# The work of R. P. Langlands

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Abel Prize Day
September 25, 2020

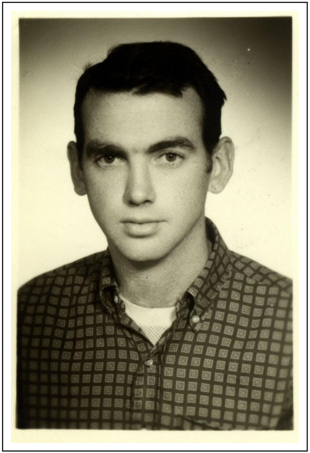
# Citation

The Norwegian Academy of Sciences and Letters has decided to award the Abel Prize for 2018 to

**Robert P. Langlands**

"for his visionary program connecting representation theory to number theory."

# Biography



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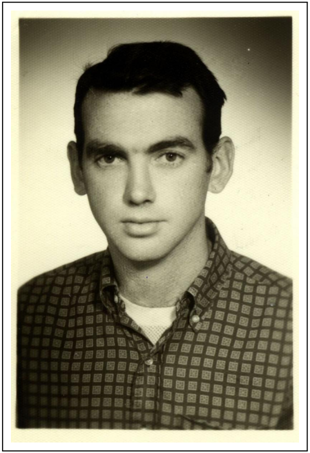

- Born 1936 in British Columbia.
- MSc, University of British Columbia, 1958

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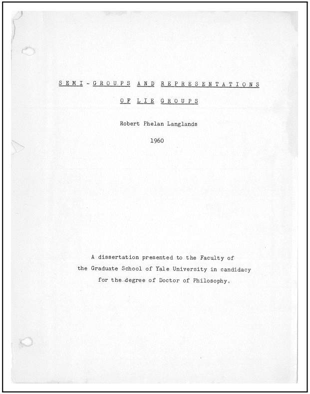
SEMI-GROUPS AND REPRESENTATIONS

OF LIE GROUPS

Robert Phelan Langlands

1960

A dissertation presented to the Faculty of
the Graduate School of Yale University in candidacy
for the degree of Doctor of Philosophy.

- Born 1936 in British Columbia.
- MSc, University of British Columbia, 1958
- PhD, Yale University, 1960; "Semi-groups and representations of Lie groups".

"Once again, there was, oddly enough, no-one to understand it, but as I know from a conversation overheard in a stairway, Browder was quite firm in defending me and my thesis in the face of another faculty member, whose stated grounds for rejecting it were solely that no-one could read it."

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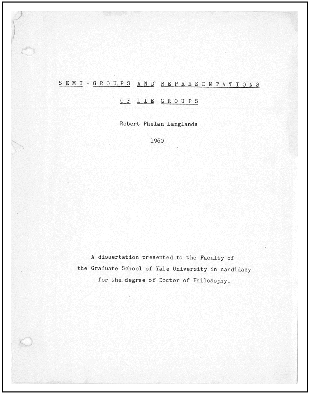

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http://publications.ias.edu/rpl/

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- Positions at Princeton, Yale and IAS.

# Cold open: Representations

*Three sentences on representation theory*

- A representation of a group $G$ on a $k$-vector space $E$ is a homomorphism $\varrho\colon G \to \mathrm{GL}(E)$.
- If $E$ is a Hilbert space and $\varrho(g)$ is always unitary, one says that $\varrho$ is a unitary representation.
- If $E \neq \{0\}$ and no (closed) proper subspace is stable, then $\varrho$ is called irreducible.

# The birth of algebraic number theory: quadratic reciprocity

For a prime $p$ and an integer $n \in \mathbf{Z}$, define the *Legendre symbol*

$$\left(\frac{n}{p}\right) = \begin{cases} 1 & \text{if } n \equiv m^2 \bmod p \text{ for some } m \text{ coprime to } p \\ 0 & \text{if } p \text{ divides } n \\ -1 & \text{otherwise.} \end{cases}$$

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**Theorem.** (Gauss) For any distinct odd primes $p$ and $q$, we have

$$\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)(-1)^{(p-1)(q-1)/2}.$$

# An application

How quickly can you compute

$$\left(\frac{5}{196561}\right) \quad ?$$

Note that by quadratic reciprocity

$$\left(\frac{5}{196561}\right) = \left(\frac{196561}{5}\right) = \left(\frac{1}{5}\right) = 1.$$

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Gödel was apparently fascinated by this example of transforming a seemingly exponential-time computation into a logarithmic-time one.

# A first interpretation

The condition

$$\left(\frac{q}{p}\right) = 1$$

means that the polynomial $X^2 - q \in \mathbf{Z}[X]$ has two roots modulo $p$: it *splits* in linear factors in $(\mathbf{Z}/p\mathbf{Z})[X]$.

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**First general question.** Given a fixed irreducible $f \in \mathbf{Z}[X]$, can one describe the primes $p$ such that $f$ splits modulo $p$? Can one describe more generally the factorization of $f$ modulo $p$?

# A second interpretation

Suppose that $q \equiv 1 \bmod 4$. Observe the following identity

$$\prod_p \left(1 - \left(\frac{q}{p}\right) p^{-s}\right)^{-1} = \prod_p \left(1 - \left(\frac{p}{q}\right) p^{-s}\right)^{-1} = \sum_{n \geqslant 1} \left(\frac{n}{q}\right) n^{-s},$$

where the second step follows from

$$\left(\frac{p_1^{n_1}}{q}\right) \cdots \left(\frac{p_k^{n_k}}{q}\right) = \left(\frac{p_1^{n_1} \cdots p_k^{n_k}}{q}\right).$$

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One such equality, first conjectured by Shimura, Taniyama and Weil is the essential step in the proof of Fermat's Great Theorem by Wiles.

# And a third...

The set $\mathbf{Q}(\sqrt{q})$ of all complex numbers of the form

$$a + b\sqrt{q}, \qquad a,\ b \text{ rational numbers},$$

is a field (one can add, multiply, divide by non-zero elements). The map $\sigma\colon a + b\sqrt{q} \mapsto a - b\sqrt{q}$ is an automorphism of this field.

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$$\eta\colon G \to \{-1, 1\} \subset \mathrm{GL}_1(\mathbf{C}).$$

This is an example of a *Galois representation*.

# Frobenius

For any prime $p$ different from $q$, the *Frobenius automorphism* $x \mapsto x^p$ modulo $p$ permutes the two roots of $X^2 - q$ in an algebraic closure of the finite field $\mathbf{Z}/p\mathbf{Z}$.

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This permutation $F_p$ is either the identity, if the roots belong to $\mathbf{Z}/p\mathbf{Z}$ (namely when $X^2 - q$ splits modulo $p$) or it exchanges the two roots. So $F_p$ may be identified with an element of $G$, which is $\sigma$ if $F_p$ exchanges the two roots.

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Quadratic Reciprocity means that, with these identifications, we have

$$\eta(F_p) = \left(\frac{q}{p}\right) = \chi(p),$$

where $\chi\colon \mathbf{Z} \to \mathrm{GL}_1(\mathbf{C})$ is defined by $\chi(n) = (n/q)$, and satisfies $\chi(nm) = \chi(n)\chi(m)$; it is a "Dirichlet character".

# Class Field Theory

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A major problem in number theory when Langlands entered the scene was to extend this beyond the abelian case.

# Galois representations

Artin had begun to study representations of the Galois group $G$ of an arbitrary Galois extension of $\mathbf{Q}$ in finite-dimensional vector spaces:

$$\varrho\colon G \to \mathrm{GL}_d(\mathbf{C}),$$

and associated to them their $L$-function

$$L(\varrho, s) = \prod_p \det(1 - \varrho(F_p)p^{-s})^{-1}.$$

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Langlands identified what should be the analogue of the Dirichlet characters in that setting: generalizations of the modular forms which were also classically studied by many 19th century mathematicians.

# A modular form

For $z \in \mathbf{C}$ with positive imaginary part, define

$$\Delta(z) = e^{2i\pi z} \prod_{n \geqslant 1} (1 - e^{2i\pi nz})^{24}.$$

We have

$$\Delta\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^{12}\Delta(z), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

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Ramanujan observed, and Hecke proved, the remarkable fact that if we expand

$$\Delta(z) = \sum_{n \geqslant 1} \tau(n) e^{2i\pi nz},$$

then the arithmetic coefficients $\tau(n) \in \mathbf{Z}$ satisfy

$$\sum_{n \geqslant 1} \tau(n) n^{-s} = \prod_p (1 - \tau(p)p^{-s} + p^{11-2s})^{-1}.$$

# Automorphic representations

There is a locally compact topological ring $\mathbf{A}$, obtained by combining all the completions of $\mathbf{Q}$ with respect to the $p$-adic and ordinary metrics. It contains $\mathbf{Q}$ as a discrete subring.

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that can be embedded in the natural representation $\mathrm{reg}$ on

$$L^2(\mathrm{GL}_d(\mathbf{Q})\backslash \mathrm{GL}_d(\mathbf{A}))$$

which is defined by

$$(\mathrm{reg}(g)f)(x) = f(xg).$$

# The Langlands correspondance

The correspondance between

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and

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should be such that

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Even for $d = 2$, this is not yet proved (when the projective image of $\varrho$ is $A_5$). If the image is $S_4$, this was proved by Langlands and Tunnell; it is one of the starting points of the work of Wiles.

# Functoriality

If we have an Artin representation

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This means that from an automorphic representation

$$\pi\colon \mathsf{GL}_d(\mathbf{A}) \to \mathrm{U}(E),$$

we should be able to construct

$$\pi'\colon \mathsf{GL}_e(\mathbf{A}) \to \mathrm{U}(F),$$

with equality of $L$-functions.

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**Theorem.** For any real numbers $-2 \leqslant a < b \leqslant 2$, we have

$$\frac{1}{\pi(x)}\mathrm{Card}\{p \leqslant x \mid a < \frac{\tau(p)}{p^{11/2}} < b\} \longrightarrow \frac{1}{\pi}\int_a^b \sqrt{1 - x^2/4}\, dx$$

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(This was proved by Clozel, Harris and Taylor in 2008.)