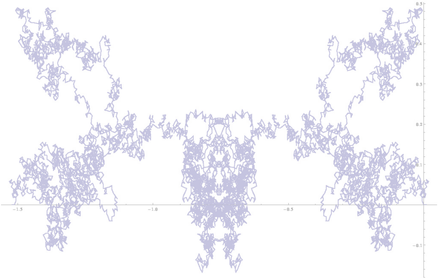
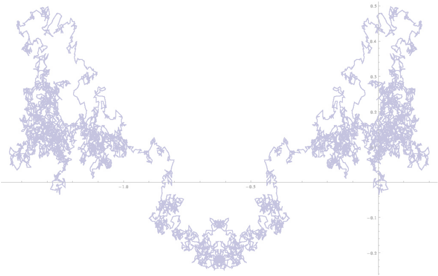
Random / not Random





E. Kowalski
ETH Zürich

September 12, 2019
Heilbronn Conference

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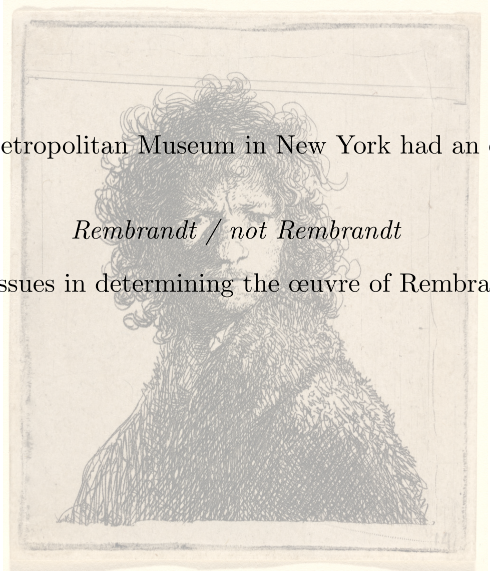

In 1995, the Metropolitan Museum in New York had an exhibition entitled

Rembrandt / not Rembrandt

exploring the issues in determining the œuvre of Rembrandt.

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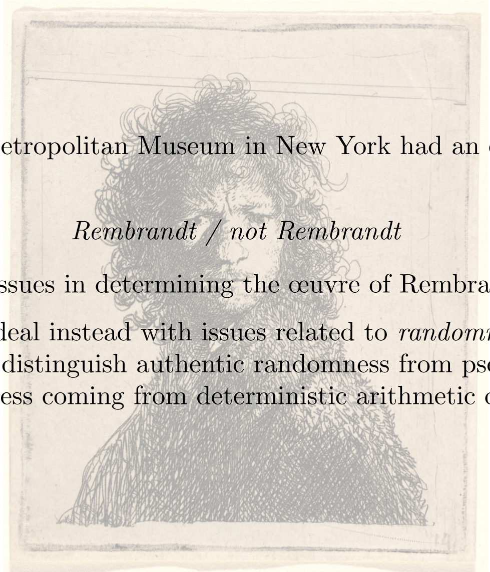

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This talk will deal instead with issues related to *randomness*, where we attempt to distinguish authentic randomness from pseudo- or quasi-randomness coming from deterministic arithmetic objects.

Values of the Euler function

Normalized Euler function: $$f(n) = \frac{\varphi(n)}{n} = \prod_{p|n}\Big(1 - \frac{1}{p}\Big).$$

Schoenberg (1928): if we look at integers $n \leq \mathrm{N}$ and let $\mathrm{N} \to +\infty$, the probability distribution of $f(n)$ converges in law to the infinite random product

$$\mathrm{F} = \prod_p \Big(1 - \frac{\mathrm{B}_p}{p}\Big)$$

where (B_p) are independent Bernoulli random variables with

$$\mathbf{P}(\mathrm{B}_p = 1) = \frac{1}{p}, \qquad \mathbf{P}(\mathrm{B}_p = 0) = 1 - \frac{1}{p}.$$

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In other words: $$\lim_{\mathrm{N}\to+\infty} \frac{1}{\mathrm{N}}|\{n \leq \mathrm{N} \mid \varphi(n) \leq \alpha n\}| = \mathbf{P}(\mathrm{F} \leq \alpha)$$

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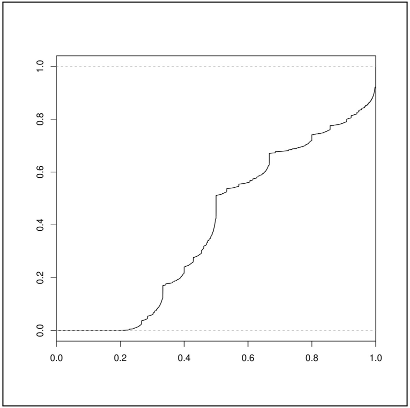

Erdős (1939): the distribution is singular; $g(\alpha) = \mathbf{P}(\mathrm{F} \leq \alpha)$ is continuous, strictly increasing, and has $g'(\alpha) = 0$ for almost all α.

Kloosterman paths

Kloosterman sums: $$\mathrm{Kl}(a;p) = \frac{1}{\sqrt{p}} \sum_{1\leq x<p} \exp\Big(2i\pi \frac{ax+\bar{x}}{p}\Big)$$

(p prime, a coprime to p, $x\bar{x} \equiv 1 \pmod{p}$)

Kloosterman paths: continuous function $\mathcal{K}_p(a)\colon [0,1] \to \mathbf{C}$ linearly interpolating

$$\frac{j}{p-1} \mapsto \frac{1}{\sqrt{p}} \sum_{1\leq x\leq j} \exp\Big(2i\pi \frac{ax+\bar{x}}{p}\Big), \qquad 0 \leq j \leq p-1.$$

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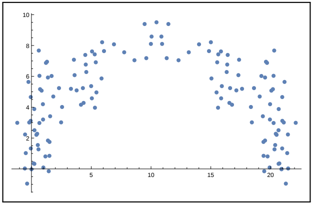
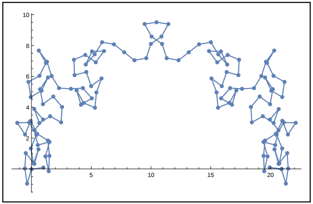




Kloosterman paths

K. & Sawin (2016): as $p \to +\infty$, if we take a modulo p uniformly at random, the Kloosterman paths $\mathcal{K}_p(a)$ converge in law to the Fourier series

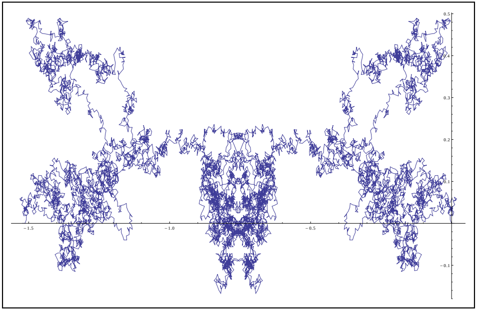
$$\mathrm{K}(t) = t\mathrm{X}_0 + \sum_{\substack{h \in \mathbf{Z} \\ h \neq 0}} \mathrm{X}_h \frac{\exp(2i\pi th) - 1}{2i\pi h}$$

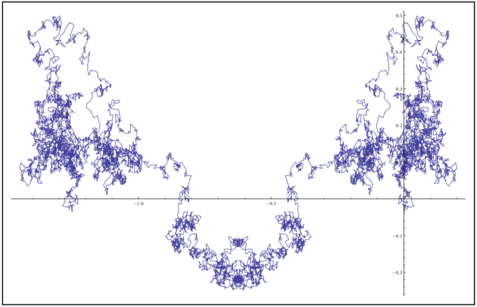
where $(\mathrm{X}_h)_{h \in \mathbf{Z}}$ are independent and distributed on $[-2, 2]$ according to the density

$$\frac{1}{\pi} \sqrt{1 - \frac{x^2}{4}} dx.$$

Picturesque randomness

One of these two pictures is a sample of the “authentic” random Fourier series, the other one is a Kloosterman path.





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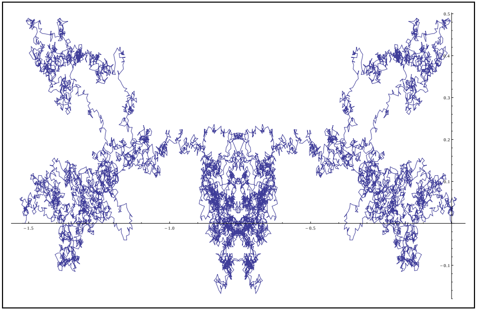
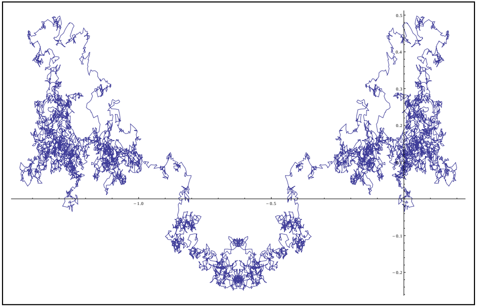




The “bat-like” shape (dixit Granville and Granville, *Prime suspects*) is due to the fact that the Fourier coefficients are purely imaginary.

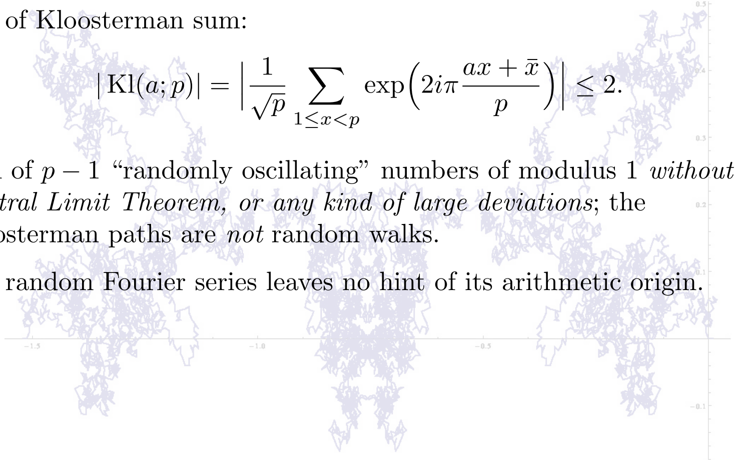
Kloosterman paths

Size of Kloosterman sum:

$$|\operatorname{Kl}(a;p)| = \Big|\frac{1}{\sqrt{p}}\sum_{1\leq x<p}\exp\Big(2i\pi\frac{ax+\bar{x}}{p}\Big)\Big| \leq 2.$$

Sum of $p-1$ "randomly oscillating" numbers of modulus 1 *without Central Limit Theorem, or any kind of large deviations*; the Kloosterman paths are *not* random walks.

The random Fourier series leaves no hint of its arithmetic origin.



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(Vexing) open question: does there exist a continuous function f in the support of K that is “space filling”?

Equivalently: is there a continuous space-filling curve f with Fourier coefficients of size $\mathrm{O}(1/h)$?

Where does the randomness come from?

Weil (1948): the full length Kloosterman sum $\mathrm{Kl}(a;p)$ is the *trace* of a matrix $\theta_{a,p} \in \mathrm{SU}_2(\mathbf{C})$; so $|\mathrm{Kl}(a;p)| \leq 2$.

Deligne / Katz (1988): these matrices are uniformly distributed, up to conjugacy, in $\mathrm{SU}_2(\mathbf{C})$.

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The (pseudo-)randomness implications of this remarkable result are certainly still very far from being exhausted.

A classical question of analysis

Consider $f\colon \mathbf{R} \to \mathbf{C}$ continuous with period 1.

Parseval formula: $$\sum_{n\in\mathbf{Z}} |c_n(f)|^2 = \int_0^1 |f(x)|^2 dx < +\infty$$

where

$$c_n(f) = \int_0^1 f(x)\exp(-2i\pi nx)dx.$$

Question: is the exponent 2 best possible? Could there exist $\delta > 0$ such that $\sum_{n\in\mathbf{Z}} |c_n(f)|^{2-\delta} < +\infty$ for any continuous f?

A classical question of analysis

Legendre symbol: p prime, $\left(\frac{n}{p}\right) = \begin{cases} 0 & p \mid n \\ 1 & n \text{ a square mod } p \\ -1 & n \text{ not a square mod } p \end{cases}$

Carleman (1917): $$f(x) = \sum_{k \geq 1} \frac{1}{k^2} f_{p_k}(x),$$

$$p_k \equiv 1 \pmod 4 \text{ such that } p_k > 4p_{k-1}^2,$$

$$f_p(x) = \frac{2}{p^{1/2}} \sum_{n=1}^{p-1} \left(1 - \frac{n}{p}\right) \left(\frac{n}{p}\right) \cos(2\pi n x).$$

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Then $$\sum_{n \in \mathbf{Z}} |c_n(f)|^{2-\delta} = +\infty \text{ for any } \delta > 0.$$

But also $$g(x) = \sum_{n \geq 2} \frac{e^{2i\pi n \log n}}{\sqrt{n}(\log n)^2} e^{2i\pi n x}$$ (cf Zygmund, p. 199).

A much more difficult question: ultraflat polynomials

$$f_{\mathrm{N}}(x) = \sum_{0 \le m \le \mathrm{N}} a(m) \exp(2i\pi m x), \qquad |a(m)| = 1$$

Question (Erdős 1957, Littlewood 1966): is it possible to find f_{N} such that $|f_{\mathrm{N}}(x)| = \sqrt{\mathrm{N}}(1 + o(1))$ for all x?

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Kahane (1980): probabilistic construction.

Bombieri–Bourgain (1999): explicit arithmetic construction with
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The proof involves again the Riemann Hypothesis over Finite Fields.

Pseudo-random functions in the sense of Gowers

Gowers norms: p prime, $f\colon \mathbf{Z}/p\mathbf{Z} \to \mathbf{C}$, $k \geq 0$

$$\|f\|_0 = \frac{1}{p}\Big|\sum_{x\in\mathbf{Z}/p\mathbf{Z}} f(x)\Big|, \quad \|f\|_{k+1}^{2^{k+1}} = \frac{1}{p}\sum_{h\in\mathbf{Z}/p\mathbf{Z}} \Big\|\Big(x \mapsto f(x)\overline{f(x+h)}\Big)\Big\|_k^{2^k}.$$

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Fouvry, K., Michel (2013): for $f(a) = \mathrm{Kl}(a;p)$, we have

$$\|f\|_k^{2^k} \leq 20^{(k+1)2^k}/p.$$

The approximation property

Question: does there exist f continuous on $[0,1]^2$ such that

$$\int_0^1 f(x,t)f(t,y)dt = 0, \qquad (x,y) \in [0,1]^2, \qquad \int_0^1 f(t,t)dt \neq 0 \quad ?$$

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Grothendieck (1955): Yes $\Longleftrightarrow$ *some* Banach space E does not have the *approximation property* (the identity on E cannot be approximated uniformly on compact sets by finite rank operators).

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Key probabilistic requirement. For $k \geq 1$, find $(\alpha_i)_{1\leq i\leq 3\cdot 2^k}$ in $\{-2,1\}$ with sum 0 such that

$$\Big|\sum_i \alpha_i \chi(i)\Big| \leq \mathrm{C}(k+1)^{1/2} 2^{k/2}$$

for all characters of $\mathbf{Z}/3\cdot 2^k\mathbf{Z}$.

Expander graphs

(Barzdin–Kolmogorov 1967; Bassalygo–Pinsker 1973):
$(\Gamma_n)_{n\geq 1}$, sequence of finite d-regular graphs with $|\Gamma_n| \to +\infty$.

δ-expander: for all $n \geq 1$, all $\emptyset \neq \mathrm{X} \subset \Gamma_n$, we have

$$(\star) \qquad \frac{|\{\text{edges of } \Gamma_n \text{ joining X to } \Gamma_n - \mathrm{X}\}|}{\min(|\mathrm{X}|, |\Gamma_n - \mathrm{X}|)} \geq \delta > 0.$$

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Existence first proved by probabilistic methods: a “random” d-regular graph with n vertices has probability at least $\geq \delta_d > 0$ of satisfying $(\star)$

Ramanujan graphs

Condition ($\star$) equivalent to

$$\frac{\sum_{x\sim y}|f(x)-f(y)|^2}{\sum_{x\in\Gamma_n}|f(x)|^2}\geq\delta'>0\qquad\Big(f\colon\Gamma_n\to\mathbf{C},\quad\sum_{x\in\Gamma_n}f(x)=0\Big)$$

(for some δ' depending on δ).

Alon–Boppana (1986): best possible δ' is $2\sqrt{d-1}$.

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Marcus–Spielman–Srivastava (2015): probabilistic construction of bipartite Ramanuajan graphs (but for all $d\geq 3$).

The Banach–Mazur compact spaces

Banach–Mazur distance: $n \geq 1$ integer; E, F complex Banach spaces of dimension n; $\log d_{\mathrm{BM}}(\mathrm{E}, \mathrm{F}) = \min_{u\colon \mathrm{E} \simeq \mathrm{F}} \|u\| \, \|u^{-1}\|$.

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Probabilistic construction: for *random* vectors $\mathrm{X} = (\mathrm{X}_k)$ on the euclidean unit sphere of $\mathbf{C}^n$, define E_{X} by the norm

$$\|x\|_{\mathrm{X}} = \inf \Big\{ \sum_k |\lambda_k| \;\Big|\; x = \sum_k \lambda_k \mathrm{X}_k \Big\}.$$

With high probability, $d_{\mathrm{BM}}(\mathrm{E}_{\mathrm{X}}, \mathrm{E}_{\widetilde{\mathrm{X}}}) \approx n$.

A conjecture about Banach–Mazur spaces

Attempt at derandomization of Gluskin's construction: take $n = p$, identify $\mathbf{C}^p$ with functions $\mathbf{Z}/p\mathbf{Z} \to \mathbf{C}$, and take families X and $\widetilde{\mathrm{X}}$ of functions

$$f(x) = \exp\Big(2i\pi \frac{\mathrm{P}(x)}{p}\Big), \qquad \widetilde{f}(x) = \Big(\frac{\mathrm{Q}(x)}{p}\Big)$$

where P and Q are polynomials of bounded degree $d \geq 2$.

Question. Is is true that $d_{\mathrm{BM}}(\mathrm{E}_{\mathrm{X}}, \mathrm{E}_{\widetilde{\mathrm{X}}}) \approx p$?

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These spaces appear in any case quite naturally in many results of Fouvry, K., Michel; do they have special properties as Banach spaces?