THE GLOBAL ROOT NUMBER FOR $J_0(p)$

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Let p be a prime, and consider $A = J_0(p)$, the Jacobian of $X_0(p)$. Its L-function is given by

$$L(A,s) = \prod_{f \in S_2^*(p)} L(f,s)$$

where $S_2^*(p)$ is the set of newforms of level p. Each $L(f,s)$ satisfies a functional equation

$$\Lambda(f,s) = \varepsilon(f) p^{1/2-s} \Lambda(f,1-s)$$

where $\Lambda(f,s) = (2\pi)^{-s}\Gamma(s+1/2)L(f,s)$ and the sign for f is

$$\varepsilon(f) = -w(f), \text{ where } f \mid w_p = w(f)f,$$

w_p being the Atkin-Lehner involution. We wish to compute the *global* sign of the functional equation for $L(A,s)$:

$$\Lambda(A,s) = \varepsilon(A) p^{g(p)(1/2-s)} \Lambda(A,1-s)$$

where $g(p) = |S_2^*(p)| = \dim A(p)$ and

$$\Lambda(A,s) = \prod_f \Lambda(f,s).$$

Lemma 1. *We have for p prime*

$$\varepsilon(A) = (-1)^{\frac{1}{2}\operatorname{Tr}(1+w_p)},$$

where the trace is acting on $S_2(p)$.

Proof. Because w_p is an involution, its eigenvalues are ± 1, and in particular one has

$$w(f) = (-1)^{\frac{1}{2}(w(f)-1)}.$$

Then taking the product over f we have

$$\varepsilon(A) = \prod_f \varepsilon(f) = (-1)^{g(p)+\frac{1}{2}(\operatorname{Tr}(w(f))-g(p))} = (-1)^{\frac{1}{2}\operatorname{Tr}(1+w_p)}.$$

□

Remark 2. In particular, notice that $\operatorname{Tr}(1+w_p) \equiv 0 \,(\operatorname{mod} 2)$, i.e.

$$\operatorname{Tr}(w_p) \equiv g(p) \,(\operatorname{mod} 2).$$

This is not obvious from the computation of $\operatorname{Tr}(w_p)$ (see below) and thus "reproves" part of genus theory for imaginary quadratic fields...

The genus $g(p)$ of $X_0(p)$ and the trace of w_p are computed classically, respectively, using the Riemann-Hurwitz formula and the Selberg trace formula.

Lemma 3. *We have*

$$g(p) = \frac{q+1}{12} - \frac{1}{4}\Big(1 + \Big(\frac{-1}{p}\Big)\Big) - \frac{1}{3}\Big(1 + \Big(\frac{-3}{p}\Big)\Big)$$

and for $p > 3$

$$\operatorname{Tr}(w_p) = 1 - \frac{1}{2} \sum_{\substack{-4p = df^2 \\ d \ disc}} h(d),$$

where the condition on d is that $d < 0$ be a discriminant of an order in an imaginary quadratic field, and $h(d)$ is the class-number of the order of discriminant d.

The restriction $p > 3$ is of course no problem since $g(3) = 0$.

Proof. For the first formula, see e.g. Shimura's book, and for the second, see e.g. Brumer's Astérisque 228 article. (The sum there has $H_1(s^2 - 4p)$ and extends to $s^2 \leqslant 4p$ such that $p \mid s$, which means $s = 0$, etc). $\square$

We can express $\mathrm{Tr}(w_p)$ more concretely by getting into congruence classes modulo 4 and 8. Namely, if $p \equiv 3 \pmod 4$, then $-p$ is the discriminant of $\mathbf{Q}(\sqrt{-p})$ and the equation $-4p = df^2$ above has two solutions, $(d, f) = (-p, 2)$ and $(d, f) = (-4p, 1)$, hence we obtain

$$\mathrm{Tr}(w_p) = 1 - \frac{1}{2}(h(-4p) + h(-p)), \text{ for } p \equiv 3 \pmod 4.$$

By a well-known formula (see e.g. Cox), we have

$$h(-4p) = 2h(-p)\Big(1 - \frac{1}{2}\Big(\frac{-p}{2}\Big)\Big) = h(-p)\Big(2 - \Big(\frac{-p}{2}\Big)\Big),$$

so using the value of the Kronecker symbol we have

$$\mathrm{Tr}(w_p) = \begin{cases} 1 - h(-p) & \text{if } p \equiv 7 \pmod 8 \\ 1 - 2h(-p) & \text{if } p \equiv 3 \pmod 8. \end{cases}$$

Only in the second case is $\mathrm{Tr}(w_p) \pmod 4$ easy to know: by genus theory, one has $h(-p) \equiv 1 \pmod 2$ hence $1 - 2h(-p) \equiv 3 \pmod 4$ if $p \equiv 3 \pmod 8$.

In the case $p \equiv 1 \pmod 4$, then $-4p$ is the discriminant of $\mathbf{Q}(\sqrt{-p})$ and the only solution is $(d, f) = (-4p, 1)$, hence

$$\mathrm{Tr}(w_p) = 1 - \frac{1}{2}h(-4p) \text{ if } p \equiv 1 \pmod 4.$$

The value of $g(p)$ modulo 4, on the other hand, depends on the class of p modulo 48.

Lemma 4. *The following table gives* $g(p) \pmod 4$*:*

$p \pmod{48}$	1	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47
$g(p) \pmod 4$	3	0	0	1	0	1	1	2	1	2	2	3	2	3	3	0

Putting together the two results, we obtain "one-fourth" of the values of $\varepsilon(A)$:

Proposition 5. *If* $p \equiv 3 \pmod 8$*, then*

$$\varepsilon(A) = \begin{cases} 1 & \text{if } p \equiv 11 \text{ or } 19 \pmod{48} \\ -1 & \text{if } p \equiv 35 \text{ or } 43 \pmod{48}. \end{cases}$$

In the other cases of $p \pmod{48}$, there does not seem to be such a simple "congruence" formula. However one can make numerical experiments and they suggest the reasonable result that the signs ± 1 should be equiprobable for $\varepsilon(J_0(p))$ as $p \to +\infty$. A proof, from the result above, is tantamount to proving two equidistribution results for $h(-p) \pmod 4$, $p \equiv 7 \pmod 8$ and $h(-4p) \pmod 8$, $p \equiv 1 \pmod 4$.

Here are some numerical data: let

$$N(x) = \sum_{p \leqslant X} \varepsilon(J_0(p)),$$

then for $X = 10^6$, one finds

$$\sup_{p \leqslant X} N(x) = 166 \text{ and } \inf_{p \leqslant X} N(x) = -195.$$

Remark 6. Henri Cohen remarked – as a result of the striking coincidence that he was computing the left-hand side just as I was coming to ask about the right-hand side... – that

$$\Gamma_p\Big(\frac{1}{2}\Big) \equiv h(-p) \pmod 4 \text{ for } p \equiv 3 \pmod 4,$$

where Γ_p is the p-adic Gamma function defined on $\mathbf{Z}_p$; the formula makes sense because $\Gamma_p(1/2)^2 = 1 \in \mathbf{Z}$, hence the value of $\Gamma_p(1/2)$ is ± 1 (see e.g. Lang, "Cyclotomic fields", vol. 2). One may wonder if more general statements of this kind exist?