SUMS OF A. IRVING

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Let p be a prime, $h \in \mathbf{F}_p^\times$ a fixed invertible element modulo p. Let χ be a fixed non-trivial multiplicative character modulo p.

We consider the function $W : \mathbf{F}_p \longrightarrow \mathbf{C}$ defined by

$$W(x) = \frac{1}{\sqrt{p}} \sum_{n \in \mathbf{F}_p} \chi(n)\overline{\chi(n+h)} e_p(nx).$$

In the notation of A. Irving's paper, we have $W(x) = p^{-1/2} W_{\chi,h}(x)$.

The goal is to study sums of products of values of W of the type

$$S(y) = \sum_{x \in \mathbf{F}_p} \prod_{i=1}^{n_1} W(x+t_i) \prod_{j=1}^{n_2} \overline{W(x+s_j)} e_p(xy)$$

where the t_i's and s_j's are in $\mathbf{F}_p$ and $y \in \mathbf{F}_p$. The sum $K_{\chi,h}(h_1,\ldots,h_k,y)$ in Irving's paper is of this type, up to a factor $p^{-2^k/2}$, with $n_1 = n_2 = 2^{k-1}$ for some integer $k \geqslant 1$ and the (t_i) (resp. (s_j)'s) are the sums

$$\sum_{i \in I} h_i$$

where I ranges over $I \subset \{1,\ldots,k\}$ with $|I|$ even (resp. odd).

We will prove:

Theorem 1. *Assume $p > 5$. We have*

$$|S(y)| \ll \sqrt{p}$$

where the implied constant depends only on n_1 and n_2 if any of the following conditions hold:

(1) *If $y + \frac{(n_2-n_1)h}{2} \neq 0$;*

(2) *If $y + (n_2 - n_1)h/2 = 0$ and there is some $x \in \mathbf{F}_p$ such that the total multiplicity*

$$|\{i \mid t_i = x\}| + |\{j \mid s_j = x\}|$$

is odd.

The proof is based on the methods of algebraic geometry described in [1], based on the Riemann Hypothesis over finite fields of Deligne and additional ideas and computations of Katz [2]. We begin by a remark that allows us to remove the complex conjugates, and which is needed to apply cleanly the results of [1]. We define

$$W'(x) = W(x)e_p(\bar{2}hx).$$

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Lemma 2. *We have*

$$S(y) = e_p(\alpha h) T\Big(y + \frac{(n_2 - n_1)h}{2}\Big)$$

where

$$\alpha = \frac{1}{2}\Big(\sum_i t_i - \sum_j s_j\Big)$$

and

$$T(y) = \sum_{x \in \mathbf{F}_p} \prod_{l=1}^{n_1+n_2} W'(x + u_l) e_p(xy)$$

where $u_l = t_l$ *for* $1 \leqslant l \leqslant n_1$ *and* $u_l = s_{l-n_1}$ *for* $n_1 + 1 \leqslant l \leqslant n_1 + n_2$.

Proof. This is a straightforward computation where the point is that $W'(x)$ is real for all $x \in \mathbf{F}_p$; this follows in turn from the relation

$$\begin{aligned} e_p(-hx)\overline{W(x)} &= \frac{1}{\sqrt{p}} \sum_{n \in \mathbf{F}_p} \overline{\chi(n)} \chi(n+h) e_p(-(n+h)x) \\ &= \frac{1}{\sqrt{p}} \sum_{m \in \mathbf{F}_p} \overline{\chi(-m-h)} \chi(-m) e_p(mx) = W(x). \end{aligned}$$

$\square$

Hence the result follows from the following result concerning sums like T:

Proposition 3. *Let* $(u_1, \ldots, u_n)$ *be distinct elements of* $\mathbf{F}_p$*, let* $\nu_i \geqslant 1$*. Let*

$$T(y) = \sum_{x \in \mathbf{F}_p} \prod_{i=1}^{n} W'(x + u_i)^{\nu_i} e_p(xy)$$

for $y \in \mathbf{F}_p$*. Then we have* $T \ll p^{1/2}$*, where the implied constant depends only on* n *and the* ν_i*, provided either that* $y \neq 0$*, or that* $y = 0$ *and some* ν_i *is odd.*

Proof. We will first apply [1, Th. 2.7] and then [1, Prop. 1.1] to the following data:

- the sheaf $\mathcal{G}$ is $\mathcal{L}_{\psi(-yX)}$, with ψ the additive character corresponding to e_p,
- the family of sheaves is $(\mathcal{F}_i)_{1 \leqslant i \leqslant n}$ with

  $$\mathcal{F}_i = [+u_i]^* \mathcal{F},$$

  where $\mathcal{F} = \mathrm{FT}_\psi(\mathcal{H}) \otimes \mathcal{L}_{\psi(hX/2)}$, with $\mathcal{H}$ the sheaf-theoretic Fourier transform of the Kummer sheaf

  $$\mathcal{L}_{\chi(X)\bar{\chi}(X+h)}.$$
- the open set U is the complement of $\{-u_i\}$ in the affine line.

By definition of the Fourier transform, the trace function of $\mathcal{F}$ is the function W' (one must check that the tensor product defining $\mathcal{H}$ is a middle-extension, but this is straightforward), and that of $\mathcal{F}_i$ is $W'(x + u_i)$, so the sum T is exactly of the type controlled by [1, Prop. 1.1] in that case.

We now check that the family $(\mathcal{F}_i)$ is strictly U-generous, in the sense of [1, Def. 2.1], which is the hypothesis in [1, Th. 2.7]. This is true because:

(1) The $\mathcal{F}_i$ are geometrically irreducible middle-extensions of weight 0, because so is $\mathcal{F}$, in turn because it is a tensor product of an Artin-Schreier sheaf (of weight 0 and of rank 1) with the Fourier transform of the middle-extension sheaf $\mathcal{H}$ of weight 0 which is geometrically irreducible and of Fourier type (not being an Artin-Schreier sheaf); as noted above, the middle-extension property is checked directly.
(2) The geometric monodromy group of $\mathcal{F}$ is equal to SL_2 for $p > 5$ (see below for the justification), so that the same holds for $\mathcal{F}_i$, and this group satisfies the second condition of loc. cit.,
(3) Any pairs of SL_2 with their standard representations is Goursat-adapted (see [2, Example 1.8.1, p. 25]), so the third condition holds;
(4) For $i \neq j$, we have no geometric isomorphism

$$\mathcal{F}_i \simeq \mathcal{F}_j \otimes \mathcal{L}, \text{ or } \mathrm{D}(\mathcal{F}_i) \simeq \mathcal{F}_j \otimes \mathcal{L},$$

of sheaves on U, for any rank 1 sheaf $\mathcal{L}$ lisse on U (see also below), where $\mathrm{D}(\mathcal{F}_i)$ is the dual of $\mathcal{F}_i$.

Leaving for a bit later the last checks indicated, we can finish the proof of the proposition first. From [1, Prop. 1.1], it is enough to show that the abstract "diagonal" classification of [1, Th. 2.7] implies that

$$H^2_c(U \times \bar{\mathbf{F}}_p, \bigotimes \mathcal{F}_i^{\otimes \nu_i} \otimes \mathrm{D}(\mathcal{G})) = 0$$

unless $y = 0$ and all ν_i are even. But [1, Th. 2.7] shows that this cohomology group vanishes *unless* we have a geometric isomorphism

$$\mathcal{G} \simeq \bigotimes \Lambda_i(\mathcal{F}_i)$$

where Λ_i is an irreducible representation of SL_2 contained in the ν_i-th tensor of its standard representation (the covering π in loc. cit. is trivial here because we are in a strictly U-generous situation). Assume we have such an isomorphism. Then, since the rank of $\mathcal{G}$ is 1, we see that Λ_i is trivial for all i, and therefore $\mathcal{G}$ is also geometrically trivial, which means that $y = 0$. Then we see that ν_i must be even for the trivial representation to be contained in the ν_i-th tensor power. (Note that this is the same argument as for sums of products of classical Kloosterman sums in [1, Cor. 3.2]).

We now finish checking the conditions (2) and (4) above. For (2), we first need to prove that the connected component of the identity G^0 of the geometric monodromy group G of $\mathcal{F}$ is SL_2. Since the tensor product with $\mathcal{L}_{\psi(hX/2)}$ does not alter this property, this is a special case of [2, Th. 7.9.4], because $\mathcal{H}$ is a tame pseudoreflection sheaf, ramified only at ∞ and at the points in $S = \{0, -h\} \subset \mathbf{A}^1$ (see also [2, Th. 7.9.6]). We then need to check that in fact G is connected so that $G = G^0 = \mathrm{SL}_2$. This is because $\mathcal{F}$ is geometrically self-dual, as follows from the formula [2, Th. 7.3.8 (2)] for the dual of a Fourier transform (intuitively, this is because W' is real-valued). Indeed, G is semisimple and hence of the form $\mu_N \, \mathrm{SL}_2$ for some $N \geqslant 1$, where μ_N is the group of N-th roots of unity. This group is self-dual in the standard representation if and only if $N \leqslant 2$, but $\mu_2 \, \mathrm{SL}_2 = \mathrm{SL}_2$.

For (4), assume first that we have a geometric isomorphism

$$\mathcal{F}_i = [+u_i]^*\mathcal{F} \simeq [+u_j]^*\mathcal{F} \otimes \mathcal{L} = \mathcal{F}_j \otimes \mathcal{L} \tag{1}$$

(on U) for some rank 1 sheaf $\mathcal{L}$.

By [2, Cor. 7.4.6] (1), the sheaf $\mathcal{F}$ is tamely ramified at 0 with drop 1, and it is unramified on $\mathbf{G}_m$ by [2, Th. 7.9.4]. If $i \neq j$, then the left-hand sheaf in (**??**) is therefore unramified at $-u_j$, since $u_i \neq u_j$. On the other hand, $\mathcal{F}_j$ is ramified at $-u_j$ and has rank 2 and drop 1. Thus the tensor product with the rank 1 sheaf $\mathcal{L}$ is ramified at $-u_j$ (the dimension of the inertial invariants is at most 1), which is a contradiction. Thus an isomorphism as above is impossible unless $i = j$.

There remains to deal with the possibility of an isomorphism

$$\mathcal{F}_i = [+u_i]^*\mathcal{F} \simeq [+u_j]^*\,\mathrm{D}(\mathcal{F}) \otimes \mathcal{L} = \mathrm{D}(\mathcal{F}_j) \otimes \mathcal{L}.$$

But we have seen that $\mathrm{D}(\mathcal{F}) \simeq \mathcal{F}$, so this is also impossible if $i \neq j$. □

References

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