# ON GROUPS WHICH ACT FREELY AND PROPERLY ON FINITE DIMENSIONAL HOMOTOPY SPHERES

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*For Urs Stammbach on his 60th birthday*

## 1. Introduction

In [28] C. T. C. Wall conjectured that if a countable group $G$ of finite virtual cohomological dimension, $\operatorname{vcd} G < \infty$, has periodic Farrell cohomology then $G$ acts freely and properly on $\mathbb{R}^n \times S^m$ for some $n$ and $m$. Obviously, if a group $G$ acts freely and properly on some $\mathbb{R}^n \times S^m$ then $G$ is countable since $\mathbb{R}^n \times S^m$ is a separable metric space. The Farrell cohomology generalizes the Tate cohomology theory for finite groups to the class of groups $G$ with $\operatorname{vcd} G < \infty$ (see for instance Ch. X of [2]). Wall's conjecture was proved by Johnson in some cases [12] and Connolly and Prassidis in general [4].

In [19] Prassidis showed that there are groups of infinite vcd which act freely and properly on some $\mathbb{R}^n \times S^m$. In particular, it follows from results of Prassidis [19] and Talelli [24] that if a countable group $G$ has periodic cohomology after 1-step then $G$ acts freely and properly on some $\mathbb{R}^n \times S^m$ [25].

A group $G$ is said to have periodic cohomology after $k$-steps if there is a positive integer $q$ such that the functors $H^i(G;\quad)$ and $H^{i+q}(G;\quad)$ are naturally equivalent for all $i > k$ (cf. [22], [26]). In [25] it was conjectured that the following statements are equivalent for a countable group $G$:

(1) $G$ acts freely and properly on some $\mathbb{R}^n \times S^m$
(2) there is an integer $q$ and an exact sequence

$$0 \to \mathbb{Z} \to A \to P_{q-2} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

with $P_i$ projective $\mathbb{Z}G$-modules, $\mathbb{Z}$ with trivial $G$-action and $\operatorname{proj.dim}_{\mathbb{Z}G} A < \infty$
(3) $G$ has periodic cohomology after some steps.

Note that if a group $G$ has periodic cohomology with period $q$ after $k$-steps and the isomorphisms are induced by cup product with an

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element $g \in H^q(G;\mathbb{Z})$, then $g$ is represented by a $q$-extension

$$0 \to \mathbb{Z} \to A \to P_{q-2} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

with $P_i$ projective $\mathbb{Z}G$-modules and $\text{proj.}\dim_{\mathbb{Z}G} A \leq k$; conversely, from (2) one can deduce that (3) holds, with the periodicity induced via a cup product (cf. [25]).
Now (1) $\Rightarrow$ (2) $\Rightarrow$ (3) (see Cor. 5.2) and from the results mentioned above (3) $\Rightarrow$ (1) if $\operatorname{vcd} G < \infty$ or if $G$ has periodic cohomology after 1-step. Also, by a result of Talelli [22], the condition (3) is equivalent for an arbitrary group $G$ to the condition

(3′) $G$ admits a periodic (complete) resolution.

The definition of complete resolutions is recalled in Section 2, where we also review the definition of generalized Tate cohomology $\hat{H}^\bullet(G;M)$ for an arbitrary group $G$. It turns out (cf. Theorem 4.1) that (3′) is equivalent for an arbitrary group $G$ to

(3″) $\hat{H}^\bullet(G;\mathbb{Z})$ contains a unit of non-zero degree.

The condition concerning units can be analyzed by considering suitable actions of $G$ on finite dimensional contractible spaces and leads us to the following

**Theorem A .** *If $G$ is a countable group in the class $\mathbf{H}\mathfrak{F}$ and there is a bound on the orders of the finite subgroups of $G$ then* (3) $\Rightarrow$ (1).

The class $\mathbf{H}\mathfrak{F}$ of *hierarchically decomposable* groups was introduced by Kropholler [13] as follows. Let $\mathbf{H}_0\mathfrak{F}$ be the class of finite groups. Now define $\mathbf{H}_\alpha\mathfrak{F}$ for each ordinal $\alpha$ inductively: if $\alpha$ is a successor ordinal then $\mathbf{H}_\alpha\mathfrak{F}$ is the class of groups which admit a finite dimensional contractible $G$-$CW$-complex with cell stabilizers in $\mathbf{H}_{\alpha-1}\mathfrak{F}$, and if $\alpha$ is a limit ordinal then $\mathbf{H}_\alpha\mathfrak{F} = \cup_{\beta<\alpha}\mathbf{H}_\beta\mathfrak{F}$. A group belongs to $\mathbf{H}\mathfrak{F}$ if it belongs to $\mathbf{H}_\alpha\mathfrak{F}$ for some $\alpha$.

**Notation.** If $\mathfrak{X}$ is a class of groups, we denote by $\mathfrak{X}_b$ the subclass consisting of those groups in $\mathfrak{X}$ for which there is a bound on the orders of their finite subgroups.

We show

**Theorem B .** *Let $G \in \mathbf{H}\mathfrak{F}_b$. Then the following statements are equivalent for $G$, and they all imply that $G \in \mathbf{H}_1\mathfrak{F}_b$ :*

(I) *there is a finite dimensional free $G$-$CW$-complex homotopy equivalent to a sphere*
(II) *there is an integer $q$ and an exact sequence*

$$0 \to \mathbb{Z} \to A \to P_{q-2} \to \cdots \to P_0 \to \mathbb{Z}$$

*with* $P_i$ *projective* $\mathbb{Z}G$*-modules,* $\mathbb{Z}$ *with trivial* $G$*-action and* $\operatorname{proj.dim}_{\mathbb{Z}G} A < \infty$

(III) $G$ *has periodic cohomology after some steps*

(IV) *there is an invertible element in the ring* $\hat{H}^\bullet(G;\mathbb{Z})$ *of non-zero degree.*

*Moreover, for* $G \in \mathbf{H}_1\mathfrak{F}_b$ *the following is equivalent to* (I), (II), (III) *and* (IV):

(V) *every finite subgroup of* $G$ *has periodic cohomology.*

Note that in case the group $G$ in Theorem B is countable, (I) gives rise to a free and proper $G$-action on $\mathbb{R}^n \times S^m$ for some $n$ and $m$ (Lemma 5.4) and therefore Theorem B implies Theorem A.

It follows from a theorem of Serre (e.g. Thm. 11.1, Ch. VIII in [2]) that the class of groups of finite vcd is contained in $\mathbf{H}_1\mathfrak{F}_b$. Connolly and Prassidis [4] proved essentially that (III) $\Rightarrow$ (I) if $\operatorname{vcd} G < \infty$, and Brown (Ch. X in [2]) that (V) $\Rightarrow$ (III) if $\operatorname{vcd} G < \infty$. Our proof of Theorem B is based on the methods developed in these papers.

Note that there are groups in $\mathbf{H}_1\mathfrak{F}$ such that (V) $\not\Rightarrow$ (III) and there are also groups in $\mathbf{H}\mathfrak{F}_b$ such that (V) $\not\Rightarrow$ (III); but there is also a family of groups in $\mathbf{H}_1\mathfrak{F} \setminus \mathbf{H}_1\mathfrak{F}_b$ such that (V) $\Rightarrow$ (III) $\Rightarrow$ (I) (for examples see Remark 4.11).

The class $\mathbf{H}_1\mathfrak{F}_b$ is a larger class than the class of groups of finite vcd. For example if $\operatorname{vcd} G_i < \infty$ $(i = 1, 2)$ and $G = G_1 *_S G_2$ then the group $G$ need not be of finite vcd [20]. However, $G \in \mathbf{H}_1\mathfrak{F}_b$. Actually if a group $G$ is the fundamental group of a finite graph of groups of finite vcd then $G \in \mathbf{H}_1\mathfrak{F}_b$; also $\mathbf{H}_1\mathfrak{F}_b$ is extension closed whereas the class of groups of finite vcd is not (see 3.9 and 3.10 for general results on groups in $\mathbf{H}_1\mathfrak{F}_b$). The Burnside group $B(d,e)$ of odd exponent $e > 665$ on $d$ generators is another example of a group of infinite vcd in $\mathbf{H}_1\mathfrak{F}_b$. It turns out that it has periodic cohomology after 2-steps and it follows from Theorem A that it acts freely and properly on some $\mathbb{R}^n \times S^m$ (cf. Cor. 5.6).

The proof of Theorem B relies on a result of Kropholler and Mislin [15] which states that if $G \in \mathbf{H}\mathfrak{F}_b$ and $\operatorname{proj.dim}_{\mathbb{Z}G} B(G,\mathbb{Z}) < \infty$, where $B(G,\mathbb{Z})$ is the $\mathbb{Z}G$-module of bounded functions from $G$ to $\mathbb{Z}$, then $G \in \mathbf{H}_1\mathfrak{F}_b$ and admits a finite dimensional $\underline{E}G$. (Recall that $\underline{E}G$, the classifying space for proper $G$-actions, is a $G$-$CW$-complex $X$ characterized up to $G$-homotopy by the requirement that for every finite subgroup $H < G$ the fixed point space $X^H$ is contractible, and for infinite $H < G$, $X^H$ is empty).

We also show that if a group $G$ contains a free abelian subgroup of infinite rank, then $G$ does not act freely and properly on any $\mathbb{R}^n \times S^m$ (cf. Cor. 5.6).

For every group $G$ there is a free $G$-$CW$-complex $S_G$ homotopy equivalent to a sphere. For example, if $Y$ is the universal cover of a $K(G,1)$ complex then $Y \times S^n$, with diagonal $G$-action, trivial on $S^n$ is such a complex $S_G$.

We believe that periodicity in cohomology after some steps is the algebraic characterization of those groups $G$ which admit a *finite dimensional* $S_G$. We prove this for $G \in \mathbf{H}\mathfrak{F}_b$.

## 2. Generalized Tate cohomology

The classical Tate cohomology for finite groups was generalized by Farrell [7] to the case of groups of finite vcd and subsequently by Ikenaga [10] to the more general class of groups $G$ admitting complete resolutions and having finite generalized cohomological dimension, $\underline{\mathrm{cd}}\, G < \infty$ (for the definition of $\underline{\mathrm{cd}}$ see below). In [17] generalized Tate cohomology groups $\hat{H}^i(G;M)$ are defined for arbitrary groups $G$ and $G$-modules $M$, specializing to the ones defined by Farrell and Ikenaga, when the latter are defined. The definition of these generalized Tate groups is as follows:

$$\hat{H}^n(G;M) := \varinjlim_{j \geq 0} S^{-j}H^{n+j}(G;M), \quad n \in \mathbb{Z}$$

with $S^{-j}H^{n+j}(G;\quad)$ denoting the $j$th left satellite of $H^{n+j}(G;\quad)$ (for details, the reader is referred to [17]; different, but equivalent definitions of generalized Tate groups can be found in [1] and [9]). The following are three of their basic properties:

(T1) $\hat{H}^i(G;P) = 0$ for every projective $P$ and $i \in \mathbb{Z}$

(T2) there is a canonical natural transformation

$$\tau : H^\bullet(G;\quad) \to \hat{H}^\bullet(G;\quad)$$

such that every natural transformation from ordinary cohomology to a cohomological functor which vanishes on projectives, factors uniquely through $\tau$

(T3) if there exists $n \in \mathbb{Z}$ such that $H^i(G;P) = 0$ for all projective $P$ and all $i > n$ then

$$\tau : H^i(G;\quad) \cong \hat{H}^i(G;\quad)$$

for all $i > n$.

Note that (T1) implies that generalized Tate cohomology is *effaceable*: there is *dimension-shifting upwards*

$$\hat{H}^i(G;M) \cong \hat{H}^{i+1}(G;\Omega M)$$

where $\Omega M$ denotes the kernel of a surjection of a projective module onto $M$.

Sometimes the generalized Tate cohomology groups can be computed using *complete resolutions*. For this we recall a few definitions.

**Definition 2.1.** A complete resolution for a group $G$ is an acyclic complex $\mathcal{F} = \{F_i, \partial_i \,|\, i \in \mathbb{Z}\}$ of projective $\mathbb{Z}G$-modules, together with a projective resolution $\mathcal{P}{=}\{P_i, d_i \,|\, i \geq 0\}$ of $G$ such that $\mathcal{F}$ and $\mathcal{P}$ coincide in sufficiently high dimensions:

$$\begin{array}{ccl} & & F_{k-1} \to \cdots \to F_0 \to F_{-1} \to \cdots \\ & \nearrow & \\ \cdots \to F_{k+1} \to F_k & & \\ & \searrow & \\ & & P_{k-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0 \end{array}$$

The number $k \in \mathbb{N}$ is called the *coincidence index* of the complete resolution.

Clearly this definition generalizes the notion of complete resolution for finite groups and groups of finite vcd (e.g. Ch. X in [2]). In an analogous way one defines a complete resolution for a particular $G$-module $M$ instead of $\mathbb{Z}$. It is easy to prove that $G$ has a complete resolution if and only if every $G$-module $M$ has a complete resolution.

We say that $M$ has a complete resolution in the *strong sense* if it has a complete resolution $\mathcal{F}$ such that the complex $\mathrm{Hom}_{\mathbb{Z}G}(\mathcal{F}, P)$ is exact for all projective $P$ (this is the way the term "complete resolution" is used in [5]). In case the trivial $G$-module $\mathbb{Z}$ has a complete resolution in the strong sense, we just say that $G$ *has a complete resolution in the strong sense.*

**Lemma 2.2.** *If $G$ admits a complete resolution and if every projective $\mathbb{Z}G$-module has finite injective dimension, then every $\mathbb{Z}G$-module admits a complete resolution in the strong sense.*

*Proof.* Let $\mathcal{F}$ be a complete resolution for $G$ (cf. 2.1). We first show how to construct a complete resolution in the strong sense for a $\mathbb{Z}$-free $\mathbb{Z}G$-module $M$. Clearly $\mathcal{F} \otimes M$ with diagonal $G$-action yields a complete resolution for $M$. We need to show that for $P$ projective, the complex $\mathrm{Hom}_{\mathbb{Z}G}(\mathcal{F}\otimes M, P)$ is exact. For this we choose an injective resolution

$$0 \to P \to I_0 \to I_1 \to \cdots \to I_n \to 0$$

and notice that for $0 \le k \le n$ the complexes $\mathrm{Hom}_{\mathbb{Z}G}(\mathcal{F}\otimes M, I_k)$ are exact, because $I_k$ is injective. Thus $\mathrm{Hom}_{\mathbb{Z}G}(\mathcal{F}\otimes M, P)$ is exact too. To treat the case of a general $M$ we choose a surjection $F \to M$, $F$ a free $\mathbb{Z}G$-module and write $\Omega M$ for the kernel, which is $\mathbb{Z}$-free. Clearly, a complete resolution of $\Omega M$ in the strong sense yields one for $M$ in an obvious way. □

**Definition 2.3.** A group $G$ admits a periodic resolution, if it admits a complete resolution $\mathcal{F}=\{F_i, \partial_i \mid i \in \mathbb{Z}\}$ such that for some $k > 0$ and all $i \in \mathbb{Z}$ one has $F_{i+k} = F_i$ and $\partial_{i+k}=\partial_i$.

It was proved by Talelli [22] that a group $G$ has period $q$ after $k$-steps if and only if there is an exact sequence

$$0 \to R_{k+q} \to P_{k+q-1} \to \ldots \to P_0 \to \mathbb{Z} \to 0$$

with all $P_j$ projective $\mathbb{Z}G$-modules and with $R_{k+q}$ isomorphic to $R_k=$ $\mathrm{Im}(P_k \to P_{k-1})$. Clearly by splicing together copies of

$$0 \to R_k \to P_{k+q-1} \to \ldots \to P_k \to R_k \to 0$$

one obtains a periodic resolution for $G$ of coincidence index $k$.

**Corollary 2.4.** *A group $G$ admits a periodic resolution if and only if $G$ has periodic cohomology after some steps.*

If $G$ has a complete resolution $\mathcal{F}$ in the strong sense, then by definition $H^\bullet(\mathrm{Hom}_{\mathbb{Z}G}(\mathcal{F}, P)) = 0$ for all projective $P$, and the universal property of the generalized Tate groups implies that one has a canonical equivalence of cohomological functors

$$\hat{H}^\bullet(G;\quad) \cong H^\bullet(\mathrm{Hom}_{\mathbb{Z}G}(\mathcal{F},\quad)).$$

We then say that "the generalized Tate cohomology can be computed using a complete resolution of $G$".

The following theorem characterizes groups for which the generalized Tate cohomology can be computed using a complete resolution of $G$ (see also Thm. 3.10 of [5]). It involves the invariants $\mathrm{spli}\, G$, which is the supremum of the projective length of injective $\mathbb{Z}G$-modules

$$\mathrm{spli}\, G := \sup\{i : \mathrm{Ext}^i_{\mathbb{Z}G}(I,\quad) \neq 0 \mid I \;\; \mathbb{Z}G\text{-injective}\},$$

and Ikenaga's *generalized* cohomological dimension $\underline{\mathrm{cd}}\, G$, which is defined by

$$\underline{\mathrm{cd}}\, G := \sup\{i : \mathrm{Ext}^i_{\mathbb{Z}G}(M, F) \neq 0 \mid M \;\; \mathbb{Z}\text{-free},\ F \;\; \mathbb{Z}G\text{-free}\}.$$

Occasionally we will also use the invariant $\operatorname{silp} G$, which is the supremum of the injective length of projective $\mathbb{Z}G$-modules

$$\operatorname{silp} G := \sup\{i : \operatorname{Ext}^i_{\mathbb{Z}G}(\quad, P) \neq 0 \,|\, P \quad \mathbb{Z}G\text{-projective}\}.$$

It is straightforward that $\underline{\operatorname{cd}}\, G$ and $\operatorname{silp} G$ are either both finite or both infinite, and more precisely

$$\underline{\operatorname{cd}}\, G \leq \operatorname{silp} G \leq 1 + \underline{\operatorname{cd}}\, G.$$

**Theorem 2.5.** *Let $G$ be an arbitrary group. Then the following conditions (1) and (2) are equivalent and they imply (3):*

(1) $\operatorname{spli} G < \infty$
(2) *$G$ admits a complete resolution and $\underline{\operatorname{cd}}\, G < \infty$*
(3) *$G$ admits a complete resolution and the generalized Tate cohomology groups of $G$ can be computed using any complete resolution of $G$.*

*Proof.* (1) $\Rightarrow$ (2): If $\operatorname{spli} G < \infty$ then $G$ admits a complete resolution by Thm. 4.1 of [8]. In general $\operatorname{silp} G \leq \operatorname{spli} G$ (cf. [8]) and $\underline{\operatorname{cd}}\, G \leq \operatorname{silp} G$ as remarked above.

(2) $\Rightarrow$ (1): If $\mathcal{F}$ denotes a complete resolution for $G$ and $\underline{\operatorname{cd}}\, G < \infty$, then $\operatorname{silp} G < 1 + \underline{\operatorname{cd}}\, G < \infty$. Therefore, by Lemma 2.2, every $\mathbb{Z}G$-module admits a complete resolution in the strong sense. By Thm. 3.10 of [5] this implies that $\operatorname{spli} G < \infty$.

(2) $\Rightarrow$ (3): If $\mathcal{F}$ denotes a complete resolution for $G$ and $\underline{\operatorname{cd}}\, G < \infty$, then $H^i(G; P) = \operatorname{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, P) = 0$ for $i > \underline{\operatorname{cd}}\, G$ and $P$ projective. As noted earlier, this implies that the generalized Tate cohomology groups can be computed using the complete resolution $\mathcal{F}$. Since $\underline{\operatorname{cd}}\, G < \infty$, complete resolutions of $G$ are unique up to chain homotopy [10], thus any complete resolution of $G$ can be used to compute the generalized Tate cohomology of $G$. ☐

Ikenaga in [10] defined a class of groups $\mathfrak{C}_\infty$ via actions on finite dimensional acyclic complexes, and he proved that the groups in this class possess complete resolutions and have finite $\underline{\operatorname{cd}}$ [10, Thm. 2], hence by 2.5 these groups satisfy $\operatorname{spli} G < \infty$.

The class $\mathfrak{C}_\infty$ is defined as follows. Let $\mathfrak{C}_0$ be the class of finite groups and for an integer $n > 0$ let $G \in \mathfrak{C}_n$ if and only if there is a finite dimensional acyclic $G$-*simplicial complex* $X$ for which

- $G_\sigma \in \mathfrak{C}_{n-1}$ for all simplices $\sigma$ of $X$
- $\sup_\sigma\{\underline{\operatorname{cd}}\, G_\sigma\} < \infty$ where $\sigma$ runs over all simplices of $X$,

and $\mathfrak{C}_\infty = \cup_n \mathfrak{C}_n$.

Ikenaga's "$G$-simplicial complexes" are such that their barycentric subdivision are $G$-$CW$-complexes and they are therefore $G$-homeomorphic to $G$-$CW$-complexes. On the other hand it is an elementary fact that every $G$-$CW$-complex is $G$-homotopy equivalent to a "G-simplicial complex" of the same dimension. If $X$ is an acyclic $G$-simplicial complex $X$ of dimension $k$, then its join $X * G$ is a contractible $G$-simplicial complex of dimension $k+1$, whose barycentric subdivision is a $G$-$CW$-complex, with point stabilizers being subgroups of the original stabilizers $G_\sigma$. As proved above, for any group $G$ with $\operatorname{spli} G < \infty$ the invariants $\underline{\mathrm{cd}}\, G$ and $\operatorname{spli} G$ differ at most by one, and clearly $\underline{\mathrm{cd}}\, G = 0$ for finite $G$. Therefore we can record the following relationship with Kropholler's classes.

**Corollary 2.6.** *Ikenaga's class $\mathfrak{C}_1$ agrees with Kropholler's class $\mathbf{H}_1\mathfrak{F}$, and for every $n \in \mathbb{N}$ one has*

$$\mathfrak{C}_n = \{G \in \mathbf{H}_n\mathfrak{F} \mid \operatorname{spli} G < \infty\}.$$

*Moreover $\mathfrak{C}_\infty$ consists of those groups in $\mathbf{H}_\omega\mathfrak{F}$ for which* $\operatorname{spli} G < \infty$*; here $\omega$ denotes the first infinite ordinal.*

*Remark* 2.7. We don't know of an example of a group $G$ with $\operatorname{spli} G < \infty$ not belonging to $\mathbf{H}_1\mathfrak{F}$; it is conceivable that $\mathfrak{C}_\infty = \mathbf{H}_1\mathfrak{F}$.

In case the generalized Tate cohomology can be computed using complete resolutions, i.e. if $\operatorname{spli} G < \infty$, the generalized Tate cohomology has many properties analogous to those of the Farrell theory, where the role of $\operatorname{vcd} G$ is played by $\underline{\mathrm{cd}}\, G$ (cf. [10]), namely:

(T4) the natural map $H^i(G; M) \to \hat{H}^i(G; M)$ is an isomorphism for $i > \underline{\mathrm{cd}}\, G$

(T5) Shapiro's Lemma holds: for any subgroup $H < G$ and $\mathbb{Z}H$-module $M$

$$\hat{H}^\bullet(H; M) \cong \hat{H}^\bullet(G; \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M))$$

(T6) $\hat{H}^\bullet(G; I) = 0$ for injective $\mathbb{Z}G$-modules $I$; hence generalized Tate cohomology is *coeffaceable* and one also has *dimension shifting downwards*

(T7) there is a cup products with the usual properties, compatible with that in ordinary cohomology:

$$\begin{array}{ccc} H^p(G; M) \otimes H^q(G; N) & \xrightarrow{\cup} & H^{p+q}(G; M \otimes N) \\ \downarrow & & \downarrow \\ \hat{H}^p(G; M) \otimes \hat{H}^q(G; N) & \xrightarrow{\cup} & \hat{H}^{p+q}(G; M \otimes N) \end{array}$$

*Remark* 2.8. Cup products as in (T7) exist for arbitrary $G$ (cf. [14]). In particular, $R := \hat{H}^0(G;\mathbb{Z})$ is a commutative ring with 1, $\hat{H}^\bullet(G;\mathbb{Z})$ is an $R$-algebra and $\hat{H}^\bullet(G;M)$ is an $R$-module. In case of $\operatorname{spli} G < \infty$ one can use a complete resolution $\mathcal{F}$ of $G$ to define a cup product using a suitable diagonal (cf. [10])

$$\Delta : \mathcal{F} \to \mathcal{F} \hat{\otimes} \mathcal{F}.$$

The following result implies that not every group has a complete resolution.

**Proposition 2.9.** *If a group $G$ has a complete resolution of coincidence index $k$, then $H^i(G;P) \neq 0$ for some projective $\mathbb{Z}G$-module $P$ and some $i \leq k$.*

*Proof.* If $G$ is finite, it admits a complete resolution of coincidence index 0 and $H^0(G;\mathbb{Z}G) \neq 0$. If $G$ is infinite and

$$\begin{array}{ccc} & & F_{k-1} \to \cdots \to F_0 \to F_{-1} \to \cdots \\ & \nearrow & \\ \cdots \to F_{k+1} \to F_k & & \\ & \searrow & \\ & & P_{k-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0 \end{array}$$

is a complete resolution with coincidence index $k$, we define $\Lambda_j = \operatorname{Ker}(F_j \to F_{j-1})$, $j \in \mathbb{Z}$, and $\Omega_i = \operatorname{Ker}(P_i \to P_{i-1})$, $i > 0$. Since $G$ is infinite $H^0(G;M) = 0$ for any submodule $M$ of a projective module. Assume that $H^i(G;P) = 0$ for all projective $P$ and all $i \leq k$. Then, by dimension shifting

$$H^0(G;\mathbb{Z}) \cong H^k(G;\Omega_k) = H^k(G;\Lambda_k) \cong H^0(G;\Lambda_{-1})$$

which is a contradiction, since $H^0(G;\mathbb{Z}) = \mathbb{Z}$ and $\Lambda_{-1}$ is a submodule of the projective module $F_{-1}$. Whence the result follows. □

**Corollary 2.10.** (i) *If a group $G$ contains a free abelian group of infinite rank then $G$ does not admit a complete resolution.*

(ii) *The Thompson group*

$$T = \langle x_0, x_1, \ldots \mid x_i x_j x_i^{-1} = x_{j+1}, \quad i < j \rangle$$

*is an example of a group of type $FP_\infty$ which does not admit a complete resolution.*

*Proof.* (i): A free abelian group $A$ of countably infinite rank satisfies $H^i(A;P) = 0$ for all $i \geq 0$ and all projective $P$. The result now follows from 2.9 since if a group $G$ has a complete resolution then every subgroup of $G$ has a complete resolution too.

(ii): Thompson's group is of type $FP_\infty$ and contains the free abelian subgroup with basis $\{x_i x_{i+1}^{-1} \mid i \in \mathbb{N}\}$ (cf. [3]). □

## 3. The stabilizer spectral sequence

The classical stabilizer spectral sequence of Farrell cohomology (cf. Ch. X in [2]) admits a generalization to the case of groups $G$ with $\operatorname{spli} G < \infty$.

**Theorem 3.1.** *Let $X$ be a finite dimensional contractible $G$-$CW$-complex and write $G_\sigma$ for the stabilizer of the cell $\sigma$ of $X$. Assume that* $\operatorname{spli} G < \infty$. *Then there is a finitely convergent spectral sequence*

$$E_1^{pq} = \prod_{\sigma \in \Sigma_p} \hat{H}^q(G_\sigma; M) \Rightarrow \hat{H}^{p+q}(G; M)$$

*where $\Sigma_p$ is a set of representatives for the $p$-cells of $X \mod G$, and $M$ is a $\mathbb{Z}G$-module.*

The spectral sequence is obtained as in the case of Farrell cohomology from the double complex

$$\operatorname{Hom}_{\mathbb{Z}G}(\mathcal{F}, C^*(X; M)),$$

where $\mathcal{F}$ denotes a complete resolution of $G$ and $C^*(X; M)$ the cellular cochain complex of $X$ with coefficients in $M$ and diagonal $G$-action. There is no need here to assume that the stabilizers $G_\sigma$ be finite; however, we will mainly be interested in that case. The *finite* convergence results from the assumption that $X$ be finite dimensional. This spectral sequence is discussed in [19]. An analysis of the first differential leads to the following useful result. For any cell $\sigma \subset X$ let

$$c(g^{-1})^* : \hat{H}^\bullet(G_\sigma; M) \to \hat{H}^\bullet(G_{g\sigma}; M)$$

be the isomorphism induced by conjugation with $g \in G$ and put $c(g^{-1})^*(u) = g \cdot u$. It follows that $E_2^{0,q}$ in 3.1 can be identified with the subgroup of *compatible families* in

$$\prod_{v \in X_0} \hat{H}^q(G_v; M), \quad X_0 \text{ the set of vertices of } X,$$

that is, the families $(u_v)$ satisfying the following conditions:

- $gu_v = u_{gv}$ for all $g \in G$ and $v \in X_0$
- if $v$ and $w$ are vertices of a 1-cell $\sigma$ of $X$, then $u_v$ and $u_w$ restrict to the same element of $\hat{H}^q(G_\sigma; M)$.

Since all groups in $\mathbf{H}_1\mathfrak{F}$ satisfy spli $< \infty$ we can argue as in the proof of Prop. 4.4, Ch. X of [2] to obtain the following.

**Theorem 3.2.** *Let $X$ be a finite dimensional contractible $G$-$CW$-complex with finite stabilizers such that for every finite subgroup $H < G$ the fixed point set $X^H$ is non-empty and connected. Let $\mathfrak{F}$ stand for the set of finite subgroups of $G$ and write*

$$\mathcal{H}^q(G;M) \subset \prod_{H \in \mathfrak{F}} \hat{H}^q(H;M)$$

*for those families $(u_H)_{H\in\mathfrak{F}}$ which are compatible with respect to the restriction maps $\hat{H}^q(H;M) \to \hat{H}^q(K;M)$ induced by embeddings $K \rightarrowtail H$ given by conjugation by elements of $G$. Then the $E_2$-term of the associated stabilizer spectral sequence 3.1 satisfies*

$$E_2^{0,q} \cong \mathcal{H}^q(G;M).$$

Note that the theorem applies in particular to groups which admit a finite dimensional classifying space for proper actions $\underline{E}G$ (the definition of $\underline{E}G$ was recalled in the Introduction).

**Corollary 3.3.** *Suppose $G$ admits a $G$-$CW$-complex $X$ as in the previous theorem and let $p$ be a prime number. Then the natural map induced by restricting to finite subgroups*

$$\rho : \hat{H}^\bullet(G;\mathbb{Z}/p\mathbb{Z}) \to \mathcal{H}^\bullet(G;\mathbb{Z}/p\mathbb{Z}) \subset \prod_{H\in\mathfrak{F}} \hat{H}^\bullet(H;\mathbb{Z}/p\mathbb{Z})$$

*has the property that every element in the kernel of $\rho$ is nilpotent, and that for any $u \in \mathcal{H}^\bullet(G;\mathbb{Z}/p\mathbb{Z})$ there is an integer $k$ such that $u^{p^k}$ lies in* Im *$\rho$ (i.e. $\rho$ is an $F$-isomorphism).*

The proof is exactly the same as the one for Prop. 4.6 of Ch. X in [2].

We will also make use of the following consequence of 3.1.

**Theorem 3.4.** *Let $G$ be a group in $\mathbf{H}_1\mathfrak{F}_b$ and let $n$ be a positive integer such that the order of any torsion subgroup of $G$ divides $n$. Then there is an integer $k$ such that*

$$n^k \cdot \hat{H}^i(G;M) = 0$$

*for all $i$ and all $\mathbb{Z}G$-modules $M$. Moreover, if $p$ is a torsion prime for $G$ then $\hat{H}^0(G;\mathbb{Z})$ contains an element of order $p$.*

*Proof.* Let $n$ be a positive integer such that the order of every torsion subgroup of $G$ divides $n$. Choose a contractible finite dimensional $G$-$CW$-complex $X$ with finite stabilizers. Then the $E_1$-term of the associated stabilizer spectral sequence is annihilated by $n$. It follows that every $\hat{H}^i(G;M)$ is annihilated by $n^k$, where $k = \dim X + 1$. It remains to show that the torsion group $\hat{H}^0(G;\mathbb{Z})$ contains an element of order $p$ if $p$ is a torsion prime for $G$. If $\mathbb{Z}/p\mathbb{Z} < G$ then the restriction map

$$\hat{H}^0(G;\mathbb{Z}) \to \hat{H}^0(\mathbb{Z}/p\mathbb{Z};\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$$

is surjective since it maps 1 to 1 and the claim follows. □

*Remark* 3.5. If there is no bound on the order of finite subgroups of $G$, then $\hat{H}^0(G;\mathbb{Z})$ is not a torsion group, because $1 \in \hat{H}^0(G;\mathbb{Z})$ restricts to a generator of $\hat{H}^0(H;\mathbb{Z}) \cong \mathbb{Z}/|H|\mathbb{Z}$ for every finite $H < G$. On the other hand if $\hat{H}^0(G;\mathbb{Z})$ is torsion then all generalized Tate groups of $G$ with coefficients in any $\mathbb{Z}G$-module are torsion, annihilated by the characteristic of the ring $\hat{H}^0(G;\mathbb{Z})$.

Note that for $G$ as in 3.4, one has

$$\hat{H}^\bullet(G;\mathbb{Z}) \cong \prod_p \hat{H}^\bullet(G;\mathbb{Z})_{(p)} \cong \bigoplus_p \hat{H}^\bullet(G;\mathbb{Z})_{(p)},$$

where $\hat{H}^\bullet(G;\mathbb{Z})_{(p)}$ stands for the $p$-primary part. This $p$-primary part can sometimes be computed up to *$F$-isomorphism* by passing to Tate cohomology with coefficients in $\mathbb{Z}/p\mathbb{Z}$.

**Lemma 3.6.** *Let $G \in \mathbf{H}_1\mathfrak{F}_b$ and $p$ a prime number. Then the natural map*

$$\alpha : \hat{H}^\bullet(G;\mathbb{Z})_{(p)} \to \hat{H}^\bullet(G;\mathbb{Z}/p\mathbb{Z})$$

*has the property that every element in the kernel of $\alpha$ is nilpotent, and for any $u \in \hat{H}^\bullet(G;\mathbb{Z}/p\mathbb{Z})$ there is an integer $k$ such that $u^{p^k}$ lies in* $\operatorname{Im}\alpha$ *(i.e. the map $\alpha$ is an $F$-isomorphism).*

The proof is the same as the one for Lemma 6.6, Ch. X in [2].

In view of our applications it is convenient to make use the following fact on groups in $\mathbf{H}\mathfrak{F}_b$, which is an easy consequence of the main theorem of [15].

**Proposition 3.7.** *For groups $G \in \mathbf{H}\mathfrak{F}_b$ the following are equivalent:*

(i) $\operatorname{spli} G < \infty$
(ii) *$G$ admits a finite dimensional $\underline{E}G$*
(iii) $G \in \mathbf{H}_1\mathfrak{F}_b$.

*Proof.* (i) $\Rightarrow$ (ii): Let $\kappa(G)$ be the supremum over the projective dimension of those $\mathbb{Z}G$-modules which have finite projective dimension when restricted to any finite subgroup of $G$. It was shown in [6, Thm. C] that for $G$ in $\mathbf{H}\mathfrak{F}$ one has $\kappa(G) = \operatorname{spli} G$. On the other hand, it is well known that for an arbitrary group $G$ the module $B(G, \mathbb{Z})$ of bounded functions $G \to \mathbb{Z}$, is free over $\mathbb{Z}H$ for any finite subgroup $H < G$, (cf. [16]). Thus if $G \in \mathbf{H}\mathfrak{F}_b$ satisfies $\operatorname{spli} G < \infty$ then $\operatorname{proj.dim}_{\mathbb{Z}G} B(G, \mathbb{Z}) < \infty$ and therefore $G$ admits by [15] a finite dimensional $\underline{E}G$.

(ii) $\Rightarrow$ (iii): This is clear since we assume that the torsion subgroups of $G$ have bounded order.

(iii) $\Rightarrow$ (i): As observed in 2.6, all groups in $\mathbf{H}_1\mathfrak{F}$ satisfy $\operatorname{spli} < \infty$. □

By combining 3.3 with 3.6 and 3.7 we obtain the following

**Corollary 3.8.** *Let $G \in \mathbf{H}\mathfrak{F}_b$ with* $\operatorname{spli} G < \infty$. *Then the natural map*

$$\hat{H}^\bullet(G; \mathbb{Z}) \to \prod_p \mathcal{H}^\bullet(G; \mathbb{Z}/p\mathbb{Z})$$

*is an $F$-isomorphism.*

The following lemma is useful for recognizing whether a group belongs to $\mathbf{H}_1\mathfrak{F}_b$.

**Lemma 3.9.** *Let $X$ be a finite dimensional contractible $G$-$CW$-complex. Then the following holds:*

- *if $X/G$ is compact and every cell stabilizer belongs to $\mathbf{H}_1\mathfrak{F}_b$ then $G \in \mathbf{H}_1\mathfrak{F}_b$*
- *if all cell stabilizers are finite of order dividing some fixed integer $n > 0$, then the order of every finite subgroup of $G$ divides $n$ and $G$ belongs to $\mathbf{H}_1\mathfrak{F}_b$.*

*Proof.* If $X/G$ is compact and $G_\sigma \in \mathbf{H}_1\mathfrak{F}_b$ is a cell stabilizer, then $\underline{\operatorname{cd}}\, G_\sigma < \infty$. Since

$$\underline{\operatorname{cd}}\, G \leq \dim X + \sup_\sigma \{\underline{\operatorname{cd}}\, G_\sigma\}$$

and the number of $G$ orbits of cells is finite, we conclude that $\underline{\operatorname{cd}}\, G$ and thus $\operatorname{spli} G$ is finite and clearly $G \in \mathbf{H}\mathfrak{F}$. To check that the order of the finite subgroups of $G$ are bounded, it suffices to check that the order of the $p$-subgroups is bounded by a bound independent of $p$. If $P < G$ is a finite $p$-subgroup of $G$ then the fixed-point space $X^P$ is not empty (cf. [2, Thm. 10.5, Ch. VIII]) which implies that $P < G_\sigma$ for some cell $\sigma$. But by assumption the order of the finite subgroups of each $G_\sigma$ is bounded and, as $X/G$ is compact, there are only finitely many $G_\sigma$'s up to isomorphism. This implies that there is a universal bound independent of $p$ for the order of the finite $p$-subgroups $P < G$. It follows that $G \in \mathbf{H}_1\mathfrak{F}_b$ by 3.7.

Next, if the order of every cell stabilizer divides $n$, then the order of every finite $p$-subgroup $P < G$ divides $n$, because $P$ is a subgroup of some cell stabilizer. As a result the order of every finite subgroup of $G$ divides $n$. □

**Corollary 3.10.** *The class of groups belonging to $\mathbf{H}_1\mathfrak{F}_b$ is extension closed. Moreover, if $G$ is the fundamental group of a finite graph of groups in $\mathbf{H}_1\mathfrak{F}_b$ then $G \in \mathbf{H}_1\mathfrak{F}_b$. In particular, $\mathbf{H}_1\mathfrak{F}_b$ is closed under amalgamated free products and $HNN$-extensions.*

*Proof.* Let $K \rightarrowtail G \twoheadrightarrow Q$ be an extension with $K$ and $Q$ in $\mathbf{H}_1\mathfrak{F}_b$. Then $G \in \mathbf{H}\mathfrak{F}_b$ since $\mathbf{H}\mathfrak{F}$ is extension closed. But by a general fact $\operatorname{spli} G \leq \operatorname{spli} K + \operatorname{spli} Q$ (cf. [8, Thm. 5.5]) and therefore $G \in \mathbf{H}_1\mathfrak{F}_b$ by 3.7.

Next, if $G$ is the fundamental group of a finite graph of groups in $\mathbf{H}_1\mathfrak{F}_b$ then $G$ acts cocompactly on a tree $T$ with stabilizers in $\mathbf{H}_1\mathfrak{F}_b$ so that $G \in \mathbf{H}_1\mathfrak{F}_b$ by 3.9. □

*Remark* 3.11. An interesting group in $\mathbf{H}\mathfrak{F}_b$ which is not of finite vcd (because it is an infinite torsion group), is the Burnside group $B(d,e)$ of odd exponent $e > 665$ on $d$ generators. It is known that there is a 2-dimensional contractible $B(d,e)$-$CW$-complex with all non-trivial stabilizers cyclic of order $e$ (e.g. [18]). By 3.9, $B(d,e)$ belongs to $\mathbf{H}_1\mathfrak{F}_b$. We will come back to this example in the next section.

## 4. Periodicity in cohomology and units

The existence of periodic resolutions is closely related to the existence of units of non-zero degree in generalized Tate cohomology. The precise relationship is as follows.

**Theorem 4.1.** *Let $G$ be an arbitrary group. Then $\hat{H}^\bullet(G;\mathbb{Z})$ contains a unit of non-zero degree if and only if $G$ admits a periodic resolution.*

*Proof.* Let
$$\ldots \to P_i \to P_{i-1} \to \ldots P_0 \to \mathbb{Z}$$
be a projective resolution and put $\Omega^i = \operatorname{Ker}(P_i \to P_{i-1})$. Let $u \in \hat{H}^k(G;\mathbb{Z})$ be a unit for some $k > 0$. According to [17] one has for all $n \in \mathbb{Z}$
$$\hat{H}^n(G;\mathbb{Z}) \cong \varinjlim_{j \geq |n|} [\Omega^{j+n}, \Omega^j]$$
where $[M,N]$ stands for $\operatorname{Hom}_{\mathbb{Z}G}(M,N)/S$, with $S$ the group of $\mathbb{Z}G$-module homomorphisms $M \to N$ which factor through a projective

module. By choosing $j$ large enough we can thus represent $u$ and $u^{-1}$ by module maps $\Omega^{j+k} \to \Omega^j$ and $\Omega^j \to \Omega^{j+k}$ such that the composites of these two maps are equal to identity maps modulo maps which factor through projectives. But this implies that there is a projective module $P$ such that

$$\Omega^j \oplus P \cong \Omega^{j+k} \oplus P.$$

Clearly, this implies that $G$ admits a periodic resolution. For the converse we may assume that there are inverse isomorphisms

$$f : \Omega^{j+k} \to \Omega^j, \qquad g : \Omega^j \to \Omega^{j+k}$$

for some $j \geq 0$ and some $k > 0$. Then $f$ and $g$ represent inverse units in $\hat{H}^\bullet(G;\mathbb{Z})$ of non-zero degree. We used here the fact that the product $xy$ of $x, y \in \hat{H}^\bullet(G;\mathbb{Z})$ is represented by $f \circ g$, if $g : \Omega^{m+r+s} \to \Omega^{m+r}$ resp. $f : \Omega^{m+r} \to \Omega^m$ represent $y$ resp. $x$ for some large $m$ [14]. □

*Remark* 4.2. It is conceivable that all groups $G$ which admit complete resolutions actually satisfy $\operatorname{spli} G < \infty$.

We will make use repeatedly of the following well-known fact on $F$-isomorphisms (see for instance the proof of Prop. 6.1, Ch. X in [2]).

**Lemma 4.3.** *Let $\phi : R \to S$ be an $F$-isomorphism of rings with 1. If $u \in S$ is a unit, then there is a $k > 0$ such that $u^k = \phi(v)$ for some unit $v \in R$.*

The following result permits in some cases to detect units in generalized Tate cohomology of $G$ by looking at finite subgroups. If $p$ is a prime we say that a finite group $H$ has $p$-periodic cohomology, if the (trivial) $\mathbb{Z}H$-module $\mathbb{Z}/p\mathbb{Z}$ has a periodic resolution. This is equivalent with the existence of $q > 0$ such that the functors $H^i(H; - \otimes \mathbb{Z}/p\mathbb{Z})$ and $H^{i+q}(H; - \otimes \mathbb{Z}/p\mathbb{Z})$ are equivalent for all $i > 0$; it is well-known that in this situation the minimal such $q > 0$ divides $2(p-1)$ for $p$ odd, or 4 for $p = 2$ (cf. [21]).

**Theorem 4.4.** *Let $p$ be a prime and let $X$ be a finite dimensional contractible $G$-CW-complex with finite stabilizers such that for all finite subgroups $H < G$ the fixed point spaces $X^H$ are non-empty and connected. Then the following statements are equivalent:*

(i) *the ring $\hat{H}^\bullet(G;\mathbb{Z}/p\mathbb{Z})$ contains a unit of non-zero degree*
(ii) *every finite subgroup $H$ of $G$ has $p$-periodic cohomology.*

*Proof.* (i) ⇒ (ii): Let $x$ be an invertible element of $\hat{H}^\bullet(G;\mathbb{Z}/p\mathbb{Z})$ of some positive degree $q$. If $H$ is a finite subgroup of $G$ then since the restriction map

$$\hat{H}^\bullet(G;\mathbb{Z}/p\mathbb{Z}) \to \hat{H}^\bullet(H;\mathbb{Z}/p\mathbb{Z})$$

is a morphism of rings with 1, the image of $x$ is a unit of $\hat{H}^\bullet(H;\mathbb{Z}/p\mathbb{Z})$ of degree $q$ and this is equivalent to $H$ having $p$-periodic cohomology with $p$-period $q$ (cf. [2], Thm 9.7, Ch. VI).

(ii) $\Rightarrow$ (i): Let $q = 2(p-1)$ for $p$ odd or $q = 4$ for $p = 2$, which is a $p$-period for every finite subgroup $H$ of $G$. Choose for each $H$ a generator

$$v_H \in \hat{H}^q(H;\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/|H|\mathbb{Z}.$$

Note that $v_H$ is unique up to a unit in the ring $\mathbb{Z}/p\mathbb{Z}$. Therefore $u = (u_H)_{H\in\mathcal{F}}$ with $u_H = v_H^{p-1}$ for all $H$ defines a compatible family and thus a unit in $\mathcal{H}^\bullet(G;\mathbb{Z}/p\mathbb{Z})$ of degree $q^{p-1}$. Since by 3.3

$$\rho : \hat{H}^\bullet(G;\mathbb{Z}/p\mathbb{Z}) \to \mathcal{H}^\bullet(G;\mathbb{Z}/p\mathbb{Z})$$

is an $F$-isomorphism the result follows from 4.3. □

**Definition 4.5.** Let $G$ be an arbitray group. Then the *finitistic dimension* $\operatorname{fin.dim} G$ is the supremum of the projective dimension of all $\mathbb{Z}G$-modules of finite projective dimension.

**Lemma 4.6.** *Let $G$ be a group such that either* $\operatorname{spli} G < \infty$ *or* $G \in \mathbf{H}\mathfrak{F}$. *Then* $\operatorname{fin.dim} G = \operatorname{spli} G$.

*Proof.* Without any assumtion on $G$ one has $\operatorname{fin.dim} G \leq \operatorname{silp} G$. Indeed, if $A$ is a module of finite projective dimension $d$, then $\operatorname{Ext}^d(A,P) \neq 0$ for some projective module $P$. It follows that the injective dimension of $P$ is at least $d$. According to [8] for any $G$ one has $\operatorname{silp} G \leq \operatorname{spli} G$. If $\operatorname{spli} G = k < \infty$, the there exits an injective module of projective dimension $k$, thus $\operatorname{fin.dim} G \geq \operatorname{spli} G$. It follows that for groups with $\operatorname{spli} G < \infty$ one has $\operatorname{fin.dim} G = \operatorname{spli} G$. If $G$ is an arbitrary group in $\mathbf{H}\mathfrak{F}$ the $\operatorname{fin.dim} G = \operatorname{spli} G$ by [6, Thm. C]. □

**Lemma 4.7.** *Let $G$ be a group with periodic cohomology after $k$-steps. Then* $\operatorname{fin.dim} G \leq k+1$.

*Proof.* By assumption there exist $k > 0$ and $q > 0$ such that the functors $H^i(G;\quad)$ and $H^{i+q}(G;\quad)$ are equivalent for all $i > k$. We claim that then $\operatorname{fin.dim} G \leq k+1$. Indeed if $M$ is a $\mathbb{Z}G$-module of finite projective dimension $m+1 > 0$, the $\Omega M$, the kernel of a surjection $P \to M$ with $P$ projective, has projective dimension $m$. Therefore there exists a $\mathbb{Z}G$-module $A$ such that $\operatorname{Ext}^m(\Omega M, A) \neq 0$. But if we had $m > k$, then

$$\operatorname{Ext}^m(\Omega M, A) \cong \operatorname{Ext}^{m+q}(\Omega M, A) \neq 0$$

which is in contradiction with $\operatorname{proj.dim} \Omega M = m$. It follows that $\operatorname{fin.dim} G \leq k+1$. □

By combining the two previous results we conclude the following.

**Corollary 4.8.** *Let $G \in \mathbf{H}\mathfrak{F}$ and assume that $G$ admits a periodic resolution. Then* $\operatorname{spli} G < \infty$.

By putting together some of our previous results we obtain

**Theorem 4.9.** *Suppose $G$ has periodic cohomology after $k$-steps. Then the following holds:*

(i) *$H^i(G; P) \neq 0$ for some $i \leq k$ and some projective $\mathbb{Z}G$-module $P$; moreover* $\operatorname{fin.dim} G \leq k+1$ *and every subgroup $H < G$ of finite cohomological dimension satisfies* $\operatorname{cd} H \leq k$

(ii) *if $G \in \mathbf{H}\mathfrak{F}$ then $H^i(G; P) = 0$ for $i > k+1$, $\operatorname{cd}_{\mathbb{Q}} G \leq k+1$ and every torsion-free subgroup $H < G$ satisfies* $\operatorname{cd} H \leq k$.

*Proof.* (i): If $G$ has periodic cohomology after $k$-steps then it admits a complete resolution of coincidence index $k$, thus $H^i(G; P) \neq 0$ for some $i \leq k$ and some projective $P$ (cf. 2.9). That $\operatorname{fin.dim} G \leq k+1$ follows from the proof of 4.7; moreover, since every subgroup $H < G$ has periodic cohomology after $k$ steps too, it follows that if $\operatorname{cd} H < \infty$ then $\operatorname{cd} H \leq k$, because there is a complete resolution with coincidence index $k$.

(ii): If $G$ is in $\mathbf{H}\mathfrak{F}$ then $\operatorname{fin.dim} G = \operatorname{spli} G$ (cf. 4.6), and obviously $H^i(G; P) = 0$ for $i > \operatorname{spli} G$ and $P$ projective, because $H^i(G; \quad)$ vanishes on injectives for $i > 0$. Moreover, for any group $G$ in $\mathbf{H}\mathfrak{F}$ one has

$$\operatorname{cd}_{\mathbb{Q}} G = \operatorname{spli}_{\mathbb{Q}} G \leq \operatorname{spli} G = \operatorname{fin.dim} G$$

and therfore $\operatorname{cd}_{\mathbb{Q}} G \leq k+1$ ($\operatorname{spli}_{\mathbb{Q}} G$ is defined like $\operatorname{spli} G$, but using $\mathbb{Q}G$-modules instead of $\mathbb{Z}G$-modules). Finally, if $H < G$ is a torsion-free subgroup, then $H$ belongs to $\mathbf{H}_1\mathfrak{F}$, since $\operatorname{spli} H \leq \operatorname{spli} G < \infty$. It then follows from (i) that $\operatorname{cd} H \leq k$. □

The following theorem corresponds to part of Theorem B of the Introduction.

**Theorem 4.10.** *Let $G \in \mathbf{H}\mathfrak{F}_b$. Then the following statements are equivalent and they all imply that $G \in \mathbf{H}_1\mathfrak{F}_b$ :*

(i) *$G$ has periodic cohomology after some steps*

(ii) *there exists an invertible element of non-zero degree in the ring $\hat{H}^\bullet(G; \mathbb{Z})$.*

*Moreover, for $G \in \mathbf{H}_1\mathfrak{F}_b$ the following is equivalent to* (i) *and* (ii):

(iii) *every finite subgroup of $G$ has periodic cohomology.*

*Proof.* (i)⇔(ii): This holds for general $G$ (cf. 4.1). Also, (i) and (ii) imply that $G$ lies in $\mathbf{H}_1\mathfrak{F}_b$ because for groups in $\mathbf{H}\mathfrak{F}$ they imply that $\operatorname{spli} G < \infty$ (cf. 4.8 and 3.7).

(i),(ii)⇒(iii): this is well-known ([2, Thm. 6.7, Ch. X]).

(iii)⇒(ii) (assuming that $G \in \mathbf{H}_1\mathfrak{F}_b$): in that case $\operatorname{spli} G < \infty$ and it follows from 3.7 that $G$ admits a finite dimensional $\underline{E}G$. We can thus apply 3.3, 3.6 and 4.4 to conclude (ii). □

*Remark* 4.11. The following is an example of a group $G \in \mathbf{H}\mathfrak{F}_b$ satisfying (iii) but not (i) or (ii). Let $G = \oplus_{\mathbb{N}}\mathbb{Z}$. Clearly $G \in \mathbf{H}\mathfrak{F}_b$ and it satisfies (iii). But $G$ does not satisfy (i) because of 2.10.

There is also an example of a group $K \in \mathbf{H}_1\mathfrak{F}$ such that (iii) does not imply (i) (see 2.2 in [23]). The group $K$ is given as the fundamental group of a certain graph of finite cyclic $p$-groups for a fixed prime $p$. Note that it follows from 4.4 that $\hat{H}^\bullet(K;\mathbb{Z}/p\mathbb{Z})$ has an invertible element of non-zero degree.

On the other hand there are also groups in $\mathbf{H}_1\mathfrak{F} \setminus \mathbf{H}_1\mathfrak{F}_b$ such that (iii) does imply (i). For instance it was shown in [23] that a countable locally finite group has period $q$ after 1-step if and only if all its finite subgroups have period $q$.

The following example illustrates 4.10

**Lemma 4.12.** *Let $B(d,e)$ be the Burnside group on $d$ generators and of odd exponent $e > 665$. Then $B(d,e) \in \mathbf{H}_1\mathfrak{F}_b$ and*

(i) *$B(d,e)$ has periodic cohomology after 2-steps*

(ii) *$H^i(B(d,e); \quad) \cong \prod_{\mathbb{N}} H^i(\mathbb{Z}/e\mathbb{Z}; \quad)$ for all $i > 2$*

(iii) *$\hat{H}^\bullet(B(d,e); \quad) \cong \prod_{\mathbb{N}} \hat{H}^\bullet(\mathbb{Z}/e\mathbb{Z}; \quad)$*

(iv) *all finite subgroups of $B(d,e)$ are cyclic of order dividing $e$.*

*Proof.* We know already that $B(d,e) =: G$ belongs to $\mathbf{H}_1\mathfrak{F}_b$ (3.11). It was shown by Ivanov [11] that $G$ has a presentation with associated relation module of the form $\oplus_{n\in\mathbb{N}}\mathbb{Z}[G/G_n]$ with each $G_n$ cyclic of order $e$. It follows that

$$H^i(G; \quad) \cong \prod_{\mathbb{N}} H^i(\mathbb{Z}/e\mathbb{Z}; \quad), \qquad \text{for all} \quad i > 2.$$

This implies in particular that $G$ has period 2 after 2-steps and that $H^i(G; \quad)$ vanishes on projectives for $i > 2$, implying (i), (ii) and (iii). It follows that $\hat{H}^2(G;\mathbb{Z})$ contains a unit and therfore, by restricting to finite subgroups, we see that every finite subgroup has periodic cohomology of period two. But finite groups of period two are known to be cyclic. Moreover, the order of the finite subgroups of $G$ divides the exponent of $\hat{H}^0(G;\mathbb{Z})$, which is $e$ and (iv) follows. □

## 5. Free actions on finite dimensional homology spheres

**Proposition 5.1.** *Let $G$ be a group which admits a finite dimensional free $G$-CW-complex $X$ such that $H^\bullet(X;\mathbb{Z}) \cong H^\bullet(S^n;\mathbb{Z})$. Then*

(i) *there is a finite dimensional free $G$-CW-complex $Y$ homotopy equivalent to $S^{2n+1}$ such that $G$ acts trivially on $H^{2n+1}(S^{2n+1};\mathbb{Z})$*

(ii) *there is an exact sequence*

$$0 \to \mathbb{Z} \to A \to P_{2n} \to \ldots \to P_0 \to \mathbb{Z} \to 0$$

*with $P_i$ projective $\mathbb{Z}G$-modules and* $\operatorname{proj.dim}_{\mathbb{Z}G} A < \infty$.

*Proof.* (i): The join $Y = X * X$ with diagonal $G$-action has the homotopy type of $S^{2n+1}$, with homologically trivial $G$-action.

(ii): Take $Y$ as in (i) and write $\{C_n, d_n \mid n \in \mathbb{N}\}$ for its cellular chain complex. Since $H^\bullet(C_*) \cong H^\bullet(S^{2n+1};\mathbb{Z})$ we obtain the following exact sequence of $\mathbb{Z}G$-modules

$$0 \to \operatorname{Ker} d_{2n+1} \to C_{2n+1} \to C_{2n} \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

and a push-out square

$$\begin{array}{ccc} \operatorname{Ker} d_{2n+1} & \longrightarrow & C_{2n+1} \\ {\scriptstyle\tau}\downarrow & & {\scriptstyle\sigma}\downarrow \\ H^{2n+1}(Y;\mathbb{Z}) & \xrightarrow{\ \rho\ } & A \end{array}$$

Then since $\operatorname{Ker}\sigma = \operatorname{Ker}\tau = \operatorname{Im} d_{2n+2}$, we obtain exact sequences

$$0 \to \mathbb{Z} \to A \to C_{2n} \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

and

$$0 \to C_{\dim Y} \to \cdots \to C_{2n+1} \to A \to 0\,,$$

which proves (ii). □

**Corollary 5.2.** *Let $G$ be an arbitrary group and consider the following statements:*

(1) *$G$ acts freely and properly on $\mathbb{R}^n \times S^m$ for some $n \geq 0$ and some $m \geq 1$*

(2) *there is an exact sequence*

$$0 \to \mathbb{Z} \to A \to P_{q-2} \to \cdots P_0 \to \mathbb{Z} \to 0$$

*with $P_i$ projective and* $\operatorname{proj.dim}_{\mathbb{Z}G} A < \infty$

(3) *$G$ has periodic cohomology after some steps,* $\operatorname{spli} G < \infty$ *and $H^i(G;P) = 0$ for $P$ $\mathbb{Z}G$-projective and $i > 1 + \operatorname{proj.dim} A$.*

*Then* (1)⇒(2)⇒(3).

*Proof.* (1) $\Rightarrow$ (2): Write $X$ for $\mathbb{R}^n \times S^m$ with the given $G$-action. Since $X/G$ is a topological manifold, it has the homotopy type of a $CW$-complex Z of dimension equal to the dimension of $X/G$. The covering space of $Z$ associated with $\pi_1(X) < \pi_1(Z)$ is a finite dimensional $G$-$CW$-complex $G$-homotopy equivalent to $X$. Thus (2) follows from the previous proposition.

(2) $\Rightarrow$ (3): Clearly, (2) implies that

$$\mathrm{Ext}^{i+q}_{\mathbb{Z}G}(\mathbb{Z},\quad) \cong \mathrm{Ext}^{i}_{\mathbb{Z}G}(\mathbb{Z},\quad)$$

for $i > \mathrm{proj.\,dim}_{\mathbb{Z}G} A$ so that $G$ has periodic cohomology with period $q$ after $(\mathrm{proj.\,dim}_{\mathbb{Z}G} A)$-steps. It remains to prove that $\mathrm{spli}\, G < \infty$. Let $I$ be an injective $\mathbb{Z}G$-module. Note that the inclusion $\mathbb{Z} \to A$ in (2) is $\mathbb{Z}$-split. Therefore, upon tensoring with $I$, we obtain an injective map $I \to A \otimes I$, which is $\mathbb{Z}G$-split, because $I$ is injective. Thus

$$\mathrm{proj.\,dim}\, I \leq \mathrm{proj.\,dim}\, A \otimes I \leq 1 + \mathrm{proj.\,dim}\, A$$

which shows that $\mathrm{spli}\, G \leq 1 + \mathrm{proj.\,dim}\, A$. Since for general $G$ one has $\mathrm{silp}\, G \leq \mathrm{spli}\, G$ we conclude that for $i > 1 + \mathrm{proj.\,dim}\, A$ and $P$ projective the cohomology groups $H^i(G; P)$ vanish. □

**Corollary 5.3.** *Let $G$ be a group of finite cohomological dimension and $\mathbb{Z} < G$ an infinite cyclic normal subgroup such that $G/\mathbb{Z}$ lies in* $\mathbf{H}\mathfrak{F}$. *Then* $\mathrm{cd}_{\mathbb{Q}}\, G/\mathbb{Z} < \infty$.

*Proof.* Let $Y$ be the universal cover of a finite dimensional $K(G,1)$. Then $Y/\mathbb{Z}$ is a finite dimensional free $(G/\mathbb{Z})$-$CW$-complex and, as $Y$ is contractible, $Y/\mathbb{Z}$ is a $K(\mathbb{Z},1)$ thus homotopy equivalent to $S^1$. By (ii) of 5.1 and making use of the implication (2) $\Rightarrow$ (3) of 5.2 we infer that the group $G/\mathbb{Z}$ has periodic cohomology after some steps. Therefore (ii) of 4.9 implies that $\mathrm{cd}_{\mathbb{Q}}\, G/\mathbb{Z} < \infty$. □

Clearly, if $G$ acts freely and properly on some $\mathbb{R}^n \times S^m$ then $G$ must be countable. Conversely, the following holds.

**Lemma 5.4.** *Let $X$ be a finite dimensional free $G$-$CW$-complex homotopy equivalent to $S^m$ and suppose that $G$ is countable. Then there exists for some $n \geq 0$ a free and proper $G$ action on $\mathbb{R}^n \times S^m$.*

*Proof.* Since $X/G$ has countable homotopy groups and is a finite dimensional $CW$-complex, it has the homotopy type of a finite dimensional locally finite countable simplicial complex $Z$. Choose a regular neighborhood $S \supset Z$ of a simplicial embedding of $Z$ into some Euclidean space $\mathbb{R}^N$. It follows that $Z$ is homotopy equivalent to the open submanifold $S \subset \mathbb{R}^N$. Using the h-cobordism theorem it follows that the universal cover of $S \times \mathbb{R}^q$ for $q$ large enough is diffeomorphic to $\mathbb{R}^n \times S^m$ for some $n \geq 0$ (see also the proof of Theorem A in [4]). □

To prove Theorem B of the introduction, we need the following.

**Proposition 5.5.** *Let $G \in \mathbf{H}\mathfrak{F}_b$ and assume that $\hat{H}^\bullet(G;\mathbb{Z})$ contains a unit of degree $k > 0$. Then for some $n > 0$ there exists a finite dimensional free $G$-$CW$-complex $E$ homotopy equivalent to $S^{nk-1}$ admitting an orientable spherical fibration*

$$\xi : S^{nk-1} \to E \to \underline{E}G$$

*whose Euler class $e(\xi)$ induces isomorphisms*

$$- \cup e(\xi) : H^i(G; \quad) \to H^{i+nk}(G; \quad)$$

*for large $i$.*

*Proof.* From 4.10 we infer that $G \in \mathbf{H}_1\mathfrak{F}_b$ and it admits therefore a finite dimensional $\underline{E}G$ (3.7). Let $u \in \hat{H}^k(G;\mathbb{Z})$ be a unit, $k > 0$. Since $G \in \mathbf{H}_1\mathfrak{F}_b$, the natural map

$$H^i(G;\mathbb{Z}) \to \hat{H}^i(G;\mathbb{Z})$$

is an isomorphism for large $i$ and we can choose $m > 0$ such that $u^m$ is the image of some $x \in H^{km}(G;\mathbb{Z})$. Choose a finite dimensional model for $\underline{E}G$. Since the finite subgroups of $G$ have bounded order, $G$ contains only finitely many isomorphism classes of finite subgroups and we can proceed as in the proof of Prop. 2.4 and Lemma 2.5 of [4] to construct an orientable spherical fibration

$$\xi : S^{kml-1} \to E \to \underline{E}G$$

for some large enough $l$, with $E$ a finite dimensional free $G$-$CW$-complex. The construction is such that the Euler class $e(\xi) \in H^{kml}(G;\mathbb{Z})$ is of the form $x^l + \nu$, where $\nu$ is a nilpotent element in the ring $H^\bullet(G;\mathbb{Z})$. It follows that

$$- \cup e(\xi) : H^i(G; \quad) \to H^{i+kml}(G; \quad)$$

is an equivalence for $i$ large, because $e(\xi)$ maps to a unit in $\hat{H}^\bullet(G;\mathbb{Z})$. □

We are now ready to prove the theorems mentioned in the introduction.

**Proof of Theorem A:** Let $G \in \mathbf{H}\mathfrak{F}_b$ be a countable group with periodic cohomology after some steps. Then, by 4.10, $\hat{H}^\bullet(G;\mathbb{Z})$ contains an invertible element of non-zero degree. From 5.5 we infer that there is a finite dimensional $G$-$CW$-complex homotopy equivalent to a sphere, and 5.4 then implies that there is a proper and free $G$-action on some $\mathbb{R}^n \times S^m$. □

**Proof of Theorem B:** Let $G \in \mathbf{H}\mathfrak{F}_b$. The assertion (I) $\Rightarrow$ (II) is a consequence of 5.1; (II) $\Rightarrow$ (III) according to 5.2 and (III) $\Rightarrow$ (IV) because of 4.10; next, (IV) $\Rightarrow$ (I) by 5.5, and 4.10 shows that (IV) implies that $G$ must belong to $\mathbf{H}_1\mathfrak{F}_b$. Finally, with the assumption that $G$ is in $\mathbf{H}_1\mathfrak{F}_b$ the assertion (IV) $\Leftrightarrow$ (V) follows from 4.10. □

**Corollary 5.6.** *Let $G$ be an arbitrary group.*

(i) *If $G$ contains a free abelian subgroup of infinite rank then $G$ does not act freely and properly on any $\mathbb{R}^n \times S^m$. In particular, the Thompson group (2.10) does not act freely and properly on any $\mathbb{R}^n \times S^m$.*

(ii) *The Burnside group $B(d,e)$ on $d$ generators and of odd exponent $e > 665$ acts freely and properly on some $\mathbb{R}^n \times S^m$.*

*Proof.* (i): If $G$ contains a free abelian subgroup of infinite rank then $G$ does not admit a complete resolution (2.10), and therefore cannot have periodic cohomology after some steps. The result now follows from 5.2. (ii): By 4.12 the group $B(d,e)$ belongs to $\mathbf{H}_1\mathfrak{F}_b$ and has periodic cohomology after 2-steps. Since $B(d,e)$ is countable, Theorem A implies the assertion. □

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