S\ell_2(\mathbb{F}_q)|}\,.$$

According to our general formula, we must have

$$\frac{1}{q^{n-1}} = (\frac{u}{q^N})^{n-1}$$

where $u = \det P\phi^*$ and $N = \dim S\ell_2$; this is indeed so, as $\det P\phi^* = q^2$ and $\dim S\ell_2 = 3$.

The Theorem of the Introduction can also be viewed as a result concerning the conjugacy relationship between the various ℓ– subgroups of $G(\mathbb{F}_q)$ and $G(\mathbb{F}_{q^n})$. For this purpose, denote by $Frob_\ell(F)$ the "Frobenius category" of finite ℓ–subgroups of the group F; its objects are the finite ℓ–subgroups of F, and morphisms are induced by inner automorphisms of F. It is a classical result that for a finite group F, the restriction maps fit together to give rise to an isomorphism

$$H^*(F;\mathbb{Z}/\ell) \to \lim_{\substack{\leftarrow \\ \pi \in Frob_\ell(F)^{op}}} H^*(\pi;\mathbb{Z}/\ell)\,.$$

Thus, any homomorphism of finite groups $\varphi : F_1 \to F_2$ inducing an equivalence of categories $Frob_\ell(F_1) \to Frob_\ell(F_2)$ will induce an isomorphism $H^*(F_2;\mathbb{Z}/\ell) \to H^*(F_1;\mathbb{Z}/\ell)$. The converse of this statement is also true according to [Mis]. Thus, our Theorem implies the following corollary, which in the case of $G = G\ell_n$ is a well-known fact on the representation theory of finite ℓ-groups in characteristic p different from ℓ.

Corollary 1.5. Let G be a connected reductive $\mathbb{F}_q$–group scheme and let ℓ be a prime different from $p = char(\mathbb{F}_q)$. Suppose that $G(\mathbb{F}_q)$ and $G(\mathbb{F}_{q^n})$ have isomorphic ℓ–Sylow subgroups. Then the inclusion $G(\mathbb{F}_q) \to G(\mathbb{F}_{q^n})$ induces an equivalence of Frobenius categories of finite ℓ–subgroups

$$Frob_\ell(G(\mathbb{F}_q)) \to Frob_\ell(G(\mathbb{F}_{q^n}))\,.$$

2. Suzuki and Ree Groups

We will be considering the three families of groups, which in [Gor] are denoted by

$${}^2B_2(2^n)\,,\ {}^2G_2(3^n)\,,\ {}^2F_4(2^n)$$

with n odd, say $n = 2m+1$; ${}^2B_2(2^n)$ is isomorphic to the Suzuki group $Sz(2^n)$, which is simple for $n > 1$, and the groups ${}^2G_2(3^n)$, ${}^2F_4(2^n)$ are the simple groups of Ree type. The orders of these groups are (cf. [Gor]):

$$|{}^2B_2(q)| = q^2(q^2+1)(q-1) \quad \text{where} \quad q = 2^{2m+1}$$

$$|{}^2G_2(q)| = q^3(q^3+1)(q-1) \quad \text{where} \quad q = 3^{2m+1}$$

$$|{}^2F_4(q)| = q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1) \quad \text{where} \quad q = 2^{2m+1}$$

Let $G = G_{\mathbb{F}_q}$ denote the split reductive group scheme over $spec(\mathbb{F}_q)$ of type B_2, G_2

respectively F_4. Then $\overline{G} = G \times_{spec\,\mathbb{F}_q} spec\,\overline{\mathbb{F}}_q$ admits an exceptional isogeny

$$\psi : \overline{G} \to \overline{G}$$

such that $\psi^2 = \phi$, the Frobenius of the $\mathbb{F}_q$–form G of $\overline{G}$. Moreover

$${}^2G(q) = \{x \in G(\mathbb{F}_q) \mid \psi x = x\}$$

agrees with ${}^2B_2(q)$, ${}^2G_2(q)$ respectively ${}^2F_4(q)$ for G of type B_2, G_2 respectively F_4 (we always assume $q = 2^{2m+1}, 3^{2m+1}$ respectively 2^{2m+1}, according to the case one considers). As in [FrMi], one gets then a commutative diagram of finite etale maps arising from Lang's construction

$$\begin{array}{ccccc} {}^2G(q) & \longrightarrow & \overline{G} & \stackrel{1/\psi}{\longrightarrow} & \overline{G} \\ \downarrow & & \| & & \downarrow \theta \\ {}^2G(q^d) & \longrightarrow & \overline{G} & \underset{1/\psi^d}{\longrightarrow} & \overline{G} \end{array}$$

where d is an odd natural number; θ is given on a point $x \in \overline{G}(\mathbb{F}_q)$ by

$$\theta(x) = x \cdot \psi(x) \cdot \psi^2(x) \ldots \psi^{d-1}(x)$$

As before, it follows that the inclusion ${}^2G(q) \to {}^2G(q^d)$ is an $H_*(\ ;\mathbb{Z}/\ell)$ isomorphism (ℓ as always prime to q) if and only if $\theta^* : H^*_{et}(\overline{G};\mathbb{Q}_\ell) \to H^*_{et}(\overline{G};\mathbb{Q}_\ell)$ has a degree prime to ℓ (the degree of θ^* is an ℓ–adic integer). There are now three cases to consider. For the computation of the relevant degrees of $(B\psi)^*$ we refer the reader to [AdMa].

(i): G of type B_2

In this case, $H^*_{et}(B\overline{G};\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell[x_4, x_8]$
and $(B\psi)^*x_4 = qx_4$, $(B\psi)^*x_8 = -q^2x_8$. Therefore

$$\begin{aligned} \deg\theta &= (1 + q + q^2 + \ldots + q^{d-1})(1 - q^2 + q^4 - \ldots + q^{2d-2}) \\ &= q^{2-2d}|{}^2B_2(q^d)|/|{}^2B_2(q)|\,. \end{aligned}$$

We conclude that $\deg\theta$ is prime to ℓ if and only if ${}^2B_2(q)$ and ${}^2B_2(q^d)$ have isomorphic ℓ–Sylow subgroups. Thus the following holds.

Proposition 2.1. Let $q = 2^{2m+1}$ and $d \geq 1$ an odd integer. Let ℓ be an odd prime. Then ${}^2B_2(q) \to {}^2B_2(q^d)$ induces an isomorphism in $\mathbb{Z}/\ell$–homology if and only if ${}^2B_2(q)$ and ${}^2B_2(q^d)$ have isomorphic ℓ–Sylow subgroups.

(ii): G of type G_2

We have $H^*_{et}(B\overline{G};\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell[x_4, x_{12}]$ and $(B\psi)^*x_4 = qx_4$, $(B\psi)^*x_{12} = -q^3x_{12}$ which yields

$$\begin{aligned}\deg\theta &= (1 + q + q^2 + \ldots + q^{d-1})(1 - q^3 + q^6 - \ldots + q^{3d-3}) \\ &= q^{3-3d}|{}^2G_2(q^d)|/|{}^2G_2(q)|\end{aligned}$$

and the next result follows.

Proposition 2.2. Let $q = 3^{2m+1}$ and d odd. Let ℓ be a prime different from 3. Then ${}^2G_2(q) \to {}^2G_2(q^d)$ induces a $\mathbb{Z}/\ell$–homology isomorphism if and only if ${}^2G_2(q)$ and ${}^2G(q^d)$ have isomorphic ℓ–Sylow subgroups.

(iii): G of type F_4

We proceed as in the other two cases and obtain:

$$\begin{aligned}&H^*(B\overline{G};\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell[x_4, x_{12}, x_{16}, x_{24}]\,, \\ &(B\psi)^*x_4 = qx_4\,, (B\psi)^*x_{12} = -q^3x_{12}\,, (B\psi)^*x_{16} = q^4x_{16}\,, \\ &(B\psi)^*x_{24} = -q^6x_{24}\,.\end{aligned}$$

This implies that

$$\deg\theta = q^{12-12d}|{}^2F_4(q^d)|/|{}^2F_4(q)|$$

and we get our final result.

Proposition 2.3. Let $q = 2^{2m+1}$, and $d \geq 1$ odd. Let ℓ be an odd prime. Then ${}^2F_4(q) \to {}^2F_4(q^d)$ induces a $\mathbb{Z}/\ell$–homology isomorphism if and only if ${}^2F_4(q)$ and ${}^2F_4(q^d)$ have isomorphic ℓ–Sylow subgroups.

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