On the classifying space for proper actions

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Abstract. We discuss conditions under which the universal proper G-CW-complex $\underline{E}G$ can be chosen to be finite dimensional. The methods we use stem from a general construction introduced in [9], involving spaces parameterized by a partially ordered set. In particular we present a construction, which turns a G-CW-complex X in a canonical way into a proper G-CW-complex $Pr(X)$ of the same homotopy type, with control on the dimension of the new space.

1. Introduction

Let G be an arbitrary discrete group. There exists up to G-homotopy a unique G-CW-complex $\underline{E}G$ such that the fixed point space $\underline{E}G^H$ is contractible for every finite $H < G$, and empty for infinite H. A G-CW-complex is called *proper* if all point stabilizers are finite (equivalently, if all its G-cells are of the form $G/H \times \sigma$ with H finite in G). The space $\underline{E}G$ is an example of a proper G-CW-complex; it is sometimes referred to as the *classifying space for proper actions*, because it enjoys the following universal property:

- For any proper G-CW-complex X there is a unique G-homotopy class of G-maps $X \to \underline{E}G$.

The following is an explicit description of a standard model for $\underline{E}G$. Let $F(G)$ denote the G-poset of finite non-empty subsets of G, the partial order being the obvious one and the G-action given by left translation. It follows that the geometric realization $|F(G)|$ of $F(G)$ is a G-CW-complex of type $\underline{E}G$. Of course $|F(G)|$ is infinite dimensional for G an infinite group. As we are only interested in $\underline{E}G$ up to G-homotopy, the following definition is useful.

Definition 1.1. *We write* $\dim_G \underline{E}G$ *for the smallest dimension in* $\mathbb{N} \cup \{\infty\}$ *of a model for* $\underline{E}G$.

Clearly $\dim_G \underline{E}G = 0$ if and only if G is finite. In case G is torsion-free, $\underline{E}G = \tilde{K}(G,1)$, the universal cover of a $K(G,1)$. It follows that for a torsion-free group, $\dim_G \underline{E}G < \infty$ is equivalent to $\operatorname{cd} G < \infty$. Since for a general G the space $\underline{E}G$ may be considered as an $\underline{E}H$ for $H < G$, the condition $\dim_G \underline{E}G < \infty$ implies that all torsion-free subgroups of G have finite cohomological dimension, universally bounded by $\dim_G \underline{E}G$, because the cellular chain complex of $\underline{E}G$ yields a free $\mathbb{Z}[H]$-resolution of $\mathbb{Z}$ for each torsion-free subgroup H of G.

A well-known theorem states that a group of finite vcd (virtual cohomological dimension) admits a finite dimensional $\underline{E}G$. The precise relationship between $\operatorname{vcd} G$ and $\dim_G \underline{E}G$ is unknown. The following is a standard conjecture (cf. K. S. Brown [4]).

Conjecture 1.2. *If* $2 < \operatorname{vcd} G < \infty$ *then* $\operatorname{vcd} G = \dim_G \underline{E}G$.

The conjecture holds in the torsion-free case, as is well-known. We excluded the case of $\operatorname{vcd} G = 2$ because there is an example of a group G with $\operatorname{vcd} G = 2$ and $\dim_G \underline{E}G = 3$ (cf. [2]). It is obvious that for a group of finite vcd one has $\operatorname{vcd} G \leq \dim_G \underline{E}G$, since for a torsion-free subgroup $H < G$ of finite index $\operatorname{cd} H = \operatorname{vcd} G$, and $\dim_G \underline{E}G$ is an upper bound for $\operatorname{cd} H$ as we remarked earlier.

A case for which the minimal dimension of $\underline{E}G$ is well understood is when $\dim_G \underline{E}G = 1$, that is an infinite group acting properly on a tree. Indeed the following theorem holds (cf. [7]).

Theorem 1.3. *For an arbitrary group G the following two conditions are equivalent:*

- $\dim_G \underline{E}G = 1$;
- $\operatorname{cd}_{\mathbb{Q}} G = 1$.

It is always true that $\operatorname{cd}_{\mathbb{Q}} G \leq \dim_G \underline{E}G$ because, upon tensoring with $\mathbb{Q}$, the cellular chain complex of $\underline{E}G$ yields a $\mathbb{Q}[G]$-projective resolution of $\mathbb{Q}$ of length $\dim_G \underline{E}G$. However, the inequality can in general not be replaced by an equality, because according to Bestvina and Mess there exist a torsion-free negatively curved group G with $\operatorname{cd}_{\mathbb{Q}} G < \operatorname{cd} G$ (cf. [1]).

A basic unsolved problem, which served as the main motivation for this note, is the following one.

Problem 1.4. *Let* $K \to G \to Q$ *be a short exact sequence of groups. If* $\dim_Q \underline{E}Q$ *and* $\dim_K \underline{E}K$ *are both finite, is* $\dim_G \underline{E}G$ *finite too?*

In the sequel we will give a partial positive answer to this question. Some of the results presented here have also been obtained by Lück [10] with other techniques.

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2. Spaces parameterized by a poset

We briefly recall a construction introduced in [9]. Let Λ be a G-poset (i.e., a partially ordered set on which the group G acts in an order preserving way). Let X be a G-CW-complex and $f : X \to \Lambda$ a (continuous) G-map, with Λ considered as a discrete topological space. Write $X(\lambda) \subset X$ for the preimage of $\lambda \in \Lambda$, and for

$$\underline{\lambda} := \lambda_0 < \lambda_1 < \ldots < \lambda_n$$

a chain of length n in Λ, put

$$X(\underline{\lambda}) = \prod_{i=0}^{i=n} X(\lambda_i).$$

The *geometric realization* $|f|$ of f is a G-CW-complex of the form

$$|f| := \Big(\coprod_{n\in\mathbb{N}} \Big(\coprod_{\underline{\lambda}\in\Lambda^{(n)}} \sigma^n \times X(\underline{\lambda})\Big)\Big)/\sim$$

with $\Lambda^{(n)}$ the set of chains of lengths n in Λ and σ^n the standard n-simplex (for details see [9]). The assignment $f \mapsto |f|$ defines a functor from the category of G-CW-complexes over Λ to the category of G-CW-complexes. It maps the terminal object $\mathrm{Id}_\Lambda : \Lambda \to \Lambda$ to the usual geometric realization, $|\mathrm{Id}_\Lambda| = |\Lambda|$, of the poset Λ. Also, the morphism $f \to \mathrm{Id}_\Lambda$ gives rise to a canonical G-map $|f| \to |\Lambda|$, which is a homotopy equivalence in case all spaces $X(\lambda)$ are contractible. In [9] the poset $\Lambda(G)$ of non-trivial finite subgroups of G was used to construct a model for $\underline{E}G$. This poset is closely related to the *singular part* $(\underline{E}G)^{\mathrm{sing}}$ of $\underline{E}G$, consisting of all points with non-trivial isotropy. This can be seen in the framework of our *spaces parameterized by posets* as follows. If one takes X to be the disjoint union of G-CW-complexes of the form $G \times_N \underline{E}G^H$ with N the normalizer of $H < G$, the union being taken over a set of orbit representatives of points H in $\Lambda(G)$, then there is an obvious G-map

$$f : X = \coprod G \times_N \underline{E}G^H \longrightarrow \Lambda(G)$$

for which each point-preimage is contractible. Thus the natural map

$$|f| \to |\Lambda(G)|$$

is a G-map and a homotopy equivalence. We claim that $|f|$ is G-homotopy equivalent to $(\underline{E}G)^{\mathrm{sing}}$ so that the following holds.

Lemma 2.1. *For an arbitrary group G the space $(\underline{E}G)^{sing}$ is homotopy equivalent to $|\Lambda(G)|$ and G-homotopy equivalent to $\underline{E}G \times |\Lambda(G)|$, with diagonal G-action.*

Proof. We use the well-known fact that a G-map between G-CW-complexes is a G-homotopy equivalence if it induces for every $H < G$ an ordinary homotopy equivalence between H-fixed point spaces. Consider the map $\alpha : |f| \to |\Lambda(G)|$ constructed above. It is a homotopy equivalence on the fixed point spaces for any finite subgroup $H < G$ (see [9, Section 9]). Because the G-action on $|f|$ is proper, there is a canonical classifying G-map $\beta : |f| \to \underline{E}G$, and it follows that

$$\{\alpha, \beta\} : |f| \to |\Lambda(G)| \times \underline{E}G$$

is a G-homotopy equivalence. Moreover the G-map $|f| \to \underline{E}G$ maps into $(\underline{E}G)^{\mathrm{sing}}$ by a G-homotopy equivalence

$$|f| \xrightarrow{\simeq} (\underline{E}G)^{\mathrm{sing}},$$

because for each $H < G$ the map $|f|^H \to \underline{E}G^H$ is a homotopy equivalence (the spaces in question are both either contractible or empty). □

Remark 2.2. A different proof of Lemma 2.1, in the special case when there is a bound on the orders of the finite subgroups of G, can be found in [5, Lemma 2.4].

Note also that Lemma 2.1 has the following obvious consequences, which are useful in studying $\underline{E}G$:

- $H_i(|\Lambda(G)|;\mathbb{Z}) = 0$ for $i > \dim_G \underline{E}G$;
- $H^*_G(|\Lambda(G)|;M) \cong H^*_G((\underline{E}G)^{\text{sing}};M)$ for any G-module M. (The *Borel Cohomology* $H^*_G(X;M)$ of a G-space X with coefficients in the G-module M is the cohomology of the cochain complex $\mathrm{Hom}_G(C_*(\tilde{K}(G,1) \times X), M))$.

In [9] the poset $\Lambda(G)$ of non-trivial finite subgroups of G was used as the starting point for the construction of $\underline{E}G$. One of the problems encountered in using $\Lambda(G)$ was caused by the fact that $|\Lambda(G)|$ is in general not contractible. In the next section we will show how to construct $\underline{E}G$ out of suitable *contractible* posets.

3. Turning G-actions into proper G-actions

A G-CW-complex is called *simplicial*, if it is a simplicial complex with G acting simplicially, such that if $g \in G$ maps a simplex to itself, it fixes it pointwise. The standard model for $\underline{E}G$ described in Section 1 is an example of a simplicial G-CW-complex.

It is an elementary fact that a general G-CW-complex is G-homotopy equivalent to a simplicial one of the same dimension. For technical reason, it is often more convenient to work with *simplicial* G-CW-complexes rather than with general G-CW-complexes. The following example shall illustrate our point. Let X be a G-CW-complex of dimension d and suppose G is a subgroup of finite index in a larger group L. The L-space $map_G(L,X)$ has a standard CW-structure with cellular L-action (see the discussion of *Serre's Theorem* in Brown's book [3]). Although this space $map_G(L,X)$ is homeomorphic in the compactly generated topology to a product of $[L:G]$ copies of X, it is *not* an L-CW-complex. However, it is L-homotopy equivalent to an L-CW-complex of dimension $d \cdot [L:G]$. To prove this, we replace X by a simplicial G-CW-complex Y of the same dimension and same G-homotopy type, and observe that it is easy to define a simplicial structure on $map_G(L,Y)$ such that it is a simplicial L-CW-complex, of the L-homotopy type of $map_G(L,X)$. One also checks that if Y is an $\underline{E}G$, then $map_G(L,Y)$ is an $\underline{E}L$.

Definition 3.1. *Let X be a simplicial G-CW-complex. We write $Po(X)$ for the associated G-poset, whose elements are the simplices of X and whose partial order is given by the inclusion relation between simplices.*

The geometric realization $|Po(X)|$ of $Po(X)$ is a simplicial G-CW-complex G-homeomorphic to X, with simplicial structure corresponding to the barycentric subdivision of X. We will use the poset $Po(X)$ to turn X into a proper G-CW-complex, by replacing each G-orbit of a simplex $\sigma \subset X$ by the proper G-space $G \times_{G(\sigma)} \underline{E}G(\sigma)$, with $G(\sigma)$ the stabilizer of σ. The precise construction is as follows.

Definition 3.2. *Let X be a simplicial G-CW-complex and write $G(\sigma)$ for the stabilizer of a cell σ. Then $Pr(X)$ denotes the proper G-CW-complex obtained as the geometric realization $|f|$ of the following map $f : Y \to Po(X)$. Choose a model $\underline{E}G(\sigma)$ for each $\sigma \in \Sigma_0$, where Σ_0 denotes a set of representatives of the G-orbits of simplices in X, and put Y to be the disjoint union*

$$\coprod_{\sigma \in \Sigma_0} G \times_{G(\sigma)} \underline{E}G(\sigma).$$

The map $f : Y \to Po(X)$ is now given by $f(g \times x) = g \cdot \sigma$, where x lies in $\underline{E}G(\sigma)$.

Corollary 3.3. *Let X be a simplicial G-CW-complex. Then*

- *the natural G-map $Pr(X) \to X$ induces a homotopy equivalence of fixed point spaces $Pr(X)^H \to X^H$ for any finite subgroup $H < G$.*
- *$Pr(X) = \underline{E}G$ if X^H is contractible for each finite subgroup $H < G$.*

Proof. The first assertion follows from [9, Lemma 8.7], using the fact that $|Po(X)^H|$ is homeomorphic to X^H. The second one follows then from the first, since $Pr(X)$ is a proper G-CW-complex. □

As a result, we obtain the following general theorem.

Theorem 3.4. *Let X be a finite dimensional G-CW-complex such that for each finite $H < G$ the fixed point space X^H is contractible. Suppose there is a universal bound $b \in \mathbb{N}$ on $\dim_{G(x)} \underline{E}G(x)$, where x runs over the vertices of X. Then*

$$\dim_G \underline{E}G < \infty.$$

Proof. We may assume that X is a simplicial G-CW-complex. The space $Pr(X)$ is then a G-CW-complex of the G-homotopy type of $\underline{E}G$ and of dimension bounded by $(b+1)(\dim X + 1) - 1$. □

4. Applications

We now turn to Problem 1.4 concerning group extensions. Recall that $\mathbf{H}_1\mathcal{F}$ stands for the class of groups which admit a finite dimensional contractible G-CW-complex with finite cell stabilizers. For example if G admits a finite dimensional $\underline{E}G$, it certainly belongs to $\mathbf{H}_1\mathcal{F}$; the converse is an open question! The class of groups $\mathbf{H}\mathcal{F}$ of *hierarchically decomposable* groups is defined to be the smallest class containing $\mathbf{H}_1\mathcal{F}$ such that a group G belongs to $\mathbf{H}\mathcal{F}$ if G admits a finite dimensional contractible G-CW-complex with cell stabilizers in $\mathbf{H}\mathcal{F}$ (for a general account of hierarchically decomposable groups the reader is referred to [8]).

Theorem 4.1. *Let $K \to G \to Q$ be a group extension with $\pi : G \to Q$ the projection. Then the following holds:*

- *if* $\underline{E}Q$ *is finite dimensional and if there is a universal bound on the dimension of* $\underline{E}\pi^{-1}\pi(H)$ *where* H *ranges over all finite subgroups of* G*, then* G *admits a finite dimensional* $\underline{E}G$.
- *if* $Q \in \mathbf{H}_1\mathcal{F}$ *and if for each finite subgroup* $H < G$ *there is a finite dimensional contractible* $\pi^{-1}\pi(H)$*-CW-complex of dimension bounded by a number independent of* H*, then* $G \in \mathbf{H}_1\mathcal{F}$.

Proof. Take $X = \underline{E}Q$ and consider it as a G-space via $\pi : G \to Q$. Then for each finite $H < G$ the space $X^H = X^{\pi(H)}$ is contractible and the result follows from 3.4. The second case is proved similarly, using for X a finite dimensional contractible Q-CW-complex instead of $\underline{E}Q$. □

Corollary 4.2. *Let* $K \to G \to Q$ *be a short exact sequence of groups and assume that* $Q \in \mathbf{H}_1\mathcal{F}$ *with* Q *admitting a bound on the order of its torsion subgroups. Then the following holds:*

- *if* $K \in \mathbf{H}_1\mathcal{F}$*, then* $G \in \mathbf{H}_1\mathcal{F}$;
- *if* $\dim_K \underline{E}K < \infty$ *then* $\dim_G \underline{E}G < \infty$.

Proof. Take a finite subgroup $H < G$ and write $\pi : G \to Q$ for the projection. Put $L = \pi^{-1}\pi(H)$ and consider the short exact sequence

$$K \to L \to \pi(H).$$

If $K \in \mathbf{H}_1\mathcal{F}$ then so is L, with associated contractible L-CW-complex of the form $map_K(L, Y)$, where Y is some finite dimensional contractible K-CW-complex. It follows that $map_K(L, Y)$ is of universally bounded dimension, because there is a bound on the index $[L : K]$ which is independent of H. Thus $G \in \mathbf{H}_1\mathcal{F}$ by 4.1. The proof of the second case is analogous; one uses that $Q \in \mathbf{H}_1\mathcal{F}$ together with the bound on the order of the torsion subgroups of Q implies that $\dim_Q \underline{E}Q < \infty$ (Corollary B of [9]). □

Using a recent result of W. Dicks and P. Kropholler [6] one obtains examples of large torsion groups in $\mathbf{H}_1\mathcal{F}$.

Lemma 4.3. (Dicks-Kropholler) *Let* G *be a locally finite group of cardinality less than* $\aleph_\omega$*. Then* G *admits a finite dimensional* $\underline{E}G$.

Combining 4.2 with 4.3 leads us to

Theorem 4.4. *Let* $K \to G \to Q$ *be a short exact sequence of groups. Assume that there is a bound on the order of the torsion subgroups of* Q *and assume that* K *is a locally finite group of cardinality* $< \aleph_\omega$*. Then the following holds:*

- *if* $Q \in \mathbf{H}_1\mathcal{F}$*, then* $G \in \mathbf{H}_1\mathcal{F}$;
- *if* $\dim_Q \underline{E}Q < \infty$ *then* $\dim_G \underline{E}G < \infty$.

As an example, the theorem can be applied to soluble as follows.

Corollary 4.5. *Let* G *be a soluble group of cardinality* $< \aleph_\omega$*. Then the following conditions on* G *are equivalent:*

(1) *G has finite torsion-free rank (Hirsch number);*
(2) $\dim_G \underline{E}G < \infty$;
(3) $\mathrm{cd}_{\mathbb{Q}} G < \infty$.

Proof. It follows from the well-known structure for soluble groups of finite torsion-free rank that there is a short exact sequence

$$K \to G \to Q$$

with K locally finite and Q of finite vcd. Thus $\dim_Q \underline{E}Q < \infty$ and the previous theorem shows that (1)⇒(2). Clearly (2)⇒(3), since one always has $\mathrm{cd}_{\mathbb{Q}} G \leq \dim_G \underline{E}G$. Finally, by a theorem of Stammbach [11] the torsion-free rank of a soluble group equals its (weak) homological dimension over $\mathbb{Q}$ and is therefore bounded by $\mathrm{cd}_{\mathbb{Q}} G$, whence (3)⇒(1). □

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