Structured eigenvalue condition numbers and linearizations for matrix polynomials

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Abstract. This work is concerned with eigenvalue problems for structured matrix polynomials, including complex symmetric, Hermitian, even, odd, palindromic, and anti-palindromic matrix polynomials. Most numerical approaches to solving such eigenvalue problems proceed by linearizing the matrix polynomial into a matrix pencil of larger size. Recently, linearizations have been classified for which the pencil reflects the structure of the original polynomial. A question of practical importance is whether this process of linearization increases the sensitivity of the eigenvalue with respect to structured perturbations. For all structures under consideration, we show that this is not the case: there is always a linearization for which the structured condition number of an eigenvalue does not differ significantly. This implies, for example, that a structure-preserving algorithm applied to the linearization fully benefits from a potentially low structured eigenvalue condition number of the original matrix polynomial.

Keywords. Eigenvalue problem, matrix polynomial, linearization, structured condition number.

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1 Introduction

Consider an $n \times n$ matrix polynomial

$$\mathrm{P}(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots + \lambda^m A_m, \tag{1}$$

with $A_0, \dots, A_m \in \mathbb{C}^{n\times n}$. An eigenvalue $\lambda \in \mathbb{C}$ of P, defined by the relation $\det(\mathrm{P}(\lambda)) = 0$, is called simple if λ is a simple root of the polynomial $\det(\mathrm{P}(\lambda))$.

This paper is concerned with the sensitivity of a simple eigenvalue λ under perturbations of the coefficients A_i. The condition number of λ is a first-order measure for the worst-case effect of perturbations on λ. Tisseur [33] has provided an explicit expression for this condition number. Subsequently, this expression was extended to polynomials in homogeneous form by Dedieu and Tisseur [9], see also [1, 5, 8], and to semi-simple eigenvalues in [22]. In the more general context of nonlinear eigenvalue problems, the sensitivity of eigenvalues and eigenvectors has been investigated in, e.g., [3, 24, 25, 26].

Loosely speaking, an eigenvalue problem (1) is called *structured* if there is some distinctive structure among the coefficients $A_0, \dots, A_m$. For example, much of the recent research on structured polynomial eigenvalue problems was motivated by the second-order T-palindromic eigenvalue problem [18, 27]

$$A_0 + \lambda A_1 + \lambda^2 A_0^T,$$

where A_1 is complex symmetric: $A_1^T = A_1$. In this paper, we consider the structures listed in Table 1. To illustrate the notation of this table, consider a T-palindromic polynomial

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	Structured Polynomial $\mathrm{P}(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$	
Structure	**Condition**	**m = 2**
T-symmetric	$\mathrm{P}^T(\lambda) = \mathrm{P}(\lambda)$	$\mathrm{P}(\lambda) = \lambda^2 A_0 + \lambda A_1 + A_2$, $A_0^T = A_0,\ A_1^T = A_1,\ A_2^T = A_2$
Hermitian	$\mathrm{P}^H(\lambda) = \mathrm{P}(\bar{\lambda})$	$\mathrm{P}(\lambda) = \lambda^2 A_0 + \lambda A_1 + A_2$, $A_0^H = A_0,\ A_1^H = A_1,\ A_2^H = A_2$
$*$-even	$\mathrm{P}^*(\lambda) = \mathrm{P}(-\lambda)$	$\mathrm{P}(\lambda) = \lambda^2 A + \lambda B + C$, $A^* = A, B^* = -B, C^* = C$
$*$-odd	$\mathrm{P}^*(\lambda) = -\mathrm{P}(-\lambda)$	$\mathrm{P}(\lambda) = \lambda^2 A + \lambda B + C$, $A^* = -A, B^* = B, C^* = -C$
$*$-palindromic	$\mathrm{P}^*(\lambda) = \lambda^m \mathrm{P}(1/\lambda)$	$\mathrm{P}(\lambda) = \lambda^2 A + \lambda B + A^*, B^* = B$
$*$-anti-palindromic	$\mathrm{P}^*(\lambda) = \lambda^m \mathrm{P}(-1/\lambda)$	$\mathrm{P}(\lambda) = \lambda^2 A + \lambda B - A^*, B^* = -B$

Table 1: Overview of structured matrix polynomials discussed in this paper. Note that $* \in \{T, H\}$ may denote either the complex transpose ($* = T$) or the Hermitian transpose ($* = H$).

characterized by the condition $\mathrm{P}^T(\lambda) = \lambda^m \mathrm{P}(1/\lambda)$. For even m, P takes the form

$$\mathrm{P}(\lambda) = A_0 + \cdots + \lambda^{m/2-1} A_{m/2-1} + \lambda^{m/2} A_{m/2} + \lambda^{m/2+1} A_{m/2+1}^T + \cdots + \lambda^m A_0^T,$$

with complex symmetric $A_{m/2}$, and for odd m, P it takes the form

$$\mathrm{P}(\lambda) = A_0 + \cdots + \lambda^{(m-1)/2} A_{(m-1)/2} + \lambda^{(m+1)/2} A_{(m+1)/2}^T + \cdots + \lambda^m A_0^T.$$

In certain situations, it is reasonable to expect that perturbations of the polynomial respect the underlying structure. For example, if a strongly backward stable eigenvalue solver was applied to a palindromic matrix polynomial then the computed eigenvalues would be the exact eigenvalues of a slightly perturbed *palindromic* eigenvalue problems. Also, structure-preserving perturbations are physically more meaningful in the sense that the spectral symmetries induced by the structure are not destroyed. Restricting the admissible perturbations might have a positive effect on the sensitivity of an eigenvalue. This question has been studied for linear eigenvalue problems in quite some detail recently [7, 12, 21, 19, 20, 22, 29, 30, 31]. It often turns out that the desirable positive effect is not very remarkable: in many cases the worst-case eigenvalue sensitivity changes little or not at all when imposing structure. Notable exceptions can be found among symplectic, skew-symmetric, and palindromic eigenvalue problems [21, 22]. Bora [6] has identified situations for which the structured and unstructured eigenvalue condition numbers for matrix polynomials are equal. In the first part of this paper, we will extend these results by providing explicit expressions for structured eigenvalue condition numbers of structured matrix polynomials.

Due to the lack of a robust genuine polynomial eigenvalue solver, the eigenvalues of P are usually computed by first reformulating (1) as an $mn \times mn$ linear generalized eigenvalue problem and then applying a standard method such as the QZ algorithm [11] to the linear problem. This process of linearization introduces unwanted effects. Besides the obvious increase of dimension, it may also happen that the eigenvalue sensitivities significantly deteriorate. Fortunately, one can use the freedom in the choice of linearization to minimize this deterioration for the eigenvalue region of interest, as proposed for quadratic eigenvalue problems in [10, 17, 33]. For the general polynomial eigenvalue problem (1), Higham et al. [16, 14] have identified linearizations with minimal eigenvalue condition number/backward error among the set of linearizations described in [28]. For structured polynomial eigenvalue problems, rather than using *any* linearization it is of course advisable to use one which has a similar structure. For example, it was shown in [27] that a palindromic matrix polynomial can usually be linearized into a palindromic or anti-palindromic matrix pencil, offering the possibility to apply structure-preserving algorithms to the linearization. It is natural to ask whether there is also a structured linearization that has no adverse effect on the structured condition number. For a small subset of structures from Table 1, this question has already been discussed in [16]. In the second part of this paper, we extend the discussion to all structures from Table 1.

The rest of this paper is organized as follows. In Section 2, we first review the derivation of the unstructured eigenvalue condition number for a matrix polynomial and then provide explicit expressions for structured eigenvalue conditions numbers. Most but not all of these expressions are generalizations of known results for linear eigenvalue problems. In Section 4, we apply these results to find good choices from the set of structured linearizations described in [27].

2 Structured condition numbers for matrix polynomials

Before discussing the effect of structure on the sensitivity of an eigenvalue, we briefly review existing results on eigenvalue condition numbers for matrix polynomials. Assume that λ is a *simple finite* eigenvalue of the matrix polynomial P defined in (1) with normalized right and left eigenvectors x and y:

$$\mathrm{P}(\lambda)x = 0, \quad y^H\mathrm{P}(\lambda) = 0, \quad \|x\|_2 = \|y\|_2 = 1. \tag{2}$$

The perturbation

$$(\mathrm{P} + \triangle\mathrm{P})(\lambda) = (A_0 + E_0) + \lambda(A_1 + E_1) + \cdots + \lambda^m(A_m + E_m)$$

moves λ to an eigenvalue $\widehat{\lambda}$ of $\mathrm{P} + \triangle\mathrm{P}$. A useful tool to study the effect of $\triangle\mathrm{P}$ is the first order *perturbation expansion*

$$\widehat{\lambda} = \lambda - \frac{1}{y^H\mathrm{P}'(\lambda)x} y^H\triangle\mathrm{P}(\lambda)x + O(\|\triangle\mathrm{P}\|^2), \tag{3}$$

which can be derived, e.g., by applying the implicit function theorem to (2), see [9, 33]. Note that $y^H\mathrm{P}'(\lambda)x \neq 0$ because λ is simple [3, 2].

To measure the sensitivity of λ we first need to specify a way to measure $\triangle\mathrm{P}$. Given a matrix norm $\|\cdot\|_M$ on $\mathbb{C}^{n\times n}$, a monotone vector norm $\|\cdot\|_v$ on $\mathbb{C}^{m+1}$ and non-negative weights $\omega_0, \ldots, \omega_m$, we define

$$\|\triangle\mathrm{P}\| := \left\| \left[\frac{1}{\omega_0}\|E_0\|_M,\ \frac{1}{\omega_1}\|E_1\|_M,\ \ldots,\ \frac{1}{\omega_m}\|E_m\|_M \right] \right\|_v. \tag{4}$$

A relatively small weight ω_i means that $\|E_i\|_M$ will be small compared to $\|\triangle\mathrm{P}\|$. In the extreme case $\omega_i = 0$, we define $\|E_i\|_M/\omega_i = 0$ for $\|E_i\|_M = 0$ and $\|E_i\|_M/\omega_i = \infty$ otherwise. If all ω_i are positive then (4) defines a norm on $\mathbb{C}^{n\times n} \times \cdots \times \mathbb{C}^{n\times n}$. See [2, 1] for more on norms of matrix polynomials.

We are now ready to introduce a condition number for the eigenvalue λ of P with respect to the choice of $\|\triangle\mathrm{P}\|$ in (4):

$$\kappa_{\mathrm{P}}(\lambda) := \lim_{\epsilon\to 0} \sup \left\{ \frac{|\hat{\lambda} - \lambda|}{\epsilon} :\ \|\triangle\mathrm{P}\| \leq \epsilon \right\}, \tag{5}$$

where $\hat{\lambda}$ is the eigenvalue of $\mathrm{P} + \triangle\mathrm{P}$ closest to λ. An explicit expression for $\kappa_{\mathrm{P}}(\lambda)$ can be found in [33, Thm. 5] for the case $\|\cdot\|_v \equiv \|\cdot\|_\infty$ and $\|\cdot\|_M \equiv \|\cdot\|_2$. In contrast, the approach used in [9] requires an accessible geometry on the perturbation space and thus facilitates the norms $\|\cdot\|_v \equiv \|\cdot\|_2$ and $\|\cdot\|_M \equiv \|\cdot\|_F$. Lemma 2.1 below is more general and includes both settings. Note that an alternative approach to the result of Lemma 2.1 is described in [1], admitting any matrix norm $\|\cdot\|_M$ and any (Hölder) p-norm $\|\cdot\|_V \equiv \|\cdot\|_p$. For stating our result, we recall that the dual to the vector norm $\|\cdot\|_v$ is defined as

$$\|w\|_d := \sup_{\|z\|_v \le 1} |w^T z|,$$

see, e.g., [13].

Lemma 2.1 *Consider the condition number $\kappa_{\mathrm{P}}(\lambda)$ defined in (5) with respect to (4). For any unitarily invariant norm $\|\cdot\|_M$ we have*

$$\kappa_{\mathrm{P}}(\lambda) = \frac{\|[\omega_0,\, \omega_1|\lambda|,\, \ldots, \omega_m|\lambda|^m]\|_d}{|y^H \mathrm{P}'(\lambda)x|} \tag{6}$$

where $\|\cdot\|_d$ denotes the vector norm dual to $\|\cdot\|_v$.

Proof. Inserting the perturbation expansion (3) into (5) yields

$$\kappa_{\mathrm{P}}(\lambda) = \frac{1}{|y^H \mathrm{P}'(\lambda)x|} \sup\left\{|y^H \triangle\mathrm{P}(\lambda)x| : \ \|\triangle\mathrm{P}\| \le 1\right\}. \tag{7}$$

Defining $b = [\|E_0\|_M/\omega_0, \ldots, \|E_m\|_M/\omega_m]^T$, we have $\|\triangle\mathrm{P}\| = \|b\|_v$. By the triangular inequality,

$$|y^H \triangle\mathrm{P}(\lambda)x| \le \sum_{i=0}^{m} |\lambda|^i |y^H E_i x|. \tag{8}$$

With a suitable scaling of E_i by a complex number of modulus 1, we can assume without loss of generality that equality holds in (8). Hence,

$$\sup_{\|\triangle\mathrm{P}\|\le 1} |y^H \triangle\mathrm{P}(\lambda)x| = \sup_{\|b\|_v \le 1} \sum_{i=0}^{m} |\lambda|^i \sup_{\|E_i\|_M = \omega_i b_i} |y^H E_i x|. \tag{9}$$

Using the particular perturbation $E_i = \omega_i b_i y x^H$, it can be easily seen that the inner supremum is $\omega_i b_i$ and hence

$$\sup_{\|\triangle\mathrm{P}\|\le 1} |y^H \triangle\mathrm{P}(\lambda)x| = \sup_{\|b\|_v \le 1} \left|[\omega_0, \omega_1|\lambda|, \ldots, \omega_m|\lambda|^m]b\right| = \|[\omega_0,\, \omega_1|\lambda|,\, \ldots, \omega_m|\lambda|^m]\|_d,$$

which completes the proof. □

From a practical point of view, measuring the perturbations of the individual coefficients of the polynomial separably makes a lot of sense and thus the choice $\|\cdot\|_v \equiv \|\cdot\|_\infty$ seems to be most natural. However, it turns out – especially when considering structured condition numbers – that more elegant results are obtained with the choice $\|\cdot\|_v \equiv \|\cdot\|_2$, which we will use throughout the rest of this paper. In this case, the expression (6) takes the form

$$\kappa_{\mathrm{P}}(\lambda) = \frac{\|[\omega_0,\, \omega_1\lambda,\, \ldots, \omega_m\lambda^m]\|_2}{|y^H \mathrm{P}'(\lambda)x|}, \tag{10}$$

see also [1, 4].

If $\lambda = \infty$ is a simple eigenvalue of P, a suitable condition number can be defined as

$$\kappa_{\mathrm{P}}(\infty) := \lim_{\epsilon\to 0} \sup\{1/|\hat{\lambda}\epsilon| : \ \|\triangle\mathrm{P}\| \le \epsilon\},$$

and, following the arguments above,

$$\kappa_{\mathrm{P}}(\infty) = \omega_m/|y^H A_{m-1}x|$$

for any p-norm $\|\cdot\|_v$. Note that this discrimination between finite and infinite disappears when homogenizing P as in [9] or measuring the distance between perturbed eigenvalues with the chordal metric as in [32]. In order to keep the presentation simple, we have decided not to use these concepts.

The rest of this section is concerned with quantifying the effect on the condition number if we restrict the perturbation $\triangle\mathrm{P}$ to a subset $\mathbb{S}$ of the space of all $n \times n$ matrix polynomials of degree at most m.

Definition 2.2 *Let λ be a simple finite eigenvalue of a matrix polynomial* P *with normalized right and left eigenvectors x and y. Then the* structured condition number *of λ with respect to $\mathbb{S}$ is defined as*

$$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) := \lim_{\epsilon\to 0} \sup \left\{ \frac{|\hat\lambda - \lambda|}{\epsilon} : \ \triangle\mathrm{P} \in \mathbb{S}, \ \|\triangle\mathrm{P}\| \le \epsilon \right\} \tag{11}$$

For infinite λ, $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) := \lim_{\epsilon\to 0} \sup\{1/|\hat\lambda\epsilon| : \ \triangle\mathrm{P} \in \mathbb{S}, \ \|\triangle\mathrm{P}\| \le \epsilon\}$.

If $\mathbb{S}$ is star-shaped, the expansion (3) can be used to show

$$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \frac{1}{|y^H \mathrm{P}'(\lambda) x|} \sup\left\{ |y^H \triangle\mathrm{P}(\lambda) x| : \ \triangle\mathrm{P} \in \mathbb{S}, \ \|\triangle\mathrm{P}\| \le 1 \right\} \tag{12}$$

and

$$\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = \frac{1}{|y^H A_{m-1} x|} \sup\left\{ |y^H E_m x| : \ \triangle\mathrm{P} \in \mathbb{S}, \ \|E_m\|_M \le \omega_m \right\}. \tag{13}$$

2.1 Structured first-order perturbation sets

To proceed from (12) we need to find the maximal absolute magnitude of elements from the set

$$\left\{ y^H \triangle\mathrm{P}(\lambda) x = y^H E_0 x + \lambda y^H E_1 x + \cdots \lambda^m y^H E_m x : \ \triangle\mathrm{P} \in \mathbb{S}, \ \|\triangle\mathrm{P}\| \le 1 \right\} \tag{14}$$

It is therefore of interest to study the nature of the set $\{y^H E x : E \in \mathbb{E}, \ \|E\|_M \le 1\}$ with respect to some $\mathbb{E} \subseteq \mathbb{C}^{n\times n}$. The following theorem by Karow [19] provides explicit descriptions of this set for certain $\mathbb{E}$. We use $\cong$ to denote the natural isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$.

Theorem 2.3 *Let $\mathbb{K}(\mathbb{E}, x, y) := \{y^H E x : E \in \mathbb{E}, \ \|E\|_M \le 1\}$ for $x, y \in \mathbb{C}^n$ with $\|x\|_2 = \|y\|_2 = 1$ and some $\mathbb{E} \subseteq \mathbb{C}^{n\times n}$. Provided that $\|\cdot\|_M \in \{\|\cdot\|_2, \|\cdot\|_F\}$, the set $\mathbb{K}(\mathbb{E}, x, y)$ is an ellipse taking the form*

$$\mathbb{K}(\mathbb{E}, x, y) \cong \mathbb{K}(\alpha, \beta) := \left\{ K(\alpha, \beta)\xi : \ \xi \in \mathbb{R}^2, \|\xi\|_2 \le 1 \right\}, \quad K(\alpha, \beta) \in \mathbb{R}^{2\times 2}, \tag{15}$$

for the cases that $\mathbb{E}$ consists of all complex ($\mathbb{E} = \mathbb{C}^{n\times n}$), real ($\mathbb{E} = \mathbb{R}^{n\times n}$), Hermitian ($\mathbb{E} =$ Herm), complex symmetric ($\mathbb{E} =$ symm), and complex skew-symmetric ($\mathbb{E} =$ skew), real symmetric (only for $\|\cdot\|_M \equiv \|\cdot\|_F$), and real skew-symmetric matrices. The matrix $K(\alpha, \beta)$ defining the ellipse in (15) can be written as

$$K(\alpha, \beta) = \begin{bmatrix} \cos\phi/2 & \sin\phi/2 \\ -\sin\phi/2 & \cos\phi/2 \end{bmatrix} \begin{bmatrix} \sqrt{\alpha + |\beta|} & 0 \\ 0 & \sqrt{\alpha - |\beta|} \end{bmatrix} \tag{16}$$

with some of the parameter configurations α, β listed in Table 2, and $\phi = \arg(\beta)$.

Note that (15)–(16) describes an ellipse with semiaxes $\sqrt{\alpha + |\beta|}$, $\sqrt{\alpha - |\beta|}$, rotated by the angle $\phi/2$. The Minkowski sum of ellipses is still convex but in general not an ellipse [23]. Finding the maximal element in (14) is equivalent to finding the maximal element in the Minkowski sum.

$\mathbb{E}$	$\|\cdot\|_M \equiv \|\cdot\|_2$ α	β	$\|\cdot\|_M \equiv \|\cdot\|_F$ α	β
$\mathbb{C}^{n\times n}$	1	0	1	0
Herm	$1-\frac{1}{2}\|y^H x\|^2$	$\frac{1}{2}(y^H x)^2$	$\frac{1}{2}$	$\frac{1}{2}(y^H x)^2$
symm	1	0	$\frac{1}{2}(1+\|y^T x\|^2)$	0
skew	$1-\|y^T x\|^2$	0	$\frac{1}{2}(1-\|y^T x\|^2)$	0

Table 2: Parameters defining the ellipse (15).

Lemma 2.4 *Let $\mathbb{K}(\alpha_0,\beta_0),\dots,\mathbb{K}(\alpha_m,\beta_m)$ be ellipses of the form (15)–(16). Define*

$$\sigma := \sup_{\substack{b_0,\dots,b_m\in\mathbb{R}\\ b_0^2+\dots+b_m^2\le 1}} \sup\left\{\|s\|_2 :\ s\in b_0\mathbb{K}(\alpha_0,\beta_0)+\dots+b_m\mathbb{K}(\alpha_m,\beta_m)\right\} \tag{17}$$

using the Minkowski sum of sets. Then

$$\sigma = \|[K(\alpha_0,\beta_0),\dots,K(\alpha_m,\beta_m)]\|_2, \tag{18}$$

and

$$\sqrt{\alpha_0+\dots+\alpha_m} \le \sigma \le \sqrt{2}\sqrt{\alpha_0+\dots+\alpha_m}. \tag{19}$$

Proof. By the definition of $\mathbb{K}(\alpha_j,\beta_j)$, it holds that

$$\begin{aligned}
\sigma &= \sup_{\substack{b_i\in\mathbb{R}\\ b_0^2+\dots+b_m^2\le 1}} \sup_{\substack{\xi_i\in\mathbb{R}^2\\ \|\xi_i\|_2\le 1}} \left\|b_0K(\alpha_0,\beta_0)\xi_0+\dots+b_mK(\alpha_m,\beta_m)\xi_m\right\|_2\\
&= \sup_{\substack{b_i\in\mathbb{R}\\ b_0^2+\dots+b_m^2\le 1}} \sup_{\substack{\tilde\xi_i\in\mathbb{R}^2\\ \|\tilde\xi_i\|_2\le b_i}} \left\|K(\alpha_0,\beta_0)\tilde\xi_0+\dots+K(\alpha_m,\beta_m)\tilde\xi_m\right\|_2\\
&= \sup_{\substack{\tilde\xi_i\in\mathbb{R}^2\\ \|\tilde\xi_0\|_2^2+\dots+\|\tilde\xi_m\|_2^2\le 1}} \left\|K(\alpha_0,\beta_0)\tilde\xi_0+\dots+K(\alpha_m,\beta_m)\tilde\xi_m\right\|_2\\
&= \left\|\left[K(\alpha_0,\beta_0),\ \dots,\ K(\alpha_m,\beta_m)\right]\right\|_2,
\end{aligned}$$

applying the definition of the matrix 2-norm. The inequality (19) then follows from the well-known bound

$$\frac{1}{\sqrt{2}}\|[K(\alpha_0,\beta_0),\dots,K(\alpha_m,\beta_m)]\|_F \le \sigma \le \|[K(\alpha_0,\beta_0),\dots,K(\alpha_m,\beta_m)]\|_F$$

and using the fact that:

$$\|[K(\alpha_0,\beta_0),\dots,K(\alpha_m,\beta_m)]\|_F^2 = \sum_{i=0}^m \|K(\alpha_i,\beta_i)\|_F^2 = \sum_{i=0}^m 2\alpha_i.$$

□

It is instructive to rederive the expression (10) for the unstructured condition number from Lemma 2.4. Starting from Equation (7), we insert the definition (4) of $\|\triangle\mathrm{P}\|$ for $\|\cdot\|_M\equiv\|\cdot\|_2$, $\|\cdot\|_v\equiv\|\cdot\|_2$, and obtain

$$\begin{aligned}
\sigma &= \sup\left\{|y^H\triangle\mathrm{P}(\lambda)x| : \|\triangle\mathrm{P}\|\le 1\right\}\\
&= \sup_{\substack{b_0^2+\dots+b_m^2\le 1\\ \|E_0\|^2\le b_0,\dots,\|E_m\|^2\le b_m}} \left|\sum_{i=0}^m \omega_i\lambda^i y^H E_i x\right|\\
&= \sup_{b_0^2+\dots+b_m^2\le 1} \sup\left\{|s| :\ s\in\sum_{i=0}^m b_i\omega_i\lambda_i\mathbb{K}(\mathbb{C}^{n\times n},x,y)\right\}.
\end{aligned} \tag{20}$$

By Theorem 2.3, $\mathbb{K}(\mathbb{C}^{n\times n}, x, y) \cong \mathbb{K}(1, 0)$ and, since a disk is invariant under rotation, $\omega_i \lambda^i \mathbb{K}(\mathbb{C}^{n\times n}, x, y) \cong \mathbb{K}(\omega_i^2 |\lambda|^{2i}, 0)$. Applying Lemma 2.4 yields

$$\sigma = \left\| \left[K(\omega_0^2, 0), K(\omega_1^2|\lambda|^2, 0), \ldots, K(\omega_m^2|\lambda|^{2m}, 0) \right] \right\|_2 = \left\| \left[\omega_0, \omega_1\lambda, \ldots, \omega_m\lambda^m \right] \right\|_2,$$

which together with (7) results in the known expression (10) for $\kappa_{\mathrm{P}}(\lambda)$.

In the following sections, it will be shown that the expressions for structured condition numbers follow in a similar way as corollaries from Lemma 2.4. To keep the notation compact, we define

$$\sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sup \left\{ |y^H \triangle\mathrm{P}(\lambda) x| : \ \triangle\mathrm{P} \in \mathbb{S}, \ \|\triangle\mathrm{P}\| \le 1 \right\}.$$

for a star-shaped structure $\mathbb{S}$. By (12), $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda)/|y^H \mathrm{P}'(\lambda) x|$. Let us recall that the vector norm underlying the definition of $\|\triangle\mathrm{P}\|$ in (4), is chosen as $\|\cdot\|_v \equiv \|\cdot\|_2$ throughout the rest of this paper.

2.2 Complex symmetric matrix polynomials

No or only an insignificant decrease of the condition number can be expected when imposing complex symmetries on the perturbations of a matrix polynomial.

Corollary 2.5 *Let $\mathbb{S}$ denote the set of complex symmetric matrix polynomials. Then for a finite or infinite, simple eigenvalue λ of a matrix polynomial $\mathrm{P} \in \mathbb{S}$,*

1. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_M \equiv \|\cdot\|_2$, *and*
2. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \frac{\sqrt{1+|y^T x|^2}}{\sqrt{2}} \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_M \equiv \|\cdot\|_F$.

Proof. Along the line of arguments leading to (20),

$$\sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sup_{b_0^2+\cdots+b_m^2 \le 1} \left\{ \|s\|_2 : \ s \in \sum_{i=0}^{m} b_i \omega_i \lambda^i \mathbb{K}(\mathsf{symm}, x, y) \right\}$$

for finite λ. As in the unstructured case, $\mathbb{K}(\mathsf{symm}, x, y) \cong \mathbb{K}(1, 0)$ for $\|\cdot\|_M \equiv \|\cdot\|_2$ by Theorem 2.3, and thus $\kappa_{\mathrm{P}}(\lambda) = \kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda)$. For $\|\cdot\|_M \equiv \|\cdot\|_F$ we have

$$\mathbb{K}(\mathsf{symm}, x, y) \cong \mathbb{K}((1 + |y^T x|^2)/2, 0) = \frac{\sqrt{1 + |y^T x|^2}}{\sqrt{2}} \mathbb{K}(1, 0),$$

showing the second part of the statement. The proof for infinite λ is entirely analogous. □

2.3 T-even and T-odd matrix polynomials

To describe the structured condition numbers for T-even and T-odd polynomials in a convenient manner, we introduce the vector

$$\Lambda_\omega = \left[\omega_m \lambda^m, \, \omega_{m-1}\lambda^{m-1}, \, \ldots, \, \omega_1 \lambda, \, \omega_0 \right]^T \tag{21}$$

along with the even coefficient projector

$$\Pi_e : \Lambda_\omega \mapsto \Pi_e(\Lambda_\omega) := \begin{cases} \left[\omega_m \lambda^m, \, 0, \, \omega_{m-2}\lambda^{m-2}, \, 0, \, \ldots, \, \omega_2 \lambda^2, \, 0, \, \omega_0 \right]^T, & \text{if } m \text{ is even}, \\ \left[0, \omega_{m-1}\lambda^{m-1}, \, 0, \, \omega_{m-3}\lambda^{m-3}, \, \ldots, \, 0, \, \omega_0 \right]^T, & \text{if } m \text{ is odd}. \end{cases} \tag{22}$$

The odd coefficient projection is defined analogously and can be written as $(1 - \Pi_e)(\Lambda_\omega)$.

Lemma 2.6 *Let $\mathbb{S}$ denote the set of all T-even matrix polynomials. Then for a finite, simple eigenvalue λ of a matrix polynomial $\mathrm{P} \in \mathbb{S}$,*

1. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sqrt{1 - |y^T x|^2 \frac{\|(1-\Pi_e)(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2}}\, \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_M \equiv \|\cdot\|_2$, *and*

2. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \frac{1}{\sqrt{2}}\sqrt{1 - |y^T x|^2 \frac{\|(1-\Pi_e)(\Lambda_\omega)\|_2^2 - \|\Pi_e(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2}}\, \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_M \equiv \|\cdot\|_F$.

For an infinite, simple eigenvalue,

3. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = \begin{cases} \kappa_{\mathrm{P}}(\infty), & \text{if } m \text{ is even}, \\ \sqrt{1-|y^T x|^2}\, \kappa_P(\infty), & \text{if } m \text{ is odd}, \end{cases}$ *for* $\|\cdot\|_M \equiv \|\cdot\|_2$, *and*

4. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = \begin{cases} \frac{1}{\sqrt{2}}\sqrt{1+|y^T x|^2}\kappa_{\mathrm{P}}(\infty), & \text{if } m \text{ is even}, \\ \frac{1}{\sqrt{2}}\sqrt{1-|y^T x|^2}\kappa_{\mathrm{P}}(\infty), & \text{if } m \text{ is odd}, \end{cases}$ *for* $\|\cdot\|_M \equiv \|\cdot\|_F$.

Proof. By definition, the even coefficients of a T-even polynomial are symmetric while the odd coefficients are skew-symmetric. Thus, for finite λ,

$$\sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sup_{b_0^2+\cdots+b_m^2 \le 1} \sup \Big\{ \|s\|_2 : \; s \in \sum_{i \text{ even}} b_i \omega_i \lambda^i \mathbb{K}(\mathsf{symm}, x, y) + \sum_{i \text{ odd}} b_i \omega_i \lambda^i \mathbb{K}(\mathsf{skew}, x, y) \Big\}.$$

Applying Theorem 2.3 and Lemma 2.4 yields for $\|\cdot\|_M \equiv \|\cdot\|_2$,

$$\begin{aligned} \sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) &= \Big\| \Big[\Pi_e(\Lambda_\omega)^T \otimes K(1,0),\, (1-\Pi_e)(\Lambda_\omega)^T \otimes K(1-|y^T x|^2, 0) \Big] \Big\|_2 \\ &= \Big\| \Big[\Pi_e(\Lambda_\omega)^T,\, \sqrt{1-|y^T x|^2}(1-\Pi_e)(\Lambda_\omega)^T \Big] \Big\|_2 \\ &= \sqrt{\|\Lambda_\omega\|_2^2 - |y^T x|^2 \|(1-\Pi_e)(\Lambda_\omega)\|_2^2}, \end{aligned}$$

once again using the fact that a disk is invariant under rotation. Similarly, it follows for $\|\cdot\|_M \equiv \|\cdot\|_F$ that

$$\begin{aligned} \sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) &= \frac{1}{\sqrt{2}} \Big\| \Big[\sqrt{1+|y^T x|^2}\Pi_e(\Lambda_\omega)^T,\, \sqrt{1-|y^T x|^2}(1-\Pi_e)(\Lambda_\omega)^T \Big] \Big\|_2 \\ &= \frac{1}{\sqrt{2}} \sqrt{\|\Lambda_\omega\|_2^2 + |y^T x|^2 \big(\|\Pi_e(\Lambda_\omega)\|_2^2 - \|(1-\Pi_e)(\Lambda_\omega)\|_2^2 \big)}. \end{aligned}$$

The result for infinite λ follows in an analogous manner. □

Remark 2.7 *Note that the statement of Lemma 2.6 does not assume that* P *itself is T-even. If we impose this condition then, for odd m,* P *has a* simple *infinite eigenvalue only if also the size of* P *is odd, see, e.g., [22]. In this case, the skew-symmetry of A_m forces the infinite eigenvalue to be preserved under arbitrary structure-preserving perturbations. This is reflected by $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = 0$.*

Lemma 2.6 reveals that the structured condition number can only be significantly lower than the unstructured one if $|y^T x|$ *and* the ratio

$$\frac{\|(1-\Pi_e)(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2} = \frac{\sum\limits_{i \text{ odd}} \omega_i^2 |\lambda|^{2i}}{\sum\limits_{i=0,\ldots,m} \omega_i^2 |\lambda|^{2i}} = 1 - \frac{\sum\limits_{i \text{ even}} \omega_i^2 |\lambda|^{2i}}{\sum\limits_{i=0,\ldots,m} \omega_i^2 |\lambda|^{2i}}$$

are close to one. The most likely situation for the latter ratio to become close to one is when m is odd, ω_m does not vanish, and $|\lambda|$ is large.

Example 2.8 ([31]) *Let*

$$\mathrm{P}(\lambda) = I + \lambda 0 + \lambda^2 I + \lambda^3 \begin{bmatrix} 0 & 1-\phi & 0 \\ -1+\phi & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{bmatrix}$$

with $0 < \phi < 1$. This matrix polynomial has one eigenvalue $\lambda_\infty = \infty$ because of the highest coefficient, which is – as any odd-sized skew-symmetric matrix – singular. The following table additionally displays the eigenvalue $\lambda_{\max}$ of largest magnitude, the eigenvalue $\lambda_{\min}$ of smallest magnitude, as well as their unstructured and structured condition numbers for the set $\mathbb{S}$ of T-even matrix polynomials. We have chosen $\omega_i = \|A_i\|_2$ and $\|\cdot\|_M \equiv \|\cdot\|_2$.

ϕ	10^0	10^{-3}	10^{-9}
$\kappa(\lambda_\infty)$	1	1.4×10^3	1.4×10^9
$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda_\infty)$	0	0	0
$\|\lambda_{\max}\|$	1.47	22.4	2.2×10^4
$\kappa_{\mathrm{P}}(\lambda_{\max})$	1.12	3.5×10^5	3.5×10^{17}
$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda_{\max})$	1.12	2.5×10^4	2.5×10^{13}
$\|\lambda_{\min}\|$	0.83	0.99	1.00
$\kappa_{\mathrm{P}}(\lambda_{\min})$	0.45	5.0×10^2	5.0×10^8
$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda_{\min})$	0.45	3.5×10^2	3.5×10^8

The entries $0 = \kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda_\infty) \ll \kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda_\infty)$ reflect the fact that the infinite eigenvalue stays intact under structure-preserving but not under general perturbations. For the largest eigenvalues, we observe a significant difference between the structured and unstructured condition numbers as $\phi \to 0$. In contrast, this difference becomes negligible for the smallest eigenvalues.

Remark 2.9 *For* even m, *the structured eigenvalue condition number of a T-even polynomial is usually close to the unstructured one. For example if all weights are equal, $\|(1-\Pi_e)(\Lambda)\|_2^2 \leq \|\Lambda\|_2^2/2$ implying $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) \geq \kappa_{\mathrm{P}}(\lambda)/\sqrt{2}$ for $\|\cdot\|_M \equiv \|\cdot\|_2$.*

For T-odd polynomials, we obtain the following analogue of Lemma 2.6 by simply exchanging the roles of odd and even in the proof.

Lemma 2.10 *Let $\mathbb{S}$ denote the set of all T-odd matrix polynomials. Then for a finite, simple eigenvalue λ of a matrix polynomial $\mathrm{P} \in \mathbb{S}$,*

1. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sqrt{1 - |y^T x|^2 \frac{\|\Pi_e(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2}}\, \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_M \equiv \|\cdot\|_2$, *and*
2. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \frac{1}{\sqrt{2}} \sqrt{1 - |y^T x|^2 \frac{\|\Pi_e(\Lambda_\omega)\|_2^2 - \|(1-\Pi_e)(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2}}\, \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_M \equiv \|\cdot\|_F$.

For an infinite, simple eigenvalue,

3. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = \begin{cases} \kappa_{\mathrm{P}}(\infty), & \text{if } m \text{ is odd}, \\ 0, & \text{if } m \text{ is even}, \end{cases}$ *for* $\|\cdot\|_M \equiv \|\cdot\|_2$, *and*
4. $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = \begin{cases} \frac{1}{\sqrt{2}}\sqrt{1 + |y^T x|^2}\kappa_{\mathrm{P}}(\infty), & \text{if } m \text{ is odd}, \\ 0, & \text{if } m \text{ is even}, \end{cases}$ *for* $\|\cdot\|_M \equiv \|\cdot\|_F$.

Similar to the discussion above, the only situation for which $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda)$ can be expected to become significantly smaller than $\kappa_{\mathrm{P}}(\lambda)$ is for $|y^T x| \approx 1$ and $\lambda \approx 0$.

2.4 T-palindromic and T-anti-palindromic matrix polynomials

For a T-palindromic polynomial it is sensible to require that the weights in the choice of $\|\triangle \mathrm{P}\|$, see (4), satisfy $\omega_i = \omega_{m-i}$. This condition is tacitly assumed throughout the entire section. The Cayley transform for polynomials introduced in [27, Sec. 2.2] defines a mapping between palindromic/anti-palindromic and odd/even polynomials. As already demonstrated in [22] for the case $m = 1$, this idea can be used to transfer the results from the previous section to the (anti-)palindromic case. For the mapping to preserve the underlying norm we have to restrict ourselves to the case $\|\cdot\|_M \equiv \|\cdot\|_F$. The coefficient projections appropriate for palindromic polynomials are given by $\Pi_\pm : \Lambda_\omega \mapsto \Pi_s(\Lambda_\omega)$ with

$$\Pi_\pm(\Lambda_\omega) := \begin{cases} \left[\omega_0 \frac{\lambda^m \pm 1}{\sqrt{2}}, \ldots, \omega_{m/2-1} \frac{\lambda^{m/2+1} \pm \lambda^{m/2-1}}{\sqrt{2}}, \omega_{m/2} \frac{\lambda^{m/2} \pm \lambda^{m/2}}{2}\right]^T & \text{if } m \text{ is even,} \\ \left[\omega_0 \frac{\lambda^m \pm 1}{\sqrt{2}}, \ldots, \omega_{(m-1)/2} \frac{\lambda^{(m+1)/2} \pm \lambda^{(m-1)/2}}{\sqrt{2}}\right]^T, & \text{if } m \text{ is odd.} \end{cases} \tag{23}$$

Note that $\|\Pi_+(\Lambda_\omega)\|_2^2 + \|\Pi_-(\Lambda_\omega)\|_2^2 = \|\Lambda_\omega\|_2^2$.

Lemma 2.11 *Let $\mathbb{S}$ denote the set of all T-palindromic matrix polynomials. Then for a finite, simple eigenvalue λ of a matrix polynomial $\mathrm{P} \in \mathbb{S}$, with $\|\cdot\|_M \equiv \|\cdot\|_F$,*

$$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \frac{1}{\sqrt{2}} \sqrt{1 + |y^T x|^2 \frac{\|\Pi_+(\Lambda_\omega)\|_2^2 - \|\Pi_-(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2}}\, \kappa_{\mathrm{P}}(\lambda).$$

For an infinite, simple eigenvalue, $\kappa_{\mathrm{P}}^{\mathbb{S}}(\infty) = \kappa_{\mathrm{P}}(\infty)$.

Proof. Assume m is odd. For $\triangle\mathrm{P} \in \mathbb{S}$,

$$\begin{aligned} \triangle\mathrm{P}(\lambda) &= \sum_{i=0}^{(m-1)/2} \lambda^i E_i + \sum_{i=0}^{(m-1)/2} \lambda^{m-i} E_i^T \\ &= \sum_{i=0}^{(m-1)/2} \frac{\lambda^i + \lambda^{m-i}}{\sqrt{2}} \frac{E_i + E_i^T}{\sqrt{2}} + \sum_{i=0}^{(m-1)/2} \frac{\lambda^i - \lambda^{m-i}}{\sqrt{2}} \frac{E_i - E_i^T}{\sqrt{2}}. \end{aligned}$$

Let us introduce the auxiliary polynomial

$$\triangle\widetilde{\mathrm{P}}(\mu) = \sum_{i=0}^{(m-1)/2} \mu^{2i} S_i + \sum_{i=0}^{(m-1)/2} \mu^{2i+1} W_i, \quad S_i = \frac{E_i + E_i^T}{\sqrt{2}}, \ W_i = \frac{E_i - E_i^T}{\sqrt{2}}.$$

Then $\widetilde{\mathrm{P}} \in \widetilde{\mathbb{S}}$, where $\widetilde{\mathbb{S}}$ denotes the set of T-even polynomials. Since symmetric and skew-symmetric matrices are orthogonal to each other with respect to the matrix inner product $\langle A, B\rangle = \mathrm{trace}(B^H A)$, we have $\|A\|_F^2 + \|A^T\|_F^2 = \|(A + A^T)/\sqrt{2}\|_F^2 + \|(A - A^T)/\sqrt{2}\|_F^2$ for any $A \in \mathbb{C}^{n\times n}$ and hence $\|\triangle\mathrm{P}\| = \|\triangle\widetilde{\mathrm{P}}\|$ for $\|\cdot\|_M \equiv \|\cdot\|_F$. This allows us to write

$$\begin{aligned} \sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) &= \sup\left\{|y^H \triangle\mathrm{P}(\lambda) x| : \ \triangle\mathrm{P} \in \mathbb{S}, \ \|\triangle\mathrm{P}\| \le 1\right\} \\ &= \sup\left\{\left|\sum \frac{\lambda^i + \lambda^{m-i}}{\sqrt{2}} y^H S_i x + \sum \frac{\lambda^i - \lambda^{m-i}}{\sqrt{2}} y^H W_i x\right| : \ \triangle\widetilde{\mathrm{P}} \in \widetilde{\mathbb{S}}, \ \|\triangle\widetilde{\mathrm{P}}\| \le 1\right\} \\ &= \frac{1}{\sqrt{2}} \sup_{b_0^2+\cdots+b_m^2 \le 1} \Big\{\|s\|_2 : \ s \in \sum b_i \omega_i (\lambda^i + \lambda^{m-i}) \mathbb{K}(\mathsf{symm}, x, y) \\ &\qquad\qquad + \sum b_{(m-1)/2+i} \omega_i (\lambda^i - \lambda^{m-i}) \mathbb{K}(\mathsf{skew}, x, y)\Big\} \\ &= \frac{1}{2} \sqrt{(1 + |y^T x|^2) \sum \omega_i^2 |\lambda^i + \lambda^{m-i}|^2 + (1 - |y^T x|^2) \sum \omega_i^2 |\lambda^i - \lambda^{m-i}|^2} \\ &= \frac{1}{\sqrt{2}} \sqrt{(1 + |y^T x|^2) \|\Pi_+(\Lambda_\omega)\|_2^2 + (1 - |y^T x|^2) \|\Pi_-(\Lambda_\omega)\|_2^2} \\ &= \frac{1}{\sqrt{2}} \sqrt{\|\Lambda_\omega\|_2^2 + |y^T x|^2 (\|\Pi_+(\Lambda_\omega)\|_2^2 - \|\Pi_-(\Lambda_\omega)\|_2^2)}, \end{aligned}$$

where we used Theorem 2.3 and Lemma 2.4.

For even m the proof is almost the same; with the only difference that the transformation leaves the complex symmetric middle coefficient $A_{m/2}$ unaltered.

For $\lambda = \infty$, observe that the corresponding optimization problem (13) involves only a single, unstructured coefficient of the polynomial and hence palindromic structure has no effect on the condition number. □

From the result of Lemma 2.11 it follows that a large difference between the structured and unstructured condition numbers for T-palindromic matrix polynomials may occur when $|y^T x|$ is close to one, *and* $\|\Pi_+(\Lambda_\omega)\|_2$ is close to zero. Assuming that all weights are positive, the latter condition implies that m is odd and and $\lambda \approx -1$. An instance of such a case is given by a variation of Example 2.8.

Example 2.12 *Consider the T-palindromic matrix polynomial*

$$\mathrm{P}(\lambda) = \begin{bmatrix} 1 & 1-\phi & 0 \\ -1+\phi & 1 & \mathrm{i} \\ 0 & -\mathrm{i} & 1 \end{bmatrix} + \lambda I + \lambda^2 I - \lambda^3 \begin{bmatrix} 1 & 1-\phi & 0 \\ -1+\phi & 1 & \mathrm{i} \\ 0 & -\mathrm{i} & 1 \end{bmatrix}$$

with $0 < \phi < 1$. An odd-sized T-palindromic matrix polynomial, P *has the eigenvalue $\lambda_{-1} = -1$. The following table additionally displays one eigenvalue λ_{close} closest to -1, an eigenvalue $\lambda_{\min}$ of smallest magnitude, as well as their unstructured and structured condition numbers for the set $\mathbb{S}$ of T-palindromic matrix polynomials. We have chosen $\omega_i = \|A_i\|_F$ and $\|\cdot\|_M \equiv \|\cdot\|_F$.*

ϕ	10^{-1}	10^{-4}	10^{-8}
$\kappa(\lambda_{-1})$	20.9	2.2×10^4	2.2×10^8
$\kappa_\mathrm{P}^{\mathbb{S}}(\lambda_{-1})$	0	0	0
$\|1+\lambda_{\text{close}}\|$	0.39	1.4×10^{-2}	1.4×10^{-4}
$\kappa_\mathrm{P}(\lambda_{\text{close}})$	11.1	1.1×10^4	1.1×10^8
$\kappa_\mathrm{P}^{\mathbb{S}}(\lambda_{\text{closer}})$	6.38	2.5×10^2	2.6×10^4
$\|1+\lambda_{\min}\|$	1.25	1.41	1.41
$\kappa_\mathrm{P}(\lambda_{\min})$	7.92	7.9×10^3	7.9×10^7
$\kappa_\mathrm{P}^{\mathbb{S}}(\lambda_{\min})$	5.75	5.6×10^3	5.6×10^7

The entries $0 = \kappa_\mathrm{P}^{\mathbb{S}}(\lambda_{-1}) \ll \kappa_\mathrm{P}(\lambda_{-1})$ reflect the fact that the eigenvalue -1 remains intact under structure-preserving but not under general perturbations. Also, eigenvalues close to -1 benefit from a significantly lower structured condition numbers as $\phi \to 0$. In contrast, only a practically irrelevant benefit is revealed for the eigenvalue $\lambda_{\min}$ not close to -1.

Structured eigenvalue condition numbers for T-anti-palindromic matrix polynomials can be derived in the same way as in Lemma 2.11.

Lemma 2.13 *Let $\mathbb{S}$ denote the set of all T-anti-palindromic matrix polynomials. Then for a finite, simple eigenvalue λ of a matrix polynomial $\mathrm{P} \in \mathbb{S}$, with $\|\cdot\|_M \equiv \|\cdot\|_F$,*

$$\kappa_\mathrm{P}^{\mathbb{S}}(\lambda) = \frac{1}{\sqrt{2}} \sqrt{1 - |y^T x|^2 \frac{\|\Pi_+(\Lambda_\omega)\|_2^2 - \|\Pi_-(\Lambda_\omega)\|_2^2}{\|\Lambda_\omega\|_2^2}}\, \kappa_\mathrm{P}(\lambda).$$

For an infinite, simple eigenvalue, $\kappa_\mathrm{P}^{\mathbb{S}}(\infty) = \kappa_\mathrm{P}(\infty)$.

2.5 Hermitian matrix polynomials

The derivations in the previous sections were greatly simplified by the fact that the first-order perturbation sets under consideration were disks. For the set of Hermitian perturbations, however, $y^H E_i x$ forms truly an ellipse. Still, a computable expression is provided by (18) from Lemma 2.4. However, the explicit formulas derived from this expression take a very technical form and provide little immediate intuition on the difference between the structured and unstructured condition number. Therefore, we will work with the bound (19) instead.

Lemma 2.14 *Let $\mathbb{S}$ denote the set of all Hermitian matrix polynomials. Then for a finite or infinite, simple eigenvalue of a matrix polynomial* $\mathrm{P} \in \mathbb{S}$,

1. $\sqrt{1 - \frac{1}{2}|y^H x|^2}\ \kappa_{\mathrm{P}}(\lambda) \leq \kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) \leq \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_M \equiv \|\cdot\|_2$, *and*

2. $\kappa_{\mathrm{P}}(\lambda)/\sqrt{2} \leq \kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) \leq \kappa_{\mathrm{P}}(\lambda)$ *for* $\|\cdot\|_M \equiv \|\cdot\|_F$.

Proof. Let $\|\cdot\|_M \equiv \|\cdot\|_F$. Then Theorem 2.3 states

$$\mathbb{K}(\mathsf{Herm}, x, y) \cong \mathbb{K}(1/2, (y^H x)^2/2).$$

Consequently,

$$\omega_i \lambda^i \mathbb{K}(\mathsf{Herm}, x, y) \cong \mathbb{K}(\omega_i^2 |\lambda|^{2i}/2, \omega_i^2 \lambda^{2i} (y^H x)^2/2),$$

which implies

$$\sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \sup_{b_0^2 + \cdots + b_m^2 \leq 1} \left\{ \|s\|_2 :\ s \in \sum_{i=0}^{m} b_i \mathbb{K}(\omega_i^2 \lambda^{2i}/2, \omega_i^2 \lambda^{2i} (y^H x)^2/2) \right\}.$$

By Lemma 2.4,

$$\frac{1}{\sqrt{2}} \|\Lambda_\omega\|_2 \leq \sigma_{\mathrm{P}}^{\mathbb{S}}(\lambda) \leq \|\Lambda_\omega\|_2.$$

The proof for the case $\|\cdot\|_M \equiv \|\cdot\|_2$ is analogous. □

Remark 2.15 *Since Hermitian and skew-Hermitian matrices are related by multiplication with* i, *which simply rotates the first-order perturbation set by* 90 *degrees, a slight modification of the proof shows that the statement of Lemma 2.14 remains true when* $\mathbb{S}$ *denotes the space of H-odd or H-even polynomials. This can in turn be used – as in the proof of Lemma 2.11 – to show that also for H-(anti-)palindromic polynomials there is at most an insignificant difference between the structured and unstructured eigenvalue condition numbers.*

3 Condition numbers for linearizations

As already mentioned in the introduction, polynomial eigenvalue problems are often solved by first linearizing the matrix polynomial into a larger matrix pencil. Of the classes of linearizations proposed in the literature, the vector spaces $\mathbb{DL}(\mathrm{P})$ introduced in [28] are particularly amenable to further analysis, while offering a degree of generality that is often sufficient in applications.

Definition 3.1 *Let* $\Lambda_{m-1} = [\lambda^{m-1}, \lambda^{m-2} \dots \lambda, 1]^T$ *and let* P *be a matrix polynomial of degree m. Then a matrix pencil* $L(\lambda) = \lambda X + Y \in \mathbb{C}^{mn \times mn}$ *is in* $\mathbb{DL}(\mathrm{P})$ *if there is a so called* ansatz vector $v \in \mathbb{C}^m$ *satisfying*

$$L(\lambda) \cdot (\Lambda_{m-1} \otimes I) = v \otimes P(\lambda) \quad \text{and} \quad (\Lambda_{m-1}^T \otimes I) \cdot L(\lambda) = v^T \otimes P(\lambda).$$

It is easy to see that the ansatz vector v is uniquely determined by $L \in \mathbb{DL}(\mathrm{P})$. In [28, Thm. 6.7] it has been shown that $L \in \mathbb{DL}(\mathrm{P})$ is a linearization of P if and only if none of the eigenvalues of P is a root of the polynomial

$$\mathsf{p}(\mu; v) = v_1 \mu^{m-1} + v_2 \mu^{m-2} + \cdots + v_{m-1}\mu + v_m \tag{24}$$

associated with the ansatz vector v. If P has eigenvalue ∞, this condition should be read as $v_1 \neq 0$. Apart from this elegant characterization, probably the most important property of $\mathbb{DL}(\mathrm{P})$ is that it leads to a simple one-to-one relation between the eigenvectors of P and $L \in \mathbb{DL}(\mathrm{P})$. To keep the notation compact, we define Λ_{m-1} as in Definition 3.1 for finite λ but let $\Lambda_{m-1} = [1, 0, \dots, 0]^T$ for $\lambda = \infty$.

Theorem 3.2 ([28]) *Let* P *be a matrix polynomial and* $L \in \mathbb{DL}(\mathrm{P})$ *with ansatz vector* v*. Then* $x \neq 0$ *is a right eigenvector of* P *associated with an eigenvalue* λ *if and only if* $\Lambda_{m-1} \otimes x$ *is a right eigenvector of* L *associated with* λ*. Similarly,* $y \neq 0$ *is a left eigenvector of* P *associated with an eigenvalue* λ *if and only if* $\overline{\Lambda}_{m-1} \otimes y$ *is a left eigenvector of* L *associated with* λ*.*

As a matrix pencil $L(\lambda) = \lambda X + Y$ is a special case of a matrix polynomial, we can use the results of Section 2 to study the (structured) eigenvalue condition numbers of L. To simplify the analysis, we will assume that the weights $\omega_0, \ldots, \omega_m$ in the definition of $\|\triangle \mathrm{P}\|$ are all equal to 1 for the rest of this paper. This assumption is only justified if P is not badly scaled, i.e., the norms of the coefficients of P do not vary significantly. To a certain extent, bad scaling can be overcome by rescaling the matrix polynomial before linearization, see [10, 14, 16, 17]. Moreover, we choose $\|\cdot\|_v \equiv \|\cdot\|_2$ and assume that $\|\cdot\|_M$ is an arbitrary but fixed unitarily invariant matrix norm. The same norm is used for measuring perturbations $\triangle L(\lambda) = \triangle X + \lambda \triangle Y$ to the linearization L. To summarize

$$\begin{aligned} \|\triangle P\| &= \sqrt{\|E_0\|_M^2 + \|E_1\|_M + \cdots + \|E_m\|_M^2}, & (25) \\ \|\triangle L\| &= \sqrt{\|\triangle X\|_M^2 + \|\triangle Y\|_M^2}, & (26) \end{aligned}$$

for the rest of this paper. For unstructured eigenvalue condition numbers, Lemma 2.1 together with Theorem 3.2 imply the following formula.

Lemma 3.3 *Let* λ *be a finite, simple eigenvalue of a matrix polynomial* P *with normalized right and left eigenvectors* x *and* y*. Then the eigenvalue condition number* $\kappa_L(\lambda)$ *for a linearization* $L \in \mathbb{DL}(\mathrm{P})$ *with ansatz vector* v *satisfies*

$$\kappa_L(\lambda) = \frac{\sqrt{1+|\lambda|^2}}{|\mathsf{p}(\lambda; v)|} \cdot \frac{\|\Lambda_{m-1}\|_2^2}{|y^H P'(\lambda) x|} = \frac{\sqrt{1+|\lambda|^2}\,\|\Lambda_{m-1}\|_2^2}{|\mathsf{p}(\lambda; v)|\,\|\Lambda_m\|_2} \kappa_{\mathrm{P}}(\lambda).$$

Proof. A similar formula for the case $\|\cdot\|_v \equiv \|\cdot\|_1$ can be found in [16, Section 3]. The proof for our case $\|\cdot\|_v \equiv \|\cdot\|_2$ is almost identical and therefore omitted. □

To allow for a simple interpretation of the result of Lemma 3.3, we define the quantity

$$\delta(\lambda; v) := \frac{\|\Lambda_{m-1}\|_2}{|\mathsf{p}(\lambda; v)|} \tag{27}$$

for a given ansatz vector v. Obviously $\delta(\lambda; v) \geq 1$. Since L is assumed to be a linearization, $\mathsf{p}(\lambda; v) \neq 0$ and hence $\delta(\lambda; v) < \infty$. Using the straightforward bound

$$1 \leq \frac{\sqrt{1+|\lambda|^2}\|\Lambda_{m-1}\|_2}{\|\Lambda_m\|_2} \leq \sqrt{2}, \tag{28}$$

the result of Lemma 3.3 yields

$$\delta(\lambda; v) \leq \frac{\kappa_L(\lambda)}{\kappa_{\mathrm{P}}(\lambda)} \leq \sqrt{2}\,\delta(\lambda; v). \tag{29}$$

This shows that the process of linearizing P invariably increases the condition number of a simple eigenvalue of P at least by a factor of $\delta(\lambda; v)$ and at most by a factor of $\sqrt{2}\delta(\lambda; v)$. In other words, $\delta(\lambda; v)$ serves as a growth factor for the eigenvalue condition number.

Since $\mathsf{p}(\lambda; v) = \Lambda_{m-1}^T v$, it follows from the Cauchy-Schwartz inequality that among all ansatz vectors with $\|v\|_2 = 1$ the vector $v = \overline{\Lambda}_{m-1}/\|\Lambda_{m-1}\|_2$ minimizes $\delta(\lambda; v)$ and, hence, for this particular choice of v we have $\delta(\lambda; v) = 1$ and

$$\kappa_{\mathrm{P}}(\lambda) \leq \kappa_L(\lambda) \leq \sqrt{2}\,\kappa_{\mathrm{P}}(\lambda).$$

Let us emphasize that this result is primarily of theoretical interest as the optimal choice of v depends on the (typically unknown) eigenvalue λ. A practically more useful recipe is to choose $v = [1, 0, \ldots, 0]^T$ if $|\lambda| \geq 1$ and $v = [0, \ldots, 0, 1]^T$ if $|\lambda| \leq 1$. In both cases, $\delta(\lambda; v) = \frac{\|\Lambda_{m-1}\|_2}{|\mathsf{p}(\lambda; v)|} \leq \sqrt{m}$ and therefore $\kappa_{\mathrm{P}}(\lambda) \leq \kappa_L(\lambda) \leq \sqrt{2m}\, \kappa_{\mathrm{P}}(\lambda)$.

In the following section, the discussion above shall be extended to structured linearizations and condition numbers.

4 Structured condition numbers for linearizations

If the polynomial P is structured, its linearization $L \in \mathbb{DL}(\mathrm{P})$ should reflect this structure. Table 3 summarizes existing results on the conditions the ansatz vector v should satisfy for this purpose. These conditions can be found in [15, Thm 3.4] for symmetric polynomials, in [15, Thm. 6.1] for Hermitian polynomials, and in [27, Tables 3.1, 3.2] for $*$-even/odd, $*$-palindromic/anti-palindromic polynomials with $* \in \{T, H\}$. The matrices $R, \Sigma \in \mathbb{R}^{m \times m}$

structure of P	structure of L	ansatz vector
T-symmetric	T-symmetric	$v \in \mathbb{C}^m$
Hermitian	Hermitian	$v \in \mathbb{R}^m$
$*$-even	$*$-even	$\Sigma v = (v^*)^T$
	$*$-odd	$\Sigma v = -(v^*)^T$
$*$-odd	$*$-even	$\Sigma v = -(v^*)^T$
	$*$-odd	$\Sigma v = (v^*)^T$
$*$-palindromic	$*$-palindromic	$Rv = (v^*)^T$
	$*$-anti-palindromic	$Rv = -(v^*)^T$
$*$-anti-palindromic	$*$-palindromic	$Rv = -(v^*)^T$
	$*$-anti-palindromic	$Rv = (v^*)^T$

Table 3: Conditions the ansatz vector v needs to satisfy in order to yield a structured linearization $L \in \mathbb{DL}(\mathrm{P})$ for a structured polynomial P. Note that $(v^*)^T = v$ if $* = T$ and $(v^*)^T = \overline{v}$ if $* = H$.

are defined as

$$R = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}, \qquad \Sigma = \mathrm{diag}\{(-1)^{m-1}, (-1)^{m-2}, \ldots, (-1)^0\}. \tag{30}$$

If, for example, a structure-preserving method is used for computing the eigenvalues of a structured linearization L then the structured condition number of L is an appropriate measure for the influence of roundoff error on the accuracy of the computed eigenvalues. It is therefore of interest to choose L such that the structured condition number is minimized.

Let us recall our choice of norms (25)–(26) for measuring perturbations. A first general result can be obtained from combining the identity $\frac{\kappa_L^{\mathbb{S}}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda)} = \frac{\kappa_L^{\mathbb{S}}(\lambda)}{\kappa_L(\lambda)} \frac{\kappa_{\mathrm{P}}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda)} \frac{\kappa_L(\lambda)}{\kappa_{\mathrm{P}}(\lambda)}$ with (29):

$$\frac{\kappa_L^{\mathbb{S}}(\lambda)}{\kappa_L(\lambda)} \frac{\kappa_{\mathrm{P}}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda)} \delta(\lambda; v) \leq \frac{\kappa_L^{\mathbb{S}}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda)} \leq \sqrt{2}\, \frac{\kappa_L^{\mathbb{S}}(\lambda)}{\kappa_L(\lambda)} \frac{\kappa_{\mathrm{P}}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda)} \delta(\lambda; v). \tag{31}$$

We will make frequent use of (31) to obtain concrete upper bounds for specific structures.

4.1 Complex symmetric matrix polynomials

For a complex symmetric matrix polynomial P, any ansatz vector v yields a complex symmetric linearization. Thus, we are free to use the optimal choice $v = \overline{\Lambda_{m-1}}/\|\Lambda_{m-1}\|_2$ from Section 3. Combined with Corollary 2.5, which states that there is (almost) no difference between structured and unstructured condition numbers, we have the following result.

Theorem 4.1 *Let $\mathbb{S}$ denote the set of complex symmetric matrix polynomials. Let λ be a finite or infinite, simple eigenvalue of a matrix polynomial* P*. Then for the linearization $L \in \mathbb{DL}(\mathrm{P})$ corresponding to an ansatz vector v, we have*

$$\delta(\lambda; v) \leq \frac{\kappa_L^{\mathbb{S}}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda)} \leq \sqrt{2}\, \delta(\lambda; v)$$

for $\|\cdot\|_M \equiv \|\cdot\|_2$ and $\|\cdot\|_M \equiv \|\cdot\|_F$. In particular, for $v = \overline{\Lambda_{m-1}}/\|\Lambda_{m-1}\|_2$, we have

$$\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) \leq \kappa_L^{\mathbb{S}}(\lambda) \leq \sqrt{2}\, \kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda). \tag{32}$$

Proof. For $\|\cdot\|_M \equiv \|\cdot\|_2$, we have $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda) = \kappa_{\mathrm{P}}(\lambda)$ and $\kappa_L^{\mathbb{S}}(\lambda) = \kappa_L(\lambda)$. Hence the result follows directly from (31). For $\|\cdot\|_M \equiv \|\cdot\|_F$, the additional factors appearing in Corollary 2.5 are the same for $\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda)$ and $\kappa_L^{\mathbb{S}}(\lambda)$. This can be seen as follows. According to Theorem 3.2, the normalized right and left eigenvectors of the linearization take the form $\tilde{x} = \Lambda_{m-1} \otimes x/\|\Lambda_{m-1}\|_2$, $\tilde{y} = \overline{\Lambda_{m-1}} \otimes y/\|\Lambda_{m-1}\|_2$. Thus,

$$\tilde{y}^T \tilde{x} = \frac{\overline{\Lambda_{m-1}}^T \Lambda_{m-1}}{\|\Lambda_{m-1}\|_2^2} y^T x = y^T x, \tag{33}$$

concluding the proof. □

4.2 T-even and T-odd matrix polynomials

In contrast to complex symmetric polynomials, structure-preserving linearizations for T-even and T-odd polynomials put a restriction on the choice of the ansatz vector: $\Sigma v = \pm v$. The following theorem shows that the increase of structured condition number can still be made nearly proportional to $\delta(\lambda; v)$.

Theorem 4.2 *Let $\mathbb{S}_e$ and $\mathbb{S}_o$ denote the sets of T-even and T-odd polynomials, respectively. Let λ be a finite or infinite, simple eigenvalue of a T-even matrix polynomial* P*. Consider the T-even and T-odd linearizations $L_e, L_o \in \mathbb{DL}(\mathrm{P})$ corresponding to ansatz vectors satisfying $\Sigma v = v$ and $\Sigma v = -v$, respectively. Then the following statements hold for $\|\cdot\|_M \equiv \|\cdot\|_2$.*

1. *If m is odd:* $\delta(\lambda; v) \leq \dfrac{\kappa_{L_e}^{\mathbb{S}_e}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)} \leq \sqrt{2}\, \delta(\lambda; v)$.

2. *If m is even and $|\lambda| \leq 1$:* $\dfrac{\delta(\lambda; v)}{\sqrt{2}} \leq \dfrac{\kappa_{L_e}^{\mathbb{S}_e}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)} \leq \sqrt{2}\, \delta(\lambda; v)$.

3. *If m is even and $|\lambda| \geq 1$:* $\dfrac{\delta(\lambda; v)}{\sqrt{2}} \leq \dfrac{\kappa_{L_o}^{\mathbb{S}_o}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)} \leq \sqrt{2}\, \delta(\lambda; v)$.

Proof. The proof makes use of the basic algebraic relation

$$\frac{|\lambda|^2}{1+|\lambda|^2} \geq \frac{\|(1-\Pi_e)(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}, \quad \text{with equality for odd } m. \tag{34}$$

1. If m is odd, (34) implies – together with Lemma 2.6 and (33) – the equality
$$\frac{\kappa_{L_e}^{\mathbb{S}_e}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)} = \frac{\sqrt{1-|y^Tx|^2\frac{|\lambda|^2}{1+|\lambda|^2}}}{\sqrt{1-|y^Tx|^2\frac{\|(1-\Pi_e)(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}} \cdot \frac{\kappa_{L_e}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)} = \frac{\kappa_{L_e}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)}. \tag{35}$$
Now the desired result follows directly from our general bounds (31).

2. If m is even and $|\lambda| \leq 1$ then
$$\frac{1}{\sqrt{2}} \cdot \frac{\kappa_{L_e}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)} \leq \frac{\kappa_{L_e}^{\mathbb{S}_e}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)} \leq \frac{\kappa_{L_e}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)},$$
where the lower and upper bounds follow from the first equality in (35) using $\frac{|\lambda|^2}{1+|\lambda|^2} \leq \frac{1}{2}$ and (34), respectively. Again, the desired result follows from (31).

3. If m is even, $|\lambda| \geq 1$ and a T-odd linearization is used then Lemma 2.10 along with (33) yield
$$\frac{\kappa_{L_o}^{\mathbb{S}_o}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)} = \frac{\sqrt{1-|y^Tx|^2\frac{1}{1+|\lambda|^2}}}{\sqrt{1-|y^Tx|^2\frac{\|(1-\Pi_e)(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}} \cdot \frac{\kappa_{L_o}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)}.$$
The relation
$$\frac{\|(1-\Pi_e)(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} \leq \frac{1}{1+|\lambda|^2} \leq \frac{1}{2}$$
then implies
$$\frac{1}{\sqrt{2}} \cdot \frac{\kappa_{L_o}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)} \leq \frac{\kappa_{L_o}^{\mathbb{S}_o}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)} \leq \frac{\kappa_{L_o}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)}.$$
Hence the result follows from (31). □

Obtaining an optimally conditioned linearization requires finding the maximum of $|\mathsf{p}(\lambda; v)| = |\Lambda_{m-1}^T v|$ among all v with $\Sigma v = \pm v$ and $\|v\|_2 \leq 1$. This maximization problem can be addressed by the following basic linear algebra result.

Proposition 4.3 *Let $\Pi_{\mathcal{V}}$ be an orthogonal projector onto a linear subspace $\mathcal{V}$ of $\mathbb{F}^m$ with $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. Then for $A \in \mathbb{F}^{l\times m}$,*
$$\max_{\substack{v\in\mathcal{V}\\ \|v\|_2\leq 1}} \|Av\|_2 = \|A\Pi_{\mathcal{V}}\|_2.$$

For a T-even linearization we have $\mathcal{V} = \{v \in \mathbb{C}^m : \Sigma v = v\}$ and the orthogonal projector onto $\mathcal{V}$ is given by the even coefficient projector Π_e defined in (22). Hence, by Proposition 4.3,
$$\max_{\substack{v=\Sigma v\\ \|v\|_2\leq 1}} |\mathsf{p}(\lambda; v)| = \max_{\substack{v=\Sigma v\\ \|v\|_2\leq 1}} |\Lambda_{m-1}^T v| = \|\Pi_e(\Lambda_{m-1})\|_2$$
where the maximum is attained by $v = \Pi_e(\overline{\Lambda}_{m-1})/\|\Pi_e(\Lambda_{m-1})\|_2$. Similarly, for a T-odd linearization,
$$\max_{\substack{v=-\Sigma v\\ \|v\|_2\leq 1}} |\mathsf{p}(\lambda; v)| = \|(1-\Pi_e)(\Lambda_{m-1})\|_2$$
with the maximum attained by $v = (1-\Pi_e)(\overline{\Lambda}_{m-1})/\|(1-\Pi_e)(\Lambda_{m-1})\|_2$.

Corollary 4.4 *Under the assumptions of Theorem 4.2, consider the specific T-even and T-odd linearizations $L_e, L_o \in \mathbb{DL}(\mathrm{P})$ belonging to the ansatz vectors $v = \Pi_e(\overline{\Lambda}_{m-1})/\|\Pi_e(\Lambda_{m-1})\|_2$ and $v = (1-\Pi_e)(\overline{\Lambda}_{m-1})/\|(1-\Pi_e)(\Lambda_{m-1})\|_2$, respectively. Then the following statements hold.*

1. *If m is odd:* $\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda) \le \kappa_{L_e}^{\mathbb{S}_e}(\lambda) \le 2\,\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)$.

2. *If m is even and $|\lambda| \le 1$:* $\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)/\sqrt{2} \le \kappa_{L_e}^{\mathbb{S}_e}(\lambda) \le 2\,\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)$.

3. *If m is even and $|\lambda| \ge 1$:* $\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)/\sqrt{2} \le \kappa_{L_o}^{\mathbb{S}_o}(\lambda) \le 2\,\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)$.

Proof. Note that, by definition, $\delta(\lambda; v) \ge 1$ and hence all lower bounds are direct consequences of Theorem 4.2. To show $\delta(\lambda; v) \le \sqrt{2}$ for the upper bounds of statements 1 and 2, we make use of the inequalities

$$\|\Pi_e(\Lambda_{m-1})\| \le \|\Lambda_{m-1}\|_2 \le \sqrt{2}\|\Pi_e(\Lambda_{m-1})\|, \tag{36}$$

which hold if either m is odd or m is even and $|\lambda| \le 1$. For statement 3, the bound $\delta(\lambda; v) \le \sqrt{2}$ is a consequence of (34). □

The morale of Theorem 4.2 and Corollary 4.4 is quickly told: There is always a "good" T-even linearization (in the sense that the linearization increases the structured condition number at most by a modest factor) if either m is odd or m is even and $|\lambda| \le 1$. In the exceptional case, when m is even and $|\lambda| \ge 1$, there is always a "good" T-odd linearization. Intuitively, the necessity of such an exceptional case becomes clear from the fact that there exists no T-even linearization for a T-even polynomial with even m and infinite eigenvalue. Even though there is a T-even linearization for even m and large but finite λ, it is not advisable to use it for numerical computations.

In practice, one does not know λ in advance and hence the linearizations used in Corollary 4.4 for which $\delta(\lambda; v) \le \sqrt{2}$ are mainly of theoretical interest. Table 4 provides practically more feasible recommendations on the choice of v, such that there is still at worst a slight increase of the structured condition number. The bounds in this table follow from Theorem 4.2 combined with $\delta(\lambda; v) \le \sqrt{m}$ for all displayed choices of v. The example linearizations are taken from [27, Tables 3.4–3.6].

m	λ of interest	v	Bound on struct. cond. of linearization	Example
odd or even	$\lvert\lambda\rvert \le 1$	e_m	$\kappa_{L_e}^{\mathbb{S}_e}(\lambda) \le \sqrt{2m}\,\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)$	$\begin{bmatrix} 0 & -A_3 & 0 \\ A_3 & A_2 & 0 \\ 0 & 0 & A_0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & A_3 \\ 0 & -A_3 & -A_2 \\ A_3 & A_2 & A_1 \end{bmatrix}$
odd	$\lvert\lambda\rvert \ge 1$	e_1	$\kappa_{L_e}^{\mathbb{S}_e}(\lambda) \le \sqrt{2m}\,\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)$	$\begin{bmatrix} A_2 & A_1 & A_0 \\ -A_1 & -A_0 & 0 \\ A_0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} A_3 & 0 & 0 \\ 0 & A_1 & A_0 \\ 0 & -A_0 & 0 \end{bmatrix}$
even	$\lvert\lambda\rvert \ge 1$	e_1	$\kappa_{L_o}^{\mathbb{S}_o}(\lambda) \le \sqrt{2m}\,\kappa_{\mathrm{P}}^{\mathbb{S}_e}(\lambda)$	$\begin{bmatrix} A_2 & 0 \\ 0 & A_0 \end{bmatrix} + \lambda \begin{bmatrix} A_1 & A_0 \\ -A_0 & 0 \end{bmatrix}$

Table 4: Recipes for choosing the ansatz vector v for a T-even or T-odd linearization L_e or L_o of a T-even matrix polynomial of degree m. Note that e_1 and e_m denote the 1st and mth unit vector of length m, respectively.

We extend Theorem 4.2 and Corollary 4.4 to T-odd polynomials.

Theorem 4.5 *Let $\mathbb{S}_e$ and $\mathbb{S}_o$ denote the sets of T-even and T-odd polynomials, respectively. Let λ be a finite or infinite, simple eigenvalue of a T-odd matrix polynomial P. Consider the T-odd and T-even linearizations $L_o, L_e \in \mathbb{DL}(\mathrm{P})$ corresponding to ansatz vectors satisfying $\Sigma v = v$ and $\Sigma v = -v$, respectively. Then the following statements hold for $\|\cdot\|_M \equiv \|\cdot\|_2$.*

1. *If m is odd:* $\delta(\lambda; v) \le \dfrac{\kappa_{L_o}^{\mathbb{S}_o}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_o}(\lambda)} \le \sqrt{2}\,\delta(\lambda; v)$.

2. *If m is even and $|\lambda| \le 1$:* $\delta(\lambda; v) \le \dfrac{\kappa_{L_o}^{\mathbb{S}_o}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_o}(\lambda)} \le 2\,\delta(\lambda; v)$.

3. If m is even and $|\lambda| \geq 1$: $\delta(\lambda; v) \leq \dfrac{\kappa_{L_e}^{\mathbb{S}_e}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_o}(\lambda)} \leq 2\,\delta(\lambda; v)$.

Proof.

1. Similar to the proof of Theorem 4.2, Lemma 2.10 yields

$$\frac{\kappa_{L_o}^{\mathbb{S}_o}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_o}(\lambda)} = \frac{\sqrt{1 - |y^T x|^2 \frac{1}{1+|\lambda|^2}}}{\sqrt{1 - |y^T x|^2 \frac{\|\Pi_e(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}} \cdot \frac{\kappa_{L_o}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)}. \tag{37}$$

   Relation (34) implies $1/(1+|\lambda|^2) = \|\Pi_e(\Lambda_m)\|_2^2 / \|\Lambda_m\|_2^2$ and hence $\frac{\kappa_{L_o}^{\mathbb{S}_o}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_o}(\lambda)} = \frac{\kappa_{L_o}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)}$, which proves the first statement by (31).

2. If m is even and $|\lambda| \leq 1$, relation (34) yields $1/(1+|\lambda|^2) \leq \|\Pi_e(\Lambda_m)\|_2^2 / \|\Lambda_m\|_2^2$, implying

$$1 \leq \frac{\sqrt{1 - |y^T x|^2 \frac{1}{1+|\lambda|^2}}}{\sqrt{1 - |y^T x|^2 \frac{\|\Pi_e(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}} \leq \frac{\sqrt{1 - \frac{1}{1+|\lambda|^2}}}{\sqrt{1 - \frac{\|\Pi_e(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}} = \frac{\|\Lambda_m\|_2}{\|\Lambda_{m-1}\|_2} \leq \sqrt{1+|\lambda|^2} \leq \sqrt{2},$$

   where the previous last bound follows from (28). Combined with (37) and (31), this shows the bounds of the second statement.

3. For m even and $|\lambda| \geq 1$, a T-even linearization gives

$$\frac{\kappa_{L_e}^{\mathbb{S}_e}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_o}(\lambda)} = \frac{\sqrt{1 - |y^T x|^2 \frac{|\lambda|^2}{1+|\lambda|^2}}}{\sqrt{1 - |y^T x|^2 \frac{\|\Pi_e(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}} \cdot \frac{\kappa_{L_e}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)}.$$

   It is not hard to verify $\frac{|\lambda|^2}{1+|\lambda|^2} \leq \frac{\|\Pi_e(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}$, implying

$$1 \leq \frac{\sqrt{1 - |y^T x|^2 \frac{|\lambda|^2}{1+|\lambda|^2}}}{\sqrt{1 - |y^T x|^2 \frac{\|\Pi_e(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}} \leq \frac{\sqrt{1 - \frac{|\lambda|^2}{1+|\lambda|^2}}}{\sqrt{1 - \frac{\|\Pi_e(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}} = \frac{\|\Lambda_m\|_2}{|\lambda| \cdot \|\Lambda_{m-1}\|_2} \leq \frac{\sqrt{1+|\lambda|^2}}{|\lambda|} \leq \sqrt{2},$$

   where the previous last bound follows again from (28). This concludes the proof by (31).
   □

Corollary 4.6 *Under the assumptions of Theorem 4.5, consider the specific T-odd and T-even linearizations $L_o, L_e \in \mathbb{DL}(\mathrm{P})$ belonging to the ansatz vectors $v = \Pi_e(\overline{\Lambda}_{m-1})/\|\Pi_e(\Lambda_{m-1})\|_2$ and $v = (1-\Pi_e)(\overline{\Lambda}_{m-1})/\|(1-\Pi_e)(\Lambda_{m-1})\|_2$, respectively. Then the following statements hold.*

1. *If m is odd:* $\kappa_{\mathrm{P}}^{\mathbb{S}_o}(\lambda) \leq \kappa_{L_o}^{\mathbb{S}_o}(\lambda) \leq 2\,\kappa_{\mathrm{P}}^{\mathbb{S}_o}(\lambda)$.

2. *If m is even and $|\lambda| \leq 1$:* $\kappa_{\mathrm{P}}^{\mathbb{S}_o}(\lambda) \leq \kappa_{L_o}^{\mathbb{S}_o}(\lambda) \leq 2\sqrt{2}\,\kappa_{\mathrm{P}}^{\mathbb{S}_o}(\lambda)$.

3. *If m is even and $|\lambda| \geq 1$:* $\kappa_{\mathrm{P}}^{\mathbb{S}_o}(\lambda) \leq \kappa_{L_e}^{\mathbb{S}_e}(\lambda) \leq 2\sqrt{2}\,\kappa_{\mathrm{P}}^{\mathbb{S}_o}(\lambda)$.

Proof. The proof follows, similar as the proof of Corollary 4.4, from Theorem 4.5 and (36).
□

We mention that Table 4 has a virtually identical analogue in the case of a T-odd matrix polynomial.

4.3 T-palindromic matrix polynomials

Turning to T-palindromic matrix polynomials, we first show that the condition number growth under T-(anti-)palindromic linearization is again governed by the quantity $\delta(\lambda; v)$ introduced in (27). Let us recall Table 3: the ansatz vector v for a T-(anti-)palindromic linearization of a T-palindromic polynomial should satisfy $Rv = v$ ($Rv = -v$) with the flip permutation R defined in (30). Our first result reveals relations between the *unstructured* condition numbers for the matrix polynomials and the *structured* condition numbers for the linearizations.

Proposition 4.7 *Let $\mathbb{S}_p$ and $\mathbb{S}_a$ denote the sets of T-palindromic and T-anti-palindromic polynomials, respectively. Let λ be a finite or infinite, simple eigenvalue of a matrix polynomial* P *and let $L_p, L_a \in \mathbb{DL}(\mathrm{P})$ be a T-palindromic and T-anti-palindromic linearizations of* P, *respectively. Then the following statements hold for $\|\cdot\|_M \equiv \|\cdot\|_F$.*

1. *If* $\mathsf{Re}(\lambda) \geq 0: \dfrac{\delta(\lambda; v)}{\sqrt{2}} \leq \dfrac{\kappa_{L_p}^{\mathbb{S}_p}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)} \leq \sqrt{2}\,\delta(\lambda; v).$

2. *If* $\mathsf{Re}(\lambda) \leq 0: \dfrac{\delta(\lambda; v)}{\sqrt{2}} \leq \dfrac{\kappa_{L_a}^{\mathbb{S}_a}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)} \leq \sqrt{2}\,\delta(\lambda; v).$

Proof. For an infinite eigenvalue, $\kappa_{L_p}^{\mathbb{S}_p}(\lambda) = \kappa_{L_p}(\lambda)$ as well as $\kappa_{L_a}^{\mathbb{S}_a}(\lambda) = \kappa_{L_a}(\lambda)$, and hence the result follows from (29). We can therefore assume λ to be finite.

1. By Lemma 2.11,

$$\kappa_{L_p}^{\mathbb{S}_p}(\lambda) = \frac{\|\Lambda_{m-1}\|_2^2 \sqrt{1 + |\lambda|^2 + 2\mathrm{Re}(\lambda)|y^T x|^2}}{\sqrt{2}\,|\mathsf{p}(\lambda; v)|\,\|\Lambda_m\|_2} \kappa_{\mathrm{P}}(\lambda).$$

   Since $\mathsf{Re}(\lambda) > 0$, it holds that $\|(1,\, \lambda)\|_2 \leq \sqrt{1 + |\lambda|^2 + 2\mathrm{Re}(\lambda)|y^T x|^2} \leq \sqrt{2}\,\|(1,\, \lambda)\|_2$. Hence the result follows from Lemma 3.3 and (28).

2. This result follows analogously from Lemma 2.13. □

In the following, we will treat the more difficult case of *structured* condition numbers for both the polynomial and its linearization.

Theorem 4.8 *Let $\mathbb{S}_p$, $\mathbb{S}_a$ be defined as in Proposition 4.7. Let λ be a finite or infinite, simple eigenvalue of a T-*palindromic *matrix polynomial* P. *Let $L_p \in \mathbb{DL}(\mathrm{P})$ and $L_a \in \mathbb{DL}(\mathrm{P})$ be T-palindromic and T-anti-palindromic linearizations of* P *corresponding to ansatz vectors satisfying $v = Rv$ and $v = -Rv$, respectively. Then the following statements hold for $\|\cdot\|_M \equiv \|\cdot\|_F$.*

1. *If m is odd:* $\dfrac{\kappa_{L_p}^{\mathbb{S}_p}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)} \leq \sqrt{2(m+1)}\,\delta(\lambda; v).$

2. *If m is even and* $\mathsf{Re}(\lambda) \geq 0$*:* $\dfrac{\delta(\lambda; v)}{\sqrt{2}} \leq \dfrac{\kappa_{L_p}^{\mathbb{S}_p}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)} \leq \sqrt{2(m+1)}\,\delta(\lambda; v).$

3. *If m is even and* $\mathsf{Re}(\lambda) \leq 0$*:* $\dfrac{\delta(\lambda; v)}{\sqrt{2}} \leq \dfrac{\kappa_{L_a}^{\mathbb{S}_a}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)} \leq \sqrt{2(m+1)}\,\delta(\lambda; v).$

Proof. With the same argument as in the proof of Proposition 4.7, we can assume w.l.o.g. that λ is finite. Moreover, since $\kappa_L^{\mathbb{S}}(\lambda)/\kappa_{\mathrm{P}}(\lambda) \leq \kappa_L^{S}(\lambda)/\kappa_{\mathrm{P}}^{\mathbb{S}}(\lambda)$ holds for any structure $\mathbb{S}$, the desired lower bounds follow from Proposition 4.7. It remains to prove the upper bounds.

1. If m is odd and $\mathsf{Re}(\lambda) \geq 0$, Lemma 2.11 implies – together with Lemma A.1.1 and (33) – the inequality

$$\frac{\kappa_{L_p}^{\mathbb{S}_p}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)} = \frac{\sqrt{1+|y^Tx|^2\frac{|1+\lambda|^2-|1-\lambda|^2}{2(1+|\lambda|^2)}}}{\sqrt{1+|y^Tx|^2\frac{\|\Pi_+(\Lambda_m)\|_2^2-\|\Pi_-(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}}\cdot\frac{\kappa_{L_p}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)} \tag{38}$$

$$\leq \frac{1}{\sqrt{1+\frac{\|\Pi_+(\Lambda_m)\|_2^2-\|\Pi_-(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}}\cdot\frac{\kappa_{L_p}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)}$$

$$= \frac{\|\Lambda_m\|_2}{\sqrt{2}\|\Pi_+(\Lambda_m)\|_2}\cdot\frac{\kappa_{L_p}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)} \leq \sqrt{m+1}\frac{\kappa_{L_p}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)}.$$

   In the case $\mathsf{Re}(\lambda) \leq 0$, Lemma A.1.2 combined with (38) directly gives $\frac{\kappa_{L_p}^{\mathbb{S}_p}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)} \leq \frac{\kappa_{L_p}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)}$. In both cases, the desired upper bound now follows from (31).

2. For even m, the result can be proven along the lines of the proof of the first statement, now using Lemma A.1.3.

3. The proof of the third statement also follows along the lines of the proof of the first statement. Lemmas 2.11, 2.13 and A.13 reveal – for even m and a T-anti-palindromic linearization – the inequality

$$\frac{\kappa_{L_a}^{\mathbb{S}_a}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)} = \frac{\sqrt{1-|y^Tx|^2\frac{|1-\lambda|^2-|1+\lambda|^2}{2(1+|\lambda|^2)}}}{\sqrt{1+|y^Tx|^2\frac{\|\Pi_+(\Lambda_m)\|_2^2-\|\Pi_-(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2}}}\cdot\frac{\kappa_{L_a}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)} \leq \sqrt{m+1}\frac{\kappa_{L_a}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)}.$$

   Again, the desired upper bound follows from (31). □

In the sense of condition numbers, the optimal linearization belongs to an ansatz vector that minimizes $\delta(\lambda; v)$ or, equivalently, maximizes $|\mathsf{p}(\lambda; v)|$. By Proposition 4.3,

$$\max_{\substack{v=Rv \\ \|v\|_2\leq 1}} |\mathsf{p}(\lambda; v)| = \|\Pi_+(\Lambda_{m-1})\|_2,$$

where the maximum is attained by v_+ defined as

$$v_\pm = \frac{\left[\frac{\lambda^{m-1}\pm 1}{2}, \ldots, \frac{\lambda^{m/2+1}\pm\lambda^{m/2}}{2}, \frac{\lambda^{m/2+1}\pm\lambda^{m/2}}{2}, \ldots, \frac{\lambda^{m-1}\pm 1}{2}\right]^T}{\|\Pi_\pm(\Lambda_{m-1})\|_2} \tag{39}$$

if m is even and as

$$v_\pm = \frac{\left[\frac{\lambda^{m-1}\pm 1}{2}, \ldots, \frac{\lambda^{(m-1)/2}\pm\lambda^{(m-1)/2}}{2}, \ldots, \frac{\lambda^{m-1}\pm 1}{2}\right]^T}{\|\Pi_\pm(\Lambda_{m-1})\|_2} \tag{40}$$

if m is odd. Similarly,

$$\max_{\substack{v=-Rv \\ \|v\|_2\leq 1}} |\mathsf{p}(\lambda; v)| = \|\Pi_-(\Lambda_{m-1})\|_2,$$

with the maximum attained by v_-.

Corollary 4.9 *Under the assumptions of Theorem 4.8, consider the specific T-palindromic and T-anti-palindromic linearizations $L_p, L_a \in \mathbb{DL}(\mathrm{P})$ belonging to the ansatz vectors v_+, v_- defined in (39)–(40), respectively. Then the following statements hold.*

1. *If m is odd: $\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq 2(m+1)\,\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$.*

2. *If m is even and* $\mathsf{Re}(\lambda) \geq 0$*:* $\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq 2(m+1)\,\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$.

3. *If m is even and* $\mathsf{Re}(\lambda) \leq 0$*:* $\kappa_{L_a}^{\mathbb{S}_a}(\lambda) \leq 2(m+1)\,\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$.

Proof. The bounds follow from Theorem 4.8 and Lemma A.1. □

Theorem 4.8 and Corollary 4.9 admit a simple interpretation. If either m is odd or m is even and λ has nonnegative real part, it is OK to use a T-palindromic linearization; there will be no significant increase of the structured condition number. In the exceptional case, when m is even and λ has negative real part, a T-anti-palindromic linearization should be preferred. This is especially true for $\lambda = -1$, in which case there is no T-palindromic linearization.

The upper bounds in Corollary 4.8 are probably too pessimistic; at least they do not fully reflect the optimality of the choice of v_+ and v_-. Yet, the heuristic choices listed in Table 5 yield almost the same bounds! These bounds are proven in the following lemma. To provide recipes for even m larger than 2, one would need to discriminate further between $|\lambda|$ close to 1 and $|\lambda|$ far away from 1, similar as for odd m.

m	λ of interest	v	Bound on struct. cond. of linearization	Example
odd	$\lvert\lambda\rvert \geq \alpha_m$ $\lvert\lambda\rvert \leq \alpha_m^{-1}$	$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$	$\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq 2\sqrt{2}(m+1)\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$	$\begin{bmatrix} A_0 & 0 & A_0 \\ A_1 - A_0^T & A_0 - A_1^T & 0 \\ A_1^T & A_1 - A_0^T & A_0 \end{bmatrix} + \lambda \begin{bmatrix} A_0^T & A_1^T - A_0 & A_1 \\ 0 & A_0^T - A_1 & A_1^T - A_0 \\ A_0^T & 0 & A_0^T \end{bmatrix}$
odd	$\lvert\lambda\rvert \leq \alpha_m$ $\lvert\lambda\rvert \geq \alpha_m^{-1}$	$e_{\frac{m-1}{2}}$	$\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq 2(m+1)\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$	$\begin{bmatrix} 0 & A_0 & 0 \\ 0 & A_1 & A_0 \\ -A_0^T & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & -A_0 \\ A_0^T & A_1^T & 0 \\ 0 & A_0^T & 0 \end{bmatrix}$
$m = 2$	$\mathsf{Re}(\lambda) \geq 0$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq 2\sqrt{3}\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$	$\begin{bmatrix} A_0 & A_0 \\ A_1 - A_0^T & A_0 \end{bmatrix} + \lambda \begin{bmatrix} A_0^T & A_1^T - A_0 \\ A_0^T & A_0^T \end{bmatrix}$
$m = 2$	$\mathsf{Re}(\lambda) \leq 0$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\kappa_{L_a}^{\mathbb{S}_a}(\lambda) \leq 2\sqrt{3}\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$	$\begin{bmatrix} -A_0 & A_0 \\ -A_1 - A_0^T & -A_0 \end{bmatrix} + \lambda \begin{bmatrix} A_0^T & A_1^T + A_0 \\ -A_0^T & A_0^T \end{bmatrix}$

Table 5: Recipes for choosing the ansatz vector v for a T-palindromic or T-anti-palindromic linearization L_e or L_o of a T-palindromic matrix polynomial of degree m. Note that $\alpha_m = 2^{1/(m-1)}$.

Lemma 4.10 *The upper bounds on $\kappa_{L_p}^{\mathbb{S}_p}(\lambda)$ and $\kappa_{L_a}^{\mathbb{S}_a}(\lambda)$ listed in Table 5 are valid.*

Proof. It suffices to derive an upper bound on $\delta(\lambda; v) = \frac{\|\Lambda_{m-1}\|_2}{\mathsf{p}(\lambda; v)}$. Multiplying such a bound by $\sqrt{2(m+1)}$ then gives the coefficient in the upper bound on the structured condition number of the linearization, see Theorem 4.8.

1. For odd m and $|\lambda| \geq \alpha_m$ or $|\lambda| \leq 1/\alpha_m$, the bound $\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq \sqrt{2}(m+1)\,\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$ follows from
$$\frac{\|\Lambda_{m-1}\|_2^2}{|\mathsf{p}(\lambda; v)|^2} \leq \frac{1 + \alpha_m^2 + \cdots + \alpha_m^{2m-2}}{|1 - \alpha_m^{m-1}|^2} = 1 + \alpha_m^2 + \cdots + \alpha_m^{2m-2} \leq 4m.$$

2. For odd m and $1/\alpha_m \leq |\lambda| \leq \alpha_m$, the bound $\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq 2(m+1)\,\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$ follows from
$$\frac{\|\Lambda_{m-1}\|_2^2}{|\mathsf{p}(\lambda; v)|^2} \leq \frac{1 + \alpha_m^2 + \cdots + \alpha_m^{2m-2}}{\alpha_m^{m-1}} = \frac{1}{2}(1 + \alpha_m^2 + \cdots + \alpha_m^{2m-2}) \leq 2m.$$

3. For $m = 2$ and $\mathsf{Re}(\lambda) \geq 0$, the bound $\kappa_{L_p}^{\mathbb{S}_p}(\lambda) \leq 2(m+1)\,\kappa_{\mathrm{P}}^{\mathbb{S}_p}(\lambda)$ follows for $|\lambda| \leq 1$ from
$$\frac{\|\Lambda_{m-1}\|_2^2}{|\mathsf{p}(\lambda; v)|^2} = \frac{1 + |\lambda|^2}{|1 + \lambda|^2} \leq 2$$

and for $|\lambda| \geq 1$ from

$$\frac{\|\Lambda_{m-1}\|_2^2}{|\mathsf{p}(\lambda; v)|^2} = \frac{|\lambda|^2}{|\lambda|^2} \frac{\frac{1}{|\lambda|^2} + 1}{|\frac{1}{\lambda} + 1|^2} \leq 2.$$

4. The proof for $m = 2$ and $\mathsf{Re}(\lambda) \leq 0$ is analogous to Part 3. □

For T-anti-palindromic polynomials, the implications of Theorems 4.8 and 4.7, Corollary 4.9 and Table 5 hold, but with the roles of T-palindromic and T-anti-palindromic exchanged. For example, if either m is odd or m is even and $\mathsf{Re}(\lambda) \geq 0$, there is always a good T-anti-palindromic linearization. Otherwise, if m is even and $\mathsf{Re}(\lambda) \leq 0$, there is a good T-palindromic linearization.

4.4 Hermitian matrix polynomials and related structures

The linearization of a Hermitian polynomial is also Hermitian if the corresponding ansatz vector v is real, see Table 3. The optimal v, which maximizes $|\mathsf{p}(\lambda; v)|$, could be found by finding the maximal singular value and the corresponding left singular vector of the real $m \times 2$ matrix $[\mathsf{Re}(\Lambda_{m-1}), \mathsf{Im}(\Lambda_{m-1})]$. Instead of invoking the rather complicated expression for this optimal choice, the following lemma uses a heuristic choice of v.

Lemma 4.11 *Let $\mathbb{S}_h$ denote the set of Hermitian polynomials. Let λ be a finite or infinite, simple eigenvalue of a Hermitian matrix polynomial* P. *Then the following statements hold for $\|\cdot\|_M \equiv \|\cdot\|_F$.*

1. *If $|\lambda| \geq 1$ then the linearization L corresponding to the ansatz vector $v = [1, 0, \ldots, 0]$ is Hermitian and satisfies $\kappa_L^{\mathbb{S}_h}(\lambda) \leq 2\sqrt{m}\kappa_{\mathrm{P}}^{\mathbb{S}_h}(\lambda)$.*
2. *If $|\lambda| \leq 1$ then the linearization L corresponding to the ansatz vector $v = [0, \ldots, 0, 1]$ is Hermitian and satisfies $\kappa_L^{\mathbb{S}_h}(\lambda) \leq 2\sqrt{m}\kappa_{\mathrm{P}}^{\mathbb{S}_h}(\lambda)$.*

Proof. Assume $|\lambda| \geq 1$. Lemma 2.14 together with Lemma 3.3 and (28) imply

$$\frac{\kappa_L^{\mathbb{S}_h}(\lambda)}{\kappa_{\mathrm{P}}^{\mathbb{S}_h}(\lambda)} \leq \sqrt{2}\, \frac{\kappa_{L_p}(\lambda)}{\kappa_{\mathrm{P}}(\lambda)} = \sqrt{2}\, \frac{\|\Lambda_{m-1}\|_2}{|\mathsf{p}(\lambda; v)|} \leq 2 \frac{\sqrt{m}|\lambda|^m}{|\lambda|^m} = 2\sqrt{m}.$$

The proof for $|\lambda| \leq 1$ proceeds analogously. □

H-even and H-odd matrix polynomials are closely related to Hermitian matrix polynomials, see Remark 2.15. In particular, Lemma 4.11 applies verbatim to H-even and H-odd polynomials. Note, however, that in the case of even m the ansatz vector $v = [1, 0, \ldots, 0]$ yields an H-odd linearization for an H-even polynomial, and vice versa. Similarly, the recipes of Table 5 can be extended to H-palindromic polynomials.

5 Summary and conclusions

We have derived relatively simple expressions for the structured eigenvalue condition numbers of certain structured matrix polynomials. These expressions have been used to analyze the possible increase of the condition numbers when the polynomial is replaced by a structured linearization. At least in the case when all coefficients of the polynomial are perturbed to the same extent, the result is very positive: There is always a structured linearization, which depends on the eigenvalue of interest, such that the condition numbers increase at most by a factor linearly depending on m. We have also provided recipes for structured linearizations, which do not depend on the exact value of the eigenvalue, and for which the increase of the condition number is still negligible. Hence, the accuracy of a strongly backward stable eigensolver applied to the structured linearization will fully enjoy the benefits of structure on the sensitivity of an eigenvalue for the original matrix polynomial. The techniques and proofs of this paper represent yet another testimonial for the versatility of the linearization spaces introduced by Mackey, Mackey, Mehl, and Mehrmann in [27, 28].

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A Appendix

The following lemma summarizes some auxiliary results needed in the proofs of Section 4.3.

Lemma A.1 *Let $\lambda \in \mathbb{C}$ and let $\Lambda_\pm$ be defined as in (23. Then the following statements hold.*

1. *Assume m is odd. If $\mathsf{Re}(\lambda) \geq 0$ then $\frac{\|\Pi_+(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} \geq \frac{1}{2(m+1)}$. If $\mathsf{Re}(\lambda) \leq 0$ then $\frac{\|\Pi_-(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} \geq \frac{1}{2(m+1)}$.*
2. *Assume m is odd. If $\mathsf{Re}(\lambda) \leq 0$ then $\frac{\|\Pi_+(\Lambda_m)\|_2^2 - \|\Pi_-(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} \geq \frac{|1+\lambda|^2 - |1-\lambda|^2}{2(1+|\lambda|^2)}$.*
3. *Assume m is even. Then $\frac{\|\Pi_+(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} \geq \frac{1}{2(m+1)}$.*

Proof.

1. For $|\lambda| \geq 1$ the statement follows from $\|\Lambda_m\|_2^2 \leq (m+1)|\lambda|^{2m}$ and

$$\begin{aligned} 2\|\Pi_+(\Lambda_m)\|_2^2 &\geq |\lambda^m + 1|^2 + |\lambda^{(m+1)/2} + \lambda^{(m-1)/2}|^2 \\ &= |\lambda|^{2m} + 2\mathsf{Re}(\lambda^m) + 1 + |\lambda|^{m-1}(|\lambda|^2 + \mathsf{Re}(\lambda) + 1) \\ &\geq |\lambda|^{2m} - 2|\lambda|^m + 1 + |\lambda|^{m-1}(|\lambda|^2 + 1) \\ &= |\lambda|^{2m} + 1 + |\lambda|^{m-1}(|\lambda| - 1)^2 \geq |\lambda|^{2m}. \end{aligned}$$

If $|\lambda| \leq 1$, we can apply an analogous argument with λ replaced by $1/\lambda$ to

$$\frac{\|\Pi_+(\Lambda_m)\|_2^2}{\|\Lambda_m\|_2^2} = \frac{|\lambda|^{2m}}{|\lambda|^{2m}} \cdot \left(\sum_{k=0}^{(m-1)/2} \frac{1}{2} \left| \frac{1}{\lambda^k} + \frac{1}{\lambda^{m-k}} \right|^2 \right) \Big/ \left(\sum_{k=0}^{m} \frac{1}{|\lambda|^{2k}} \right).$$

The second part follows from the first part by the substitution $\lambda \to -\lambda$.

2. Using $(1+|\lambda|^2)(1+|\lambda|^4+\cdots+|\lambda|^{(2m-2)}) = \|\Lambda_m\|_2^2$, we prove the equivalent statement

$$\|\Pi_+(\Lambda_m)\|_2^2 \geq (|1+\lambda|^2 - |1-\lambda|^2)(1+|\lambda|^4+\cdots+|\lambda|^{(2m-2)}).$$

Assume $|\lambda| \leq 1$. Then the statement follows if we can show

$$\|\Pi_+(\Lambda_m)\|_2^2 - \|\Pi_-(\Lambda_m)\|_2^2 \geq \frac{1}{4}(|1+\lambda|^2 - |1-\lambda|^2)(m+1). \tag{41}$$

Inserting $\lambda = |\lambda|(\cos(\phi) + \mathrm{i}\sin(\phi))$, we expand

$$\begin{aligned} \|\Pi_+(\Lambda_m)\|_2^2 - \|\Pi_-(\Lambda_m)\|_2^2 &= \frac{1}{2} \sum_{k=0}^{(m-1)/2} |\lambda^{m-k} + \lambda^k|^2 - |\lambda^{m-k} - \lambda^k|^2 \\ &= 2|\lambda|^m \sum_{k=0}^{(m-1)/2} (\cos((m-k)\phi)\cos(k\phi) + \sin((m-k)\phi)\sin(k\phi)) \\ &= 2|\lambda|^m \sum_{k=0}^{(m-1)/2} \cos((m-2k)\phi) = 2|\lambda|^m \sum_{k=0}^{(m-1)/2} \cos(\phi + 2k\phi) \\ &= 2|\lambda|^m \frac{\sin(\frac{m+1}{2}\phi)\,\cos(\frac{m+1}{2}\phi)}{\sin\phi} = 2|\lambda|^m \frac{\sin((m+1)\phi)}{\sin\phi}. \end{aligned}$$

On the other hand,

$$|1+\lambda|^2 - |1-\lambda|^2 = 4|\lambda| \frac{\sin(2\phi)}{\sin\phi}.$$

Thus (41) is equivalent to

$$|\lambda|^{m-1}\frac{\sin((m+1)\phi)}{\sin\phi} \geq \frac{m+1}{2}\frac{\sin(2\phi)}{\sin\phi}$$

Dividing by $\cos(\phi) \leq 0$ on both sides, this is in turn equivalent to

$$|\lambda|^{m-1}\frac{\sin((m+1)\phi)}{\sin(2\phi)} \leq \frac{m+1}{2}.$$

Finally, using $|\lambda| \leq 1$, the last inequality follows from the basic trigonometric inequality $\frac{\sin((m+1)\phi)}{\sin(2\phi)} \leq \frac{m+1}{2}$. For $|\lambda| \geq 1$, we can use the same trick as in the proof of statement 1 and replace λ by $1/\lambda$.

3. Again as in the proof of statement 1, we can assume w.l.o.g. $|\lambda| \geq 1$. Then $\|\Lambda_m\|_2^2 \leq (m+1)|\lambda|^{2m}$ and

$$\begin{aligned} 2\|\Pi_+(\Lambda_m)\|_2^2 &\geq |\lambda^m + 1|^2 + 2|\lambda|^m \\ &= |\lambda|^{2m} + 2\mathsf{Re}(\lambda^m) + 1 + 2|\lambda|^m \geq |\lambda|^{2m}, \end{aligned}$$

concluding the proof.

□