FROM ACYCLIC GROUPS
TO
THE BASS CONJECTURE FOR AMENABLE GROUPS

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Abstract. We prove that the Bost Conjecture on the ℓ^1-assembly map for countable discrete groups implies the Bass Conjecture. It follows that all amenable groups satisfy the Bass Conjecture.

1. Introduction

Throughout this paper, let G be a discrete group. For each finitely generated projective (left) $\mathbb{Z}G$-module P, there exists an idempotent matrix $(m_{ij}) = M \in M_n(\mathbb{Z}G)$ such that P is isomorphic to the image under right multiplication $\mathbb{Z}G^n \to \mathbb{Z}G^n$ by M. Writing $[\mathbb{Z}G, \mathbb{Z}G]$ for the additive subgroup of $\mathbb{Z}G$ generated by the elements $gh - hg$ $(g, h \in G)$, we identify $\mathbb{Z}G/[\mathbb{Z}G, \mathbb{Z}G]$ with $\bigoplus_{[s]\in[G]} \mathbb{Z} \cdot [s]$, where $[G]$ is the set of conjugacy classes of elements of G. The *Hattori-Stallings rank* r_P is then defined by

$$r_P = \sum_{i=1}^{n} m_{ii} + [\mathbb{Z}G, \mathbb{Z}G] = \sum_{[s]\in[G]} r_P(s)[s] \in \bigoplus_{[s]\in[G]} \mathbb{Z} \cdot [s].$$

In his seminal paper [1], H. Bass made the following conjecture.

Conjecture 1.1 (Classical Bass Conjecture)**.** *For any finitely generated projective $\mathbb{Z}G$-module P, the values $r_P(s) \in \mathbb{Z}$ of the Hattori-Stallings rank r_P are zero for $s \in G \setminus \{1\}$.*

One of the striking results of [1] is a proof of this conjecture for the case of torsion-free linear groups (for the case of arbitrary linear groups see [29]). Using methods of cyclic homology, Eckmann in [17] and Emmanouil in [19] proved the conjecture for many more groups (for a recent survey see Eckmann [18]). The case of solvable groups was settled only very recently by Farrell and Linnell [20], where they prove the classical Bass conjecture for all elementary amenable groups.

We show below that groups that satisfy the *Bost conjecture* satisfy the classical Bass conjecture too, and indeed more general versions thereof that we call the ℓ^1 Bass conjecture (2.2) and $\mathbb{C}G$ Bass conjecture. Combining with known information concerning the Bost conjecture gives the result of the title.

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Theorem 1.2. *Amenable groups satisfy the classical Bass conjecture.*

The class of amenable groups includes the class of elementary amenable groups, which is the class obtained from finite and abelian groups by means of subgroups, quotients, extensions and increasing unions. This inclusion is strict [21].

The proof is obtained via the following chain of deductions, describing a tour from geometric functional analysis, through operator algebra K-theory, algebraic topology and combinatorial group theory, and ultimately to general linear algebra.

Theorem 1.3 (Lafforgue [28])**.** *For any countable discrete group with the Haagerup property (for example, any countable amenable group* [5]*), the Bost assembly map*

$$\beta_*^G : K_*^G(\underline{E}G) \to K_*^{\mathrm{top}}(\ell^1(G))$$

is an isomorphism.

We refer to Lafforgue's work in [28] for the definition of the Bost assembly map. The Bost conjecture (see [36]) is that the isomorphism holds for all countable discrete groups.

The brunt of our paper consists in proving the following key link between these conjectures.

Theorem 1.4. *Let G be a countable discrete group for which the Bost assembly map is rationally an epimorphism in degree* 0*. Then the ℓ^1 Bass conjecture 2.2 holds for G.*

The proof of this theorem involves a natural embedding of G in an acyclic group $A(G)$ that is injective on conjugacy classes, with the centralizer of any finitely generated abelian subgroup of $A(G)$ acyclic as well (cf. Theorem 4.9 below). This allows us to control the image of the universal trace

$$T^1 : K_0(\ell^1(G)) \to HH_0(\ell^1(G)).$$

The proof of the classical Bass conjecture is then clinched by an easy lemma.

Lemma 1.5. (a) *The ℓ^1 Bass conjecture 2.2 holds for a group G if it holds for all its countable subgroups.*

(b) *The ℓ^1 Bass conjecture 2.2 implies the $\mathbb{C}G$ Bass conjecture 2.1.*

(c) *The $\mathbb{C}G$ Bass conjecture 2.1 implies the classical Bass conjecture.*

Theorem 1.4 also provides information concerning the non-existence of idempotents in $\mathbb{C}G$ (Kaplansky conjecture), and indeed an analogous result for the case of $\ell^1(G)$. Of course, here one needs to assume the group in question to be torsion-free.

Corollary 1.6. *Let G be a torsion-free countable group and assume that the Bost assembly map β_0^G is rationally surjective. Then $\ell^1(G)$ contains no idempotent other than* 0 *and* 1.

For G a torsion-free hyperbolic group, Ji proved already in [22] that $\ell^1(G)$ does not have any idempotent other than 0 and 1. The analogous result for $C_r^*(G)$ was recently proved by Puschnigg [35] as well as Mineyev and Yu [33]. As pointed out to us by A. Valette, this corollary can easily be deduced using the Atiyah L^2-index theorem, but our proof avoids it, and as a matter of fact our techniques allow a new proof for this theorem, see [14]. As an application of our methods, we obtain a new proof of a recent result of Lück [30], which is a bound for the image of the composite of the Kaplansky trace following the Baum-Connes assembly map (see Section 6).

For a discussion of other groups to which our argument applies, see Section 7 below. As well as groups that have the Haagerup property, they include discrete subgroups of virtually connected semisimple linear Lie groups, hyperbolic groups and cocompact CAT(0)-groups, to name a few.

2. Traces on $\mathbb{C}G$ and $\ell^1(G)$

Let A be an algebra over $\mathbb{C}$ and M be a $\mathbb{C}$-module. A *trace map* over A is a $\mathbb{C}$-linear map $t : A \to M$ satisfying $t(ab) = t(ba)$ for any $a, b \in A$. There is a *universal trace*

$$T : A \to HH_0(A) = A/[A, A],\ a \mapsto a + [A, A]$$

where $HH_0(A)$ stands for the 0-th Hochschild homology group, and $[A, A]$ denotes the $\mathbb{C}$-submodule of A, generated by the elements of the form $ab-ba$ (for $a, b \in A$). In fact, every trace factors uniquely through T. From consideration of idempotent matrix representatives as in the Introduction, any trace gives rise to a well-defined map $K_0(A) \to M$, where $K_0(A)$ is the Grothendieck group of finitely generated projective A-modules. In particular, the universal trace induces

$$T : K_0(A) \to HH_0(A).$$

We focus on the cases where A is the group algebra $\mathbb{C}G$ or the Banach algebra $\ell^1(G)$ of summable series $a = \sum_{g\in G} a_g g$, with ℓ^1-norm $\|a\|_1 = \sum_{g\in G} |a_g| < \infty$, for G a discrete group. We sometimes refer to elements $a = \sum_{g\in G} a_g g \in \ell^1(G)$ as functions $G \to \mathbb{C}$, $g \mapsto a_g$. In this way $\mathbb{C}G \subset \ell^1(G)$ corresponds to the functions with finite support. For $A = \mathbb{C}G$ or $\ell^1(G)$ we now define the *Kaplansky trace*

$$\kappa : A \to \mathbb{C}$$
$$a \mapsto a_1$$

if $a = \sum_{g\in G} a_g g$ and 1 denotes the neutral element in G; note that indeed $\kappa(ab) = \kappa(ba)$. We use the same notation for the induced map

$\kappa : K_0(A) \to \mathbb{C}$. Another example of trace in these cases is given by the *augmentation trace*

$$\begin{aligned} \epsilon : A &\to \mathbb{C} \\ a &\mapsto \sum_{g \in G} a_g. \end{aligned}$$

The special feature of that trace is that it is an algebra homomorphism, and thus, $K_0(\ )$ and $HH_0(\ )$ being covariant functors from the category of algebras to the category of abelian groups, we get the following commutative diagram:

$$\begin{tikzcd} K_0(A) \arrow[r, "T"] \arrow[d, "K_0(\epsilon)"'] & HH_0(A) \arrow[d, "HH_0(\epsilon)"] \\ K_0(\mathbb{C}) \arrow[r, "T"] & HH_0(\mathbb{C}) = \mathbb{C}. \end{tikzcd}$$

Since $K_0(\mathbb{C}) = \mathbb{Z}$, we deduce that the induced map $\epsilon : K_0(A) \to \mathbb{C}$ has image $\mathbb{Z} \subset \mathbb{C}$.

We consider a third trace, which we discuss first for $A = \mathbb{C}G$. We call as usual *Hattori-Stallings trace* the map

$$\begin{aligned} HS : \mathbb{C}G &\to \bigoplus_{[G]} \mathbb{C} \\ a &\mapsto \sum_{[x] \in [G]} \epsilon_{[x]}(a)[x] \end{aligned}$$

where $[x]$ stands for the conjugacy class of $x \in G$, $[G]$ the set of conjugacy classes of G and $\epsilon_{[x]}(a) = \sum_{g \in [x]} a_g$. This is in fact nothing else but the universal trace for the case $A = \mathbb{C}G$. The $[x]$-component $HS_{[x]}$ of the associated trace

$$HS : K_0(\mathbb{C}G) \to \bigoplus_{[G]} \mathbb{C}$$

satisfies $HS_{[x]}([\mathbb{C}G \otimes_G P]) = r_P(x)$, where P denotes a finitely generated projective $\mathbb{Z}G$-module. We recall that, by a result of Linnell (see [29], Lemma 4.1), one has $r_P(x) = 0$ for $x \in G \setminus \{1\}$ of finite order. Thus we consider the following version of the Bass conjecture.

Conjecture 2.1 ($\mathbb{C}G$ Bass Conjecture)**.** *The Hattori-Stallings trace*

$$HS : K_0(\mathbb{C}G) \to \bigoplus_{[G]} \mathbb{C}$$

takes its values in the $\mathbb{C}$*-vector space spanned by the conjugacy classes of elements of finite order.*

Proof of Lemma 1.5(c). To deduce from the above conjecture the classical Bass conjecture concerning $\mathbb{Z}G$, one considers the natural map

$$i_* : K_0(\mathbb{Z}G) \to K_0(\mathbb{C}G)$$

and it suffices (because of Linnell's result) to show that, for $[P] \in K_0(\mathbb{Z}G)$ and for $x \in G$ of infinite order, $HS_{[x]}(i_*[P]) = 0$. Therefore Conjecture 2.1 implies the classical Bass conjecture concerning $\mathbb{Z}G$. $\square$

In the case where $A = \ell^1(G)$, per definition, the universal trace T on $K_0(\ell^1(G))$ takes its values in $HH_0(\ell^1(G))$; the topology on $\ell^1(G)$ does not enter here. Now consider the Banach space $\ell^1([G])$, the completion of the vector space $\bigoplus_{[G]} \mathbb{C}$ with respect to the ℓ^1-norm. Again, we think of elements $\sum_{[x]\in[G]} a_{[x]}[x]$ in $\ell^1([G])$ as functions $[G] \to \mathbb{C}$, $[x] \mapsto a_{[x]}$. If we write $\mathrm{FC}(G) \subset [G]$ for the subset of conjugacy classes $[g] \in [G]$ with g of finite order, we have

$$\bigoplus_{\mathrm{FC}(G)} \mathbb{C} \subset \bigoplus_{[G]} \mathbb{C} \subset \ell^1([G]).$$

For $a = \sum_{g\in G} a_g g \in \ell^1(G)$, we define $\epsilon_{[x]}(a) \in \mathbb{C}$ by $\epsilon_{[x]}(a) = \sum_{g\in[x]} a_g$, which is a convergent series because $\sum_{g\in G} |a_g| < \infty$. The map

$$\begin{aligned} p : \ell^1(G) &\to \ell^1([G]) \\ a &\mapsto \sum_{[x]\in[G]} \epsilon_{[x]}(a)[x] \end{aligned}$$

is a trace and therefore induces a well-defined map

$$\overline{p} : HH_0(\ell^1(G)) \to \ell^1([G]).$$

Indeed, any $z \in [\ell^1(G), \ell^1(G)]$ has the form

$$z = \sum_{i=1}^{n} (a^i b^i - b^i a^i) = \sum_{i=1}^{n} \sum_{g,h\in G} a^i_g b^i_h (gh - hg)$$

for $a^i = \sum_{g\in G} a^i_g g$ and $b^i = \sum_{h\in G} b^i_h h$ elements in $\ell^1(G)$; thus, since $[gh] = [hghh^{-1}] = [hg]$, for all $x \in G$ the $[x]$-component of $p(z)$ is given by the absolutely convergent series $\sum_{i=1}^{n} \sum_{gh\in[x]} a^i_g b^i_h - a^i_g b^i_h = 0$. Hence the trace p yields an ℓ^1-version of the Hattori-Stallings trace

$$HS^1 := \overline{p} \circ T : K_0(\ell^1(G)) \to \ell^1([G])$$

and leads us to the following ℓ^1-version of the Bass conjecture.

Conjecture 2.2 (ℓ^1 Bass Conjecture). *The ℓ^1 Hattori-Stallings trace*

$$HS^1 : K_0(\ell^1(G)) \to \ell^1([G])$$

takes its values in the subspace $\bigoplus_{\mathrm{FC}(G)} \mathbb{C}$ of functions finitely supported by the conjugacy classes of elements of finite order.

Proof of Lemma 1.5(b). With $j : \mathbb{C}G \hookrightarrow \ell^1(G)$, the commutative diagram

$$\begin{array}{ccc} K_0(\mathbb{C}G) & \xrightarrow{HS} & \bigoplus_{[G]} \mathbb{C} \\ {\scriptstyle j_*}\downarrow & & \downarrow \\ K_0(\ell^1(G)) & \xrightarrow{HS^1} & \ell^1([G]) \end{array}$$

shows that if HS^1 maps into $\bigoplus_{\mathrm{FC}(G)} \mathbb{C}$, then so does HS. □

Certain arguments in the sequel require G to be a countable group. For our applications to the Bass conjecture, this does not create any problem, in view of the following.

Proof of Lemma 1.5(a). Notice that an idempotent matrix in $M_n(\ell^1(G))$ representing a finitely generated projective $\ell^1(G)$-module P involves only countably many elements from G. So, for some countable subgroup G_α of G and finitely generated projective $\ell^1(G_\alpha)$-module Q, P is of the form $\ell^1(G) \otimes_{\ell^1(G_\alpha)} Q$. Thus the inclusion maps $G_\alpha \hookrightarrow G$ of all countable subgroups G_α of G induce an epimorphism

$$\bigoplus_\alpha K_0(\ell^1(G_\alpha)) \to K_0(\ell^1(G)),$$

and naturality of HS^1 finishes the argument. □

Of course, although not needed here, a similar argument also holds for Conjecture 2.1.

3. $K_0^G\Lambda$-discrete spaces

We recall some more terminology (discussed further in, for example, [34]).

The *reduced C^*-algebra* $C_r^*(G)$ is the completion of $\ell^1(G)$ with respect to the operator norm, where $\ell^1(G)$ acts on the Hilbert space $\ell^2(G)$ of square summable functions via its regular representation. Recall that $C_r^*(\ )$ is not a functor on the category of groups, but for $H < G$ a subgroup, $C_r^*(H)$ is in a natural way a subalgebra of $C_r^*(G)$, so that for an injective group homomorphism one does obtain an induced morphism of reduced C^*-algebras.

Let $KK_*^G(A, B)$ denote the equivariant Kasparov K-groups of the pair of separable G-C^*-algebras A, B (see [25]). Recall that a G-CW-complex X is said to be *proper* if the stabilizer of each vertex is finite. For X a proper G-CW-complex with G a *countable* group, the equivariant K-homology groups $RK_*^G(X)$ are then defined by

$$RK_*^G(X) = \mathrm{colim}_{\{Y \subset X | Y \text{ cocompact}\}} KK_*^G(C_0Y, \mathbb{C}),$$

where Y runs over the cocompact G-subcomplexes of X, $\mathbb{C}$ is considered as a C^*-algebra with trivial G-action, and C_0Y denotes the C^*-algebra of continuous functions $Y \to \mathbb{C}$ on the locally compact CW-complex Y that vanish at infinity.

These homology groups RK_*^G turn out to be representable in the following sense. We write $\mathfrak{O}(G)$ for the orbit category of G (the objects are the cosets G/H and morphisms are G-maps). There exists an $\mathfrak{O}(G)$-spectrum representing a homology theory K_*^G on the category of *all* G-CW-complexes such that for $H < G$ one has

$$K_*^G(G/H) = K_*^{\mathrm{top}}(C_r^* H),$$

where $K_*^{\mathrm{top}}(C_r^* H)$ is the (topological) algebraic K-theory of the Banach algebra $C_r^* H$ (note that $K_0^{\mathrm{top}}(C_r^* G) = K_0(C_r^* G)$, the projective class group of the ring $C_r^* G$ – the topology does not enter in this case, see [24]). Moreover, the homology theory K_*^G is such that for all *proper* G-CW-complexes X and countable groups G one has a natural isomorphism

$$K_*^G(X) \cong RK_*^G(X).$$

We emphasize that the right-hand side is defined only in case G is countable, whereas the left-hand side is defined for any discrete group G. Note that, because K_*^G is defined by a spectrum, it is fully additive:

$$K_*^G(\coprod X_\alpha) \simeq \bigoplus K_*^G(X_\alpha).$$

For details concerning the $\mathfrak{O}(G)$-spectrum representing equivariant K-homology, the reader is referred to Davis and Lück [16].

Definition 3.1. Let Λ be a (unital) subring of $\mathbb{C}$. A proper G-CW-complex X is called $K_0^G\Lambda$-*discrete* if the natural map induced by the inclusion $\iota : X^0 \hookrightarrow X$ of the 0-skeleton

$$\iota_* : K_0^G(X^0) \to K_0^G(X)$$

is an epimorphism after tensoring with Λ.

Clearly, being $K_0^G\Lambda$-discrete depends only on the G-homotopy type of the G-CW-complex X.

Remark 3.2. If X is a proper G-CW-complex, then $X^0 = \coprod G/F_\alpha$ for some set of finite subgroups $F_\alpha \subset G$. Therefore

$$K_0^G(X^0) = K_0^G(\coprod G/F_\alpha) = \bigoplus K_0^G(G/F_\alpha)$$

and, since G/F_α is the G-space induced from the proper F_α-space $\{\mathrm{pt}\}$,

$$K_0^G(G/F_\alpha) \cong K_0^{F_\alpha}(\{\mathrm{pt}\}) \cong R_{\mathbb{C}}(F_\alpha),$$

where $R_{\mathbb{C}}(F_\alpha)$ is the complex representation ring of the finite group F_α. Thus one has a natural isomorphism of abelian groups

$$\beta : K_0^G(X^0) \to \bigoplus R_{\mathbb{C}}(F_\alpha).$$

Recall that a group G is called *acyclic* if the classifying space $K(G,1) = BG$ satisfies $H_*(BG;\mathbb{Z}) = 0$ for $* > 0$. Equivalently, G is acyclic if the suspension ΣBG of BG is contractible. As usual, we write EG for the universal cover of BG. It is a free G-CW-complex, and so proper.

Lemma 3.3. *If G is acyclic, then EG is $K_0^G\mathbb{Z}$-discrete.*

Proof. Using a suitable model for EG we may assume that $EG^0 = G$ as a discrete G-space, and therefore

$$K_0^G(EG^0) = K_0^G(G) = K_0(\{\mathrm{pt}\}) \cong \mathbb{Z}.$$

On the other hand, $K_0^G(EG) \cong K_0(BG)$ and, as here the suspension ΣBG is contractible, the inclusion $\{\mathrm{pt}\} \hookrightarrow BG$ induces an isomorphism

$$K_0^G(EG^0) \cong K_0(\{\mathrm{pt}\}) \cong K_0(BG) \cong K_0^G(EG),$$

showing that EG is $K_0^G\mathbb{Z}$-discrete. $\square$

The universal proper G-CW-complex $\underline{E}G$ is characterized up to G-homotopy by the property that

$$(\underline{E}G)^H \simeq \begin{cases} \{\mathrm{pt}\} & \text{if } |H| < \infty \\ \varnothing & \text{otherwise.} \end{cases}$$

This implies that for any finite subgroup $H < G$, with centralizer $C_G(H) < G$, the $C_G(H)$-CW-complex $(\underline{E}G)^H$ is a model for $\underline{E}C_G(H)$. For a discussion of $\underline{E}G$ and its properties, see for instance [27]. If G is torsion-free, then EG is a model for $\underline{E}G$, so that $\underline{E}G$ is $K_0^G\mathbb{Z}$-discrete when G is also acyclic. In the context of the Bost conjecture or Baum-Connes conjecture (6.2 below), we are interested in $\underline{E}G$, which differs from EG as soon as G is not torsion-free. So, to deal with groups with torsion, we need a stronger version of acyclicity, which in particular takes into account the centralizers of finite order elements in the group.

Definition 3.4. A group G is called *pervasively acyclic*, if for every finitely generated abelian subgroup $A < G$ the centralizer $C_G(A)$ is an acyclic group.

We also need a way of keeping track of the torsion in G.

Definition 3.5. For any group G, Λ_G denotes the subring of $\mathbb{Q}$ generated by the elements $1/|H|$, where H runs over the finite subgroups of G.

The following lemma is useful later.

Lemma 3.6. *For a group G, the G-map $EG \to \underline{E}G$ induces an isomorphism*

$$H_*(EG/G;\Lambda) \to H_*(\underline{E}G/G;\Lambda)$$

for any abelian group Λ such that $\Lambda_G \subset \Lambda \subset \mathbb{Q}$.

Proof. The Brown spectral sequence (see [13], VII, 7.10) for the G-CW-complex $\underline{E}G$ takes the form

$$E^1_{p,q} = \bigoplus_{\sigma\in\Sigma_p} H_q(BG_\sigma;\Lambda) \Longrightarrow H_{p+q}(BG;\Lambda)$$

with G_σ the (finite) stabilizer of the p-cell $\sigma \subset \underline{E}G$ and Σ_p a set of representatives of orbits of p-cells. Our assumption on Λ implies that $E^1_{p,q} = 0$ for $q > 0$, so that $E^1_{p,*} = E^1_{p,0} \cong C_p(\underline{E}G/G;\Lambda)$, the cellular chain complex of $\underline{E}G/G$ with coefficients in Λ. Thus

$$E^2_{p,*} = E^2_{p,0} \cong H_p(\underline{E}G/G;\Lambda) \cong E^\infty_{p,0} \cong H_p(BG;\Lambda),$$

with the isomorphism being induced by the edge-homomorphism. □

Our aim now is to show that, for any pervasively acyclic group G, its classifying space for proper actions $\underline{E}G$ is $K^G_0\Lambda_G$-discrete.

We first recall the use of Bredon homology in the context of equivariant K-homology (we refer to [34] for a more detailed exposition of these techniques). Let G be any group and $\mathfrak{F}$ the set of finite subgroups of G. We write $\mathfrak{O}(G,\mathfrak{F})$ for the full subcategory of the orbit category $\mathfrak{O}(G)$ with objects G/H, $H \in \mathfrak{F}$. If X is a G-CW-complex, its cellular chain complex C_*X gives rise to a contravariant functor into the category Ab_* of chain complexes of abelian groups,

$$C^{\mathfrak{F}}_*X : \mathfrak{O}(G,\mathfrak{F}) \longrightarrow \mathrm{Ab}_*, \qquad G/H \longmapsto C_*X^H.$$

If $M : \mathfrak{O}(G,\mathfrak{F}) \longrightarrow \mathrm{Ab}$ is a (covariant) functor with target Ab (the category of abelian groups), the Bredon homology $H^{\mathfrak{F}}_*(X;M)$ is defined to be the homology of the chain complex of abelian groups $C^{\mathfrak{F}}_*X \otimes_{\mathfrak{F}} M$. The latter is defined as

$$\sum_{H\in\mathfrak{F}} C_*X^H \otimes M(G/H)/\sim,$$

where $\sim$ is the equivalence relation induced by $f^*x\otimes y \sim x\otimes f_*y$, for $f : G/H \to G/K$ running over all morphisms of $\mathfrak{O}(G,\mathfrak{F})$.

Remark 3.7. In the case of equivariant K-homology, the functor $M = K^G_j(?)$ is of particular interest. For j even, $K^G_j(G/H) \cong R_{\mathbb{C}}(H)$, the complex representation ring of the finite group H, and for j odd $K^G_j(G/H) = 0$. A morphism $\varphi : G/H \to G/K$ gives rise to

$$\varphi_* : K^G_0(G/H) \to K^G_0(G/K),$$

which corresponds to the map $R_{\mathbb{C}}(H) \to R_{\mathbb{C}}(K)$ induced by $h \mapsto x^{-1}hx$, if $\varphi(H) = xK$. In this way the automorphism group

$$\mathrm{map}_G(G/H, G/H) \cong N_G(H)/H$$

acts on $K^G_0(G/H) = R_{\mathbb{C}}(H)$; in particular $C_G(H)$ acts trivially on $R_{\mathbb{C}}(H)$ via $C_G(H) \to N_G(H)/H$. If X is a proper G-CW-complex, there is an Atiyah-Hirzebruch spectral sequence (cf. Lück's Remark 3.9 in [31])

$$E^2_{i,j} = H^{\mathfrak{F}}_i(X;\, K^G_j(?)) \Longrightarrow K^G_{i+j}(X).$$

Theorem 3.8. *Let X be a proper G-CW-complex. If for every finite cyclic subgroup $C < G$ one has $H_{2i}(X^C/C_G(C); \Lambda_G) = 0$ for all $i > 0$, then X is $K_0^G\Lambda_G$-discrete.*

Proof. As in Lück [31] we write $\mathfrak{Sub}(G, \mathfrak{F})$ for the quotient category of $\mathfrak{O}(G, \mathfrak{F})$ with the same objects as $\mathfrak{O}(G, \mathfrak{F})$, but with morphisms from G/H to G/K given by $\mathrm{map}_G(G/H, G/K)/\, C_G(H)$, where the centralizer $C_G(H)$ acts via

$$C_G(H) \to N_G(H)/H \cong \mathrm{map}_G(G/H, G/H)$$

on G-maps $G/H \to G/K$. The cellular chain complex of X gives rise to a contravariant functor to the category of chain complexes of abelian groups:

$$C^{\mathfrak{Sub}}_* X : \mathfrak{Sub}(G, \mathfrak{F}) \to \mathrm{Ab}_*, \quad G/H \mapsto C_*(X^H/C_G(H)).$$

As before, if $M : \mathfrak{Sub}(G, \mathfrak{F}) \to \mathrm{Ab}$ is any (covariant) functor, Bredon-type homology groups $H^{\mathfrak{Sub}}_*(X; M)$ are defined. Note that the projection $\pi : \mathfrak{O}(G, \mathfrak{F}) \to \mathfrak{Sub}(G, \mathfrak{F})$ induces a functor $\pi^* M := M \circ \pi$. There is a natural isomorphism (cf. [31] (3.6))

$$H^{\mathfrak{F}}_*(X;\, \pi^* M) \cong H^{\mathfrak{Sub}}_*(X; M).$$

For the representation ring functor $R_{\mathbb{C}} : \mathfrak{O}(G, \mathfrak{F}) \to \mathrm{Ab}$, $G/H \mapsto R_{\mathbb{C}}(H) = K_0^G(G/H)$, the centralizer $C_G(H)$ acts trivially on $R_{\mathbb{C}}(H)$. Thus $R_{\mathbb{C}}$ factors through $\mathfrak{Sub}(G, \mathfrak{F})$, to yield a functor still denoted by $R_{\mathbb{C}}$, and hence we can replace the left-hand side of the Atiyah-Hirzebruch spectral sequence by $H^{\mathfrak{Sub}}_i(X; K^G_j(?))$. Denote by $\Lambda_G\mathfrak{S}$ the category of functors $\mathfrak{Sub}(G, \mathfrak{F}) \to \Lambda_G$-Mod; an object of $\Lambda_G\mathfrak{S}$ is called a $\Lambda_G\mathfrak{S}$-*module*. According to Lück [30], Theorem 3.5 (b), $\Lambda_G \otimes R_{\mathbb{C}}$ is a projective $\Lambda_G\mathfrak{S}$-module. This implies that for any proper G-CW-complex X

$$H^{\mathfrak{Sub}}_*(X; \Lambda_G \otimes R_{\mathbb{C}}) \cong H^{\Lambda_G\mathfrak{S}}_*(X) \otimes_{\mathfrak{Sub}} (\Lambda_G \otimes R_{\mathbb{C}}),$$

where $H^{\Lambda_G\mathfrak{S}}_*(X)$ denotes the $\Lambda_G\mathfrak{S}$-module $G/H \mapsto H_*(X^H/C_G(H); \Lambda_G)$ (the 'tensor product' $- \otimes_{\mathfrak{Sub}} -$ is, as before, defined by taking the sum over all objects

$$\sum_{H \in \mathfrak{F}} H_*(X^H/C_G(H); \Lambda_G) \otimes (\Lambda_G \otimes R_{\mathbb{C}}(H))$$

and dividing out by the equivalence relation generated by $f^*x \otimes y \sim x \otimes f_* y$, with f running over the morphisms of $\mathfrak{Sub}(G, \mathfrak{F})$).

According to Artin's theorem, for $H < G$ a finite subgroup, every $y \in \mathbb{Z}[1/\,|H|] \otimes R_{\mathbb{C}}(H)$ is a $\mathbb{Z}[1/\,|H|]$-linear combination of images $f_* y_\alpha$

with $y_\alpha \in \mathbb{Z}[1/|H|] \otimes R_{\mathbb{C}}(C_\alpha)$, and $C_\alpha < H$ a cyclic subgroup. This means that for $H \in \mathfrak{F}$ one has:

$$\begin{aligned} &H_*(X^H/C_G(H);\Lambda_G) \otimes (\Lambda_G \otimes R_{\mathbb{C}}(H))/\sim \\ &= \sum H_*(X^C/C_G(C);\Lambda_G) \otimes (\Lambda_G \otimes R_{\mathbb{C}}(C))/\sim \end{aligned}$$

where the sum is taken over finite cyclic subgroups of H, and thus

$$H_{2i}^{\mathfrak{Sub}}(X;\, \Lambda_G \otimes R_{\mathbb{C}}) = 0, \quad \text{for all } i > 0$$

since according to our assumption, the groups $H_{2i}(X^C/C_G(C);\Lambda_G)$ are zero for all $i > 0$. This implies that the Atiyah-Hirzebruch spectral sequence for X collapses when the coefficients are tensored with Λ_G, since all differentials either originate or end up in $\{0\}$. In particular, the edge homomorphism

$$c_0(X) : H_0^{\mathfrak{F}}(X;\, \Lambda_G \otimes R_{\mathbb{C}}) \longrightarrow K_0^G(X) \otimes \Lambda_G$$

is an isomorphism. Now, from the definitions and the fact that for a subgroup H of G we have $(X^0)^H = (X^H)^0$, we readily obtain the surjectivity of

$$\iota_* : H_0^{\mathfrak{F}}(X^0;\, \Lambda_G \otimes R_{\mathbb{C}}) \longrightarrow H_0^{\mathfrak{F}}(X;\, \Lambda_G \otimes R_{\mathbb{C}})$$

from its counterpart in ordinary homology. Hence the commutative diagram expressing naturality

$$\begin{array}{ccc} H_0^{\mathfrak{F}}(X^0;\Lambda_G \otimes R_{\mathbb{C}}) & \xrightarrow{c_0(X^0)} & K_0^G(X^0) \otimes \Lambda_G \\ {\scriptstyle \iota_*}\big\downarrow & & {\scriptstyle \iota_*}\big\downarrow \\ H_0^{\mathfrak{F}}(X;\Lambda_G \otimes R_{\mathbb{C}}) & \xrightarrow{c_0(X)} & K_0^G(X) \otimes \Lambda_G \end{array}$$

implies the surjectivity of

$$\iota_* : K_0^G(X^0) \otimes \Lambda_G \longrightarrow K_0^G(X) \otimes \Lambda_G\,,$$

which establishes that X is $K_0^G\Lambda_G$-discrete. □

Corollary 3.9. *For any pervasively acyclic group G, $\underline{E}G$ is $K_0^G\Lambda_G$-discrete.*

Proof. For every finite cyclic subgroup $C < G$, $(\underline{E}G)^C$ is a model for $\underline{E}C_G(C)$. The natural map $EC_G(C) \to \underline{E}C_G(C)$ gives rise to a map

$$BC_G(C) = EC_G(C)/\,C_G(C) \to \underline{E}C_G(C)/\,C_G(C) = (\underline{E}G)^C/\,C_G(C)$$

which induces an isomorphism

$$H_*(BC_G(C);\Lambda_G) \cong H_*((\underline{E}G)^C/\,C_G(C);\Lambda_G),$$

by Lemma 3.6 applied to the group $C_G(C)$. Thus, since the centralizers $C_G(C)$ are acyclic, the result follows by choosing $X = \underline{E}G$ in the previous theorem. □

4. Pervasively acyclic groups

In this section we introduce a functorial embedding of a given group G in an acyclic group $A(G)$ that has further strong properties required for our arguments. Embeddings into acyclic groups have historically been important for algebraic K-theory [37] and algebraic topology [23]. One of our further requirements for $A(G)$ is that the centralizers in $A(G)$ of finitely generated abelian subgroups of $A(G)$ should also be acyclic (so that $A(G)$ is pervasively acyclic, cf. Definition 3.4). In the extreme case where G is itself abelian, a weaker form of this was already known to be possible by making G the centre of an acyclic group [4], [7], [8]. A prominent class of acyclic groups suited to our purpose comprises the binate groups [8]. We now recall the definition.

Definition 4.1. A group G is said to be *binate* if for any finitely generated subgroup H of G there is a homomorphism $\varphi_H : H \to G$ and an element $u_H \in G$ such that for all h in H we have

$$h = [u_H, \varphi_H(h)] = u_H \varphi_H(h) u_H^{-1} \varphi_H(h)^{-1}.$$

Obviously φ_H is injective, while the fact that it is a homomorphism implies, from the usual product formula for commutators, that its image commutes with H. So this apparatus embeds a pair of commuting copies of each finitely generated subgroup of G in G (*binate* = arranged in pairs). The key property of binate groups is their acyclicity.

Theorem 4.2 (see [6], (11.11))**.** *Every binate group is acyclic.*

In general, there is a construction for embedding a given group in a binate group, the *universal binate tower* [8]. It is universal in the sense that it maps to any other such construction [9]. See [10] for justification of the role of binate groups in this context. The construction we now give is an adaptation of the universal binate tower.

In the following, we use the notation

$$\begin{aligned} \Delta_G &= \{(g, g) \in G \times G \mid g \in G\}\,, \\ \Delta'_F &= \left\{(f^{-1}, f) \in F \times F \mid f \in F\right\}. \end{aligned}$$

Definition 4.3. Let H be a group with $\{F_i\}_{i\in I}$ as the set of all its finitely generated abelian subgroups. For $i \in I$, write C_i for the centralizer in H of F_i. Then in $H \times H$ the subgroups $(1 \times F_i) = \{(1, f) \mid f \in F_i\}$ and Δ_{C_i} commute, so that their product $(1 \times F_i) \cdot \Delta_{C_i}$ is also a subgroup of $H \times H$. Likewise, $\Delta'_{F_i} \cdot (1 \times C_i)$ is also a subgroup, and the obvious bijection

$$\begin{aligned} (1 \times F_i) \cdot \Delta_{C_i} &\longrightarrow \Delta'_{F_i} \cdot (1 \times C_i) \\ (k, fk) &\longmapsto (f^{-1}, fk) \end{aligned}$$

is a group isomorphism. Now define $A_1(H)$ to be the generalized HNN extension

$$A_1(H) = \mathrm{HNN}(H \times H;\ (1 \times F_i) \cdot \Delta_{C_i} \cong \Delta'_{F_i} \cdot (1 \times C_i),\ t_i)_{i \in I}$$

meaning that, whenever $f \in F_i$ and $k \in C_i$,

$$(k, fk) = t_i(f^{-1}, fk)t_i^{-1}.$$

Lemma 4.4. *The inclusion $h \mapsto (h, 1)$ of H in $H \times H$ as $H \times 1$ extends to a functorial inclusion of H in $A_1(H)$.*

Proof. That we obtain an inclusion is an application of the Higman-Neumann-Neumann Embedding Theorem for HNN extensions to the present situation. Functoriality is a routine check, since homomorphisms map finitely generated abelian subgroups to finitely generated abelian subgroups and centralizers into centralizers. □

The inclusion of H in $A_1(H)$ is used implicitly in the sequel. Note that an element in $A_1(H)$ can be written as a word involving elements of $H \times H$ and stable letters.

Lemma 4.5. *For all i in I, t_i centralizes F_i, that is, $t_i \in C_{A_1(H)}(F_i)$.*

Proof. From the construction we have, for any $f \in F_i$,

$$t_i(f, 1)t_i^{-1} = t_i(f, f^{-1}f)t_i^{-1} = (f, f^{-1}f) = (f, 1).$$

□

Lemma 4.6. *If two elements of H are conjugate in $A_1(H)$, then they are conjugate in H.*

Proof. Fix $h \in H$. Our aim is to contradict the assertion that there exist elements of $H \times H$ that are conjugate to h in $A_1(H)$ but not in $H \times H$. This contradiction gives the required result, for it means that whenever (h', h'') in $H \times H$ is conjugate to $h = (h, 1)$, then there exists $x = (x', x'') \in H \times H$ such that

$$(h', h'') = (x', x'')(h, 1)(x', x'')^{-1}.$$

(Of course, we are primarily interested in the case where we begin with the further assumption that $h'' = 1$; however, the more general case is used later.) The above equation yields that indeed $h'' = 1$ and so $(h', h'') = h' = x'h(x')^{-1}$, making h and xhx^{-1} conjugate in H itself. Therefore, in order to establish a contradiction, let us take s to be the minimal number of indices $i \in I$ of stable letters t_i occurring in any expression of x as a word in $A_1(H)$, where x varies among all elements of $A_1(H)$ for which xhx^{-1} lies in $H \times H$ but fails to be conjugate to h in $H \times H$. The fact that $x \notin H \times H$ means $s \geq 1$. Let w be such a word involving precisely s distinct stable letters.

Write $\bar{h} = whw^{-1}$, and let $i(1), \ldots, i(s) \in I$ be the indices of stable letters t_i occurring in w. We argue as in Step 2 of the proof of Britton's

Lemma [12]. We first observe that the trivial word $\bar{h}^{-1}whw^{-1}$ lies in the subgroup

$$B_s = \mathrm{HNN}(H \times H;\ (1 \times F_{i(j)}) \cdot \Delta_{C_{i(j)}} \cong \Delta'_{F_{i(j)}} \cdot (1 \times C_{i(j)}),\ t_{i(j)})_{j=1,\dots,s}$$

of $A_1(H)$. We put $B_0 = H \times H$, and, for $r = 1, \dots, s$, write

$$B_r = \mathrm{HNN}(B_{r-1};\ (1 \times F_{i(r)}) \cdot \Delta_{C_{i(r)}} \cong \Delta'_{F_{i(r)}} \cdot (1 \times C_{i(r)}),\ t_{i(r)}).$$

Then, once again using the Embedding Theorem, we have

$$H = H \times 1 < H \times H = B_0 < B_1 < \dots < B_{r-1} < B_r < \dots < B_s.$$

Clearly it suffices to establish the following.

Claim. *If h is conjugate in B_r $(1 \le r \le s)$ to an element g of B_{r-1}, then it is conjugate in B_{r-1} to g.*

To prove this, we write $F = F_{i(r)}$, $C = C_{i(r)}$, $t = t_{i(r)}$. By the Normal Form Theorem [32] p.182 for B_r as an HNN extension of B_{r-1}, we may suppose that n is minimal among all $x \in B_r$ with $xhx^{-1} = g$ and

$$x = g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} g_2 \cdots t^{\varepsilon_n} g_n$$

reduced. That is, each $g_i \in B_{r-1}$, each $\varepsilon_i \in \{\pm 1\}$, and there is no subword of the form

$$t g_i t^{-1} \quad \text{with} \quad g_i \in \Delta'_F \cdot (1 \times C)$$

nor

$$t^{-1} g_i t \quad \text{with} \quad g_i \in (1 \times F) \cdot \Delta_C.$$

To show that in fact $x \in B_{r-1}$ we contradict the minimality of n when $n \ge 1$. Since in B_r

$$1 = g^{-1} g_0 t^{\varepsilon_1} g_1 \cdots t^{\varepsilon_n} g_n h g_n^{-1} t^{-\varepsilon_n} \cdots g_1^{-1} t^{-\varepsilon_1} g_0^{-1},$$

by Britton's Lemma (see [32]) we must have either

$$\varepsilon_n = 1 \quad \text{and} \quad g_n h g_n^{-1} \in \Delta'_F \cdot (1 \times C)$$

or

$$\varepsilon_n = -1 \quad \text{and} \quad g_n h g_n^{-1} \in (1 \times F) \cdot \Delta_C.$$

Since $g_n \in B_{r-1} \le B_{s-1}$ and $g_n h g_n^{-1} \in H \times H$, by minimality of s we must have

$$g_n h g_n^{-1} = y h y^{-1} \quad \text{with} \quad y \in H \times H.$$

As noted at the beginning, this forces $g_n h g_n^{-1} \in H$. Therefore

$$g_n h g_n^{-1} \in H \cap ((\Delta'_F \cdot (1 \times C)) \cup ((1 \times F) \cdot \Delta_C)) = F.$$

However, by Lemma 4.5, $t \in C_{B_r}(F)$. Hence

$$\begin{aligned} xhx^{-1} &= g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} g_2 \cdots t^{\varepsilon_{n-1}} g_{n-1} g_n h g_n^{-1} g_{n-1}^{-1} t^{-\varepsilon_{n-1}} \cdots g_1^{-1} t^{-\varepsilon_1} g_0^{-1} \\ &= z h z^{-1} \end{aligned}$$

where $z = g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} g_2 \cdots t^{\varepsilon_{n-1}} g_{n-1} g_n$ involves fewer than n occurrences of t, thereby contradicting the minimality of n. □

The final property that we need is that no new primary torsion is created in the construction.

Lemma 4.7. *$A_1(H)$ contains an element of prime power order p^k, if and only if H also contains an element of order p^k.*

Proof. It is evident that H and $H \times H$ have the same prime powers arising as orders of elements. The Torsion Theorem for HNN extensions [32] p.185, after iteration as in the proof of Lemma 4.6, shows that $H \times H$ and $A_1(H)$ share the same finite orders of elements. (Of course $H \times H$, and hence $A_1(H)$, may have non-prime-power orders not found in H itself.) $\square$

Definition 4.8. Let G be a group. We write $A_1 = A_1(G)$, and for $n \geq 2$ inductively define $A_n(G) = A_1(A_{n-1})$, which we also write as A_n. Thus by Lemma 4.4 $A_{n-1} \leq A_n$, and we put $A = A(G) = \cup A_n$.

Theorem 4.9. *The homomorphism $G \to A(G)$ has the following properties.*

(a) *It is a functorial inclusion.*
(b) *Every finitely generated abelian subgroup of $A(G)$ has its centralizer in $A(G)$ binate, hence acyclic. In particular, $A(G)$ is pervasively acyclic.*
(c) *If two elements of G are conjugate in $A(G)$, then they are conjugate in G itself.*
(d) *The prime powers that occur as orders of elements of $A(G)$ are precisely those that occur as orders of elements of G.*
(e) *If G is countable, $A(G)$ is too.*

Proof. (a), (c) and (d) are easy consequences of Lemmas 4.4, 4.6 and 4.7 respectively. So we concentrate on proving (b). In view of Theorem 4.2, we show that, for each finitely generated abelian subgroup F of $A(G)$, the centralizer $C_A(F)$ is binate. Therefore let H be a finitely generated subgroup of $C_A(F)$. It follows that we can find n with both H and F subgroups of A_n, and so that $F = F_i$ for some suitable member i of the index set of finitely generated abelian subgroups of A_n. Then the homomorphism φ sending $h \in H$ to $(1, h) \in A_{n+1}$ maps into $C_A(F)$. By Lemma 4.5 we have also $t_i \in C_A(F)$. Thus the fact that, for any $h \in H$,

$$h = (h, h)(1, h)^{-1} = t_i(1, h)t_i^{-1}(1, h)^{-1} = [t_i,\, \varphi(h)]$$

reveals $C_A(F)$ to be binate, as required.

For (e) one notices that a countable group has only countably many finitely generated subgroups. $\square$

5. The Bass conjecture for amenable groups: proof of Theorem 1.4

In this section we show how the use of pervasively acyclic groups allows us to deduce the Bass conjecture from the Bost conjecture on the assembly map

$$\beta_*^G : K_*^G(\underline{E}G) \to K_*^{\mathrm{top}}(\ell^1(G)).$$

Notice that in the case where $* = 0$ and A is any Banach algebra, the groups $K_0(A)$ and $K_0^{\mathrm{top}}(A)$ agree, as they both are defined to be the Grothendieck group of finitely generated projective A-modules – the topology does not enter in this case.

Proof of Theorem 1.4. Consider the embedding $G \to A$ where $A = A(G)$ is the pervasively acyclic group of Definition 4.8. This embedding, as well as the inclusion $\iota : \underline{E}A^0 \to \underline{E}A$, yields the following commutative diagram:

$$\begin{tikzcd}
K_0^G(\underline{E}G) \arrow[r, "\beta_0^G"] \arrow[d] & K_0(\ell^1(G)) \arrow[r, "HS^1"] \arrow[d] & \ell^1([G]) \arrow[d] \\
K_0^A(\underline{E}A) \arrow[r, "\beta_0^A"] & K_0(\ell^1(A)) \arrow[r, "HS^1"] & \ell^1([A]) \\
K_0^A(\underline{E}A^0) \arrow[r, "\beta"] \arrow[u, "\iota_*"] & \bigoplus R_{\mathbb{C}}F_\alpha \arrow[r, "\gamma"] \arrow[u] & \bigoplus \ell^1([F_\alpha]). \arrow[u, "j"']
\end{tikzcd}$$

The map β arises from Remark 3.2, by writing $\underline{E}A^0$ as a disjoint union of orbits A/F_α (where the F_α are the finite subgroups of A). We now discuss the maps γ and j. For $F_\alpha < A$ finite, we have a commutative diagram

$$\begin{tikzcd}
K_0(\ell^1(A)) \arrow[r, "HS^1"] & \ell^1([A]) \\
R_{\mathbb{C}}(F_\alpha) = K_0(\ell^1(F_\alpha)) \arrow[r, "HS^1"] \arrow[u] & \ell^1([F_\alpha]) \arrow[u, "j_\alpha"']
\end{tikzcd}$$

Here j_a is induced by $[F_\alpha] \to [A]$, and therefore maps into $\bigoplus_{\mathrm{FC}(A)} \mathbb{C}$, where $\mathrm{FC}(A)$ denotes the set of conjugacy classes of finite order elements of A. The map γ is just the sum of Hattori-Stallings traces (on the F_α), and $j : \bigoplus \ell^1([F_\alpha]) \to \ell^1([A])$ is the map that restricts on each $\ell^1([F_\alpha])$ to j_α.

After tensoring the left two columns of this diagram with $\mathbb{Q}$, by assumption the Bost assembly map $\beta_0^G \otimes \mathrm{Id}_{\mathbb{Q}}$ becomes surjective, and the map $\iota_* \otimes \mathrm{Id}_{\mathbb{Q}}$ becomes an epimorphism because of Corollary 3.9. It is a consequence of Theorem 4.9 part (c) that $[G] \subseteq [A]$, and thus

$$\ell^1([G]) \subset \ell^1([A]) \supset \ell^1\left(FC(A)\right) \supset \bigoplus_{\mathrm{FC}(A)} \mathbb{C},$$

with

$$\bigoplus_{\mathrm{FC}(G)} \mathbb{C} = \ell^1([G]) \bigcap \bigoplus_{\mathrm{FC}(A)} \mathbb{C}.$$

Therefore the map

$$HS^1 : K_0(\ell^1(G)) \to \ell^1([G])$$

takes its values in the $\mathbb{C}$-vector space spanned by the conjugacy classes of elements of finite order. □

6. On the image of the Kaplansky trace

We now turn to the idempotent conjecture for $\ell^1(G)$ and the proof of Corollary 1.6. Using similar arguments to those that have been used in [34], Lemma 6.1, one could deduce the idempotent conjecture for $\ell^1(G)$ from the surjectivity of the Bost assembly map. However, our aim is to prove something stronger, namely the idempotent conjecture for $\ell^1(G)$ out of the *rational surjectivity* of the Bost assembly map. We start with some standard facts concerning the Kaplansky trace.

Recall that $C^*_r(G)$ acts on $\ell^2(G)$. Considering $1 \in G$ as an element of $\ell^2(G)$, one defines for $f \in C^*_r(G)$ the *Kaplansky trace*

$$\kappa(f) = \langle f(1), 1\rangle \in \mathbb{C}.$$

This defines a trace $\kappa : C^*_r(G) \to \mathbb{C}$ which extends the Kaplansky trace $\ell^1(G) \to \mathbb{C}$ described earlier, in the sense that the resulting map in K-theory (which we still denote by κ) fits into a commutative diagram

$$\begin{array}{ccc} K_0(\ell^1(G)) & \xrightarrow{\kappa} & \mathbb{C} \\ \big\downarrow & & \big\| \\ K_0(C^*_r(G)) & \xrightarrow{\kappa} & \mathbb{C}. \end{array}$$

Moreover, κ is natural in the sense that for $H < G$ one has a commutative diagram

$$\begin{array}{ccc} K_0(C^*_r(H)) & \xrightarrow{\kappa} & \mathbb{C} \\ \big\downarrow & & \big\| \\ K_0(C^*_r(G)) & \xrightarrow{\kappa} & \mathbb{C}. \end{array}$$

Combined with Theorem 1.4, the following proposition completes the proof of Corollary 1.6.

Proposition 6.1. *Let G be a torsion-free group satisfying Conjecture 2.2. Then $\ell^1(G)$ contains no idempotent other than* 0 *and* 1.

Proof. We first show that in the case where Conjecture 2.2 holds and G is torsion-free, both the Kaplansky trace κ and the augmentation trace ϵ, considered as maps $K_0(\ell^1(G)) \to \mathbb{C}$, coincide. On the one hand, for G torsion-free, Conjecture 2.2 says that $HS^1 = \epsilon_{[1]} = \epsilon$ (since G being torsion-free implies that $\mathrm{FC}(G) = \{1\}$), and on the other hand as observed earlier one has that $\kappa = \epsilon_{[1]}$. Since the augmentation trace $\epsilon : K_0(\ell^1(G)) \to \mathbb{C}$ is known to assume integral values only (recall from Section 2 that it factors through $K_0(\mathbb{C})$), this means that the Kaplansky trace κ has integral range as well.

If $a \in \ell^1(G) \subset C_r^*(G)$ is an idempotent, then the projective ideal $P = C_r^*(G) \cdot a \subset C_r^*(G)$ satisfies $\kappa([P]) = 0$ or 1, so that $P = 0$ or $C_r^*(G)$. Therefore $a = 0$ or 1 by the usual argument concerning idempotents in $C_r^*(G)$ (cf. [3], Proposition 7.16). $\square$

Using similar ideas to those in the previous section, we now give another application of pervasively acyclic groups and Proposition 3.9, but in the context of the Baum-Connes conjecture. Here we recall:

Conjecture 6.2 (Baum-Connes)**.** *Let G be a countable discrete group. The Baum-Connes assembly map*

$$\mu_*^G : K_*^G(\underline{E}G) \to K_*^{\mathrm{top}}(C_r^*(G))$$

is an isomorphism.

We refer to Baum, Connes and Higson [3], as well as Lafforgue's work [28] for the construction of the Baum-Connes assembly map, and for considerable partial information concerning the validity of the Baum-Connes conjecture. Our techniques allow us to give a new proof of a recent theorem of Lück [30].

Theorem 6.3. *Let $\kappa : K_0(C_r^*(G)) \to \mathbb{C}$ be the Kaplansky trace and $\mu_0^G : K_0^G(\underline{E}G) \to K_0(C_r^*(G))$ the Baum Connes assembly map. Then the image of $\kappa \circ \mu_0^G$ is contained in Λ_G.*

Proof. Consider the embedding $G \to A$ where $A = A(G)$ denotes the pervasively acyclic group of Definition 4.8. Notice that $\Lambda_G = \Lambda_{A(G)}$ by Theorem 4.9 (d). This equality, as well as the inclusion $\iota : \underline{E}A^0 \to \underline{E}A$, yields the commutative diagram

$$\begin{array}{ccccc}
K_0^G(\underline{E}G) & \xrightarrow{\mu^G} & K_0(C_r^*(G)) & \xrightarrow{\kappa} & \mathbb{C} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\
K_0^A(\underline{E}A) & \xrightarrow{\mu^A} & K_0(C_r^*(A)) & \xrightarrow{\kappa} & \mathbb{C} \\
{\scriptstyle \iota_*}\uparrow & & \uparrow & & \uparrow \\
K_0^A(\underline{E}A^0) & \xrightarrow{\beta} & \bigoplus R_{\mathbb{C}}F_\alpha & \xrightarrow{\sigma} & \Lambda_G
\end{array}$$

where β is as in the proof of Theorem 1.4, and σ is given as follows. If $\{x_\alpha\}$ is an element in $\bigoplus R_{\mathbb{C}}F_\alpha$, then

$$\sigma(\{x_\alpha\}) = \sum_\alpha \kappa_\alpha(x_\alpha)$$

where $\kappa_\alpha : R_{\mathbb{C}}F_\alpha \to \mathbb{C}$ is the Kaplansky trace on $K_0(C_r^*(F_\alpha)) = R_{\mathbb{C}}F_\alpha$, which takes its values in $\frac{1}{|F_\alpha|}\mathbb{Z} \subset \Lambda_G$. Indeed, the groups F_α are finite, and for a finitely generated projective $\mathbb{C}F_\alpha$-module P one has, according to Bass ([1], Corollary 6.3), by comparing the Kaplansky trace of P with the Kaplansky trace of P considered as a module over $\mathbb{C}\{1\}$,

$$\kappa([P]) = \frac{1}{|F_\alpha|}\dim_{\mathbb{C}} P \in \frac{1}{|F_\alpha|}\mathbb{Z}.$$

Now write M for the image of $\kappa \circ \mu_0^G$. As the map $\iota_* \otimes \mathrm{Id}_{\Lambda_G}$ becomes an epimorphism because of Corollary 3.9, commutativity of the diagram now shows that $M \otimes \Lambda_G$ lies in $\Lambda_G \otimes \Lambda_G$. (This is essentially Lück's theorem.) The $\mathbb{Z}$-flatness of submodules of $\mathbb{C}$ then gives injections

$$M \cong M \otimes \mathbb{Z} \to M \otimes \Lambda_G \to \Lambda_G \otimes \Lambda_G \cong \Lambda_G,$$

and so the result. □

7. New examples of groups satisfying the Bass conjecture

We describe a wide class of groups for which the Bost conjecture is known. This class, which Lafforgue in [28] called $\mathcal{C}'$, includes all discrete countable groups acting metrically properly and isometrically on one of the following spaces:

(a) an affine Hilbert space (those groups are said to have the *Haagerup property*, or to be *a-T-menable*);
(b) a uniformly locally finite, weakly δ-geodesic and strongly δ-bolic space (we will see that cocompact CAT(0)-groups satisfy this assumption);
(c) a non-positively curved Riemannian manifold, with curvature bounded from below and bounded derivative of the curvature tensor (with respect to the connection induced from the Levi-Civita connection on the tangent bundle).

Theorem 7.1. (Lafforgue [28]) *The Bost conjecture holds for any group in the class $\mathcal{C}'$.*

We now discuss in turn the three classes of groups specified by (a), (b) and (c) above.

Class (a). Here, we have Theorem 1.3 as a special case. We recall that the class of groups satisfying the Haagerup property has the following closure properties (see [15]).

– The Haagerup property is closed under taking subgroups, and direct products.
– If G acts on a locally finite tree with finite edge stabilizers, and with the vertex stabilizers having the Haagerup property, then so does G.
– If $G = \bigcup_{n\geq 0} G_n$, with $G_i < G_{i+1}$ for all i, and each G_i has the Haagerup property, then so does G.

– If G and H are countable amenable groups and C is central in H and G then $G *_C H$ has the Haagerup property (in particular, free products of countable amenable groups have the Haagerup property).

Class (b). We now turn to the second class of groups contained in $\mathcal{C}'$.

Definition 7.2. A metric space (X, d) is called *uniformly locally finite* if, for any $r \geq 0$, there exists $k \in \mathbb{N}$ such that any ball of radius r contains at most k points (notice that this forces X to be discrete). For $\delta > 0$, the metric space X is termed *weakly δ-geodesic* if, for any $x, y \in X$ and $t \in [0, d(x, y)]$, there exists $a \in X$ such that

$$d(a, x) \leq t + \delta, \quad d(a, y) \leq d(x, y) - t + \delta.$$

The two conditions above are automatically satisfied by any finitely generated group G endowed with the word metric associated to any finite generating set, and by the orbit of a point in a Riemannian manifold with non-positive curvature, under a group acting properly, isometrically and cocompactly on the manifold. The following definitions are taken from [26] and [28].

Definition 7.3. Given $\delta > 0$, a metric space (X, d) is said to be *weakly δ-bolic* if the following conditions are satisfied:

(b_1) For any $r > 0$, there exists $R = R(\delta, r) > 0$ such that, for any four points x_1, x_2, y_1, y_2 in X satisfying $d(x_1, y_1) + d(x_2, y_2) \leq r$ and $d(x_1, y_2) + d(y_1, x_2) \geq R$, one has:

$$d(x_1, x_2) + d(y_1, y_2) \leq d(x_1, y_2) + d(y_1, x_2) + \delta.$$

(b_2) There exists a map $m : X \times X \to X$ such that:

(i) For all $x, y \in X$,

$$d(m(x, y), x) \leq \frac{d(x, y)}{2} + \delta \quad \text{and} \quad d(m(x, y), y) \leq \frac{d(x, y)}{2} + \delta.$$

(ii) For all $x, y, z \in X$,

$$d(m(x, y), z) \leq \max\{d(x, z), d(y, z)\} + 2\delta.$$

(iii) For all $p \geq 0$, there exists $N = N(p) \geq 0$ such that, for any $n \geq N$ and $x, y, z \in X$ with $d(x, z) \leq n$, $d(y, z) \leq n$ and $d(x, y) > n$, one has

$$d(m(x, y), z) < n - p.$$

Given $\delta > 0$, a metric space (X, d) is called *strongly δ-bolic* if it is weakly δ-bolic and if condition (b_1) is satisfied for any $\delta > 0$.

Mineyev and Yu in [33] showed that a hyperbolic group G can be endowed with a left G-invariant metric such that there exists $\delta > 0$ for which G is strongly δ-bolic.

We recall that in a geodesic metric space (X, d) a *geodesic triangle* Δ consists of three points a, b, c with three (possibly non-unique) geodesics joining them, and a *comparison triangle* $\overline{\Delta}$ for Δ is a euclidean triangle with side lengths $d(a,b), d(b,c), d(c,a)$. We write $\overline{a}, \overline{b}$ and $\overline{c}$ for the vertices of $\overline{\Delta}$, and if x is a point in Δ (say on a geodesic between a and b), we write $\overline{x}$ for a comparison point for x, namely a point in $\overline{\Delta}$ such that $d_E(\overline{x}, \overline{a}) = d(x,a)$ (where d_E denotes the euclidean distance in $\mathbb{R}^2$). A geodesic metric space (X,d) is termed CAT(0) if for all geodesic triangles Δ in X and all $x, y \in \Delta$,

$$d(x,y) \leq d_E(\overline{x}, \overline{y})$$

where $\overline{x}, \overline{y}$ are any two comparison points in any euclidean comparison triangle $\overline{\Delta}$ for Δ. The following is an easy fact.

Proposition 7.4. *CAT(0) metric spaces are strongly δ-bolic for any $\delta > 0$.*

Proof. We leave to the reader the verification that $\mathbb{R}^2$ with its euclidean distance is a strongly δ-bolic space for any $\delta > 0$ (the rest of the proof relies on this fact, see also [26] Proposition 2.4.). Let (X,d) be a CAT(0) metric space. We start by checking that condition (b_2) holds. The map m is defined as follows:

$$m(x,y) = \gamma(t)$$

where $t = d(x,y)/2$ and $\gamma : [0, d(x,y)] \to X$ is the unique geodesic from x to y. Point (i) is satisfied by assumption on m, (ii) holds since in CAT(0) spaces the metric is strictly convex, and (iii) follows from the CAT(0) inequality. It remains to prove condition (b_1) for any $\delta > 0$. To do this, we follow Bridson's advice and use the CAT(0) 4-points condition (see Bridson-Haefliger's book [11] p.164) which says that in a CAT(0) space, every 4-tuple of points x_1, x_2, y_1, y_2 has a *subembedding* in $\mathbb{R}^2$, meaning that there exist 4 points $\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2$ in $\mathbb{R}^2$ such that $d_E(\overline{x}_i, \overline{y}_j) = d(x_i, y_j)$ for $i = 1, 2$ and $d(x_1, x_2) \leq d_E(\overline{x}_1, \overline{x}_2)$, $d(y_1, y_2) \leq d_E(\overline{y}_1, \overline{y}_2)$. We now take $\delta, r \geq 0$ and $R = R(\delta, r)$ as for $\mathbb{R}^2$. By definition of subembedding, the 4-tuple $\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2$ satisfies the assumptions of (b_1) as soon as the 4-tuple x_1, x_2, y_1, y_2 does, and we conclude with

$$\begin{aligned}
&d(x_1, x_2) + d(y_1, y_2) \\
&\leq d_E(\overline{x}_1, \overline{x}_2) + d_E(\overline{y}_1, \overline{y}_2) \leq d_E(\overline{x}_1, \overline{y}_2) + d_E(\overline{x}_2, \overline{y}_1) + \delta \\
&= d(x_1, y_2) + d(y_1, x_2) + \delta.
\end{aligned}$$

□

If a countable discrete group G acts properly and cocompactly on a CAT(0) metric space X, then there exists a $\delta > 0$ such that, for any $x_0 \in X$, $Y = Gx_0 \subset X$ endowed with the induced metric from X is weakly δ-geodesic and strongly δ-bolic. Indeed, one may choose $\delta = 2R$, where R is a positive real number (existing by cocompactness) such that $X = \bigcup_{g \in G} B(gx_0, R)$. Finally, G is finitely generated (see [11]

p. 439). Thus Y is automatically uniformly locally finite and, since G obviously acts properly on Y, this shows that cocompact CAT(0) groups are in the class $\mathcal{C}'$.

We recall that that the class of cocompact CAT(0) groups (that is, discrete groups acting properly, isometrically and cocompactly on a CAT(0) metric space) is closed under the following operations (see [11] p. 439):

- direct products;
- HNN extensions along finite subgroups;
- free products with amalgamation along virtually cyclic subgroups.

Class (c). The group $\mathrm{SL}_n(\mathbb{Z})$ (and more generally any discrete subgroup of a virtually connected semisimple linear Lie group) is in the class $\mathcal{C}'$, as any non-positively curved symmetric space is a Riemannian manifold satisfying the required assumptions on the curvature.

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