

Mapping Class Groups, Characteristic Classes and Bernoulli Numbers

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Introduction

The mapping class group Γ_g of a closed, connected and oriented surface S_g of genus g is defined as the group of connected components of the group of orientation preserving diffeomorphisms of S_g . This group has been the object of many recent studies. Of particular interest are its finite subgroups; these are precisely the finite groups which occur as groups of symmetries of the surface S_g equipped with a complex structure (a Riemann surface). The interplay of algebra, topology and analysis in the study of Γ_g make it one of the most fascinating groups. As it is the case for classical arithmetic groups, the finite subgroups of Γ_g are related to certain concepts in number theory. We shall discuss in this essay invariants of Γ_g which are related to number theory via Bernoulli numbers. The invariants we have in mind are firstly certain characteristic classes, associated with a natural flat vector bundle over $B\Gamma_g$, secondly, the orbifold Euler characteristic of the group Γ_g and thirdly its Yagita invariant. The characteristic classes are related to the denominators of Bernoulli numbers, the Euler characteristic involves the whole Bernoulli numbers, and our theorems concerning the Yagita invariant have to do with the notion of regular primes, which is expressible in terms of numerators of Bernoulli numbers. Although the three concepts which we study seem rather unrelated from the point of view of their definitions, the fact that they all are tightly linked to properties of finite subgroups and their normalizers and centralizers in Γ_g renders it plausible, that the resulting invariants must be somehow linked. The precise relationship, however, remains for the time being a mystery.

We have tried to make these notes easy to read for non-experts. We therefore recall in Sections 1 through 3 many facts and definitions, and state the relevant properties of the mapping class group without proofs, in form of a survey. We also have included a substantial bibliography, helping the reader to find the proofs of the basic theorems of the subject, which are scattered through the literature and which are crucial for analyzing homological properties of the mapping class group. For the classical part, not dealing with the cohomology of the mapping class group, the reader should consult Birman's book [Bi]. Section 4 contains a short introduction to the theory of characteristic classes for group representations, and in Section 5 we

compute the order of the Euler class $e_{2g}(\Gamma_g)$, associated with the flat bundle over $B\Gamma_g$ induced by the action of Γ_g on the homology group $H_1(S_g; \mathbb{R})$. In Section 6 this Euler class is related to the Euler characteristic $\chi(\Gamma_g)$ of the group Γ_g , and in Section 7 we discuss periodicity phenomena of Γ_g as well as the Yagita invariant.

1. The Definition of the Mapping Class Group

Let S_g denote a closed, connected and oriented topological surface of genus g . It is well-known that S_g admits a unique smooth structure; we shall also write S_g for the corresponding smooth (oriented) manifold. There are four basic ways of viewing the mapping class group Γ_g of the surface S_g , one being purely topological, the second more geometric in nature, the third homotopical and the fourth algebraic, involving the fundamental group of the surface in question only. The definitions we have in mind have the following form :

- (I) $\Gamma_g = \text{Homeo}_+(S_g)/\text{Homeo}^0(S_g)$
- (II) $\Gamma_g = \text{Diffeo}_+(S_g)/\text{Diffeo}^0(S_g)$
- (III) $\Gamma_g = \text{Hoequ}_+(S_g)/\text{Hoequ}^0(S_g)$
- (IV) $\Gamma_g = \text{Out}_+(\pi_1(S_g, s_0))$.

We shall first give some background information and comments concerning these equivalent definitions. Let $s_0 \in S_g$ denote a basepoint. The fundamental group of S_g has a presentation

$$\pi_1(S_g, s_0) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod [a_i, b_i] \rangle$$

and thus $\pi_1(S_g, s_0)_{ab} \cong H_1(S_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$. Since S_g is assumed to be orientable one has $H_2(S_g; \mathbb{Z}) \cong \mathbb{Z}$. A map $f : S_g \rightarrow S_g$ is said to be orientation preserving, if the induced map

$$H_2(f) : H_2(S_g; \mathbb{Z}) \rightarrow H_2(S_g; \mathbb{Z})$$

is the identity map. It is useful to notice that this is equivalent to the requirement that the determinant of

$$H_1(f) : H_1(S_g; \mathbb{Z}) \rightarrow H_1(S_g; \mathbb{Z})$$

equals one (the multiplicative structure of the cohomology ring $H^*(S_g; \mathbb{Z})$ reveals that $H^2(f)$ is multiplication by $\det H^1(f)$). Let $\text{Homeo}(S_g)$ denote the topological group of homeomorphisms of S_g , with the compact-open topology. We shall write Homeo_+ for the subgroup of orientation preserving homeomorphisms, and Homeo^0 for the connected component of the identity (we use the “+” here as a subscript rather than as a superscript, to avoid confusion with Quillen’s *plus*-construction). The mapping class group Γ_g of the surface S_g is then defined as the discrete group of connected components

$$(I) \quad \Gamma_g = \text{Homeo}_+(S_g)/\text{Homeo}^0(S_g).$$

We will consider (I) as our basic definition for Γ_g , and want to compare it with (II),(III) and (IV). Consider now S_g as a smooth oriented manifold. In accordance to the notation used above, we write $\text{Diffeo}_+(S_g)$ for the group of orientation preserving diffeomorphisms of S_g with the C^∞ -topology, and $\text{Diffeo}^0(S_g)$ for the connected component of the identity. It was proved by Dehn [De] that

$Homeo_+(S_g)/Homeo^0(S_g)$ is generated by “*Dehn twists*”, which are diffeomorphisms obtained by splitting S_g along a simple closed smooth curve, rotating one part by 2π , and gluing the surface back together. It follows that the natural map

$$Diffeo_+(S_g) \rightarrow Homeo_+(S_g)/Homeo^0(S_g)$$

is surjective. The kernel is precisely $Diffeo^0(S_g)$; namely, if $f: S_g \rightarrow S_g$ is a diffeomorphism isotopic to the identity (i.e. $f \in Homeo^0(S_g)$), then f is a fortiori homotopic to the identity, and therefore, according to Earle and Eells [**Ea-Ee**], the map f can be connected by a path in $Diffeo_+(S_g)$ to the identity map. We have thus established that

$$(II) \quad \Gamma_g = Diffeo_+(S_g)/Diffeo^0(S_g).$$

In case $g = 0$, that is $S_0 = S^2$ the 2-sphere, $Diffeo_+(S^2)$ is connected; the inclusion of $SO(3)$ in $Diffeo_+(S^2)$ is actually a homotopy equivalence by Smale’s result [**Sm**]. Thus $\Gamma_0 = \{e\}$. For $g > 0$ however, the mapping class groups Γ_g turn out to be all non-trivial. The group Γ_1 can be most easily understood using the definitions (III) and (IV) respectively, which we shall discuss now. Let $Hoequ_+(S_g)$ be the topological monoid of orientation preserving homotopy equivalences of S_g with the compact–open topology, and $Hoequ^0(S_g)$ the connected component of the identity. By a result due to Nielsen ([**Ni1**]), the natural map

$$Homeo_+(S_g) \rightarrow Hoequ_+(S_g)/Hoequ^0(S_g)$$

is surjective, and Baer proved [**Ba**] that any homeomorphism which is homotopic to the identity, is actually isotopic to the identity, showing that the kernel is precisely $Homeo^0(S_g)$, (compare also Mangler [**Ma**]). Therefore, we conclude:

$$(III) \quad \Gamma_g = Hoequ_+(S_g)/Hoequ^0(S_g).$$

Denote the set of free homotopy classes of maps between the spaces X and Y by $[X, Y]$. We have then a natural map

$$\Lambda : Hoequ_+(S_g) \rightarrow [S_g, S_g].$$

The homotopy set $[S_g, S_g]$ may be identified with the set of orbits of the usual $\pi_1(S_g, s_0)$ -action on the pointed homotopy set $[(S_g, s_0), (S_g, s_0)]_*$ of pointed homotopy classes of pointed maps. Since for $g > 0$ the surface S_g has a contractible universal covering space, there is a natural bijection

$$[(S_g, s_0), (S_g, s_0)]_* \cong Hom(\pi_1(S_g, s_0), \pi_1(S_g, s_0)), \quad (g > 0).$$

Passing to orbit spaces with respect to the $\pi_1(S_g, s_0)$ -action, we obtain

$$[S_g, S_g] \cong Rep(\pi, \pi),$$

where $Rep(\pi, \pi)$ stands for the set of conjugacy class of homomorphisms $\pi \rightarrow \pi$, with $\pi = \pi_1(S_g, s_0)$. Homotopy equivalences correspond then to automorphisms modulo inner automorphisms of π . If we denote by $Out(\pi)$ the group of outer automorphisms of π , we can view this group as a subset of $Rep(\pi, \pi)$, and the map Λ defined above yields a surjective homomorphism

$$Hoequ(S_g) \rightarrow Out(\pi_1(S_g, s_0)) \subset [S_g, S_g].$$

The kernel consists of course of all homotopy equivalences homotopic to the identity. If we write Out_+ for the “orientation-preserving” outer automorphisms, that is, the subgroup of $Out(\pi_1(S_g, s_0))$ consisting of those elements which act on the abelianized fundamental group $\pi_1(S_g, s_0)_{ab}$ by a homomorphism of determinant one, then we infer that

$$Hoequ_+(S_g)/Hoequ^0(S_g) \cong Out_+(S_g).$$

Note that the formula is also correct in case $g = 0$. From (III) we conclude then that

$$(IV) \quad \Gamma_g \cong Out_+(\pi_1(S_g, s_0)).$$

For the case of $g = 1$ one has $S_1 = S^1 \times S^1$ a torus, and therefore Γ_1 is isomorphic to $Out_+(\mathbb{Z} \oplus \mathbb{Z}) \cong Sl_2(\mathbb{Z})$.

2. Some Algebraic Properties of the Mapping Class Group

As mentioned earlier, Γ_g is generated by Dehn twists, associated with (isotopy classes) of simple closed curves on S_g and it follows that Γ_g is finitely generated, see Dehn [De]. An explicit finite set of generators is described in Lickorish [Li]. Actually, Γ_g is finitely presented. This is obvious for Γ_1 . Indeed, $\Gamma_1 \cong Sl_2(\mathbb{Z})$, so that one obtains a finite presentation for Γ_1 from the well-known decomposition of $Sl_2(\mathbb{Z})$ as an amalgamated free product :

$$Sl_2(\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} \underset{\mathbb{Z}/2\mathbb{Z}}{*} \mathbb{Z}/6\mathbb{Z}.$$

Note also that this implies

$$(\Gamma_1)_{ab} \cong \mathbb{Z}/12\mathbb{Z}.$$

Birman and Hilden described in [Bi-Hi1] an explicit finite presentation for Γ_2 , which can be used to show that

$$(\Gamma_2)_{ab} \cong \mathbb{Z}/10\mathbb{Z}.$$

It was proved by Powell [Po] that, for higher genus the mapping class group is always perfect :

$$(\Gamma_g)_{ab} = 0 \quad \text{for } g > 2.$$

That Γ_g is finitely presented for general g was first proved by McCool in [McCo], (see also Hatcher–Thurston [Ha-Th], as well as Wajnryb [Wa] for explicit presentations).

A lot is known about finite subgroups of Γ_g . It follows from Harvey [Harv2] that the number of conjugacy classes of finite subgroups of Γ_g is finite. The individual finite subgroups of Γ_g can be described as follows. Let τ denote a complex structure on S_g compatible with the smooth structure. Then the group of holomorphic automorphisms $Aut(S_g, \tau)$ is a subgroup of $Diffeo(S_g)$. It is a classical result that for $g > 1$ the group $Aut(S_g, \tau)$ is finite and the induced map

$$\theta_\tau : Aut(S_g, \tau) \rightarrow \Gamma_g \quad (g > 1),$$

is injective (cf. Farkas–Kra [Fa-Kr]). According to Kerckhoff [Ke], the finite subgroups of Γ_g are precisely the subgroups of the form $\theta_\tau(F)$ for F a finite group of holomorphic automorphisms of (S_g, τ) for some τ , amounting to a positive solution of the *Nielsen realization problem*; for an account on the long history of this problem

as well as the partial results proved earlier, the reader should consult Zieschang's book [Zi]. Another classical result, due to Hurwitz [Hu], states that

$$|Aut(S_g, \tau)| \leq 84(g-1), \quad (g > 1).$$

As a consequence, all finite subgroups $F \subset \Gamma_g$ satisfy the *Hurwitz bound*

$$|F| \leq 84(g-1), \quad (g > 1).$$

For a finite cyclic subgroup $F \subset \Gamma_g$ this bound can be improved to

$$|F| \leq 4g + 2,$$

and this bound is sharp, that is, Γ_g always contains a cyclic subgroup of order $4g+2$, see Wiman [Wi]. Note also that every finite group F admits an embedding into some Γ_g since, as is well-known, every finite group occurs as a group of symmetries of some Riemann surface (see Broughton [Br] for an explicit condition, in terms of a presentation of F , for the existence of an embedding of F in Γ_g). It is even the case that every finite group is isomorphic to the full automorphism group $Aut(S_g, \tau)$ of some closed Riemann surface, a result due to Greenberg [Gre]. The minimal $g = g(F)$ such that F admits an embedding into Γ_g is called the *genus* of F . It is an interesting problem to determine $g(F)$ for certain families of finite groups F . For the case of a finite cyclic group F of prime power order $|F| > 2$ the problem is quite elementary and one finds $g(F) = \frac{1}{2}\phi(|F|)$, where ϕ denotes the Euler-function, see Glover–Mislin [Gl-Mi2]; the case of general finite cyclic groups was settled by Harvey [Harv1]. For a more involved example, the reader should consult Glover–Sjerve [Gl-Sj], where the genus of the group $PSl_2(\mathbb{F}_q)$ is computed for \mathbb{F}_q an arbitrary finite field.

Maps from Γ_g to finite groups were studied by Grossman in [Gro]. She proved that Γ_g is residually finite (which means that Γ_g admits an embedding into a product of finite groups). In particular, the elements of Γ_g may therefore be separated by means of finite dimensional linear representations. However, it is still an open question whether Γ_g admits a *faithful* finite-dimensional linear representation. But it is known that for $g > 1$, Γ_g is not isomorphic to an arithmetic group; for a discussion of this fact see Ivanov [Iv2] or Harer [Ha5].

Another important result is that Γ_g contains a torsion-free subgroup of finite index. This can be seen in an elementary way as follows. From our definition (IV) for Γ_g , there is a natural map $\rho: \Gamma_g \subset Out(\pi) \rightarrow Aut((\pi)_{ab})$, where π denotes the fundamental group of S_g , and thus $\pi_{ab} \cong \mathbb{Z}^{2g}$ the abelianized fundamental group. Hence there is a left exact sequence

$$T\Gamma_g = \ker \rho \rightarrow \Gamma_g \rightarrow Gl_{2g}(\mathbb{Z}),$$

where $T\Gamma_g$ denotes the *Torelli* group, which is easily seen to be torsion-free (every element of finite order in Γ_g can be realized as a holomorphic automorphism on some Riemann surface (S_g, τ) , and these act non-trivially on $H_1(S_g; \mathbb{Z})$). Since $Gl_{2g}(\mathbb{Z})$ contains a torsion-free subgroup of finite index, it follows that Γ_g too possesses one.

To conclude this Section, we want to mention two additional basic results, concerning infinite subgroups of Γ_g . The first one is an analogue of a Theorem of Tits on linear groups. According to McCarthy [McC] and Ivanov [Iv1] the following *Tits-alternative* holds for the mapping class group: every subgroup of Γ_g either contains a free subgroup on two generators, or a solvable subgroup of finite

index. Solvable subgroups of Γ_g were analyzed in Birman–Lubotzky–McCarthy [B-L-M]. They proved that every solvable subgroup of Γ_g is virtually abelian, and that the maximal rank of a free abelian subgroup in Γ_g is, for $g > 1$, equal to $3g - 3$.

3. Some Cohomological Results on Γ_g

Since $Diffeo^0(S_g)$ is a contractible space for $g > 1$ (see Earle–Eells [Ea-Ee]), the natural map of classifying spaces $BDiffeo_+(S_g) \rightarrow B\Gamma_g$ is a homotopy equivalence, and thus

$$H^*(\Gamma_g; \mathbb{Z}) \cong H^*(BDiffeo_+(S_g); \mathbb{Z}), \quad g > 1.$$

One can therefore think of the cohomology elements of Γ_g as universal characteristic classes for smooth orientable S_g -bundles. The most important tool for studying the cohomology of Γ_g is its action on *Teichmüller space* T_g , which is for $g > 1$ a smooth manifold homeomorphic to \mathbb{R}^{6g-6} . Teichmüller space is a parameter space for complex structures on the oriented closed smooth surface S_g , where two complex structures on S_g are considered as equivalent if and only if there exists a diffeomorphism $f : S_g \rightarrow S_g$ diffeotopic to the identity (i.e. $f \in Diffeo^0(S_g)$), carrying one complex structure to the other. According to Earle–Eells [Ea-Ee], one can describe T_g as follows. Consider the space $CS(S_g)$ of complex structures on S_g compatible with the smooth structure and orientation; it carries a natural topology and an obvious action of $Diffeo_+(S_g)$. Then one has

$$T_g = CS(S_g)/Diffeo^0(S_g).$$

There remains a natural action of $\Gamma_g = Diffeo_+(S_g)/Diffeo^0(S_g)$ on T_g , which is known to be properly discontinuous. The orbit space

$$M_g = T_g/\Gamma_g = CS(S_g)/Diffeo_+(S_g)$$

is called the *moduli space* of S_g . It has the structure of a complex variety and its points correspond to conformal equivalence classes of complex structures on S_g . Because the action of Γ_g on T_g is properly discontinuous, the stabilizers of points of T_g are finite. The natural projection

$$T_g \rightarrow M_g$$

is a branched covering space, and it can be thought of as a resolution of the singularities for the variety M_g . In particular, T_g inherits a natural complex structure such that Γ_g acts by complex automorphisms. By a result due to Royden [Roy], Γ_g is actually the full automorphism group of Teichmüller space with this complex structure. If $t \in T_g$ is fixed by the (finite) subgroup $F \subset \Gamma_g$, then we can think of F as a group of symmetries for a complex structure on S_g . On the other hand, by the positive solution of the *Nielsen realization problem* mentioned earlier, every finite subgroup $F \subset \Gamma_g$ is a group of symmetries of some complex structure on S_g and thus has a fixed point when acting on T_g . Harvey proved in [Harv2] that there is a contractible simplicial complex \overline{T}_g containing T_g such that the Γ_g -action extends to a proper simplicial action on \overline{T}_g , with compact quotient \overline{T}_g/Γ_g . From general principles (see Brown’s book [Brow1]), this immediately implies a wealth of finiteness properties for Γ_g , the first of which has also been discussed in the previous Section:

- (1) Γ_g is finitely presented.

- (2) Γ_g is of finite virtual cohomological dimension (vcd); actually it follows that $vcd(\Gamma_g) \leq 6g - 6 = \dim T_g$, ($g > 1$). Note also that $vcd(\Gamma_g) \geq 3g - 3$, since Γ_g possesses a free abelian subgroup of rank $3g - 3$. It has been proved by Harer in [**Ha3**] that, more precisely, one has

$$vcd(\Gamma_g) = 4g - 5, \quad g > 1.$$

Of course, $vcd(\Gamma_1) = 1$, since $\Gamma_1 \cong Sl_2(\mathbb{Z})$ is an amalgamated free product of finite groups.

- (3) Γ_g is of type *WFL* (i.e. all torsion-free subgroups of finite index admit finitely generated free resolutions of finite length).
- (4) For every subgroup $H \subset \Gamma_g$ of finite index, the homology groups $H_i(H; \mathbb{Z})$ are finitely generated, and they are finite if $i > vcd(\Gamma_g)$. In particular, the *naive Euler characteristic*

$$\tilde{\chi}(H) = \sum (-1)^i \text{rk } H_i(H; \mathbb{Z})$$

is well defined.

REMARK. There are of course many other ways to define Teichmüller space T_g . We just want to mention one different definitions, which is discussed in Goldman's paper [**Go**], and which is particularly attractive. By choosing a complex structure on S_g one obtains an embedding θ in $\pi_1(S_g)$ into $PSl_2(\mathbb{R})$, which is the group of isometries of the upper half-plane. Teichmüller space can then be identified with the component of θ in $Rep(\pi_1(S_g), PSl_2(\mathbb{R}))$, the space of conjugacy classes of homomorphisms from $\pi_1(S_g)$ to $PSl_2(\mathbb{R})$. Since $\pi_1(S_g)$ is finitely presented, this space admits a natural embedding as a real algebraic subvariety in a quotient of a product of copies of $PSl_2(\mathbb{R})$. The action of Γ_g on T_g is just the one induced by the action of the group of orientation preserving automorphisms of $\pi_1(S_g)$ on the space $Hom(\pi_1(S_g), PSl_2(\mathbb{R}))$.

3.1 Rational cohomology. Next, we shall describe some results concerning the rational cohomology of Γ_g . We first recall that, because the stabilizers of the Γ_g -action on the contractible space T_g are all finite, the rational cohomology of the moduli space M_g satisfies

$$H^*(M_g; \mathbb{Q}) \cong H^*(\Gamma_g; \mathbb{Q}),$$

which explains the great interest in the rational cohomology of the mapping class group. Obviously, Γ_1 is \mathbb{Q} -acyclic, being an amalgamated free product of finite groups. By a result due to Igusa [**Ig**], Γ_2 is \mathbb{Q} -acyclic too, but this is not the case for $g \geq 3$. Of course, for $g \geq 3$, $H^1(\Gamma_g; \mathbb{Q})$ is trivial, since Γ_g is perfect. However, Harer proved in [**Ha4**] that

$$H^2(\Gamma_g; \mathbb{Q}) \cong \mathbb{Q} \quad \text{for } g \geq 3,$$

and he also showed that

$$H^3(\Gamma_g; \mathbb{Q}) = 0 \quad \text{for } g \geq 6.$$

For all values of g one can describe as follows a part of the rational cohomology of Γ_g whose dimension increases rapidly with g . According to Miller [**Mil**] and Morita [**Mo**] there are classes y_j in $H^{2j}(\Gamma_g; \mathbb{Q})$, $j \geq 1$, such that the induced map of the polynomial ring

$$\Lambda : \mathbb{Q}[y_1, y_2, y_3 \dots] \rightarrow H^*(\Gamma_g; \mathbb{Q})$$

is injective in dimensions less than $g/3$. The classes y_j for odd j can be described as symplectic characteristic classes in the following way, an interpretation which is useful in many contexts. Consider the natural map

$$B\Gamma_g \rightarrow BSp_{2g}(\mathbb{R})$$

induced by the action of Γ_g on $H^1(S_g; \mathbb{R})$, viewed as a symplectic space using the cup product. Since a maximal compact subgroup in $Sp_{2g}(\mathbb{R})$ is isomorphic to $U(g)$, one has

$$H^*(BSp_{2g}(\mathbb{R}); \mathbb{Z}) = \mathbb{Z}[d_1, d_2, \dots, d_g],$$

with $d_j \in H^{2j}(BSp_{2g}(\mathbb{R}); \mathbb{Z})$ the *universal symplectic characteristic classes*, characterized by the property that d_j restricts to the universal Chern class c_j of the maximal compact subgroup $U(g) \subset Sp_{2g}(\mathbb{R})$. Each even indexed class d_{2j} restrict in $H^{4j}(\Gamma_g; \mathbb{Q})$ to the image of a polynomial in d_i 's involving only odd i 's; this can easily be seen using the fact that the rational Pontrjagin classes of a flat real vector bundle vanish, and that the Pontrjagin class $p_i \in H^{4i}(BGl_{2g}(\mathbb{R}); \mathbb{Z})$ restricts with respect to the inclusion of $Sp_{2g}(\mathbb{R})$ in $Gl_{2g}(\mathbb{R})$ according to the formula

$$\text{im}(1 - p_1 + p_2 - \dots) = (1 + d_1 + d_2 + \dots)(1 - d_1 + d_2 - \dots) \in H^*(BSp_{2g}(\mathbb{R}); \mathbb{Z}).$$

From a rational point of view, the only part of the map

$$H^*(BSp_{2g}(\mathbb{R}); \mathbb{Z}) \rightarrow H^*(\Gamma_g; \mathbb{Z})$$

which is of any relevance, is therefore the induced map

$$H^*(BSp_{2g}(\mathbb{R}); \mathbb{Q}) \supset \mathbb{Q}[d_1, d_3, d_5, \dots] \xrightarrow{\Phi} H^*(\Gamma_g; \mathbb{Q}).$$

It is shown in Miller [Mi] and Morita [Mo] that the image of this map Φ agrees with the image of the restriction of the map Λ to $\mathbb{Q}[y_1, y_3, y_5, \dots]$, and in particular Φ is therefore injective in dimensions less than $g/3$ too.

3.2 Mod- p cohomology in the stable range. Again, we can look at the representation of Γ_g obtained by letting Γ_g act on $H_1(S_g; \mathbb{R})$, yielding a homomorphisms

$$\Gamma_g \rightarrow Gl_{2g}(\mathbb{R}) \rightarrow Gl_{2g}(\mathbb{C}).$$

Recall that

$$H^*(BGl_{2g}(\mathbb{R}); \mathbb{F}_2) = \mathbb{F}_2[w_1, w_2, \dots, w_{2g}],$$

a polynomial algebra in the universal Stiefel–Whitney classes. Kaufmann showed in [Kau] that the odd Stiefel–Whitney classes w_{2j+1} restrict to zero in $H^*(\Gamma_g; \mathbb{F}_2)$, and that the induced map

$$H^*(BGl_{2g}(\mathbb{R}); \mathbb{F}_2) \supset \mathbb{F}_2[w_2, w_4, w_6, \dots] \rightarrow H^*(\Gamma_g; \mathbb{F}_2)$$

is injective in dimensions less than $g/3$. He also proved (loc. cit.) a corresponding result for the the case of an odd prime p , by considering

$$H^*(BGl_{2g}(\mathbb{C}); \mathbb{F}_p) = \mathbb{F}_p[\bar{c}_1, \bar{c}_2, \dots, \bar{c}_{2g}],$$

where \bar{c}_i stands for the mod- p reduction of the universal Chern class c_i . The result is that the mod- p Chern classes \bar{c}_i restrict to zero in $H^*(\Gamma_g; \mathbb{F}_p)$ in case $i \not\equiv 0 \pmod{p-1}$, whereas the induced map

$$\mathbb{F}_p[\bar{c}_{p-1}, \bar{c}_{2(p-1)}, \dots] \rightarrow H^*(\Gamma_g; \mathbb{F}_p)$$

is injective in dimensions less than $g/3$ (the dimension condition should be thought of as a stability condition, cf. (STAB) in Section 3.4.) Since [Kau] is not so readily available, we give a short outline of his proof. First, the vanishing conditions follow from standard results on characteristic classes of classical groups. Namely, the natural inclusion of $Sp_{2g}(\mathbb{R})$ in $Gl_{2g}(\mathbb{R})$ induces a map

$$H^*(BGl_{2g}(\mathbb{R}); \mathbb{F}_2) \rightarrow H^*(BSp_{2g}(\mathbb{R}); \mathbb{F}_2)$$

for which the odd universal Stiefel–Whitney classes w_{2i+1} restrict to zero; the corresponding result for Γ_g then follows, since the $Gl_{2g}(\mathbb{R})$ -representation of Γ_g considered above factors through $Sp_{2g}(\mathbb{R})$. For the case of the Chern classes one considers the restriction map

$$H^*(BGl_{2g}(\mathbb{C}); \mathbb{Z}) \rightarrow H^*(BGl_{2g}(\mathbb{Z}); \mathbb{Z})$$

which maps the universal Chern class c_j to $c_j(\mathbb{Z}) \in H^{2j}(BGl_{2g}(\mathbb{Z}); \mathbb{Z})$, a torsion class of order prime to p if $j \not\equiv 0 \pmod{p-1}$ (see [Eck-Mi2]). The mod p Chern classes \bar{c}_j restrict therefore already in $H^*(BGl_{2g}(\mathbb{Z}); \mathbb{F}_p)$ to zero, if j is not divisible by $(p-1)$; thus \bar{c}_j restricts to zero in $H^{2j}(\Gamma_g; \mathbb{F}_p)$ too, if $j \not\equiv 0 \pmod{p-1}$. For the part of Kaufmann’s result which deals with injectivity, one proceeds as follows. One makes use of the mapping class groups $\Gamma_{g,1}$ of “oriented surfaces of genus g , with one boundary component”, which can be arranged to form a natural increasing sequence

$$\Gamma_{g,1} \hookrightarrow \Gamma_{g+1,1}$$

and one defines the *stable* mapping class group by putting

$$\Gamma_\infty := \cup_g \Gamma_{g,1}.$$

The point is that both natural maps

$$\Gamma_g \leftarrow \Gamma_{g,1} \rightarrow \Gamma_\infty$$

induce integral cohomology isomorphisms in dimensions less than $g/3$, see Harer [Ha2], so that Γ_∞ is suitable for computing the cohomology of Γ_g in that range. Note also that $\Gamma_{g,1}$ acts on the homology of the surface $S_{g,1} = S_g \setminus D$, with D the interior of a closed disk. Because the inclusion $S_{g,1} \subset S_g$ induces an isomorphism in H^1 , one obtains natural representations $\Gamma_{g,1} \rightarrow Sl_{2g}(\mathbb{Z})$, which are compatible with the corresponding representations of Γ_g . Moreover, there are natural pairings

$$\Gamma_{g,1} \times \Gamma_{h,1} \rightarrow \Gamma_{g+h,1}$$

compatible with the usual pairings

$$Sl_{2g}(\mathbb{Z}) \times Sl_{2h}(\mathbb{Z}) \rightarrow Sl_{2g+2h}(\mathbb{Z}),$$

inducing H -space structures on $B\Gamma_\infty^+$ and $BSl(\mathbb{Z})^+$ respectively, where the plus stands for Quillen’s *plus*-construction. By naturality, the induced map

$$B\Gamma_\infty^+ \rightarrow BSl(\mathbb{Z})^+$$

is an H -map, thus inducing morphisms of Hopf-algebras

$$H^*(BGl(\mathbb{C}); \mathbb{F}_p) \rightarrow H^*(BGl(\mathbb{R}); \mathbb{F}_p) \rightarrow H^*(B\Gamma_\infty; \mathbb{F}_p),$$

with the Hopf–algebra structure on $H^*(BGl(\mathbb{C}); \mathbb{F}_p)$ and $H^*(BGl(\mathbb{R}); \mathbb{F}_p)$ induced by the Whitney sum construction. We assume now that p is an odd prime; in case $p = 2$ one argues similarly. The Hopf–algebra structure on

$$A^* = H^*(BGl(\mathbb{C}); \mathbb{F}_p)$$

is given by

$$\Delta(\bar{c}_k) = \sum_{i=0}^k \bar{c}_{k-i} \otimes \bar{c}_i.$$

Now consider $B^* = \mathbb{F}_p[\bar{c}_{p-1}, \bar{c}_{2(p-1)}, \dots]$, with a Hopf–algebra structure defined by

$$\Delta \bar{c}_{k(p-1)} = \sum_{i=0}^k \bar{c}_{(k-i)(p-1)} \otimes \bar{c}_{i(p-1)}.$$

Although the inclusion $B^* \subset A^*$ is not a morphism of Hopf–algebras, any Hopf–algebra map $A^* \rightarrow H^*(\Gamma_\infty; \mathbb{F}_p)$ which maps \bar{c}_j to zero for j not divisible by $(p-1)$, will restrict to a morphism of Hopf–algebras on B^* . Note also that a morphism of (graded) Hopf–algebras with domain B^* is injective, if it is injective when restricted to the subspace PB^* of primitive elements of B^* . One checks that the graded vector space PB^* has a basis consisting of the Newton polynomials $N_k = N_k(\bar{c}_{p-1}, \bar{c}_{2(p-1)}, \dots)$, given by the usual recursion formula:

$$N_1 = \bar{c}_{p-1}$$

and, for $k > 1$

$$N_k = \bar{c}_{p-1} N_{k-1} - \bar{c}_{2(p-1)} N_{k-2} + \dots + (-1)^{k-2} \bar{c}_{(k-1)(p-1)} N_1 + (-1)^{k-1} k \bar{c}_{k(p-1)}.$$

Kaufmann proves then that these classes $N_k \in H^{2k(p-1)}(BGl(\mathbb{C}); \mathbb{F}_p)$ do not restrict to zero in $H^{2k(p-1)}(\Gamma_\infty; \mathbb{F}_p)$, by evaluating them on a suitable subgroup of order p in $\Gamma_{(p^n-1)(p-1)/2}$, with $n \gg k$. His result then follows.

3.3 The case of genus less than three. Since Γ_1 is isomorphic to an amalgamated free product of a cyclic group of order four and a cyclic group of order six over a cyclic group of order two, the Mayer–Vietoris sequence reveals that

$$H^*(\Gamma_1; \mathbb{Z}) = \mathbb{Z}[x]/(12x),$$

with $x \in H^2(\Gamma_1; \mathbb{Z})$. An explicit description of x can be given as follows. Consider the flat real vector bundle over $B\Gamma_1$, induced by the inclusion of $\Gamma_1 = Sl_2(\mathbb{Z})$ into $Sl_2(\mathbb{R})$. One checks that the universal Euler class e_2 in $H^2(BSl_2(\mathbb{R}); \mathbb{Z})$ restricts to a generator of $H^2(C; \mathbb{Z})$ for any cyclic subgroup $C \subset Sl_2(\mathbb{R})$. It therefore restricts to a generator $e_2(\Gamma_1) = x$ in $H^2(\Gamma_1; \mathbb{Z})$. Also, it follows that the projection $\Gamma_1 \rightarrow (\Gamma_1)_{ab}$, a cyclic group of order 12, induces in integral cohomology an isomorphism.

As mentioned earlier, Γ_2 is \mathbb{Q} -acyclic and therefore, because the integral cohomology groups of Γ_g are finitely generated, $H^k(\Gamma_2; \mathbb{Z})$ is a finite group for $k > 0$. Moreover, Lee–Weintraub proved in [Le–We] that Γ_2 is \mathbb{F}_p -acyclic for all primes $p > 5$. For the remaining primes $p = 2, 3$ and 5 it is useful to study the short exact sequence

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \Gamma_2 \rightarrow \Gamma_0^6 \rightarrow 0,$$

where Γ_0^6 is the mapping class group of the 2-sphere with 6 punctures, that is

$$\Gamma_0^6 = \pi_0(Diff eo_+(S^2 \setminus \{x_1, \dots, x_6\})).$$

This short exact sequence is discussed in Birman–Hilden [Bi-Hi1]. It is obtained by considering the genus two surface S_2 as a branched covering space of the 2-sphere S^2 with six branch points, by forming the quotient surface $S_2 / \langle \tau \rangle$, τ the hyperelliptic involution, which is known to generate the center of Γ_2 . There is an obvious map $\Gamma_0^6 \rightarrow \Sigma_6$, the symmetric group on six letters, with kernel denoted by K_6 . An analysis of K_6 led Cohen and Benson [Co4][Be][Be-Co] to the following results concerning the mod- p cohomology of Γ_2 .

($\Gamma_2 \bmod p$)

- a) There is a subgroup $\mathbb{Z}/5\mathbb{Z} \subset \Gamma_2$ such that the restriction map induces an isomorphism

$$H^*(\Gamma_2; \mathbb{F}_5) \cong H^*(\mathbb{Z}/5\mathbb{Z}; \mathbb{F}_5).$$

- b) There are elements x, y, z and w of degree 3, 4, 4 and 5, such that

$$H^*(\Gamma_2; \mathbb{F}_3) \cong \mathbb{F}_3[x, y, z, w] / \langle x^2, xz, z^2, zw, w^2, yz - xw \rangle.$$

- c) The Poincaré series of $H^*(\Gamma_2; \mathbb{F}_2)$ is given by

$$\frac{1 + t^2 + 2t^3 + t^4 + t^5}{(1-t)(1-t^4)} = 1 + t + 2t^2 + 4t^3 + 6t^4 + \dots.$$

A discussion of the integral cohomology of Γ_2 is presented in Cohen [Co3], see also [Co1-2]; we want to mention in particular the following result, which we will use later on:

$$120 \cdot H^i(\Gamma_2; \mathbb{Z}) = 0 \quad \text{for } i > 0.$$

Away from the prime 2 the cohomology is particularly easy to describe, because it is “periodic with period 4 from dimension four on”. It can be expressed as follows:

$$H^*(\Gamma_2; \mathbb{Z}[1/2]) = \mathbb{Z}[1/2][x, y, z] / \langle 5x, 3y, 3z, z^2 \rangle,$$

with $x \in H^2$, $y \in H^4$ and $z \in H^5$. The groups in low dimension are

$$H^i(\Gamma_2; \mathbb{Z}) = \begin{cases} 0, & \text{for } i = 1 \\ \mathbb{Z}/10\mathbb{Z}, & \text{for } i = 2 \\ \mathbb{Z}/2\mathbb{Z}, & \text{for } i = 3 \\ \mathbb{Z}/120\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2, & \text{for } i = 4 \\ \mathbb{Z}/6\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2, & \text{for } i = 5. \end{cases}$$

3.4 Torsion in the cohomology of the mapping class group. The Bernoulli numbers B_n are rational numbers defined recursively by the formula

$$(B+1)^{\downarrow n} - B_n = 0, \quad n \geq 2,$$

where the exponent “ $\downarrow n$ ” means that after evaluating the n 'th power of the monomial, one replaces the power B^k by B_k . For $n = 2$ this yields $B_2 + 2B_1 + 1 - B_2 = 0$, hence $B_1 = -1/2$. It turns out that for odd $n > 1$ one always has $B_n = 0$ and, as already observed by Euler, the B_{2k} 's are related to the Taylor series of $\tan(x)$, which is given by

$$x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 \dots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!} x^{2k-1}.$$

Thus $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, \dots , $B_{12} = -\frac{691}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 13}$, \dots and so on. There are several conflicting notations in use concerning Bernoulli numbers. The one we use here differs by a sign from the one used in [G1-Mi1] and [G1-Mi2], but agrees with the one used in [Brow] and [Ha-Za]. The Bernoulli numbers turn up in number theory in several places. Our convention is such that for any integer $k > 0$, the Riemann zeta function satisfies the equation

$$\zeta(1 - 2k) = -B_{2k}/2k.$$

In Glover-Mislin [G1-Mi2] it is proved that for $g > (8m)^2$ the cohomology group $H^{4m}(\Gamma_g; \mathbb{Z})$ contains an element of order

$$E_{2m} = \text{den}(B_{2m}/2m),$$

the denominator of $B_{2m}/2m$, expressed as a fraction in lowest terms ($E_2 = 12$, $E_4 = 120$, $E_6 = 252, \dots$). On the other hand, Harer proved in [Ha2] a *stability result* for Γ_g :

$$\text{(STAB)} \quad H^k(\Gamma_g; \mathbb{Z}) \quad \text{is for } g > 3k \text{ independent of } g.$$

It follows then that :

$$\text{(TOR)} \quad H^{4m}(\Gamma_g; \mathbb{Z}) \quad \text{contains for } g > 12m \text{ an element of order } E_{2m}.$$

This result will be improved in Section 5. In [G1-Mi2] the torsion result (TOR) was used to construct *strange* torsion in the cohomology of Γ_g (torsion, which is not present in the group Γ_g itself). It is shown that for p a prime larger than 13 and $g = g(p) = (p^2 - 4p + 1)/2$, the mapping class group $\Gamma_{g(p)}$ is p -torsion-free, but $H^{2(p-1)}(\Gamma_{g(p)}; \mathbb{Z})$ contains an element of order p .

REMARK. An alternative way to construct torsion in $H^*(\Gamma_g; \mathbb{Z})$ is to use a result due to Charney-Cohen [Ch-Co], which states that, in the stable range, the cohomology of Γ_g contains a direct summand isomorphic to the cohomology of $\text{Im } J_{1/2}$, where $\text{Im } J_{1/2}$ is a space which is a factor of $BGL(\mathbb{Z})^+$, usually referred to as “the image of the J -homomorphism localized away from 2”.

3.5 Periodicity and Krull dimension. It is well-known that for a finite group F the cohomology ring $H^*(F; \mathbb{F}_p)$ is noetherian. More generally, Quillen’s Proposition 14.5 of [Qu] implies that if Γ denotes a group of finite virtual cohomological dimension acting simplicially on a finite dimensional contractible simplicial complex, with compact quotient and finite stabilizers, then $H^*(\Gamma; \mathbb{F}_p)$ is noetherian. In particular, this implies that, by considering the action of Γ_g on \overline{T}_g :

$$\text{(NOETH)} \quad H^*(\Gamma_g; \mathbb{F}_p) \quad \text{is a noetherian ring.}$$

To get an idea of the growth rate of H^n as $n \rightarrow \infty$, one considers the Krull dimension. Recall that the Krull dimension of a commutative ring R with 1 is defined as the supremum of the lengths n of chains of distinct prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_n.$$

For an arbitrary (discrete) group Γ and prime p , the Krull dimension of Γ at p , $\kappa(\Gamma, p)$, is per definition the Krull dimension of the commutative ring $H^{ev}(\Gamma; \mathbb{F}_p)$ of even dimensional cohomology classes. If $H^*(\Gamma; \mathbb{F}_p)$ is noetherian, standard results

from commutative algebra imply that $\kappa(\Gamma, p)$ is the smallest integer $\kappa \geq 0$ such that there is a constant $C > 0$ satisfying for all $n \geq 0$

$$\sum_{i \leq n} \dim_{\mathbb{F}_p} H^i(\Gamma; \mathbb{F}_p) \leq C \cdot n^\kappa.$$

Note that in this last formula we did not restrict to the even cohomology; indeed one easily checks that for a finitely generated graded-commutative \mathbb{F}_p -algebra H^* satisfying for all $n \geq 0$ the condition $\sum_{2i \leq 2n} \dim H^{2i} \leq C(2n)^\kappa$ with $C > 0$, one can find a constant $D > 0$ such that $\sum_{i \leq n} \dim H^i \leq Dn^\kappa$, and conversely. In particular, if $H^*(\Gamma; \mathbb{F}_p)$ is noetherian then

$$\kappa(\Gamma, p) = 0 \iff \dim_{\mathbb{F}_p} H^*(\Gamma; \mathbb{F}_p) < \infty.$$

For groups Γ of finite virtual cohomological dimension, the prime ideals of the ring $H^{ev}(\Gamma; \mathbb{F}_p)$ are intimately linked to the elementary abelian subgroups of Γ ; in case Γ has only finitely many conjugacy classes of elementary abelian p -subgroups, the minimal prime ideals are in one-one correspondence with the conjugacy classes of maximal elementary abelian p -subgroups (see Quillen [Qu]). It was moreover proved in [Qu] that for a finite group F the Krull dimension $\kappa(F, p)$ equals the maximal rank of an elementary abelian p -subgroup of F . This result still holds for certain infinite groups, in particular for Γ_g , as was proved by Broughton in [Br]. Note that from our description of the cohomology of Γ_1 and Γ_2 one readily sees that:

- $\kappa(\Gamma_1, 2) = \kappa(\Gamma_1, 3) = 1$, and $\kappa(\Gamma_1, p) = 0$ for $p > 3$,
- $\kappa(\Gamma_2, 2) = 2$, $\kappa(\Gamma_2, 3) = \kappa(\Gamma_2, 5) = 1$, and $\kappa(\Gamma_2, p) = 0$ for $p > 5$.

In [Br] Broughton established an explicit general formula for the Krull dimension of Γ_g , by determining the maximal rank of an elementary abelian p -subgroup contained in the mapping class group:

(KRULL-DIM I)

Let $g > 1$. Then the Krull dimension $\kappa(\Gamma_g, p)$ is the largest integer κ such that there are nonnegative integers $k \neq 1$ and h satisfying:

- 1) $2g - 2 = p^\kappa(2h - 2) + p^{\kappa-1}(p - 1)k$, and
- 2) $\kappa \leq 2h$ if $k = 0$, and
- 3) $\kappa < 2h + k$ if $k > 1$.

It will be useful to record the following immediate consequences.

(KRULL-DIM II)

Let $g > 1$. Then the Krull dimension of Γ_g satisfies the following:

- 1) Case $p = 2$: $\kappa(\Gamma_g, 2) \geq 2$.
 - 1.1) if g is even, $\kappa(\Gamma_g, 2) = 2$
 - 1.2) if g is odd, $\kappa(\Gamma_g, 2) \geq 3$
- 2) Case p odd:
 - 2.1) if $g \not\equiv 1 \pmod{p}$, then $\kappa(\Gamma_g, p) \leq 1$
 - 2.2) if $g \equiv 1 \pmod{p}$, then $\kappa(\Gamma_g, p) \geq 1$ and if we write g in the form $l \cdot p^\alpha + 1$ with l prime to p and $\alpha > 0$, then $\kappa(\Gamma_g, p) \leq \alpha + 1$; moreover, if $l \gg p$ one has $\kappa(\Gamma_g, p) = \alpha + 1$.

A particularly interesting case arises when $\kappa(\Gamma_g, p) = 1$. It was observed by Venkov [Ve] that if Γ is a group of finite virtual cohomological dimension with $\kappa(\Gamma, p) \leq 1$, then $H^*(\Gamma, \mathbb{F}_p)$ is “periodic in sufficiently high dimensions”, meaning that

$$\exists k > 0, \forall i \gg 0 : \quad H^i(\Gamma; \mathbb{F}_p) \cong H^{i+k}(\Gamma; \mathbb{F}_p).$$

Periodicity phenomena are much easier to handle using *Farrell* cohomology instead of ordinary cohomology. We will write

$$\hat{H}^i(\Gamma; M), \quad i \in \mathbb{Z}$$

for the i -th Farrell cohomology group [Fa] of the group Γ of finite virtual cohomological dimension, and M a Γ -module. For an in depth discussion of the properties of these Farrell cohomology groups, see Brown’s book [Brow1]. The interested reader might also consult [Mis2], where “Tate cohomology groups” $\hat{H}^i(\Gamma; M)$ are defined for *arbitrary* groups Γ , in a way that for groups of finite virtual cohomological dimension one obtains the Farrell cohomology groups, and thus for finite groups the classical Tate cohomology groups. Here are some basic facts concerning these generalized Tate groups.

(TATE-GROUPS)

- a) For an arbitrary group Γ and projective Γ -module P one has $\hat{H}^*(\Gamma; P) = 0$
- b) $\hat{H}^0(\Gamma; \mathbb{Z}) = 0 \iff cd(\Gamma) < \infty$
- c) Suppose $vcd(\Gamma) < \infty$. Then
 - c1) $\hat{H}^i(\Gamma; M)$ is a torsion group for all i and all Γ -modules M
 - c2) if $i > vcd(\Gamma)$, the natural map $H^i(\Gamma; M) \rightarrow \hat{H}^i(\Gamma; M)$ is an isomorphism for all Γ -modules M .
 - c3) the following conditions are equivalent :
 - c3.1) every abelian p -subgroup of Γ has rank ≤ 1
 - c3.2) there exists a $d > 0$ such that $\hat{H}^i(\Gamma; M)_{(p)} \cong \hat{H}^{i+d}(\Gamma; M)_{(p)}$ for all Γ -modules M and all $i \in \mathbb{Z}$ (the smallest such d is called the p -period of Γ and is denoted by $p(\Gamma)$; for an abelian torsion group A and prime p we write $A_{(p)}$ for its p -torsion subgroup)
 - c3.3) the ring $\hat{H}^*(\Gamma; \mathbb{Z})_{(p)}$ contains an invertible element of positive degree.

Property a) results from the definition of these general Tate groups,, which we will not recall here; the interesting fact b) was proved by Kropholler [Kr1-2], and the list c) just recalls some of the basic results on Farrell cohomology. A group satisfying one of the equivalent conditions listed under (c3) above will be called p -periodic (see Xia [Xi1] for a discussion of that concept). For a group like Γ_g , being p -periodic is therefore equivalent to the condition that its Krull dimension be less than or equal to one. A basic example of an infinite p -periodic group is the semidirect product

$$S(n,p) := \mathbb{Z}/p^n\mathbb{Z} \rtimes \mathbb{Z},$$

with p -period equal $2(p-1)p^{n-1}$ in case p is an odd prime, $n \geq 1$, and \mathbb{Z} is acting by means of a surjective map $\mathbb{Z} \rightarrow \text{Aut}(\mathbb{Z}/p^n\mathbb{Z})$. It was proved in [G-M-X1] that in general, the p -period of a p -periodic group Γ divides $2(p-1)p^n$ for some $n \geq 0$. For Γ_g one has a stronger result. It was shown in [G-M-X1] that the p -period of a p -periodic Γ_g actually always divides $2(p-1)$, and the exact value of the p -period

as a function of p and g was computed in an explicit way. Moreover, the values of g for which Γ_g is p -periodic were determined by Xia in [Xi2]. The results on the p -period of Γ_g may be summarized as follows:

(p-PERIOD)

Let $g > 1$. Then

a) Γ_g is never 2-periodic

b) for an odd prime p , Γ_g is p -periodic if and only if one of the following two condition holds:

b1) $g \not\equiv 1 \pmod{p}$

b2) g is of the form $kp + 1$ with $k \not\equiv 0, -1 \pmod{p}$ and the interval $[(2k + 3)/p, (2k + 2)/(p - 1)]$ does not contain any integer

c) The p -period of p -periodic Γ_g , denoted by $p(\Gamma_g)$, is given by

$$p(\Gamma_g) = \text{lcm}\{2[N(\pi) : C(\pi)] \mid \pi \in P\},$$

where π ranges over the set P of subgroups of order p in Γ_g , and $N(\pi)$ (respectively $C(\pi)$) denotes the normalizer (respectively centralizer) of π in Γ_g . In particular, the p -period $p(\Gamma_g)$ divides $2(p - 1)$.

We use the convention that $\text{lcm}\{2[N(\pi) : C(\pi)] \mid \pi \in P\} = 1$ in case P is the empty set; in that case $p(\Gamma_g) = 1$ too, according to our definition. It is possible to convert the general formula for $p(\Gamma_g)$ in an explicit formula in terms of g and p as follows. If $\pi \subset \Gamma_g$ is a subgroup of order p then, as discussed earlier, one can lift π to a subgroup of $\text{Diffeo}_+(S_g)$, and it is a classical result, that the number of fixed points $n(\pi)$ of this π -action on S_g does not depend on the lift chosen. It was proved in [G-M-X1] that for $g > 1$ one has

$$\text{lcm}\{2[N(\pi) : C(\pi)] \mid \pi \in P\} = \text{lcm}\{\text{gcd}(2(p - 1), 2n(\pi)) \mid \pi \in P\}.$$

According to Xia [Xi2], the numbers $n(\pi)$, which occur as cardinalities of such fixed-point sets, form a set denoted by $B_{g,p}$, which is given by the following formula. We may restrict to the case p odd, since Γ_g is never 2-periodic for $g > 1$. Write $2g - 2$ in the form $mp - i$ with $0 \leq i < p$. Then

$$B_{g,p} = \begin{cases} \{i, i + p, \dots, i + [2g/(p - 1) - m]p\}, & \text{if } i \not\equiv 1 \pmod{p} \\ \{1 + p, \dots, 1 + [2g/(p - 1) - m]p\}, & \text{if } i \equiv 1 \pmod{p}. \end{cases}$$

As usual, the notation $[x]$ stands for the integral part of the rational number x . We also use the natural convention that $\text{gcd}(2(p - 1), 0) = 2(p - 1)$ and, as before, the lcm of an empty set of numbers is understood to equal 1. For example, one easily checks that for $p = 3$ the sets $B_{g,p}$ contain always an even number. We also note, by using (KRULL-DIM I) or by checking the condition (b2) of (p-PERIOD), that for $g > 1$ and $g \equiv 1 \pmod{3}$, Γ_g is never 3-periodic. This shows that the following holds.

(3-PERIOD)

Suppose $g > 1$. Then Γ_g is 3-periodic if and only if $g \not\equiv 1 \pmod{3}$ and the 3-period of Γ_g is always 4.

It is easy to see from the definition of the p -period that a subgroup of a p -periodic group Γ is also p -periodic, with p -period dividing the p -period of Γ . As a result, one concludes for example:

- The groups $S(2,3) = \mathbb{Z}/9\mathbb{Z} \rtimes \mathbb{Z}$ cannot be embedded into any mapping class group Γ_g with $g \not\equiv 1 \pmod{3}$.

Indeed, using the same notation as earlier, $S(2,3)$ is 3-periodic with 3-period 12, which is larger than 4, the 3-period of a 3-periodic Γ_g . Returning to the case of an arbitrary prime p , we like to mention one more result, which follows from (p-PERIOD). Consider Γ_g with $g \equiv 1 \pmod{p}$. As mentioned earlier, there can only be a finite number of such Γ_g 's which are p -periodic (p a fixed prime). On the other hand, because S_g can be considered as an unramified covering space of S_h with h given by $p(2-2h) = 2-2g$, it follows that Γ_g contains a subgroup π of order p satisfying $n(\pi) = 0$, corresponding to a fixed-point-free action. From our formula for the p -period we infer thus:

- Suppose Γ_g is p -periodic and $g \equiv 1 \pmod{p}$. Then its p -period is $2(p-1)$.

We would like to conclude this Section by mentioning computations, which involve the mapping class groups $\Gamma_{(p-1)/2}$, respectively $\Gamma_{(p-1)}$, and which demonstrate the power of using p -periodicity and the p -period in the course of computing Farrell cohomology. For an odd prime p , the smallest value $g \geq 1$ such that Γ_g has p -torsion, is $g = (p-1)/2$, and $g = p-1$ is the second smallest such value. The Krull dimension at $p > 2$ is 1 for $\Gamma_{(p-1)/2}$ and Γ_{p-1} , so that these groups are p -periodic. The p -primary part of the Farrell cohomology has been completely computed for these two families of examples by Xia in his papers [Xi3-4].

4. Characteristic Classes for Group Representations

In Eckmann-Mislin [Eck-Mi1-5] characteristic classes of group representations where discussed in relationship with their field of definition. In our applications here we will mainly be concerned with representations defined over \mathbb{Q} . But first we shall recall some basic facts; a general reference on characteristic classes of representations is Thomas' book [Th1].

4.1 Chern classes. If $\rho : G \rightarrow Gl_n(\mathbb{C})$ denotes a complex representation of the (discrete) group G , then the induced map of classifying spaces

$$(B\rho)^* : H^*(BGl_n(\mathbb{C}); \mathbb{Z}) \rightarrow H^*(G; \mathbb{Z})$$

maps the universal Chern classes $c_i \in H^{2i}(BGl_n(\mathbb{C}); \mathbb{Z})$ to the Chern classes of ρ , defined by

$$c_i(\rho) := (B\rho)^* c_i \in H^{2i}(G; \mathbb{Z}).$$

To understand the Chern classes of representations of finite groups, it is useful to analyze first the case of cyclic groups, which we will quickly review. The boundary homomorphism associated to the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \xrightarrow{\text{exp}} Gl_1(\mathbb{C}) \rightarrow 1$$

induces an isomorphism

$$c_1 : \text{Hom}(\mathbb{Z}/n\mathbb{Z}, Gl_1(\mathbb{C})) \cong H^2(\mathbb{Z}/n\mathbb{Z}; \mathbb{Z}),$$

mapping a one dimensional representation to its first Chern class. Since every complex representation ρ of $\mathbb{Z}/n\mathbb{Z}$ decomposes as a sum $\oplus \rho_i$ of one dimensional representations, we can express the total Chern class

$$c(\rho) = 1 + c_1(\rho) + c_2(\rho) + \dots$$

as

$$c(\rho) = \prod c(\rho_i) = \prod (1 + c_1(\rho_i)).$$

Let $\phi(n)$ denote the number of generators of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ (the Euler ϕ -function). Note that there are $\phi(n)$ faithful irreducible \mathbb{C} -representations of $\mathbb{Z}/n\mathbb{Z}$, and their sum, which we denote by σ_n , is a representation of degree $\phi(n)$, which is defined over \mathbb{Z} ; the representation σ_n is sometimes called the *cyclotomic* representation, since as a \mathbb{Q} -representation it is equivalent to the one obtained from the Galois action of $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ on the cyclotomic extension of \mathbb{Q} gotten by adjoining a primitive n -th root of unity ζ_n . It is obvious that any faithful irreducible representation of $\mathbb{Z}/n\mathbb{Z}$ over \mathbb{Q} must involve σ_n and thus σ_n is characterized as being the smallest faithful irreducible \mathbb{Q} -representation of $\mathbb{Z}/n\mathbb{Z}$. Note that its top Chern class

$$c_{\phi(n)}(\sigma_n) \in H^{2\phi(n)}(\mathbb{Z}/n\mathbb{Z}; \mathbb{Z})$$

has (maximal) order n , because the total Chern class $c(\sigma_n)$ is given by

$$\prod_{j,(j,n)=1} (1 + jc_1(\sigma_n)) = 1 + \dots + \prod_{j,(j,n)=1} j \cdot c_1(\sigma_n)^{\phi(n)}.$$

Recall that E_{2m} denotes the denominator of $B_{2m}/2m$. From the well-known divisibility properties of Bernoulli numbers one can infer that a prime power $p^k > 1$ divides E_{2m} if and only if $(p-1)p^{k-1}$ divides $2mi$ (von Staudt's Theorem). Thus

$$E_{2m} = \text{lcm}\{n \mid 2m \equiv 0 \pmod{\phi(n)}\}.$$

Therefore, if $p^s > 1$ denotes the highest power of a prime p dividing E_{2m} , we can write $2m$ in the form $(p-1)p^{s-1} \cdot t$ with t prime to p . If σ denotes the representation given by taking a t -fold sum of the cyclotomic representation of $\mathbb{Z}/p^s\mathbb{Z}$ composed with the projection of $\mathbb{Z}/E_{2m}\mathbb{Z}$ onto $\mathbb{Z}/p^s\mathbb{Z}$, then $c_{2m}(\sigma)$ will have order p^s . Taking a sum of representations of this type, one for each prime divisor of E_{2m} , one ends up with a representation

$$\theta_m : \mathbb{Z}/E_{2m}\mathbb{Z} \rightarrow \text{Gl}(\mathbb{Z})$$

with $c_{2m}(\theta_m) \in H^{4m}(\mathbb{Z}/E_{2m}\mathbb{Z}; \mathbb{Z})$ of (maximal) order E_{2m} . The main result of [Eck-Mi1] says that this order is optimal for \mathbb{Q} -representations of *arbitrary* finite groups, in the following sense:

- If $\rho : F \rightarrow \text{Gl}(\mathbb{Q})$ denotes an arbitrary representation of a finite group F over \mathbb{Q} , then the order of $c_i(\rho)$ is at most two for i odd, and it divides E_i for $i > 0$ even.

The bound is actually also best possible for \mathbb{Q} -representations of infinite groups, up to possibly a factor 2. This follows from [Eck-Mi4] in conjunction with Arlettaz's result [Ar] who proved that the universal Chern classes restrict to torsion classes in the cohomology of the group $\text{Gl}(\mathbb{Q})$ considered as a discrete group (the complication to overcome is the fact that the integral homology of $\text{Gl}_n(\mathbb{Q})$ is not finitely generated); see also [Mis1] for a general torsion result on characteristic classes.

4.2 The Euler class. It is well-known that the integral cohomology ring of $BSl_n(\mathbb{R})$ is generated by elements of order two, the Euler class

$$e_n \in H^n(BSl_n(\mathbb{R}); \mathbb{Z}),$$

and the Pontrjagin classes

$$p_j \in H^{4j}(BSl_n(\mathbb{R}); \mathbb{Z}), \quad 2j \leq n.$$

For n odd, the Euler class has order 2. If n is even, say $n = 2m$, one has a relation of the form

$$e_{2m}^2 = p_m = (-1)^m \text{res}(c_{2m})$$

with $c_{2m} \in H^{4m}(BGl(\mathbb{C}); \mathbb{Z})$ the universal Chern class, and $\text{res}(c_{2m})$ the image under the restriction map induced by the inclusion of $Sl_{2m}(\mathbb{R})$ in $Gl(\mathbb{C})$. For any subring R of \mathbb{R} , we will write $e_n(R)$ for the restriction $\text{res}(e_n) \in H^n(BSl_n(R)^\delta; \mathbb{Z})$, where $BSl_n(R)^\delta$ stands for the classifying space of the group $Sl_n(R)$, considered as a discrete group. In case that R is a discrete subring of \mathbb{R} , we will omit the superscript δ ; a similar convention is used for the case of the Chern classes. The Euler class

$$e_{2m}(\mathbb{Q}) \in H^{2m}(BSl_{2m}(\mathbb{Q})^\delta; \mathbb{Z})$$

has infinite order (see Milnor [Miln]), but, according to Sullivan [Su]

- $e_{2m}(\mathbb{Z}) \in H^{2m}(BSl_{2m}(\mathbb{Z}); \mathbb{Z})$ is a torsion class.

The order of $e_{2m}(\mathbb{Z})$ is — the same number is coming up again — E_{2m} or $2E_{2m}$. For the restriction of $e_{2m}(\mathbb{Q})$ to the cohomology of a finite subgroup $Sl_{2m}(\mathbb{Q})$ it was proved in [Eck-Mi1] that the “universal bound” for the order is precisely E_{2m} :

- If ρ denotes a representation $F \rightarrow Sl_{2m}(\mathbb{Q})$ of a finite group F , then the order of $e_{2m}(\rho) := (B\rho)^* e_{2m}(\mathbb{Q})$ divides E_{2m} , and E_{2m} is the best universal bound for the order of the Euler class of such representations.

One should note that a finite subgroup of $Sl_n(\mathbb{Q})$ is not necessarily conjugate to a subgroup of $Sl_n(\mathbb{Z})$!

5. Torsion in Γ_g and the Homology Representation

Our goal in this Section is to improve the (TOR)–result concerning torsion in the integral cohomology of the mapping class group, stated in Section 3.3. The techniques we use are essentially the ones used in Glover–Mislin [Gl-Mi2], just a little bit refined.

5.1 Fixed point data. A basic invariant of an orientation preserving diffeomorphism of finite order $n > 1$, $f \in Diff_{eo_+}(S_g)$, is its *fixed point data*. It is defined as follows. Because f preserves orientation, the singular set of f , that is, the points $x \in S_g$ for which the orbit $\{f^k(x) | k \in \mathbb{Z}\}$ has fewer than n elements, is necessarily a finite set. Let $\{x_i\}$ be a set of representatives of the singular orbits of f and write n_i for the order of $stab_f(x_i)$, the stabilizer of $\langle f \rangle$ at $x_i \in S_g$, where $\langle f \rangle$ denotes the subgroup generated by f . Then f^{n/n_i} generates $stab_f(x_i)$ and, with respect to a fixed Riemannian structure, the differential of f^{n/n_i} acts by rotation on the tangent space at x_i . Let k_i be an integer such that $f^{k_i n/n_i}$ acts by rotation through $2\pi/n_i$. The number k_i is well defined modulo n_i , and k_i is prime to n_i . The fixed point data of f , denoted by $\delta(f)$, is then the collection

$$\delta(f) = \langle g, n | k_1/n_1, \dots, k_q/n_q \rangle$$

where g is the genus of the surface S_g , n the order of f , and q the number of singular orbits of the f -action; the numbers $k_1/n_1, \dots, k_q/n_q$ are unique up to order, if we choose k_i so that $1 \leq k_i < n_i$.

A classical theorem of Nielsen [Ni2] states that two diffeomorphisms of finite order are conjugate in $Diffeo_+(S_g)$ if and only if they have the same fixed point data. Symonds [Sy] proved that the fixed point data of a diffeomorphism of finite order depends only upon its isotopy class, that is, its image in Γ_g . By the classical case of the ‘‘Nielsen Realization Theorem’’ (cf. Fenchel [Fe]), every element of finite order of Γ_g can be represented by a diffeomorphism of the same order. It follows that one can define the fixed point data $\delta(x)$ for a torsion element $x \in \Gamma_g$ by putting $\delta(x) = \delta(f)$, where f denotes any lift of x to $Diffeo_+(S_g)$ of the same order. As a consequence we infer:

- Two elements of finite order in Γ_g are conjugate if and only if they have the same fixed point data.

If $x \in \Gamma_g$ has finite order and has fixed point data

$$\delta(x) = \langle g, n \mid k_1/n_1, \dots, k_q/n_q \rangle$$

and if f denotes a diffeomorphism of order n representing x , then the orbit space $S_g / \langle f \rangle$ is a surface S_h such that the natural projection $\pi : S_g \rightarrow S_h$ is an n -sheeted branched covering, with q branch points in S_h corresponding to the singular orbits of the action of the group $\langle f \rangle$ generated by f . The order of a branch point $P \in S_h$ is defined as $n/|\pi^{-1}(P)|$, and these orders correspond therefore to the orders of stabilizers, denoted by n_i above. From covering space theory one sees that the genus h is determined by g and the fixed point data $\delta(x)$ via the *Riemann–Hurwitz Relation*:

$$(R-H) \quad 2g - 2 = n((2h - 2) + \sum_{i=1}^q (1 - 1/n_i)).$$

REMARK. The branching orders can be computed by counting the numbers of fixed points of the different powers of the map f . The number of such fixed points can be computed from the representation of the group $\langle f \rangle \cong \mathbb{Z}/n\mathbb{Z}$ on $H_1(S_g; \mathbb{R})$, by the Lefschetz–Hopf trace formula. From character theory it is then plain that the conjugacy class of the image of f in $Gl_{2g}(\mathbb{R})$ with respect to this representation, is determined by the branching numbers $\{n_i\}$. We could also look at the conjugacy class of f in $Sp_{2g}(\mathbb{Z})$, by considering the action of f on the symplectic space $H_1(S_g; \mathbb{Z})$, the symplectic structure being given by intersection product. In case of $n = p$ a prime, this conjugacy question was analyzed by Edmonds and Ewing [E-E]. The fixed point data of an $x \in \Gamma_g$ of order p has the form

$$\delta(x) = \langle g, p \mid k_1/p, \dots, k_q/p \rangle$$

where $0 < k_i < p$; the number q here equals the cardinality of the fixed point set of any diffeomorphism of order p representing x , and k_i is called the *type* of the i 'th fixed point. Note that q can be computed as the *Lefschetz number* $\Lambda(x)$ of x , that is

$$\Lambda(x) = \sum (-1)^i \text{trace}(x_* : H_i(S_g; \mathbb{Q}) \rightarrow H_i(S_g; \mathbb{Q})) = q.$$

The fixed point types come up in the formula for the *signature* of x , given by

$$\text{sign}(x) = \sum_{0 < k < p} N_k (\zeta^k + 1) / (\zeta^k - 1)$$

with N_k denoting the number of fixed points of type k , and $\zeta = \exp(2\pi\sqrt{-1}/p)$. This signature invariant corresponds to the *equivariant signature* of the linear transformation induced by x on the hermitian space $H_1(S_g; \mathbb{C})$; see also Ewing's paper [Ew] for the relationship with the *Eichler Trace Formula*. The main result of [E-E] states that:

- Two elements of prime order in Γ_g have conjugate images in $Sp_{2g}(\mathbb{Z})$ if and only if they have the same Lefschetz number and signature.

Closely related results were also proved by Symonds [Sy].

5.2 The homology representation of Γ_g . We consider again the natural action of Γ_g on the symplectic space $H_1(S_g; \mathbb{Z})$, yielding the canonical representation

$$\rho_g : \Gamma_g \rightarrow Sp_{2g}(\mathbb{Z}).$$

We will write

$$c_i(\Gamma_g) \in H^{2i}(\Gamma_g; \mathbb{Z}),$$

respectively

$$e_{2g}(\Gamma_g) \in H^{2g}(\Gamma_g; \mathbb{Z})$$

for the images of the universal Chern classes, respectively the universal Euler class, under the restriction maps induced via

$$\Gamma_g \rightarrow Sp_{2g}(\mathbb{Z}) \subset Sl_{2g}(\mathbb{R}) \subset Gl(\mathbb{C}).$$

From our earlier discussions it is plain that the Chern classes $c_i(\Gamma_g)$ for $i > 0$, and the Euler class $e_{2g}(\Gamma_g)$, are torsion classes. We want to establish the following result concerning their order.

TOR-THEOREM. *Denote as before the denominator of $B_{2m}/2m$ by E_{2m} , so that $E_2 = 12$, $E_4 = 120$, $E_6 = 252$ and so on. Then the following holds.*

- (1) $c_2(\Gamma_1) \in H^4(\Gamma_1; \mathbb{Z})$ and $e_2(\Gamma_1) \in H^2(\Gamma_1; \mathbb{Z})$ both have order 12.
- (2) $c_4(\Gamma_2) \in H^8(\Gamma_2; \mathbb{Z})$ and $e_4(\Gamma_2) \in H^4(\Gamma_2; \mathbb{Z})$ both have order 120.
- (3) For $g > 2$, the order of $c_{2g}(\Gamma_g) \in H^{4g}(\Gamma_g; \mathbb{Z})$ and $e_{2g} \in H^{2g}(\Gamma_g; \mathbb{Z})$ is either E_{2g} or $2E_{2g}$.

PROOF. Because of the results on the order of $c_i(\mathbb{Z})$ and $e_n(\mathbb{Z})$ mentioned earlier, it suffices to show that $c_{2g}(\Gamma_g)$ has order at least E_{2g} . It follows then that $e_{2g}(\Gamma_g)$ has order at least E_{2g} , as

$$e_{2g}(\Gamma_g)^2 = (-1)^g c_{2g}(\Gamma_g).$$

Also, for $g \leq 2$ the computations concerning $H^*(\Gamma_g; \mathbb{Z})$ rule out the existence of elements of order $2E_{2g}$ in $H^*(\Gamma_g; \mathbb{Z})$. To get the general lower bound on the order of $c_{2g}(\Gamma_g)$, we proceed as follows. Suppose that p^β (with $\beta \geq 1$) is the largest power of a prime p , which divides E_{2g} . Then, by von Staudt's theorem, $p^{\beta-1}(p-1)$ divides $2g$, say

$$2g = l \cdot p^{\beta-1}(p-1).$$

Next, we construct a subgroup

$$\pi \subset \Gamma_g, \quad \pi \cong \mathbb{Z}/p^\beta \mathbb{Z}$$

such that the restriction $c_{2g}(\pi) \in H^{2g}(\pi; \mathbb{Z})$ of $c_{2g}(\Gamma_g)$ has order p^β . To this end, we consider the branched covering space

$$S_g \rightarrow S^2$$

with two branch points of order p^β and l of order p . We recall the classical construction of such a covering space. One begins by deleting $2 + l$ points from the 2-sphere to get

$$X = S^2 \setminus \{x_1, x_2, y_1, \dots, y_l\}$$

and chooses a suitable surjective homomorphism

$$\partial : \pi_1(X) \rightarrow \mathbb{Z}/p^n\mathbb{Z}.$$

To describe ∂ more explicitly, we choose a presentation of $\pi_1(X)$ of the form

$$\langle u_1, u_2, v_1, \dots, v_l \mid u_1 u_2 v_1 \dots v_l = 1 \rangle$$

and a generator $x \in \mathbb{Z}/p^\beta\mathbb{Z}$. Let's restrict to the case p odd and $\beta > 1$; the other cases are similar. Put $\partial(v_i) = y$ for $1 \leq i \leq l$, where y denotes a fixed element of $\mathbb{Z}/p^\beta\mathbb{Z}$ of order p , and define $\partial(u_1) = x$ respectively $\partial(u_2) = -(x + ly)$ so that ∂ is well defined and surjective. By compactifying the regular covering space associated to the kernel of ∂ , one obtains a branched covering $S_g \rightarrow S^2$, with genus g determined by the Riemann–Hurwitz relation (H–R),

$$2g - 2 = p^\beta(-2 + (2 - 2/p^\beta) + (l - l/p)),$$

yielding $2g = l(p - 1)p^{\beta-1}$, as desired. The associated covering transformation group is cyclic of order p^β and acts with $2 + l$ branch points, two of order p^β and l of order p . It defines therefore a subgroup $\pi \subset \Gamma_g$, generated by an element z with fixed point data

$$\delta(z) = \langle l(p - 1)p^{\beta-1}/2, p^\beta \mid k_1/p^\beta, k_2/p^\beta, k_3/p, \dots, k_{2+l}/p \rangle.$$

Consider now the representation of π gotten by the composite

$$\rho : \pi \subset \Gamma_g \rightarrow Sp_{2g}(\mathbb{Z}) \subset Gl_{2g}(\mathbb{C}).$$

We claim that ρ is equivalent the sum of l copies of the cyclotomic representation σ_{p^β} . This is established by comparing characters. It is easy to see that σ_{p^β} is induced from the reduced regular representation of a subgroup of order p (just use the fact that the cyclotomic representation is the unique faithful irreducible representation over \mathbb{Q}). Thus, by the well-known formula for the character of an induced representation, one infers that the character χ of σ_{p^β} is given by

$$\chi(w) = \begin{cases} 0, & \text{if } pw \neq 0 \\ -p^{\beta-1}, & \text{if } pw = 0 \text{ but } w \neq 0 \\ p^{\beta-1}(p - 1), & \text{if } w = 0. \end{cases}$$

Here w denotes a general element in $\mathbb{Z}/p^\beta\mathbb{Z}$. On the other hand, if \tilde{z} denotes a lift of order p^β in $Diff_{eo_+}(S_g)$ of the generator $z \in \pi \subset \Gamma_g$, then by construction \tilde{z} has precisely 2 fixed points (we are still assuming that $\beta > 1$). More generally we can say that if \tilde{z}^j has order greater than p , then it has precisely two fixed points, and when its order is p , it has $2 + lp^{\beta-1}$ fixed points. The Lefschetz number computes the number of fixed points $FP(\tilde{z}^j)$ of the map \tilde{z}^j , in case \tilde{z}^j is not the identity map, by

$$\Lambda(z^j) = 2 - \text{trace}(z_*^j \mid H_1(S_g; \mathbb{Q})) = FP(\tilde{z}^j).$$

We infer readily that

$$\text{trace}(z_*^j | H_1(S_g; \mathbb{Q})) = \begin{cases} 0, & \text{if } z^{pj} \neq 1 \\ -lp^{\beta-1}, & \text{if } z^{pj} = 1 \text{ but } z^j \neq 1 \\ 2g, & \text{if } z^j = 1. \end{cases}$$

Comparing with the computation for the trace of the cyclotomic representation, we see that ρ is equivalent to the sum of l copies of σ_{p^β} . Since the top Chern class of σ_{p^β} has order p^β (compare Section 4.1), we conclude that the top Chern class

$$c_{2g}(\rho) \in H^{4g}(\pi; \mathbb{Z})$$

is the l 'th power of an element of order p^β and has therefore order p^β . This implies that $c_{2g}(\Gamma_g)$ has order at least p^β , and we are done.

6. The Euler Characteristic

A group Γ is said to be of *finite homological type* if it has finite virtual cohomological dimension and if for every Γ -module M which is finitely generated as an abelian group, $H_i(\Gamma; M)$ is finitely generated for every i . For instance, if Γ has finite *vcd* and acts properly and simplicially on a finite dimensional simplicial complex, with compact quotient, then Γ is homologically of finite type. Thus, Γ_g is of finite homological type, as we see from its action on the extended Teichmüller space \overline{T}_g .

If Γ is an arbitrary group of finite homological type, its Euler characteristic is defined by

$$\chi(\Gamma) = \frac{\chi(\Delta)}{[\Gamma : \Delta]},$$

where $\Delta \subset \Gamma$ denotes a torsion-free subgroup of finite index $[\Gamma : \Delta]$ in Γ , and where

$$\chi(\Delta) = \sum (-1)^i \dim_{\mathbb{Q}} H_i(\Delta; \mathbb{Q})$$

is the topological Euler characteristic of the classifying space of Δ . One checks that $\chi(\Gamma)$ is well-defined (cf. Brown's book [**Brow1**]); for background and various results concerning the Euler characteristic, see also Brown [**Brow2-4**]. Notice that for a finite group F one has obviously

$$\chi(F) = \frac{1}{|F|},$$

and for a free group L of rank n , whose classifying space is homotopy equivalent to a wedge of n circles, one has

$$\chi(L) = 1 - n.$$

It follows then that for the case of $\Gamma_1 \cong Sl_2(\mathbb{Z})$, whose commutator subgroup is free of rank two and whose abelianized group is of order 12, the Euler characteristic is given by

$$\chi(Sl_2(\mathbb{Z})) = -\frac{1}{12}.$$

The denominator of the Euler characteristic is also in the general case closely linked to the torsion of the group. Indeed, a basic result [**Brow2**] is the following :

- If a prime power p^n divides the denominator of $\chi(\Gamma)$, then Γ possesses a subgroup of order p^n

One also defines the *naive* Euler characteristic of a group Γ of finite homological type by

$$\tilde{\chi}(\Gamma) = \sum (-1)^i \dim_{\mathbb{Q}} H_i(\Gamma; \mathbb{Q}).$$

Note that for the mapping class group $\tilde{\chi}(\Gamma_g)$ is just the topological Euler characteristic of the moduli space M_g . Brown proved in [Brow4] the following general relationship between χ and $\tilde{\chi}$:

- Let Γ be a group such that for all elements of finite order $x \in \Gamma$ the centralizers $C(x)$ have finite homological type. Then

$$\chi(\Gamma) = \tilde{\chi}(\Gamma) - \sum_{x \in T} \chi(C(x)),$$

where T denotes a set of representatives for the conjugacy classes of non-trivial torsion elements of Γ .

Using this result, and the fact that for an $x \in \Gamma_g$ of finite order the centralizer $C(x)$ can be identified with a mapping class group, Harer-Zagier [Ha-Za] were able to prove the following amazing result.

$$\text{(Ha-Za 1)} \quad \chi(\Gamma_g) = \frac{1}{2-2g} \zeta(1-2g), \quad g > 1.$$

It is convenient, and natural from the point of view of the formula (Ha-Za 1), to work with the *pointed* mapping class group $\tilde{\Gamma}_g$, which is defined to be the group of connected components of $\text{Diffeo}_+(S_g, s_0)$, the group of *pointed* orientation preserving diffeomorphisms of S_g . The obvious map

$$\tilde{\Gamma}_g \rightarrow \text{Aut}_+(\pi_1(S_g, s_0))$$

is an isomorphism, compare with (IV) of Section 1. Thus

$$\tilde{\Gamma}_1 \cong \Gamma_1 \cong \text{Sl}_2(\mathbb{Z})$$

and for $g > 1$ one has a natural short exact sequence

$$1 \rightarrow \pi_1(S_g, s_0) \rightarrow \tilde{\Gamma}_g \rightarrow \Gamma_g \rightarrow 1,$$

where we have identified the group of inner automorphisms of $\pi_1(S_g, s_0)$ with $\pi_1(S_g, s_0)$, a group with trivial center $g > 1$. It follows then readily, by taking in account that $\chi(\pi_1(S_g)) = 2 - 2g$, that

$$\chi(\tilde{\Gamma}_g) = (2 - 2g)\chi(\Gamma_g), \quad g > 1.$$

Thus we can rewrite (Ha-Za 1) to get

$$\text{(Ha-Za 2)} \quad \chi(\tilde{\Gamma}_g) = \zeta(1-2g), \quad g \geq 1.$$

Note that the formula is indeed also correct for $g = 1$, as both sides of the equation equal then $-\frac{1}{12}$. It is a classical result that the values of the ζ -function at negative integers can be expressed in terms of Bernoulli numbers as follows:

$$\zeta(1-2g) = -\frac{B_{2g}}{2g}, \quad g \geq 1.$$

Our computation of the order of $e_{2g}(\Gamma_g)$ shows now that the denominator

$$\text{den}(\chi(\tilde{\Gamma}_g)) = \text{den}\left(-\frac{B_{2g}}{2g}\right) = E_{2g}$$

corresponds up to possibly a factor two to the order of $e_{2g}(\Gamma_g)$:

- For $g \geq 1$ the denominator of $\chi(\tilde{\Gamma}_g)$ equals the order of $e_{2g}(\Gamma_g)$ or half that order.

In this statement, we could have used equally well the Euler class $e_{2g}(\tilde{\Gamma}_g)$ in place of $e_{2g}(\Gamma_g)$, i.e., the Euler class of the flat bundle induced by the composite map

$$\tilde{\Gamma}_g \rightarrow \Gamma_g \rightarrow Sp_{2g}(\mathbb{Z}) \subset Sl_{2g}(\mathbb{R}).$$

Indeed, the cyclic subgroups $\pi \subset \Gamma_g$ of order p^β , which were used to detect the order of $e_{2g}(\Gamma_g)$, are generated by an element which lifts to a periodic diffeomorphism of S_g with a fixed point, as we see by looking at its fixed point data. It follows that π lifts to $\tilde{\Gamma}_g$ and detects the order of $e_{2g}(\tilde{\Gamma}_g)$ as well. It would be good to have a more direct way of understanding the relationship between the Euler class and the Euler characteristic. Is the order of $e_{2g}(\Gamma_g)$ precisely the denominator of $\chi(\tilde{\Gamma}_g)$? For small values of g , the explicit computations show that this is indeed the case: $e_2(\Gamma_1)$ has order 12 and $e_4(\Gamma_2)$ has order 120; on the other hand,

$$\chi(\tilde{\Gamma}_1) = -\frac{1}{12}, \quad \text{and} \quad \chi(\tilde{\Gamma}_2) = \frac{1}{120}.$$

7. The Yagita Invariant of the Mapping Class Group

The invariant, which we call the *Yagita invariant*, was first introduced by Yagita [Ya] in the case of finite groups, and in Thomas [Th2] for more general groups. It is related to group actions on products of spheres in a similar way as finite groups with periodic cohomology are related to actions on spheres (a more precise statement is given below). We recall the definition of the Yagita invariant and some background.

Let Γ be a group of finite virtual cohomological dimension and $\pi \subset \Gamma$ any subgroup of prime order p . Because π injects into any finite quotient of the form Γ/Δ , where Δ is a torsion-free normal subgroup of finite index in Γ , the image $\text{Im}(H^k(\Gamma; \mathbb{Z}) \rightarrow H^k(\pi; \mathbb{Z}))$ of the restriction map in cohomology is non-zero for some degree $k > 0$. Reduction mod- p maps $H^*(\pi; \mathbb{Z})$ onto $\mathbb{F}_p[u] \subset H^*(\pi; \mathbb{F}_p)$ with u a generator in $H^2(\pi; \mathbb{F}_p)$. Thus, there exists a maximum value $m = m(\pi, \Gamma)$ such that

$$\text{Im}((H^*(\Gamma; \mathbb{Z}) \rightarrow H^*(\pi; \mathbb{F}_p))) \subset \mathbb{F}_p[u^m] \subset H^*(\pi; \mathbb{F}_p).$$

Note that $m(\pi, \Gamma)$ is bounded by $m(\pi, \Gamma/\Delta)$, where Δ denotes as before a torsion-free normal subgroup of finite index. Since Γ/Δ is finite, we conclude that $m(\pi, \Gamma)$ is bounded by a bound depending on Γ only. The Yagita invariant $p(\Gamma)$ of Γ with respect to the prime p is then defined to be the least common multiple of values $2m(\pi)$, where π ranges over all subgroups of order p of Γ . We use the convention that $p(\Gamma) = 1$ if Γ is p -torsion-free. The invariant $p(\Gamma)$ agrees with the p -period of a p -periodic group and one can show [G-M-X2] that $p(\Gamma)$ has in general the form $l \cdot p^k$ with l dividing $2(p-1)$; in particular, for the prime 2 the Yagita invariant is a power of 2.

The interest in $p(\Gamma)$ stems from the fact that it provides a lower bound for the dimension of a complex, which admits a certain type of action of Γ . For instance,

using the same reasoning as in [Ya], where only finite groups were considered, one finds that if Γ acts properly discontinuously on $\mathbb{R}^n \times (S^m)^k$ and trivially on $H^*(\mathbb{R}^n \times (S^m)^k; \mathbb{Z})$, in a way that the stabilizer of any point $x \in \mathbb{R}^n \times (S^m)^k$ is a p -torsion-free group, then $m + 1$ is a multiple of the Yagita invariant $p(\Gamma)$.

The mapping class group Γ_g is never 2-periodic for $g > 1$, since the Krull dimension $\kappa(\Gamma_g, 2)$ is at least two. For an odd prime p and p -periodic Γ_g we have described $p(\Gamma_g)$ earlier. We recall that for an odd prime p and genus $g \not\equiv 1 \pmod{p}$, Γ_g is always p -periodic; thus for the discussion of the Yagita invariant of Γ_g we will only need to be concerned with the case $g \equiv 1 \pmod{p}$. In [G-M-X2] a complete result is given in case p is an odd regular prime, and partial results for general primes. Recall that a prime p is called *regular* if it does not divide the class number of the cyclotomic field $\mathbb{Q}(\exp(2\pi\sqrt{-1}/p))$. A famous criterion of Kummer states that a prime p is regular if and only if p does not divide the numerator of any Bernoulli number B_{2i} with $2 \leq 2i < p - 1$. Thus 691, the numerator of B_{12} , is an example of an *irregular* prime; the first three irregular primes are 37, 59 and 67. The following terminology was introduced in [G-M-X2].

- Let p be a prime. We say that an integer g satisfies the (p) -condition if and only if g is of the form $lp^\alpha + 1$ with l prime to p , $\alpha > 0$, and $2l = p(2h - 2) + k(p - 1)$ for some integers $h > 0$, $k \geq 0$ with $k \neq 1$.

We can now state the main results concerning the Yagita invariant of the mapping class group, as proved in [G-M-X2].

(Y-INVARIANT 1)

Let p be an odd regular prime and assume that $g = lp^\alpha + 1$ with l prime to p and $\alpha > 0$. Then the Yagita invariant $p(\Gamma_g)$ is determined as follows.

- If g does not satisfy the (p) -condition, then $p(\Gamma_g)$ equals $2(p - 1)p^{\alpha-1}$.
- If g satisfies the (p) -condition, then $p(\Gamma_g)$ equals $2(p - 1)p^\alpha$.

For the case of a general odd prime, only a partial result is available, which however underlines the role of the (p) -condition.

(Y-INVARIANT 2)

Let p be an odd prime and $g = lp^\alpha + 1$ with l prime to p and $\alpha > 0$. Then the following holds.

- $p(\Gamma_g)$ has the form $2(p - 1)p^\alpha$ or $2(p - 1)p^{\alpha-1}$.
- If g satisfies the (p) -condition, then $p(\Gamma_g) = 2(p - 1)p^\alpha$.
- If $1 < 2l < p - 1$ then $p(\Gamma_g) = 2(p - 1)p^{\alpha-1}$.

For the prime 2, the following result on the Yagita invariant is due to Xia [Xi5].

- For even genus g the Yagita invariant $p(\Gamma_g)$ at the prime 2 equal 4.

We want to sketch the strategy involved in the proofs concerning the Yagita invariant of Γ_g . As explained, we can assume that p is an odd prime and $g = lp^\alpha + 1$, $\alpha > 0$ and l prime to p .

First step:

- Show that $p(\Gamma_g)$ is of the form $2(p - 1)p^\beta$ with $\beta \geq \alpha - 1$.

For this, one observes that Γ_g contains a subgroup π isomorphic to $\mathbb{Z}/p^\alpha\mathbb{Z}$ suitable to provide a lower bound for $p(\Gamma_g)$. Indeed, the covering transformation group in the cyclic p^α -sheeted unramified covering space

$$S_{lp^\alpha+1} \rightarrow S_{l+1},$$

which one can construct by mapping $\pi_1(S_{l+1})$ onto $\mathbb{Z}/p^\alpha\mathbb{Z}$, is suitable. All generators of that group $\pi \subset \Gamma_g$ have the same fixed-point data, and they are therefore conjugate in Γ_g . This implies that the restriction map

$$H^*(\Gamma_g; \mathbb{Z}) \rightarrow H^*(\pi; \mathbb{F}_p)$$

is zero in dimensions not divisible by $2(p-1)p^{\alpha-1}$, by looking at the action of $\text{Aut}(\pi)$ on the cohomology of π . Therefore, $p(\Gamma_g)$ is a multiple of $2(p-1)p^{\alpha-1}$. On the other hand, as mentioned earlier, the Yagita invariant of any group of finite vc d is always a factor of $2(p-1)p^n$ for some $n \geq 0$, and the result follows.

Second step:

- Show that $p(\Gamma_g)$ divides $2(p-1)p^\alpha$.

It suffices to show that for every subgroup $\pi \subset \Gamma_g$ of order p one can find a number $j(\pi)$ prime to p such that the restriction map

$$H^{2j(\pi)p^\alpha}(\Gamma_g; \mathbb{Z}) \rightarrow H^{2j(\pi)p^\alpha}(\pi; \mathbb{Z})$$

is non-trivial. This can be achieved, for details consult [G-M-X2]. Besides of studying the restriction of various characteristic classes, one also makes use of the action of Γ_g on the complement of the union of the singular sets in Teichmüller space of the actions of all subgroups of Γ_g which are conjugate to π . This complement is a smooth non-compact manifold, which is used to construct a cohomology element in $H^*(\Gamma_g)$, whose restriction to the cohomology of π is non-zero.

Third step:

- Settle the case when g satisfies the p -condition.

The p -condition is just the condition needed to be able to construct a subgroup $\mathbb{Z}/p^{\alpha+1}\mathbb{Z}$ in Γ_g , with a fixed point data for a generator x to be of the form

$$\delta(x) = \langle g, p^{\alpha+1} \mid k_1/p, \dots, k_l/p \rangle,$$

a fixed point data which can be used in a manner similar to the argument in step one to show that $p(\Gamma_g)$ is a multiple of p^α . It follows then that $p(\Gamma_g)$ must be equal to $2(p-1)p^\alpha$.

Fourth step:

- Show that if p is a regular prime and g does not satisfy the p -condition, then $p(\Gamma_g) < 2(p-1)p^\alpha$.

At this point we know that $p(\Gamma_g)$ is either equal to $2(p-1)p^\alpha$ or $2(p-1)p^{\alpha-1}$. It was proved in [G-M-X2] that for every subgroup π of order p in Γ_g , p a regular prime, one can find a symplectic characteristic class $d_i(\Gamma_g)$ which restricts non-trivially to $H^*(\pi; \mathbb{Z})$ and which satisfies $0 < i < p^\alpha$. This implies that $p(\Gamma_g) \neq 2(p-1)p^\alpha$ and thus $p(\Gamma_g) = 2(p-1)p^{\alpha-1}$. The regularity condition on p is used as follows. The case of $p = 3$ can be checked directly, so we can assume $p \geq 5$. To get the non-vanishing of the symplectic characteristic class one is led to consider the $(p-1)/2 \times (p-1)/2$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 2 & 4 & 6 & \dots & p-1 \\ 3 & 6 & 9 & \dots & \\ \vdots & \vdots & & & \\ \frac{p-1}{2} & p-1 & \dots & & \end{pmatrix}.$$

with the (i, j) entry in the rows below the first row being the the smallest positive residue of the product $ij \bmod p$. This matrix, which is very closely related to the matrix considered by Carlitz and Olson in [Ca-Ol] turns out to have

$$|\det(A)| = 2p^{(p-5)/2} \cdot h_1, \quad p \geq 5,$$

where h_1 denotes a well understood divisor of the class number of the field $\mathbb{Q}(\exp(2\pi\sqrt{-1}/p))$. It is well-known that p is irregular if and only if p divides h_1 , which is usually called the first factor of the class number of $\mathbb{Q}(\exp(2\pi\sqrt{-1}/p))$. But the argument concerning the symplectic characteristic classes requires that

$$p^{(p-3)/2} \nmid \det A,$$

which is equivalent with the condition that p be regular.

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