

VARIANTS OF RECOGNITION PROBLEMS FOR MODULAR FORMS

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ABSTRACT. We consider the problem of distinguishing two modular forms, or two elliptic curves, by looking at the coefficients of their L -functions for small primes (compared to their conductor). Using analytic methods based on large-sieve type inequalities we give various upper bounds on the number of forms having the first few coefficients equal to those of a fixed one. In addition, we consider similar questions of “recognizing” symmetric squares and CM forms from the behavior of small primes.

1. INTRODUCTION

This paper is an addition to [DK], together with some variations on the problem considered there. It was motivated by discussions with F. Brumley who recently approached this question using a very nice method based on density theorems for zeros of L -functions.

Recall the setting, which generalizes the classical question of the least quadratic non-residue modulo a prime p : given two primitive holomorphic cusp forms f, g of the same weight $k \geq 1$ and nebentypus ε , with conductors $q(f), q(g)$ respectively, how large should M be (in terms of the parameters described) so that the equality

$$\lambda_f(p) = \lambda_g(p)$$

for all primes $p \leq M$ implies that $f = g$? Here

$$f(z) = \sum_{n \geq 1} \lambda_f(n) n^{(k-1)/2} e(nz), \quad g(z) = \sum_{n \geq 1} \lambda_g(n) n^{(k-1)/2} e(nz)$$

are the Fourier expansions of f and g , or in other words $\lambda_f(n)$ is the n -th Hecke eigenvalue for f .

The Generalized Riemann Hypothesis for Rankin-Selberg L -functions shows that one can take $M = C(\log kq(f)q(g))^2$, where $C > 0$ is some absolute (and effective) constant (see e.g. [IK, Prop. 5.22]).

The following is a reformulation of the main result of [DK]:

Theorem 1.1. *Let $\alpha > 0$ be any positive number and let $Q \geq 2$. For any primitive form f of weight k and conductor $q \leq Q$, the number N of primitive non-CM modular forms g of weight k with conductor $q(g) \leq Q$ such that*

$$(1) \quad \lambda_g(p) = \lambda_f(p) \text{ for } p \leq (\log Q)^\alpha$$

is bounded by

$$N \ll Q^{10\alpha^{-1} + 1/2 + \varepsilon}$$

for any $\varepsilon > 0$, the implied constant depending only on α and ε .

Since the number of possible forms g is $\gg Q^2$, it follows that for $\alpha > 10$ the probability of g satisfying (1) tends to 0 as $Q \rightarrow +\infty$. However, the exponent $\frac{1}{2}$ is unsatisfactory as it prevents the same conclusion to be reached if we look only for g in a sub-family with less than about $Q^{1/2}$ elements of conductor $\leq Q$. In [DK], it is also shown that this exponent can be removed when looking at families with q squarefree.

In [DK], this exponent arises from the need to use the symmetric square L -functions when the Fourier coefficients considered in (1) are “too small”, so that one ends up counting not really the modular forms g but their symmetric squares. As shown by Ramakrishnan in the Appendix to [DK], we have $\text{Sym}^2 f = \text{Sym}^2 g$ if and only if $f = g \otimes \chi$ for some primitive real character χ (the twist $g \otimes \chi$ is taken to be the primitive form associated to the modular form with Fourier coefficients $\lambda_g(n)\chi(n)n^{(k-1)/2}$, which may be of smaller level). Thus the exponent $1/2$ arose by simply counting the possible number of characters χ such that the twist has conductor $\leq Q$.

In the next section we improve slightly the analysis of this case, removing the exponent $1/2$, but at the cost of introducing a dependency on f in the final estimate. We will see that it does not seem

easy to remove this dependency in general.¹ However in Section 3, we get some uniform results for elliptic curves, using some cute arguments on group representations. In Section 4, we discuss two new variants: distinguishing symmetric squares from small primes, and CM forms from small primes inert in a quadratic field.

Notation. In general we denote by $\lambda_f(p)$ the Hecke eigenvalues for a primitive automorphic form, normalized so that the critical line for $L(f, s)$ is $\text{Re}(s) = \frac{1}{2}$, independently of the weight. If f is associated to an elliptic curve E/\mathbf{Q} , we write $a_E(p)$ for the Fourier coefficients so $a_E(p) = \sqrt{p}\lambda_f(p)$. We denote by $q(f)$ the conductor of a primitive form, and by $q(\chi)$ that of a Dirichlet character.

Finally, the notation $f \ll g$ and $f = O(g)$ are synonymous for us and refer to bounds for all x in an explicitly specified set X (sometimes clear from the context), with a well-defined implied constant.

2. RECOGNIZING A FIXED FORM

Instead of fixing a form, it is maybe clearer to formulate the result as a multiplicity bound for a generic sequence of would-be eigenvalues. This viewpoint is inspired by that taken by Sarnak [Sa].

Proposition 2.1. *Let $\Lambda = (\lambda(p))$ be a fixed sequence of complex numbers indexed by prime numbers. For any $\alpha > 0$, any $Q \geq 2$, the number N of primitive modular forms g of weight 2 with conductor $q(g) \leq Q$ such that*

$$\lambda_g(p) = \lambda(p) \text{ for } p \leq (\log Q)^\alpha$$

is bounded by

$$N \ll Q^{12\alpha^{-1} + \varepsilon}$$

for any $\varepsilon > 0$, the implied constant depending on ε and Λ .

We first state a slightly more precise version of the counting lemma that was used in [DK] and will be used repeatedly here.

Lemma 2.2. *Let $Q \geq 2$, $\alpha > 2$, $\beta > 0$, $q \geq 1$. Let $m \geq 1$ be maximal such that $(\log Q)^{m\alpha} \leq Q^\beta$. Let \mathcal{P} be a set of primes $p \leq (\log Q)^\alpha$ such that*

$$(2) \quad |\mathcal{P}| \geq \delta \frac{(\log Q)^\alpha}{\log\{\log Q\}^\alpha}.$$

Then the number N of integers $n \leq Q^\beta$ such that n is squarefree, has m prime factors, all of which belong to \mathcal{P} , satisfies

$$Q^{\beta - \beta\alpha^{-1} - \varepsilon} \ll N \ll Q^{\beta - \beta\alpha^{-1} + \varepsilon}$$

for any $\varepsilon > 0$, the implied constants depending only on δ , α and ε .

Proof. We have by Stirling's formula

$$N = \binom{|\mathcal{P}_q|}{m} \gg m^{-1/2} \left(\frac{|\mathcal{P}_q|}{m} \right)^m$$

with an absolute implied constant, where \mathcal{P}_q is the set of primes $p \in \mathcal{P}$ not dividing q (use $m^2 \leq N$). We have

$$|\mathcal{P}_q| \gg \frac{(\log Q)^\alpha}{\log\{\log Q\}^\alpha}$$

by (2), the implied constant depending only on δ since q has $\ll \log \log Q$ prime factors.

Hence we see that

$$N \gg m^{-1/2} \left(\frac{(\log Q)^\alpha}{\log\{\log Q\}^\alpha} \frac{\alpha \log \log Q}{\beta \log Q} \right)^m \gg Q^{-\varepsilon} \left((\log Q)^{\alpha-1} \right)^{\frac{\beta(\log Q)}{(\alpha \log \log Q)} - 1} \gg Q^{\beta \frac{\alpha-1}{\alpha} - \varepsilon}.$$

On the other hand, Stirling's formula again yields similarly

$$N \leq \binom{|\mathcal{P}|}{m} \ll \left(\frac{|\mathcal{P}|}{m} \right)^m \ll Q^{\beta \frac{\alpha-1}{\alpha} + \varepsilon}.$$

□

¹ Note that also in the best individual and unconditional bounds currently known, e.g. in [KMV], [Ric] one also has to fix the form to which the others are compared.

Proof of Proposition 2.1. Let S be the set of forms g that is being counted. One can assume S not empty. The argument of [DK] gives the estimate

$$N' \ll Q^{10\alpha^{-1}+\varepsilon}$$

for the number N' of symmetric squares of the forms $g \in S$, and by Ramakrishnan's theorem, two elements g_1, g_2 of S have the same symmetric square if and only if $g_1 = g_2 \otimes \chi$ for some real primitive quadratic character χ .

Fix a form $g \in S$ with symmetric square having maximal multiplicity. For any $g_1 \in S$ with the same symmetric square as g , we have $g_1 = g \otimes \chi$ for some χ hence

$$\lambda_{g_1}(p) = \chi(p)\lambda_g(p)$$

for any p coprime with $q(g_1)$ and $q(g)$. Using the assumption it follows that

$$\lambda(p) = \chi(p)\lambda(p)$$

for $p \leq (\log Q)^\alpha$ such that $(p, q(g_1)q(g)) = 1$, so that either $\lambda(p) = 0$ or $\chi(p) = 1$ for those primes.

By results of Serre [Se1], the set of primes p for which $\lambda_g(p) \neq 0$ has positive density $\geq \frac{1}{2}$ (in fact, density 1 if g is not a CM form, and density $\frac{1}{2}$ if g is a CM form). Since $S \neq \emptyset$ by assumption, this means that we have

$$(3) \quad |\{p \leq X \mid \lambda(p) \neq 0\}| \gg \frac{X}{\log X}$$

for $X = (\log Q)^\alpha$, the implied constant depending on Λ . Let $\mathcal{P} = \{p \leq (\log Q)^\alpha \mid \lambda(p) \neq 0\}$ be this set of primes. For any χ as above we have $\chi(p) = 1$ for $p \in \mathcal{P}$ coprime with $q(g)q(\chi)$.

To estimate the number of χ , we are therefore led back to Linnik's original problem, precisely to a simple variant since the set of primes involved is not the initial segment. For any primitive Dirichlet character ψ , write

$$L(\psi) = \sum_{n \leq Q^2} a_n \psi(n)$$

where a_n is the characteristic function of the set T of integers $n \leq Q^2$ such that n is squarefree, has a fixed number m of prime factors, and $p \mid n$ implies $p \in \mathcal{P}$ and $(p, q(g)) = 1$. We have moreover

$$L(\chi) = |\{n \in T \mid (n, q(\chi)) = 1\}|$$

if $g \otimes \chi \in S$. By Lemma 2.2 twice (with the same m , but with different q), if m is suitably chosen, we have

$$|T| \geq L(\chi) \gg Q^{2(1-\alpha^{-1})-\varepsilon}, \quad |T| \ll Q^{2(1-\alpha^{-1})+\varepsilon}, \quad \text{so } L(\chi) \gg |T|^{1-\varepsilon}$$

for any $\varepsilon > 0$, the implied constants depending on α, ε and Λ (not on χ).

On the other hand, by positivity and the multiplicative large sieve inequality we have

$$|T|^{2-\varepsilon} |\{\chi \mid g \otimes \chi \in S\}| \leq \sum_{q \leq Q} \sum_{\chi \pmod{q}}^* |L(\chi)|^2 \leq Q^2 \sum_{n \leq Q^2} |a_n|^2 \leq Q^2 |T|$$

so the number of characters is $\ll Q^{2\alpha^{-1}+\varepsilon}$ for any $\varepsilon > 0$, the implied constant depending on α, ε and Λ .

This bounds the maximal possible multiplicity of the symmetric square, hence

$$N \ll N' Q^{2\alpha^{-1}+\varepsilon} \ll Q^{12\alpha^{-1}+\varepsilon}$$

for any $\varepsilon > 0$, the implied constant depending on Λ, α and ε . □

To remove the dependency of the result on the sequence Λ amounts to giving a uniform version of (3) for $\lambda(p) = \lambda_f(p)$. This is certainly difficult because the primes involved are very small, so even the existence of one $p \leq (\log Q)^\alpha$ with $\lambda_f(p) \neq 0$ is an extremely strong result for f of conductor about Q . And since, if a form exists with many zero coefficients, then many twists may have the same symmetric square and the same first Fourier coefficients, one cannot expect to deal with g on average only.

The case of Maass forms, also considered by Brumley, can not be treated as above because an analogue of (3) is not known in this case; one only knows that there are $\gg_g x/\log x$ values of $n \leq x$ such that $\lambda_f(n) \neq 0$ (see Proposition 3 of [KRW]).

3. RECOGNIZING ELLIPTIC CURVES

In the case of elliptic curves, one can derive a strong uniform version on the assumption of the Riemann Hypothesis, not for all L -functions (since GRH for Rankin-Selberg L -functions gives more than what we want!), but for real characters and for holomorphic modular forms of weight 1 (with solvable image, if one wishes). Without GRH, a much weaker uniform result follows by the standard zero-free regions:

Proposition 3.1. *Let $\alpha > 0$, $Q \geq 2$ and let X be a set with maximal cardinality of isogeny classes of elliptic curves E/\mathbf{Q} with conductors $\leq Q$ such that*

$$a_E(p) = a_F(p) \text{ for } E, F \in X, p \leq (\log Q)^\alpha.$$

(1) *Assuming GRH for Dirichlet L -functions of real characters and holomorphic modular forms of weight 1, we have*

$$|X| \ll Q^{12\alpha^{-1} + \varepsilon}$$

for any $\varepsilon > 0$, the implied constant depending on α and ε only.

(2) *Unconditionally, we have*

$$|X| \ll Q^{12\alpha^{-1} + \varepsilon}$$

provided X contains at least one curve with conductor $\leq \log\{\log Q\}^\alpha$. The implied constant depends on α and ε only.

We need a lemma (probably well-known) to bound the conductors of some Artin L -functions. In what follows, we denote by $G_{\mathbf{Q}}$ the Galois group of a separable closure of \mathbf{Q} .

Lemma 3.2. *Let E/\mathbf{Q} be an elliptic curve, ℓ be a prime number, $\rho_\ell : G_{\mathbf{Q}} \rightarrow GL(2, \mathbf{Z}/\ell\mathbf{Z})$ the Galois representation given by the action on the ℓ -torsion points of E . For any irreducible linear representation $\pi : GL(2, \mathbf{Z}/\ell\mathbf{Z}) \rightarrow GL(m, \mathbf{C})$, the Artin conductor of the representation $\pi \circ \rho_\ell$ is $\leq c(\ell)q^m$, where q is the conductor of E and $c(\ell) \geq 1$ is an integer depending only on ℓ .*

Proof. For any fixed prime p , notice that for any real number $v \geq -1$, the upper-numbering ramification group I_p^v (normalized as in [DDT, §2.1], so I_p^v is the inertia group at p for $v \leq 0$) acts trivially on ρ_ℓ and on $\pi \circ \rho_\ell$ if it acts trivially on the ℓ -torsion points of E . For $p \neq \ell$, the exponent of the Artin conductor of $\pi \circ \rho_\ell$ at p is given by

$$v_p(\pi \circ \rho_\ell) = \int_{-1}^{+\infty} \text{codim}(\pi \circ \rho_\ell)^{I_p^v} dv$$

(see e.g. [DDT, p. 49]). Let V be such that I_p^v acts trivially on $E[\ell]$ for $v > V$ and non-trivially for $v < V$. Then we have

$$(4) \quad v_p(\pi \circ \rho_\ell) = \int_{-1}^V \text{codim}(\pi \circ \rho_\ell)^{I_p^v} dv \leq m(V + 1)$$

and similarly the exponent of the conductor for the representation modulo ℓ is

$$v_p(\rho_\ell) = \int_{-1}^V \text{codim}(\rho_\ell)^{I_p^v} dv \geq V + 1,$$

since $\text{codim}(\rho_\ell)^{I_p^v}$ must be ≥ 1 if the action is non-trivial. Since $p^{v_p(\rho_\ell)} \mid q$ (see e.g. [DDT, Lemma 2.7, Remark 2.14]), we obtain the result at p .

For $p = \ell$, let $K_\ell = \mathbf{Q}(E[\ell])$ be the field generated by coordinates of ℓ -torsion points of E and $n = [K_\ell : \mathbf{Q}]$. The exponent of ℓ in the discriminant of K_ℓ/\mathbf{Q} is the same as the exponent of the conductor of the regular representation $\omega : G_{\mathbf{Q}} \rightarrow \mathbf{C}[\text{Gal}(K_\ell/\mathbf{Q})] \simeq GL(n, \mathbf{C})$ (see e.g. [Se3, VI.3, Cor. 1]). Hence as above it is given by

$$v_\ell(K_\ell/\mathbf{Q}) = \int_{-1}^{+\infty} \text{codim}(\omega)^{I_\ell^v} dv.$$

Let V be as above for ℓ , so also I_ℓ^v acts trivially on ω for $v > V$. We have

$$v_\ell(\pi \circ \rho_\ell) = \int_{-1}^V \text{codim}(\pi \circ \rho_\ell)^{I_\ell^v} dv \leq m(V + 1)$$

$$v_\ell(K_\ell/\mathbf{Q}) = \int_{-1}^V \text{codim}(\omega)^{I_\ell^v} dv \geq V + 1.$$

Hence to prove the desired result it is enough to find an upper bound for $v_\ell(K_\ell/\mathbf{Q})$ which is independent of E . Such a bound follows, for instance, from the fact that the degree of K_ℓ is bounded independently of E , and from universal bounds for the discriminant of extensions of local fields of bounded degree (see, e.g., [Se1, Cor. to Prop. 2]). \square

Remark 3.3. For our application, one could also simply bound the conductor of $\pi \circ \rho_\ell$ by the discriminant of K_ℓ and apply a general bound like [Se1, Prop. 6].

Although we will use only the case $q = 3$ of the next lemma, it is nice enough to state generally.

Lemma 3.4. *Let \mathbf{F}_q be a field with q elements of characteristic $\neq 2$. Let N be the set of elements in $GL(2, \mathbf{F}_q)$ which are not diagonalizable over $\bar{\mathbf{F}}_q$, f_N the characteristic function of N . Then we have*

$$(5) \quad qf_N = 1 + \chi \circ \det + \sum_{\psi^2 \neq 1} \chi_{\psi, \psi^{-1}} - \sum_{\varphi | \mathbf{F}_q = 1} \chi_\varphi$$

where χ is the quadratic character of \mathbf{F}_q , ψ runs over characters of \mathbf{F}_q^\times and $\chi_{\psi, \psi^{-1}}$ is the character of the associated irreducible representations of degree $q + 1$, φ runs over characters of $\mathbf{F}_{q^2}^\times$ with $\varphi^q \neq \varphi$ and χ_φ is the character of the associated irreducible representation of degree $q - 1$; see the discussion below for details.

Proof. The set N is the union of the $q - 1$ conjugacy classes of the elements

$$n_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$$

with $x \neq 0$, each of which has $q^2 - 1$ elements. To decompose f_N in terms of characters, we need only look at the corresponding column of the character table, which we quote with the notation of [FH, p. 70] (transposed for easier reading, and with the degree of the representations for reference):

	U_α	V_α	$W_{\alpha, \beta}$	X_φ
degree	1	q	$q + 1$	$q - 1$
n_x	$\alpha(x^2)$	0	$\alpha(x)\beta(x)$	$-\varphi(x)$

(U_α is the character $\alpha(\det(x))$, V_α is the irreducible component of degree q of the permutation representation on $\mathbf{P}^1(\mathbf{F}_q)$, twisted by α , $W_{\alpha, \beta}$ is induced from the character (α, β) of the Borel subgroup (up to permutation) with $\alpha \neq \beta$, and X_φ , for φ a character of $\mathbf{F}_{q^2}^\times$ such that $\varphi^q \neq \varphi$, is the (a priori virtual) representation $V_1 \otimes W_{1, \varphi} - W_{1, \varphi} - \text{Ind}(\varphi)$).

The obvious point now is that for each irreducible character χ , χ restricted to $n_x \in N$ is, as function of x , a constant multiple of some multiplicative character of \mathbf{F}_q^\times . Hence the scalar product $\langle f_N, \chi \rangle$ vanishes by orthogonality unless the corresponding character is trivial on \mathbf{F}_q^\times . This immediately implies (5) by inspection. \square

Remark 3.5. In particular, among the $q^2 - 1$ irreducible representations of $GL(2, \mathbf{F}_q)$, only q occur in the decomposition of f_N . One can prove more conceptually that characters restricted to N are multiples of multiplicative characters, as pointed out by D. Bump: the matrix n_x is conjugate to

$$\begin{pmatrix} x & x \\ 0 & x \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

hence, if ψ is the central character, we have

$$\chi(n_x) = c\psi(x) \text{ with } c = \chi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right).$$

Proof of Proposition 3.1. By modularity of elliptic curves and the reasoning at the beginning of the proof of Proposition 2.1, it suffices to prove a uniform lower bound for the number of primes $\leq (\log Q)^\alpha$ where $a_E(p) = 0$.

For any prime ℓ , the Galois representation $\rho_\ell : G_{\mathbf{Q}} \rightarrow GL(2, \mathbf{Z}/\ell\mathbf{Z})$ on ℓ -torsion points satisfies $a_E(p) \equiv \text{Tr } \rho_\ell(\text{Fr}_p) \pmod{\ell}$ for any unramified prime $p \neq \ell$. In particular, if $\rho_\ell(\text{Fr}_p)$ is in the set of conjugacy classes with non-zero trace in $\mathbf{Z}/\ell\mathbf{Z}$, we have $a_E(p) \neq 0$.

If E does not have non-trivial rational 2-torsion points, one can use $\ell = 2$, but if $E[2](\mathbf{Q}) \neq 0$ we have $a_E(p) \equiv 0 \pmod{2}$ for any odd prime. To avoid considering an unnecessary case, we take $\ell = 3$.

We distinguish two cases, depending on whether $G_3 = \rho_3(G_{\mathbf{Q}}) \subset GL(2, \mathbf{Z}/3\mathbf{Z})$ has order divisible by 3 or not. If G_3 has order divisible by 3, it must contain elements of the set N described in Lemma 3.4

for $q = 3$. For any $x \in N$, we have $\text{Tr } x = \pm 1 \neq 0$, so any prime p for which $\rho_3(\text{Fr}_p) \in G_3 \cap N$ satisfies $a_E(p) \neq 0$, and we know a priori from the Chebotarev density theorem that this set of primes has positive density.

By Lemma 3.4, all characters of \mathbf{F}_3^\times being quadratic, the characteristic function f_N of N is given by

$$(6) \quad 3f_N = 1 + \det -\chi_2,$$

where 1 denotes the character of the trivial representation, \det is the determinant quadratic character, and χ_2 is the character of an irreducible representation of degree 2 (precisely of X_φ where φ is the character of $\mathbf{F}_9^\times = \mathbf{F}_3[X]^\times$, with $X^2 = -1$, that maps the primitive root $1 + X \in \mathbf{F}_9^\times$ to $i \in \mathbf{C}$).

It might be that χ_2 restricted to G_3 is not irreducible, in which case it splits into a direct sum of two characters of degree 1, say $\varepsilon_1, \varepsilon_2$. Since the set of primes we're counting has density > 0 by assumption, the trivial representation occurs with coefficient > 0 in the decomposition of f_N so one has $\varepsilon_i \neq 1$ (see (6)).

The Artin L -functions of 1, \det (and $\varepsilon_1, \varepsilon_2$ possibly) are Dirichlet L -functions, and since G_3 is solvable, the Artin L -function of χ_2 is the L -function of a primitive holomorphic cusp form of weight 1 by the Langlands-Tunnell Theorem. The conductors of all those L -functions are $\ll Q^2$ by Lemma 3.2 and we obtain

$$\begin{aligned} 3 \sum_{\substack{p \leq X \\ a_E(p) \neq 0}} \log p &\geq 3 \sum_{p \leq X} f_N(\rho_3(\text{Fr}_p)) \log p \\ &= \sum_{p \leq X} \log p + \sum_{p \leq X} \det(\rho_3(\text{Fr}_p)) \log p - \sum_{p \leq X} \chi_2(\rho_3(\text{Fr}_p)) \log p + O(1). \end{aligned}$$

By GRH for 1, \det, χ_2 (or $\varepsilon_1, \varepsilon_2$) we get for $X \geq 2$ that

$$3 \sum_{\substack{p \leq X \\ a_E(p) \neq 0}} \log p \geq X + O(\sqrt{X}(\log QX)^2),$$

with an absolute implied constant (see e.g. [IK, Th. 5.15]). Taking $X = (\log Q)^\alpha$, it follows by partial summation that for $\alpha > 5$ we have

$$(7) \quad |\{p \leq (\log Q)^\alpha \mid a_E(p) \neq 0\}| \gg \frac{(\log Q)^\alpha}{\log\{(\log Q)^\alpha\}}$$

with an absolute implied constant, which is the required uniform version of (3). If $\alpha \leq 5$, the statement of Proposition 3.1 is trivial in any case.

If G_3 has order divisible by 3 (which means that the discriminant of E is a cube, see e.g. [Se2, p. 305]), it only contains semisimple elements and its order is at most 16. It is either abelian or dihedral and its irreducible representations are thus of degree 1 or 2. Decomposing the characteristic function of the identity (with trace $-1 \neq 0$) in characters, one gets a lower bound like (7) using L -functions of degree 1 or 2 again.

We now come to what can be proved without GRH. For any primitive Dirichlet character χ modulo q we have

$$\sum_{p \leq X} \chi(p) \log p = \delta(\chi)X + O\left(\frac{\sqrt{q}X}{(\log X)^A}\right)$$

for $X \geq 2$ and $A > 0$, the implied constant depending on A only. For a holomorphic cusp form f of weight 1 with conductor q we have

$$\sum_{p \leq X} \lambda_f(p) \log p \ll \frac{\sqrt{q}X}{(\log X)^A}$$

for $X \geq 2$ and $A > 0$, the implied constant depending on A only (see e.g. [IK, Cor. 5.29, Th. 5.40]; by a result of Stark [St] for Artin L -functions of degree > 1 , or of Hoffstein and Ramakrishnan [HR] for cusp forms, the L -function of the forms that occur here do not have exceptional zeros). Denoting by q the conductor of E and taking again the case of $\ell = 3$ with G_3 of order divisible by 3, we have now

$$3 \sum_{\substack{p \leq X \\ a_E(p) \neq 0}} \log p \geq X + O\left(\frac{qX}{(\log X)^A}\right).$$

by (6) and the upper bound for the conductors of the L -functions for \det and χ_2 . Take $A = 2$. If $X = (\log Q)^\alpha$ and $q \leq \log X$, we obtain

$$|\{p \leq (\log Q)^\alpha \mid a_E(p) \neq 0\}| \gg \frac{(\log Q)^\alpha}{\log\{(\log Q)^\alpha\}}$$

where the implied constant is absolute. The other case is obviously similar. \square

4. VARIATIONS: RECOGNIZING SYMMETRIC SQUARES AND CM FORMS

The theorem of Ramakrishnan characterizing the non-injectivity of the symmetric square on modular forms suggests the following problem: assume that f and g are primitive forms (with same weight k) and that $\lambda_f(p^2) = \lambda_g(p^2)$ for $p \leq M$. How large should M be so that $\text{Sym}^2 f = \text{Sym}^2 g$, or in other words, so that $f = g \otimes \chi$ for some real primitive character χ ?

By GRH for the Rankin-Selberg L -functions $L(\text{Sym}^2 f \otimes \text{Sym}^2 g, s)$ and $L(\text{Sym}^2 f \otimes \text{Sym}^2 f, s)$, it follows that if f and g are not CM forms, the conclusion $\text{Sym}^2 f = \text{Sym}^2 g$ follows for $M = C(\log kq(f)q(g))^2$ for some absolute constant $C > 0$. (In the CM case, one gets a bound of same size by working over the CM field, using the corresponding Hecke characters).

Using the Linnik method we get:

Theorem 4.1. *Let $\alpha > 0$ be any positive number and let $Q \geq 2$. For any weight 2 primitive form f of conductor $q \leq Q$, the number N of symmetric squares of primitive non-CM modular forms g of weight 2 and conductor $q(g) \leq Q$ such that*

$$(8) \quad \lambda_g(p^2) = \lambda_f(p^2) \text{ for } p \leq (\log Q)^\alpha$$

is bounded by

$$N \ll Q^{24\alpha^{-1} + \varepsilon}$$

for any $\varepsilon > 0$, the implied constant depending only on α and ε .

There is no extraneous exponent here, but as the proof will clearly show this is because we fix the nebentypus to be trivial. The proof follows [DK] again, but this time we must use a mean-value estimate for the symmetric fourth power L -functions (available from [DK] thanks to Kim's result [K] that for a non-CM modular form g , its fourth symmetric power $\text{Sym}^4 g$ is a cuspidal automorphic representation on $GL(5)$ with conductor $\leq q^4 \leq Q^4$) if the $\lambda_f(p^2)$ are too small for small p . An issue of multiplicity arises for the symmetric fourth power. The simplest case, which suffices our purpose, is as follows:

Lemma 4.2. *For any two holomorphic primitive forms f and g of weight ≥ 2 and trivial nebentypus, we have $\text{Sym}^4 f = \text{Sym}^4 g$ if and only if $\text{Sym}^2 f = \text{Sym}^2 g$.*

This follows either from Proposition 5.1 of [CM], or from more general results of Rajan [R] on recovering ℓ -adic representations. This case is an easy application of Serre's ℓ -adic methods, and the non-expert reader is invited to see it as an interesting exercise (it boils down to showing that $\lambda_f(p^2) = 1 - \lambda_g(p^2)$ can not occur very often). Note that other cases of this lemma are a little bit different. With non-trivial nebentypus, further multiplicity comes trivially from quartic twists $f \otimes \chi_4$, where χ_4 is of order 4 exactly. More interestingly, for forms of weight 1 there is a different source of multiplicity: if f corresponds to an icosahedral Galois representation, then f has a Galois conjugate f^τ (where τ basically is the non-trivial automorphism of $\mathbf{Q}(\sqrt{5})/\mathbf{Q}$), with f^τ not a quartic twist of f , and $\text{Sym}^4 f^\tau = \text{Sym}^4 f$. (The proof above fails because $(1 - \sqrt{5})/2 = 1 - (1 + \sqrt{5})/2 \dots$)

This in particular means that the corresponding statement for Maass forms is certainly quite deep as the Langlands-type correspondance is not known to exist for even icosahedral representations.

Proof. Let p be an unramified prime for f , and let α and β be the usual local roots of f at p , so that $\alpha + \beta = \lambda_f(p)$ and $\alpha\beta = 1$. We have $\lambda_f(p^2) = \alpha^2 + \beta^2 + 1$ and

$$(9) \quad \lambda_f(p^4) = \alpha^4 + \alpha^2 + 1 + \beta^2 + \beta^4 = (\alpha^2 + \beta^2 + 1)^2 - \alpha^2 - \beta^2 - 2 = \lambda_f(p^2)^2 - \lambda_f(p^2) - 1.$$

In particular if $|\lambda_f(p^2)| < \frac{1}{2}$, we have $|\lambda_f(p^4)| \geq 1/4$. If more than half the primes $p \leq (\log Q)^\alpha$ satisfy $|\lambda_f(p^2)| < \frac{1}{2}$, then we derive

$$N \ll Q^{10\alpha^{-1} + \varepsilon}$$

from the argument of [DK].

Otherwise we have $|\lambda_f(p^4)| \geq 1/4$ for at least half the primes $\leq (\log Q)^\alpha$. For the usual parameters $X \geq 2$ and $m \geq 1$ to be chosen later, denote

$$L(g) = \sum_{n \leq X} b_n \lambda_g(n^4)$$

where $b_n = a_n \lambda_f(n^4)$, and a_n is the characteristic function of those integers $n \leq X$ such that n is squarefree and has m prime factors, all unramified for f , with $|\lambda_f(p^4)| \geq 1/4$. For any g where $\text{Sym}^2 g$ contributes to N we have

$$L(g) \geq \frac{1}{16^m} \sum_{n \leq X} a_n.$$

By positivity we have

$$\frac{N}{256^m} \left(\sum_{n \leq X} a_n \right)^2 \leq \sum_{q \leq Q} \sum_g |L(g)|^2$$

where g runs over non-CM forms of weight 2 and conductor q . Let

$$L(\text{Sym}^4 g, s) = \sum_{n \geq 1} \nu_g(n) n^{-s}$$

be the Dirichlet series expansion of the fourth symmetric power L -function of f . By the above properties of $\lambda_f(p^4)$ we can write

$$\sum_{q \leq Q} \sum_g |L(g)|^2 = \sum_{q \leq Q} \sum_g \left| \sum_{n \leq X} a_n \nu_g(n) \right|^2.$$

By Kim's functoriality theorem and Theorem 4 of [DK], we have

$$\sum_{q \leq Q} \sum_{\text{Sym}^4 g} \left| \sum_{n \leq X} a_n \nu_g(n) \right|^2 \ll X^{1+\varepsilon} \sum_{n \leq X} |a_n|^2 = X^{1+\varepsilon} \sum_{n \leq X} a_n$$

if $X > Q^\beta$ with $\beta > 24$, for any $\varepsilon > 0$, the implied constant depending on β and ε . (The exponent 24 arises because the conductor is $\leq Q^4$, and the number of forms $\ll Q^2 = Q^{4/2}$ so $d = 1/2$, $n = 5$, in the notation of loc. cit.)

Note the last sum is over the symmetric fourth powers of g . By Lemma 4.2, we can rewrite the sum over $\text{Sym}^4 g$ as one over $\text{Sym}^2 g$, which is precisely what we are counting. By Lemma 2.2, for suitable m (maximal such that $(\log Q)^{m\alpha} \leq Q^{24}$), we have for any $\varepsilon > 0$ the lower bound

$$\sum_{n \leq X} a_n \gg Q^{24(1-\alpha^{-1})-\varepsilon}$$

when choosing m maximal with $(\log Q)^{\alpha m} \leq Q^\beta$, $\beta > 24$, the implied constant depending only on α and ε . Hence the result follows as before, m being small enough that $256^m \ll X^\varepsilon$ for any $\varepsilon > 0$. \square

The final problem we consider is motivated now by the result of Serre used in the proof of Proposition 2.1: let k/\mathbf{Q} be a fixed quadratic field, and let f be primitive form of weight $k \geq 2$; assume that $\lambda_f(p) = 0$ for all $p \leq M$ which are inert in k . How large should M be so that this implies that f is a CM form?

Denote by χ the quadratic character associated with k and by D the discriminant of k . The assumption implies that $\lambda_f(p) = \chi(p) \lambda_f(p) = \lambda_{f \otimes \chi}(p)$ for all $p \leq M$ unramified in k . Hence by GRH again, it follows that $f = f \otimes \chi$ if $M \geq C(\log |D|q(f))^2$ with an absolute constant $C \geq 0$. Since this condition implies that f is a CM form, this solves the problem.

Here is a Linnik-style result about this problem. Since the condition $\lambda_f(p) = 0$ is stable by any twist by a character, it is natural to count the forms up to twist; fixing the nebentypus, this means up to quadratic twist (to put it differently, a twist of a CM form is itself a CM form, but the nebentypus changes except for quadratic twists). So it is natural to count only the number of possible symmetric squares.

Theorem 4.3. *Let $\alpha > 0$ be any positive number and let $Q \geq 2$. For any quadratic field k/\mathbf{Q} , the number N of symmetric squares of primitive modular forms of weight 2 and conductor $\leq Q$ such*

$$(10) \quad \lambda_f(p) = 0 \text{ for } p \leq (\log Q)^\alpha \text{ inert in } k$$

is bounded by

$$N \ll Q^{10\alpha^{-1}+\varepsilon}$$

for any $\varepsilon > 0$, the implied constant depending only on α and ε .

Proof. If f satisfies (10), we have $\lambda_f(n^2) = 1$ for all n squarefree which is divisible only by primes $p \leq (\log Q)^\alpha$ which are inert in k , and unramified for f . Thus with

$$L(f) = \sum_{n \leq X} a_n \lambda_f(n^2)$$

where a_n is the characteristic function of numbers of this type having exactly m prime factors, we have for such f

$$L(f) = \sum_{n \leq X} a_n,$$

and by positivity we have

$$N\left(\sum_{n \leq X} a_n\right)^2 \leq \sum_f^* |L(f)|^2,$$

where \sum^* indicates that we sum over the distinct symmetric squares only. By the large-sieve type inequality for symmetric square L -functions, we have

$$\sum_f^* |L(f)|^2 \ll X^{1+\varepsilon} \sum_{n \leq X} |a_n|^2 = X^{1+\varepsilon} \sum_{n \leq X} a_n,$$

if $X > Q^\beta$ for any $\beta > 10$ and $\varepsilon > 0$, the implied constant depending only on β and ε .

To estimate the number of n with $a_n = 1$, we have the analogue of (3)

$$|\{p \leq X \mid p \text{ is inert in } k\}| \gg \frac{X}{\log X},$$

for $X \geq 2$, the implied constant depending on k . Applying Lemma 2.2 to this situation (again for m as described in the statement), this gives the bound $Q^{10\alpha-1+\varepsilon}$ for the number of symmetric squares. \square

REFERENCES

- [CM] J. Cogdell and P. Michel: *On the complex moments of symmetric power L -functions at $s = 1$* , I.M.R.N 2004, no. 31, 1561–1617.
- [DDT] H. Darmon, F. Diamond and M. Taylor: *Fermat's Last Theorem*, Current Developments in Math., International Press (1995), 1–154.
- [DK] W. Duke and E. Kowalski: *A problem of Linnik for elliptic curves and mean-value estimates for automorphic representations*, Invent. math. 139 (2000), 1–39.
- [FH] W. Fulton and J. Harris: *Representation theory. A first course*, GTM 129, Springer 1991.
- [HR] J. Hoffstein and D. Ramakrishnan: *Siegel zeros and cusp forms*, I.M.R.N 1995, No.6, 279–308 (1995).
- [IK] H. Iwaniec and E. Kowalski: *Analytic Number Theory*, A.M.S Colloquium Series 53, 2004.
- [K] H. Kim: *Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2* , with appendix 1 by D. Ramakrishnan and appendix 2 by H. Kim and P. Sarnak, J. Amer. Math. Soc. 16 (2003), no. 1, 139–183.
- [KMV] E. Kowalski, P. Michel and J. VanderKam: *Rankin-Selberg L -functions in the level aspect*, Duke Math. J. 114 (2002), no. 1, 123–191.
- [KRW] E. Kowalski, O. Robert and J. Wu: *Small gaps in coefficients of L -functions and \mathfrak{B} -free numbers in small intervals*, preprint (2003).
- [R] C.S. Rajan: *Recovering ℓ -adic representations*, preprint (2002).
- [Ric] G. Ricotta: *Zéros réels et taille des fonctions L de Rankin-Selberg par rapport au niveau*, thèse de doctorat, Université Montpellier II (2004).
- [Sa] P. Sarnak: letter to Zeev Rudnick.
- [Se1] J-P. Serre: *Quelques applications du théorème de densité de Chebotarev*, Publ. Math. I.H.E.S 54 (1981), 123–201.
- [Se2] J-P. Serre: *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. math. 15 (1972), 259–331.
- [Se3] J-P. Serre: *Corps locaux*, 3ème ed., Hermann 1968.
- [St] H. Stark: *Some effective cases of the Brauer-Siegel theorem*, Invent. math. 23 (1974), 135–152.

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