ABOUT THE MINIMAL VALUE OF SOME POLYNOMIALS

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Statements

Here are two problems raised by E. Kowalski; solving the second solves trivially the first one. They apperared as Problem 11155 in the May, 2005, issue of the American Math. Monthly.

(1) Let P_n denote the polynomial

$$P_n(x) = \sum_{j=0}^{n} \binom{n}{j}^2 x^{2j}(1-x)^{2(n-j)}.$$

Show that the minimal value of P_n on the closed interval $[0,1]$ is attained at $x = \frac{1}{2}$.

(2) With P_n as above and $Q_n(u) = P_n(u + \frac{1}{2})$, show that in the Taylor expansion

$$Q_n(u) = \sum_{p=0}^{2n} Q_{n,p} u^p$$

one has $Q_{n,p} = 0$ if p is odd and $Q_{n,p} \geqslant 0$ if p is even.

These polynomials arise as the variance of the derivative of the $(n+1)$-st Bernstein polynomial of a standard Brownian motion $B(t)$, $0 \leqslant x \leqslant 1$. See [1, §5] for this and the application of the problem to a new simple proof that for fixed x, Brownian motion is almost surely not differentiable at x (with a proof of a statement weaker than (1) which is sufficient for this purpose).

Solution

Here is a solution to the second problem, due to T. Rivoal.

Start from the identity

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}^2 x^j (1-x)^{n-j} = \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{n+j}{n} x^j,$$

(for instance, the right-hand side is the hypergeometric function or polynomial $F(n+1, -n; 1; x)$ where F is the Gauss hypergeometric function, and the left-hand side is seen to be $L_n(1-2x)$, where L_n is the Legendre polynomial, hence both are seen to satisfy the same differential equation

$$x(1-x)u'' + (1-2x)u' + n(n+1)u = 0$$

by definition, and one concludes easily). Making the change of variable

$$\frac{x}{x-1} = \frac{u^2}{(1-u)^2}$$

one gets after simplification that

$$P_n(u) = (-1)^n \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{n+j}{n} u^{2j}(2u-1)^{n-j}.$$

Hence

$$\begin{aligned}
Q_n(v) &= P_n\left(v+\frac{1}{2}\right) = (-1)^n \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{n+j}{n}\left(v+\frac{1}{2}\right)^{2j} (2v)^{n-j} \\
&= (-1)^n \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{n+j}{n} \sum_{\ell=0}^{2j} \binom{2j}{\ell} 2^{\ell+(n-j)} v^{(2j-\ell)+(n-j)} \\
&= \sum_{p=0}^{n} \left[(-1)^n \, 2^p \sum_{j=|n-p|}^{n} \binom{n}{j}\binom{n+j}{n}\binom{2j}{n+j-p} (-4)^{-j} \right] v^p.
\end{aligned}$$

The coefficient $Q_{n,p}$ of v^p is then identified with a value of a hypergeometric function

$$Q_{n,p} = \frac{(-1)^p}{2^{2|n-p|-p}} \binom{n}{|n-p|}\binom{n+|n-p|}{n}\binom{2|n-p|}{n-p+|n-p|} \times {}_4F_3\left[\begin{matrix} 1, \frac{1}{2}+|n-p|, 1+n+|n-p|, |n-p|-n \\ 1+|n-p|, 1+n-p+|n-p|, 1+p-n+|n-p| \end{matrix}\right].$$

(with the convention that in the hypergeometric function the value of z is 1), e.g. using Maple's call

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sum(binomial(n,j)*binomial(n+j,n)*
        binomial(2*j,n+j-p)*(-4)^(-j),j=abs(n-p)..infinity)
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This ${}_4F_3$ value is in fact a ${}_3F_2$ because one of the bottom parameters is always equal to 1 and simplifies with the top 1.

We then need to discuss two cases, depending on whether $n \geqslant p$ or $n \leqslant p$.

If $n \geqslant p$, we have

$$Q_{n,p} = \frac{(-1)^p}{2^{2n-3p}} \binom{n}{p}\binom{2n-p}{n} {}_3F_2\left[\begin{matrix} -p,\ 1+2n-p,\ \frac{1}{2}+n-p \\ 1+n-p,\ 1+2n-2p \end{matrix}\right].$$

If $n \leqslant p$, we have

$$Q_{n,p} = \frac{(-1)^p}{2^{p-2n}} \binom{n}{p-n}\binom{p}{n} {}_3F_2\left[\begin{matrix} -2n+p,\ 1+p,\ \frac{1}{2}+p-n \\ 1+p-n,\ 1+2p-2n \end{matrix}\right].$$

In both expressions one can use the summation formula of Watson (see e.g. [2, III.23, p. 245]):

$${}_3F_2\left[\begin{matrix} a,\ b,\ c \\ 1+\frac{1}{2}a+\frac{1}{2}b,\ 2c \end{matrix}\right] = \Gamma\left[\begin{matrix} \frac{1}{2},\ c+\frac{1}{2},\ \frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b, \frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c \\ \frac{1}{2}+\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}b, \frac{1}{2}-\frac{1}{2}a+c, \frac{1}{2}-\frac{1}{2}b+c \end{matrix}\right],$$

where the right-hand side is interpreted as

$$\frac{\Gamma(1/2)\Gamma(c+1/2)(\cdots)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)(\cdots).}$$

This formula is valid under certain conditions on the complex parameters a, b, c which are satisfied here. In particular, if a $\Gamma(-m)$ occurs in the denominateur for some integer $m \geqslant 0$, this means the value of the expression is 0.

If $n \geqslant p$, we use Watson's formula with $a = -p$, $b = 1+2n-p$ and $c = \frac{1}{2}+n-p$ and if $n \leqslant p$, we use it with $a = p-2n$, $b = 1+p$ et $c = \frac{1}{2}+p-n$. In the end, after simplification, we get

$$Q_{n,p} = \frac{(-1)^p}{2^{2n-3p}} \frac{\Gamma(1+2n-p)\Gamma\left(\frac{1}{2}\right)^2}{\Gamma(1+p)\Gamma\left(1+\frac{2n-p}{2}\right)^2\Gamma\left(\frac{1}{2}-\frac{p}{2}\right)^2} \quad \text{if} \quad n \geqslant p$$

$$Q_{n,p} = \frac{(-1)^p}{2^{p-2n}} \frac{\Gamma(1+p)\Gamma\left(\frac{1}{2}\right)^2}{\Gamma(1+2n-p)\Gamma\left(1+\frac{p}{2}\right)^2\Gamma\left(\frac{1}{2}-\frac{2n-p}{2}\right)^2} \quad \text{if} \quad n \leqslant p.$$

If n is odd, $Q_{n,p} = 0$ since there is a $\Gamma(-m)^2$ with m integer in the denominator, and if n is even, we see that $Q_{n,p} > 0$.

References

[1] E. Kowalski, *Bernstein polynomials and Brownian motion*, American Math. Monthly, to appear.

[2] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, Cambridge, 1966.

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