COMPLEMENTS TO FOUVRY-KATZ-LAUMON STRATIFICATION

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1. Fouvry-Katz-Laumon stratification

The goal of this note is to provide some complements to the stratification results for exponential sums given by Fouvry and Katz in [2] (building on the work of Katz and Laumon [7]; see also Fouvry's paper [1] for the first applications of these results to analytic number theory).

We recall the statements of Theorems 1.1 and 1.2 in [2] (compare with [1, Prop. 1.0]).

Theorem 1 (Fouvry–Katz, Th. 1.1)**.** *Let d and n be positive integers. Let V be a locally closed subscheme of $\mathbf{A}^n_{\mathbf{Z}}$ such that $\dim V_{\mathbf{C}} \leqslant d$. Let $f \in \mathbf{Z}[X_1, \ldots, x_n]$ be given.*

Then there exists a constant C, depending on (n, d, V, f) closed subschemes $X_j \subset \mathbf{A}^n_{\mathbf{Z}}$ for $1 \leqslant j \leqslant n$, of relative dimension $\leqslant n-j$, such that

$$\mathbf{A}^n_{\mathbf{Z}} \supset X_1 \supset \cdots \supset X_n$$

with the property that: for any invertible function g on V, for any prime number p, for any $h \in (\mathbf{A}^n - X_j)(\mathbf{F}_p)$, for any non-trivial additive character ψ of $\mathbf{F}_p$ and for any multiplicative character χ of $\mathbf{F}_p^\times$, we have

$$\Big| \sum_{x \in V(\mathbf{F}_p)} \chi(g(x))\psi(f(x) + h_1x_1 + \cdots + h_nx_n) \Big| \leqslant Cp^{d/2+(j-1)/2}. \tag{1}$$

We will denote below by $C(V, f) \geqslant 0$ any constant C for which Theorem 1 holds for the data (V, f) (the dependency on n and d is left implicit).

Theorem 2 (Fouvry–Katz, Th. 1.2)**.** *Let d, n and D be positive integers. Let V be a closed subscheme of $\mathbf{A}^n_{\mathbf{Z}[1/D]}$ such that $V_{\mathbf{C}}$ is irreducible and smooth of dimension d. Suppose that $A(V, k, \psi) \geqslant 1$ for all finite fields k of sufficiently large characteristic and for all $\bar{\mathbf{Q}}_\ell^\times$-valued non-trivial additive characters ψ of k. Then:*

(1) *There exists a constant C, depending only on V, closed subschemes $X_j \subset \mathbf{A}^n_{\mathbf{Z}[1/D]}$ for $1 \leqslant j \leqslant n$, of relative dimension $\leqslant n-j$, such that*

$$\mathbf{A}^n_{\mathbf{Z}[1/D]} \supset X_1 \supset \cdots \supset X_n$$

such that for any $h \in (\mathbf{A}^n - X_j)(\mathbf{F}_p)$ we have

$$\Big| \sum_{x \in V(\mathbf{F}_p)} \psi(h_1x_1 + \cdots + h_nx_n) \Big| \leqslant Cp^{\max(d/2,(d+j-2)/2)}. \tag{2}$$

(2) *Morever, we may choose the closed subschemes X_j to be defined by the vanishing of homogeneous forms.*

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We will denote below by $C'(V) \geqslant 0$ any constant C for which Theorem 2 holds for V.

Motivated by a question of L. Pierce and F. Thorne, we wish to consider some uniformity aspects of these statements in the situation where V ranges over a family of varieties V_a defined by equations $\Delta(x) = a$, where $a \in \mathbf{Z}$ is a parameter. As we will explain, one can indeed obtain uniform estimates for the constant C in Theorems 1 and 2, as well as for the subschemes X_j. We will do this for both theorems in turn in the next sections.

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2. Uniformity in Theorem 1

We consider a locally closed subscheme

$$W \subset \mathbf{A}^n_{\mathbf{Z}}$$

of relative dimension $\leqslant d+1$, given with an arbitrary morphism $\Delta : W \longrightarrow \mathbf{A}^1_{\mathbf{Z}}$ such that $\dim V_{a,\mathbf{C}} \leqslant d$ for all a, where $V_a = \Delta^{-1}(a)$ is the fiber of Δ above a (viewed as a point of $\mathbf{A}^1_{\mathbf{Z}}$). We may then apply Theorem 1 for any $a \in \mathbf{Z}$ to the data (V_a, f).

We first make an important technical remark: when dealing with this situation, we must be precise concerning the variation of the function g, since it depends on V (as being the place where it is defined). There are two variants we will deal with:

(1) Statements valid for all a, with g allowed to depend on a with no restriction, among functions invertible on V_a;
(2) Statements valid for all a with g ranging over invertible functions on W (i.e., the restriction of g to V_a varies "algebraically").

Our first result addresses the uniformity of the constant C, and allows arbitrary variation of g with a, but with a complexity restriction.

Proposition 3. *With notation as above, for the sums*

$$\sum_{x \in V_a(\mathbf{F}_p)} \chi(g(x))\psi(f(x) + h_1x_1 + \cdots + h_nx_n) \tag{3}$$

the constants $C(V_a, f)$ in Theorem 1 may be bounded independently of $a \in \mathbf{Z}$, provided *we only consider functions g of the type $g = g_1/g_2$ where $g_i \in \mathbf{Z}[X_1, \ldots, X_n]$ are polynomials such that* $\deg(g_i)$ *is bounded.*

Proof. The point is that any specific instance of sums (1) is obtained from an application of the Lefschetz trace formula, which expresses a sum

$$S = \sum_{x \in V(\mathbf{F}_p)} K(x)$$

where $V_{\mathbf{F}_p}$ is an algebraic variety and $K : \mathbf{F}_p \longrightarrow \mathbf{C}$ is the trace function of some étale sheaf $\mathcal{F}$ of weight 0, as the sum

$$S = \sum_{i=0}^{2\dim(V)} (-1)^i \operatorname{Tr}(\mathrm{Fr} \mid H^i_c(V \times \bar{\mathbf{F}}_p, \mathcal{F})),$$

where Fr denotes the geometric Frobenius automorphism of $\mathbf{F}_p$.

Both V and K may depend on parameters; in the case of (3), $V = V_a$ depends on a and K depends on (a, h, χ, f, g, ψ).

The Katz-Laumon and Fouvry-Katz stratification estimates are based on combining, for suitable instances, some vanishing condition

$$H_c^i(V \times \bar{\mathbf{F}}_p, \mathcal{F}) = 0 \tag{4}$$

for $i \geqslant 2d - k + 1$, for some integer $k \geqslant 0$, and the estimates

$$|\operatorname{Tr}(\mathrm{Fr} \mid H_c^i(V \times \bar{\mathbf{F}}_p, \mathcal{F}))| \leqslant p^{i/2} \dim H_c^i(V \times \bar{\mathbf{F}}_p, \mathcal{F})$$

that follow from Deligne's Riemann Hypothesis (and the fact that we assume that $\mathcal{F}$ is of weight 0).

So the constant C in (1) can be taken to be some upper-bound for

$$\sum_{i=0}^{2d-k} \dim H_c^i(V_a \times \bar{\mathbf{F}}_p, \mathcal{F}),$$

taken over the ranges of instances (a, h, χ, g, h, ψ) of the sums (3) which are considered. (In [2], the parameter denoted k here is related to the parameter j in the statement of Theorem 1.)

The point is then that, for the type of trace functions K used in Theorem 1, a result of Katz [4] gives explicit uniform estimates for

$$\sum_{i=0}^{2d} \dim H_c^i(V_a \times \bar{\mathbf{F}}_p, \mathcal{F}),$$

independently of the vanishing condition (4). Precisely, one applies [4, Th. 12], which gives a bound depending (explicitly) on:

- The dimension N of the affine space in which V_a is embedded;
- The number of polynomial equations (and non-vanishing conditions) defining V_a;
- The degree of these equations and non-vanishing conditions defining V_a;
- The degree of the polynomials f, g_1 and g_2 in the representation

$$K(x) = \chi(g_1(x))\overline{\chi(g_2(x))}\psi(f(x_1, \ldots, x_n) + h_1 x_1 + \cdots + h_n x_n).$$

In particular, we see that when all other parameters are fixed, this bound by Katz is the same for all varieties V_a for any $a \in \mathbf{Z}$, which leads to a uniform bound on C, provided that g_1 and g_2 are polynomials of bounded degree. □

Remark 4. In applications, the restriction we make on the degree of g_1 and g_2 is unlikely to be a serious one, and it might be that some additional refinement of [4] would allow us to avoid it.

We now consider the issue of uniformity with respect to a of the subschemes X_j of Theorem 1. In this context, we consider functions $g : W \longrightarrow \mathbf{A}^1_{\mathbf{Z}}$ that are invertible on all of W.

Remark 5. In practice, if $g : W \longrightarrow \mathbf{A}^1_{\mathbf{Z}}$ is not invertible, one would replace W by the open subscheme $W[g^{-1}]$ where g is invertible (although this introduces a dependency on g in the estimates).

Proposition 6. *With notation as above, assume further that Δ is the restriction to W of a function $\Delta : \mathbf{A}^n_{\mathbf{Z}} \longrightarrow \mathbf{A}^1_{\mathbf{Z}}$. Then there exist closed subschemes Y_j of $\mathbf{A}^{n+1}_{\mathbf{Z}}$ for $1 \leqslant j \leqslant n+1$ such that*

$$\mathbf{A}^{n+1}_{\mathbf{Z}} \supset Y_1 \supset \cdots \supset Y_{n+1}$$

with relative dimension $\leqslant n+1-j$, and with the property that Theorem 1 holds for (V_a, f) with the closed subschemes

$$X_j(a,f) = \{h \in \mathbf{A}^n_{\mathbf{Z}} \mid (a,h) \in Y_j\}$$

for all but finitely many $a \in \mathbf{Z}$, the exceptional a depending only on (W, Δ, f), provided the sums

$$\sum_{x \in V_a(\mathbf{F}_p)} \chi(g(x))\psi(f(x) + h_1x_1 + \cdots + h_nx_n)$$

are considered only for $g : W \longrightarrow \mathbf{A}^1_{\mathbf{Z}}$ invertible on W.

In particular, in this situation, we have $|X_j(a,f)(\mathbf{F}_p)| \ll p^{n-j}$ where the implied constant depends only on (W, Δ, f).

In other words, one can find the subschemes X_j in Theorem 1 in such a way that they vary "algebraically" with a, up to maybe allowing finitely many exceptions. Note that each exception can be handled independently by the "fixed a" version of Theorem 1, so this exceptional set is unlikely to create problems in applications.

Remark 7. The assumption on Δ is always true if W is closed in $\mathbf{A}^n_{\mathbf{Z}}$, since the ring of functions on W is a quotient of $\mathbf{Z}[X_1, \ldots, X_n]$ in that case.

Proof. The strategy is to prove a variant of [2, Th. 3.1] and then to deduce the statement from that. The key idea is to first construct the stratification using [2, Th. 2.1], which uses a first description of the relevant exponential sums. Then an alternate description, using the Fourier transform, gives cancellation using the formal properties of (semi)perverse sheaves.

We apply [2, Th. 2.1] with the following data:

- $T = \mathbf{A}^n_{\mathbf{Z}} \times \mathbf{A}^1_{\mathbf{Z}}$, with coordinates $(h,a) = (h_1, \ldots, h_n, a)$;
- $X = W \times \mathbf{A}^n_{\mathbf{Z}}$, with coordinates $(x,h) = (x_1, \ldots, x_n, h_1, \ldots, h_n)$; for the stratification $\mathcal{X}$, we take $\{X\}$ alone;
- $\pi : X \longrightarrow T$ is given by $\pi(x,h) = (h, \Delta(x))$;
- the function f is the function $F : X \longrightarrow \mathbf{A}^1$ given by $F(x,h) = f(x) + \sum x_ih_i$ (viewed as a T-morphism $X \longrightarrow \mathbf{A}^1_T$).

Theorem 2.1 of [2] gives data $(N, C, \mathcal{H})$ where $N \geqslant 1$ is an integer, $C \geqslant 0$ is a real number and $\mathcal{H} = (H_i)_{i \in I}$ is a finite stratification of T, all of which depends only on (W, Δ, f). (Note that C is mentioned in the statement of [2, Th. 2.1], but does not appear in the statements of the properties (1) and (2) that $(N, C, \mathcal{H})$ are stated to satisfy; this is a typographical mistake, and the right-hand side of the main inequality in property (2) should be $C \sup_{x \in X_t} \|L\|(x)$ instead of $\sup_{x \in X_t} \|L\|(x)$).

Consider an object K of $D^b_c(W[1/M\ell], \bar{\mathbf{Q}}_\ell)$ for some $M \geqslant 1$ and some prime ℓ, adapted to the stratification $\{W\}$ of W. For a finite field k of characteristic not dividing $NM\ell$, a given parameter tuple $t = (a,h) \in T(k)$, a non-trivial additive character $\psi : k \longrightarrow \bar{\mathbf{Q}}_\ell$ of k, and a direct factor L of $K \otimes k$, the associated "standard sum" S (see [2, p. 120]) is the trace function of the object

$$R = R\pi_{k,!}(p^*_{1,k}L \otimes \mathcal{L}_{\psi(F)}) \tag{5}$$

where $p_1 : X \longrightarrow W$ is the first projection. The pullback $p_{1,k}^* L$ is an object of $D_c^b(X_k, \bar{\mathbf{Q}}_\ell)$ which is a direct factor of $(p_1^* K) \otimes k$, and the object R belongs to $D_c^b(T_k, \bar{\mathbf{Q}}_\ell)$.

The trace function τ_R of R is given at $t = (h, a) \in T(k)$ by

$$\begin{aligned}
\tau_R(t) &= \sum_{(x,h) \in \pi^{-1}(t)(k)} \psi(F_t(x)) \operatorname{Tr}(\mathrm{Fr}_{k,(x,h)} \mid p_1^* L_t) \\
&= \sum_{\substack{x \in W(k) \\ \Delta(x) = a}} \psi(f(x) + h_1 x_1 + \cdots + h_n x_n) \operatorname{Tr}(\mathrm{Fr}_{k,(x,h)} \mid p_1^* L) \\
&= \sum_{x \in V_a(k)} \psi(f(x) + h_1 x_1 + \cdots + h_n x_n) \operatorname{Tr}(\mathrm{Fr}_{k,x} \mid L),
\end{aligned}$$

which is the basic sum of interest for V_a.

The second key point is another description of the same family of sums as a Fourier transform; this will lead to the stratification estimates, through the perversity properties of the Fourier transform.

To do so, we define a function $G : \mathbf{A}_{\mathbf{Z}}^{n+1} \longrightarrow \mathbf{A}_{\mathbf{Z}}^1$ by

$$G(x, b) = -b\Delta(x).$$

We denote by $i_W : W_k \longrightarrow \mathbf{A}_k^n$ the natural immersion (over the given finite field k). Then we define the object

$$M = q_1^*(i_{W,!}(L) \otimes \mathcal{L}_{\psi(f)})[1] \otimes \mathcal{L}_{\psi(G)} \tag{6}$$

in $D_c^b(T_k, \bar{\mathbf{Q}}_\ell)$, where $q_1 : T_k = \mathbf{A}_k^n \times \mathbf{A}_k^1 \longrightarrow \mathbf{A}_k^n$ is the first projection (the final $[1]$ denotes the shift operation in $D_c^b(T_k, \bar{\mathbf{Q}}_\ell)$). Note in particular that q_1 is a smooth morphism of relative dimension 1.

We now compute the trace function τ_N of the Fourier transform $N = \mathrm{FT}_\psi(M)$ of M, which belongs to $D_c^b(\mathbf{A}_k^{n+1}, \bar{\mathbf{Q}}_\ell)$: if $(a, h) \in \mathbf{A}^{n+1}(k)$ denote the Fourier variables, we have

$$\tau_N(a, h) = (-1)^{n+1} \sum_{(x,b) \in k^{n+1}} \psi(ab + h_1 x_1 + \cdots + h_n x_n) \operatorname{Tr}(\mathrm{Fr}_{k,(x,b)} \mid M) \tag{7}$$

$$= (-1)^n \sum_{\substack{(x,b) \in k^{n+1} \\ x \in W(k)}} \psi(ab + h_1 x_1 + \cdots + h_n x_n) \operatorname{Tr}(\mathrm{Fr}_{k,x} \mid L) \psi(f(x)) \psi(-b\Delta(x)) \tag{8}$$

$$= (-1)^n \sum_{x \in W(k)} \psi(f(x) + h_1 x_1 + \cdots + h_n x_n) \operatorname{Tr}(\mathrm{Fr}_{k,x} \mid L) \sum_{b \in k} \psi(b(a - \Delta(x))) \tag{9}$$

$$= (-1)^n |k| \sum_{x \in V_a(k)} \psi(f(x) + h_1 x_1 + \cdots + h_n x_n) \operatorname{Tr}(\mathrm{Fr}_{k,x} \mid L) \tag{10}$$

by orthogonality. Up to the factor $(-1)^n |k|$, this is the same as the standard sum. This translates the fact that $R \simeq N$, up to a shift and a Tate twist (accounting for the sign and the factor $|k|$). More precisely, we claim that $N \simeq R[n+2](-1)$. We will give the proof in Lemma 8 below.

Now assume that K and L are fibrewise semiperverse and fibrewise mixed of weight $\leqslant d+1$, as in [2, Th. 3.1] (except that there the weight is $\leqslant d$; this shift reflects the fact that $\dim W$ is $\leqslant d+1$, which influences the normalization). Then M is semiperverse and mixed of weight

$\leqslant d+2$. Indeed, the first factor is so as in [2, p. 123, last paragraph], using the fact that the operation $q_1^*(\cdot)[1]$ preserves semiperversity and adds 1 to the weight (the relative dimension of q_1; see e.g. [7, 1.3.2 (4)]). Then we tensor it by the lisse sheaf $\mathcal{L}_{\psi(G)}$ of weight 0 on $\mathbf{A}_k^{n+1}$, which preserves semiperversity and the weight. Now, by the theory of the Fourier transform, N is therefore also semiperverse, and it is mixed of weight $\leqslant d+2+n+1 = d+n+3$ (see e.g. [7, Cor. 2.1.5 (iii), Th. 2.2.1]).

Let η_i be the dimension of a strat H_i in $\mathcal{H}$. Translating the semiperversity and weight condition in terms of the Lefschetz trace formula, as done in [2, p. 124, 125], leads to the property that

$$|k|\Big|\sum_{x\in V_a(k)} \psi(f(x)+h_1x_1+\cdots+h_nx_n)\operatorname{Tr}(\mathrm{Fr}_{k,x}\mid L)\Big| \leqslant C\Big(\sup_{v\in W(k)} \|L\|(v)\Big)|k|^{(d+n+3-\eta_i)/2}$$

for $(a,h)\in H_i(k)$. Cancelling the factor $|k|$ on both sides, we get

$$\Big|\sum_{x\in V_a(k)} \psi(f(x)+h_1x_1+\cdots+h_nx_n)\operatorname{Tr}(\mathrm{Fr}_{k,x}\mid L)\Big| \leqslant C\Big(\sup_{v\in W(k)} \|L\|(v)\Big)|k|^{(d+n+1-\eta_i)/2}$$

for $(a,h)\in H_i(k)$.

Let's check this for consistency: there is a unique i such that the "generic" strat H_i has relative dimension $\eta_i = \dim_{\mathbf{Z}} T = n+1$; for (a,h) in this strat we get a sum over V_a of size $|k|^{d/2}$, which is square-root cancellation.

To finish, we must handle the possibility that, for some strat H_i and some $a\in\mathbf{Z}$, the fiber $H_{i,a} = \{h \mid (a,h)\in H_i\}$ could still be of relative dimension η_i instead of $\eta_i - 1$ (as is needed to obtain $X_j(a,f)$ of dimension $\leqslant n-j$). We work around this possibility as follows: for each i, we consider the projection

$$\pi_i : H_i \longrightarrow \mathbf{A}^1_{\mathbf{Z}}$$

on the first coordinate a, so that $H_{i,a} = \pi_i^{-1}(a)$. Let $J\subset I$ be the subset of those $j\in I$ where π_j is *not* dominant, i.e., such that the image of π_j is not Zariski-dense in $\mathbf{A}^1_{\mathbf{Z}}$. For $j\in J$, the Zariski-closure $A_j\subset \mathbf{A}^1_{\mathbf{Z}}$ of the image of π_j is such that $A_{j,\mathbf{C}}(\mathbf{C})$ is a finite set of values. We denote by A_0 the union of the sets $A_{j,\mathbf{C}}(\mathbf{C})$ for $j\in J$. This is a finite set, since the stratification is finite, and it depends only on (W,Δ,f).

For $i\notin J$, the map π_i is dominant. Then there exists a dense open subset $U_i\subset\mathbf{A}^1_{\mathbf{Z}}$ such that $\dim_{\mathbf{Z}} H_{i,a} = \eta_i - 1$ for all $a\in U_i$, by standard algebraic geometry (see, e.g., [3, Cor. 14.116 (i)]). The complement $\tilde{A}_i$ of U_i is such that $\tilde{A}_{i,\mathbf{C}}(\mathbf{C})$ is finite. We denote by A_1 the union of $\tilde{A}_{i,\mathbf{C}}(\mathbf{C})$ for $i\notin J$, and finally we let $A = A_0\cup A_1$. This is again a finite set, depending only on (W,f).

Let $a\in\mathbf{Z}$. If $a\notin A$ and $i\notin J$, the fiber $H_{i,a}$ is then of relative dimension η_i-1 by the above.

Denote then

$$Y_j = \overline{\bigcup_{\eta_i\leqslant n+1-j} \bar{H}_i}$$

as in [2], the schematic closure of the Zariski closure in $T[1/N]$ of the union of the strat with relative dimension $\leqslant n+1-j$.

For all integers $a\notin A$, the fibers $Y_{j,a}$ of Y_j are then of relative dimension $\leqslant n-j$, by construction of A. Moreover, for any prime p large enough and for any integer $a\notin A$, we derive

$$\Big|\sum_{x\in V_a(k)}\psi(f(x)+h_1x_1+\cdots+h_nx_n)\,\mathrm{Tr}(\mathrm{Fr}_{\mathbf{F}_p,x}\mid L)\Big|\leqslant C\Big(\sup_{v\in W(k)}\|L\|(v)\Big)|k|^{d/2+(j-1)/2}$$

if $(a,h)\in\mathbf{F}_p^{n+1}-Y_j(\mathbf{F}_p)$.

Assume χ is of order κ. We can take the object

$$K=g^*([x\mapsto x^\kappa]_*\bar{\mathbf{Q}}_\ell)[n+1]$$

(which is adapted to $\{W\}$) and the shifted Kummer sheaf $L=\mathcal{L}_{\chi(g)}[d+1]$ as direct factor of $K\otimes\mathbf{F}_p$. Since L satisfies

$$\mathrm{Tr}(\mathrm{Fr}_{\mathbf{F}_p,x}\mid L)=\chi(g(x))$$

for all $x\in W(\mathbf{F}_p)$, and since $\|L\|(v)=1$ for all $v\in W(\mathbf{F}_p)$, we obtain our statement.

To conclude, the bound

$$|X_j(a,f)|\ll p^{n-j}$$

follows for $a\notin A$ from standard uniform bounds for point-counting in algebraic families, and the implied constant may be adjusted to include the finitely many $a\in A$. □

Here is the lemma we used. Such computations are quite standard (see for instance in the works [5] or [6] of Katz), and follow line by line the computation from (7) to (10) using the function-sheaf dictionary (but involve a fair amount of notational bookkeeping).

Lemma 8. *With notation as in the proof, we have* $N\simeq R[n+2](-1)$.

Proof. In this proof, all morphisms and schemes are viewed as defined over the given finite field k. Recall that by definition (see [8, Def. 1.2.1.1]) we have

$$N=R\pi_{2,!}(\pi_1^*M\otimes\mathcal{L}_{\psi(x\cdot h+ba)})[n+1]$$

where π_1 and π_2 are the two projections

$$\begin{cases}(\mathbf{A}^n\times\mathbf{A}^1)\times(\mathbf{A}^n\times\mathbf{A}^1)\longrightarrow\mathbf{A}^n\times\mathbf{A}^1\\ (x,b,h,a)\overset{\pi_1}{\longmapsto}(x,b)\\ (x,b,h,a)\overset{\pi_2}{\longmapsto}(h,a).\end{cases}$$

and $x\cdot h=x_1h_1+\cdots+x_nh_n$. (This is the analogue of (7).)

Let $\tilde\pi_1$ and $\tilde\pi_2$ denote the restrictions of π_1 and π_2 to $W\times\mathbf{A}^1\times\mathbf{A}^n\times\mathbf{A}^1$, and $\tilde q_1$ the projection $W\times\mathbf{A}^1\longrightarrow W$. Then the presence of the $i_{W,!}$ term in the definition (6) of M gives

$$N\simeq R\tilde\pi_{2,!}(\tilde\pi_1^*(\tilde q_1^*(L\otimes\mathcal{L}_{\psi(f)})[1]\otimes\mathcal{L}_{\psi(G)})\otimes\mathcal{L}_{\psi(x\cdot h+ba)})[n+1]$$

(where $\mathcal{L}_{\psi(f)}$ and $\mathcal{L}_{\psi(G)}$ denote here the restriction to W and $W\times\mathbf{A}^1$ of the corresponding sheaves on $\mathbf{A}^n$ and $\mathbf{A}^{n+1}$). This formula is the analogue of (8).

Next we factor $\tilde\pi_2=\beta\circ\alpha$ where

$$\alpha\,:\,W\times\mathbf{A}^1\times\mathbf{A}^n\times\mathbf{A}^1\longrightarrow W\times\mathbf{A}^n\times\mathbf{A}^1$$
$$\beta\,:\,W\times\mathbf{A}^n\times\mathbf{A}^1\longrightarrow\mathbf{A}^n\times\mathbf{A}^1$$

are given by $\alpha(x,b,h,a)=(x,h,a)$ and $\beta(x,h,a)=(h,a)$. Correspondingly we get $N=R\tilde\beta_!(N_1)[n+1]$ where

$$N_1=R\alpha_!(\tilde\pi_1^*(\tilde q_1^*(L\otimes\mathcal{L}_{\psi(f)})[1]\otimes\mathcal{L}_{\psi(G)})\otimes\mathcal{L}_{\psi(x\cdot h+ba)}).$$

Note that $N_1 = R\alpha_!(\alpha^* N_2 \otimes \mathcal{L}_{\psi(b(a-\Delta(x)))})$ with

$$N_2 = r_1^*(L \otimes \mathcal{L}_{\psi(f)})[1] \otimes \mathcal{L}_{\psi(x\cdot h)}$$

where $r_1 : W \times \mathbf{A}^n \times \mathbf{A}^1 \longrightarrow W$ is the obvious projection. By the projection formula (see, e.g., [9, Th. 7.4.7]), we get

$$N_1 \simeq r_1^*(L \otimes \mathcal{L}_{\psi(f)})[1] \otimes \mathcal{L}_{\psi(x\cdot h)} \otimes R\alpha_!(\mathcal{L}_{\psi(b(a-\Delta(x)))})$$

and hence

$$N \simeq R\beta_! \left(r_1^*(L \otimes \mathcal{L}_{\psi(f)})[1] \otimes \mathcal{L}_{\psi(x\cdot h)} \otimes R\alpha_!(\mathcal{L}_{\psi(b(a-\Delta(x)))}) \right) [n+1],$$

which is the analogue of (9).

By standard properties of the Artin-Schreier sheaf we have

$$R\alpha_!(\mathcal{L}_{\psi(b(a-\Delta(x)))}) \simeq i_{\Gamma,!}\bar{\mathbf{Q}}_\ell(-1)$$

where $i_\Gamma : \Gamma \longrightarrow W \times \mathbf{A}^n \times \mathbf{A}^1$ is the natural immersion of

$$\Gamma = \{(x,h,a) \in W \times \mathbf{A}^n \times \mathbf{A}^1 \mid \Delta(x) = a\}.$$

Precisely, consider the cartesian diagram

$$\begin{array}{ccc}
W \times \mathbf{A}^n \times \mathbf{A}^1 \times \mathbf{A}^n & \xrightarrow{\xi} & \mathbf{A}^1 \times \mathbf{A}^1 \\
(x,b,h,a) & \longmapsto & (a-\Delta(x), b) \\
\downarrow \alpha & & \downarrow p_1 \\
W \times \mathbf{A}^1 \times \mathbf{A}^n & \xrightarrow{\theta} & \mathbf{A}^1 \\
(x,h,a) & \longmapsto & a - \Delta(x).
\end{array}$$

Consider the lisse sheaf $\mathcal{L}_{\psi(vw)}$ on $\mathbf{A}^1 \times \mathbf{A}^1$ in the top-right corner; applying the proper base change theorem (see, e.g., [9, Th. 7.4.4]) we get an isomorphism

$$\theta^* R p_{1,!} \mathcal{L}_{\psi(vw)} \simeq R\alpha_! \xi^* \mathcal{L}_{\psi(vw)} = R\alpha_!(\mathcal{L}_{\psi(b(a-\Delta(x)))}).$$

Then we need only know that $Rp_{1,!}\mathcal{L}_{\psi(vw)} = i_{0,!}\bar{\mathbf{Q}}_\ell(-1)$, where $i_0 : \{0\} \longrightarrow \mathbf{A}^1$ is the immersion, to deduce that

$$R\alpha_!(\mathcal{L}_{\psi(b(a-\Delta(x)))}) \simeq \theta^* i_{0,!}\bar{\mathbf{Q}}_\ell(-1) = i_{\Gamma,!}\bar{\mathbf{Q}}_\ell(-1).$$

This step translates the use of the orthogonality relation to go from (9) to (10). Coming back to N, we therefore have

$$\begin{aligned}
N &\simeq R\beta_! \left(r_1^*(L \otimes \mathcal{L}_{\psi(f)})[1] \otimes \mathcal{L}_{\psi(x\cdot h)} \otimes i_{\Gamma,!}\bar{\mathbf{Q}}_\ell(-1) \right) [n+1] \\
&\simeq R\tilde{\beta}_!(\tilde{r}_1^* L \otimes \mathcal{L}_{\psi(f(x)+x\cdot h)})[n+2](-1)
\end{aligned}$$

where $\tilde{\beta}$ is the restriction of β to Γ and $\tilde{r}_1 : \Gamma \longrightarrow W$ the restriction of r_1. But now observe that we have an isomorphism

$$\gamma \begin{cases} X \longrightarrow \Gamma \\ (x,h) \mapsto (x,h,\Delta(a)) \end{cases}$$

such that $\tilde{\beta} \circ \gamma = \pi$ and $\tilde{r}_1 \circ \gamma = p_1$. It follows that

$$N \simeq R\pi_!(p_1^* L \otimes \mathcal{L}_{\psi(f(x)+x\cdot h)})[n+2](-1) \simeq R[n+2](-1)$$

as claimed. □

3. Uniformity in Theorem 2

We now address similar uniformity issues for Theorem 2. We consider now a closed subscheme $W \subset \mathbf{A}^n_{\mathbf{Z}}$ such that $W_{\mathbf{C}}$ is irreducible and smooth of dimension $d+1$, and a function $\Delta : W \longrightarrow \mathbf{A}^1_{\mathbf{Z}}$. There is then some $D \geqslant 1$ such that W_S/S is smooth with geometrically connected fibers of dimension $d+1$, where $S = \mathrm{Spec}(\mathbf{Z}[1/D])$.

We assume that there exists a finite set $F \subset \mathbf{Z}$ and $M \geqslant 1$ such that for $a \notin F$, the following holds:

- the fiber V_a is such that $V_{a,\mathbf{C}}$ is smooth of dimension d;
- we have $A(V_a, k, \psi) \geqslant 1$ for all finite fields k of characteristic $p \nmid M$ and for all $\bar{\mathbf{Q}}_\ell^\times$-valued non-trivial additive characters ψ of k.

Then Theorem 2 applies to all V_a for $a \notin F$. The analogue of Proposition 3 is very simple:

Proposition 9. *With notation and assumptions as above, for the sums*

$$\sum_{x \in V_a(\mathbf{F}_p)} \psi(h_1x_1 + \cdots + h_nx_n) \tag{11}$$

the constants $C'(V_a)$ in Theorem 2 may be bounded independently of $a \notin F$.

Proof. This follows from Theorem 2 and [4] exactly as in Proposition 3, noting that no restriction on g is needed since this parameter does not occur. □

We conclude with the analogue of Proposition 6:

Proposition 10. *With notation as above, there exist closed subschemes Y_j of $\mathbf{A}^{n+1}_S$ for $1 \leqslant j \leqslant n+1$ such that*

$$\mathbf{A}^{n+1}_S \supset Y_1 \supset \cdots \supset Y_{n+1}$$

with relative dimension $\leqslant n+1-j$, and with the property that Theorem 2 holds for V_a with the closed subschemes

$$X_j(a) = \{h \in \mathbf{A}^n_{\mathbf{Z}} \mid (a,h) \in Y_j\}$$

for all but finitely many $a \in \mathbf{Z}$, the exceptional a depending only on (W, Δ).

In particular, in this situation, we have $|X_j(a)(\mathbf{F}_p)| \ll p^{n-j}$ where the implied constant depends only on (W, Δ).

Proof. We begin first as in the proof of Proposition 6 with $f = 0$, constructing varieties Y_j, obtaining an integer $N \geqslant 1$ and a stratification $\mathcal{H}$ indexed by I, with H_i of relative dimension η_i. We define the set J and the finite set A as in the final steps of that proof, so that for $a \notin A$ and $i \notin J$, the fibers $H_{i,a}$ are of relative dimension $\eta_i - 1$. These sets depend only on (W, Δ).

Consider a finite field k of characteristic $p \nmid NMD$ and non-trivial additive character k. Let then $L = K = \bar{\mathbf{Q}}_\ell[d]$, and define the object R, as in the proof of Proposition 6 in this special case (see (5)).

The point is that, for $a \notin A \cup F$, if i_a denotes the closed immersion $\mathbf{A}^n \hookrightarrow \mathbf{A}^n \times \mathbf{A}^1$ given by $i_a(h) = (h, a)$, then by proper base change (see, e.g., [9, Th. 7.4.4]), we have $i_a^*R \simeq R_a$, where R_a is the object used in [2, proof of Th. 1.1] for $(V_a, 0)$ and $K = L = \bar{\mathbf{Q}}_\ell[d]$.

By the proof of [2, Th. 3.1], R_a is semiperverse, geometrically irreducible and geometrically non-zero (because it is the Fourier transform of $i_!\bar{\mathbf{Q}}_\ell[d]$, where here $i : V_a \hookrightarrow \mathbf{A}^n$, which is geometrically irreducible, and because of the condition on the A-number, by its very definition [2, p. 127]). Since R_a is adapted to the stratification $(i_a^*\mathcal{H}_i)_{i\notin J}$, the same reasoning as in the proof of Theorem 4.4 of [2] gives the additional vanishing of cohomology that implies the uniformity for the first part of Theorem 2.

For the second part (the homogeneity of the varieties $X_j(a)$), for $a \notin A \cup F$, we see that the argument in [2, p. 131] applies without change. $\square$

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