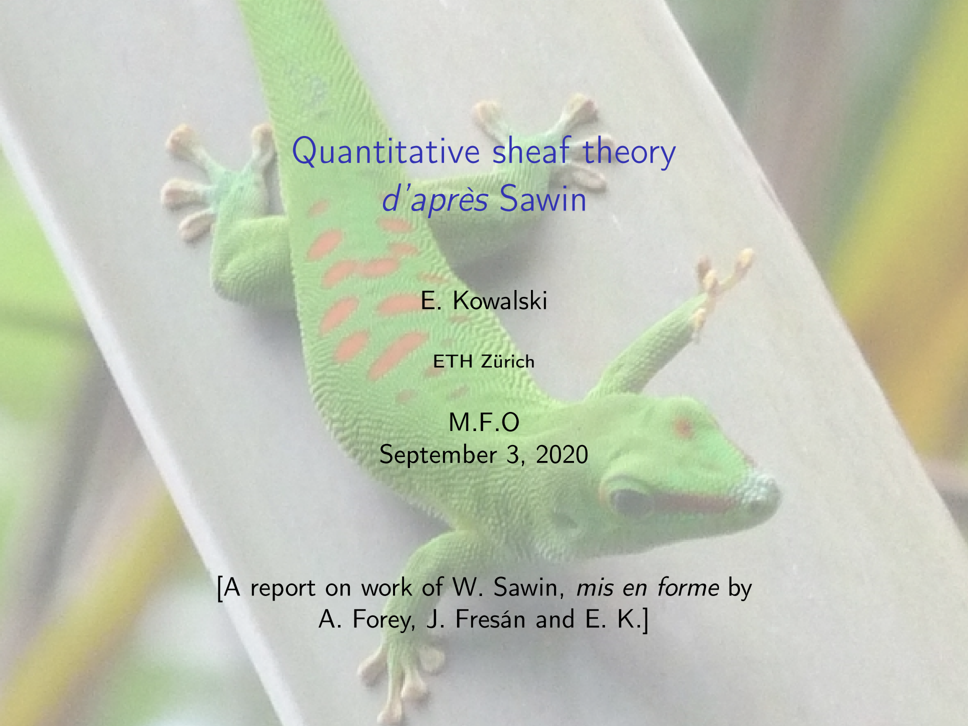

Quantitative sheaf theory *d'après* Sawin

E. Kowalski

ETH Zürich

M.F.O
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[A report on work of W. Sawin, *mis en forme* by A. Forey, J. Fresán and E. K.]

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For this particular case, bounds for N_p are due to Bombieri, Adolphson–Sperber and especially Katz.

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Or

$$\sum_{0 \leqslant x_i \leqslant X} e\Big(\frac{f(x)}{p}\Big) \lambda_1(x_1) \cdots \lambda_n(x_n)$$

for some other interesting arithmetic functions λ_i.

One-variable sums

This problem was particularly evident in the papers of Fouvry, Michel and myself, where we consider general one-variable trace functions and analytic expressions like

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We defined (Feb. 28, 2012) a "complexity" invariant c that turns out to give a good theory for one-variable sums, in the sense that in analytic estimates such as

$$\sum_{n \leqslant X} \lambda_f(n) t(n) \ll_t \Big(1 + \frac{X}{q}\Big) q^{1-1/8+\varepsilon},$$

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Although we speak informally of the complexity of t, it is really a complexity for the underlying geometric object (sheaf).

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However, the proof is very special to this case, exploiting Laumon's subtle local theory of the algebraic Fourier transform. Replacing the kernel $e(xy/p)$ by another is impossible, unless one finds some relation to the Fourier transform (e.g., convolution).

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- It essentially solves, "once and for all", the "N_p-problem" for analytic number theory (over $\mathbf{Q}$)....
- ... but that should be considered as less important as the problem of proving cancellation (showing that certain cohomology groups vanish)!

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These are parallel to constructions in algebraic geometry that provide an extremely flexible formalism.

Higher-dimensional trace functions: examples

Let $n \geqslant 1$ be an integer. Let ψ (resp. χ) be a character of $\mathbf{F}$ (resp. of $\mathbf{F}^{\times}$). Put $\chi(0) = 0$ if χ is non-trivial, and otherwise $\chi(0) = 1$.

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The following are trace functions on $\mathbf{A}^n$:

(AS) For any polynomial $f \in \mathbf{F}[x_1, \ldots, x_n]$, the function $t_1(x) = \psi(f(x))$.

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(D) The complex conjugate $\overline{t}_1$ is a trace function.

Fourier transform

Denote $x \cdot y = x_1 y_1 + \cdots + x_n y_n$.

Corollary. Let t be a trace function in n variables. The Fourier transform

$$\widehat{t}(y) = \frac{1}{|\mathbf{F}|^{n/2}} \sum_{x \in \mathbf{F}^n} t(x) \psi(x \cdot y)$$

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Indeed:

- $\psi(x \cdot y)$ is a trace function in $2n$ variables (x, y) (rule **AS**).
- $t(x)$ is a trace function in $2n$ variables (rule **PB** applied to $(x, y) \mapsto x$).
- $t(x)\psi(x \cdot y)$ is a trace function in $2n$ variables (rule **TP**).
- the sum over x of $t(x)\psi(x \cdot y)$ is a trace function in n variables y (rule **DI** applied to $(x, y) \mapsto y$).
- and dividing by $|\mathbf{F}|^{n/2}$ is allowed (rule **TT**).

Exercise

Consider a family of elliptic curves

$$E_u \colon y^2 = x^3 + a(u)x + b(u)$$

with a, $b \in \mathbf{F}[u_1, \ldots, u_n]$ (with non-zero discriminant).

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with a, $b \in \mathbf{F}[u_1, \ldots, u_n]$ (with non-zero discriminant).
Show that

$$t(u) = |E_u(\mathbf{F})| - (|\mathbf{F}| + 1)$$

is a trace function in n variables.

Dictionary

Algebraic geometers use different notation; here is a partial dictionary:

Analytic number theory	Algebraic geometry
$\psi(f(x))$	$\mathscr{L}_{\psi(f)}$
$t_1 t_2$	$\mathscr{F}_1 \otimes \mathscr{F}_2$
$t \circ g$	$g^* \mathscr{F}$
(DI) applied to t and h	$\mathrm{R} h_! \mathscr{F}$
$\lvert\mathbf{F}\rvert^{h/2} t(x)$	$\mathscr{F}(-h/2)$
$\frac{1}{\lvert\mathbf{F}\rvert^{n/2}} \sum_{x \in \mathbf{F}^n} t(x)\psi(x \cdot y)$	$\mathrm{R} p_{2,!}(p_1^* \mathscr{F} \otimes \mathscr{L}_{\psi(x \cdot y)})(n/2)$
	where $p_1(x,y) = x$, $p_2(x,y) = y$.

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These measure the *complexity* of the trace function, or of the morphism.

Main result

General principle. In all operations on trace functions (and polynomials), the complexity "after" is bounded in terms of the complexity "before" – this is a form of *continuity*.

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Moreover, in most cases, the complexity can increase at most linearly.

And the complexity controls the "number of roots" and other analytic invariants of the trace functions. (So putting $c(\mathscr{F}) = 0$ would not be wise...)

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(D) $c(\bar{t}) \ll c(t)$.

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- Under suitable conditions, the Riemann Hypothesis becomes

$$\Big|\frac{1}{|\mathbf{F}|^n}\sum_{x\in\mathbf{F}^n} t_1(x)\overline{t_2(x)} - (\text{main term})\Big| \ll c(\mathscr{F}_1)c(\mathscr{F}_2)|\mathbf{F}|^{-1/2}.$$

Control properties

Given a trace function t in n variables (associated to an object $\mathscr{F}$), we have:

- The "total number of roots" of $\mathscr{F}$ is $\leqslant c(\mathscr{F})$.
- This means that the L-function of $\mathscr{F}$ (constructed using extensions of $\mathbf{F}$) can be written as f_1/f_2 for polynomials f_1 and f_2 with $\deg(f_1) + \deg(f_2) \leqslant c(\mathscr{F})$.
- Under suitable conditions, the Riemann Hypothesis becomes

$$\Big|\frac{1}{|\mathbf{F}|^n}\sum_{x\in\mathbf{F}^n} t_1(x)\overline{t_2(x)} - (\text{main term})\Big| \ll c(\mathscr{F}_1)c(\mathscr{F}_2)|\mathbf{F}|^{-1/2}.$$

- For one variable trace functions

$$c_{\mathrm{fkm}}(\mathscr{F}) \leqslant c(\mathscr{F}) \leqslant 3c_{\mathrm{fkm}}(\mathscr{F})^2.$$

A two-variable example

Take $f \in \mathbf{F}(x, y)$ and for a trace function t in one variable, define

$$T_f(t)(y) = \frac{1}{|\mathbf{F}|^{1/2}} \sum_{x \in \mathbf{F}} t(x) e(f(x, y)/p).$$

This is a trace function in one variable and $c(T_f(t)) \ll c(f)c(t)$.

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More generally, for a trace function t in n variables, we get

$$c(\widehat{t}) \ll c(t)$$

for the Fourier transform, where the implied constant depends only on n.

Application 1: equidistribution along primes

Combining the formalism of complexity with Deligne's Riemann Hypothesis, we can for instance prove the following equidistribution result, which answers a question of Katz:

Theorem. Let $n \geqslant 1$ and $e \geqslant 1$ be integers. Let $P(n,e)$ be the set of polynomials of degree e in n variables. For $f \in P(n,e)(\mathbf{F}_p)$, let

$$S(f;p) = \frac{1}{p^{n/2}} \sum_{x \in \mathbf{F}_p^n} e\Big(\frac{f(x)}{p}\Big).$$

The families $(S(f;p))_{f \in P(n,e)(\mathbf{F}_p)}$ become equidistributed as $p \to +\infty$ with respect to the measure which is the image under the trace of the probability Haar measure on $\mathrm{U}_{(e-1)^n}(\mathbf{C})$.

Application 2: more equidistribution

In work in progress of Forey, Fresán, K., we generalize Katz's work on Mellin transforms to other groups (e.g. to exponential sums parameterized by tuples $(\chi_1, \ldots, \chi_n)$ of multiplicative characters, or by pairs (χ, ψ) of multiplicative and additive characters).

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For instance, we can get "vertical" equidistribution statements for

$$S(\chi, \psi; \mathbf{F}) = \frac{1}{|\mathbf{F}|^{1/2}} \sum_{x \in \mathbf{F}} \chi(x)\psi(x)t(x)$$

for suitable trace functions t.

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We can also obtain applications to things like the variance of arithmetic functions for twists of higher-degree L-functions over $\mathbf{F}[u]$ (generalizing work of Hall, Keating and Roditty–Gershon).