

# Explicit points on elliptic curves of high rank over function fields

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Random matrices,  $L$ -functions, and primes  
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If  $p > 2$ , let

$$P(v) = (X(v), Y(v)) = \left( \frac{v^q(v^q - v)}{1 + 4v^q}, \frac{1}{2} \left[ \frac{v^{2q}}{(1 + 4v)^{q-1}} + \frac{v^{2q}(1 + 2v + 2v^q)}{(1 + 4v)^{(3q-1)/2}} \right] \right)$$

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Let  $u = t^{1/d}$  and  $K_d = k(u)$ . Then for  $i = 0, \dots, d - 1$

$$P(\zeta_d^i u) \in E(K_d),$$

they are almost independent (1 relation), and they generate a finite index subgroup of  $E(K_d)$  which has rank  $d - 1$ .

## Motivation

- Berger's construction
- Explicit Berger
- First example
- Second example

## An example

- High analytic ranks
- Less ubiquitous BSD
- Sketch of 4-monomial proof

Goal is to explain a systematic construction of such examples.

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Engineering applications?

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For example, if  $p > 3$  and  $E$  is an elliptic curve over  $\mathbb{F}_p(t)$  with an odd number of places of multiplicative reduction away from  $t = 0$  and  $t = \infty$ , then  $\text{ord}_{s=1} L(E/K_d, s)$  is unbounded as  $d$  varies.

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Example:

$$y^2 = x^{2g+2} + x^{2g+1} + t$$

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This gives many examples of (simple, non-isotrivial) abelian varieties of every dimension with large rank.

## Sketch of proof:

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Put it all together

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Project: Find less rigid constructions of surfaces DPCT.

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**Theorem** (Berger, UA thesis 2007, JNT 2008): Under mild hypotheses on  $f$  and  $g$ ,

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**Cor:** For  $k = \mathbb{F}_q$ , there are families with parameters of elliptic curves over  $\mathbb{F}_q(t)$  with arbitrarily large rank in the tower  $\mathbb{F}_q(t^{1/d})$ .

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$$(\mathcal{C}_d \times \mathcal{D}_d) / \mu_d \xrightarrow{\sim} \mathcal{S}_d$$

Working out the geometry leads to the following *rank formula* for  $J = \text{Jac}(X)$ :

$$\text{Rank } J(K_d) = \text{Rank } \text{hom}_{k-av}(J_{C_d}, J_{D_d})^{\mu_d} - c_1 d + c_2$$

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The numerical formula comes from a connection between homomorphisms and points which in good cases can be made very explicit.

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$\mathcal{C}_d \cong \mathcal{D}_d$  in a way that anti-commutes with  $\mu_d$  actions. So,

$$\text{Rank } E(K_d) = \text{Rank } \text{End}_{k-av}(J_{\mathcal{C}_d})^{anti-\mu_d}$$

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If  $k$  is finite of characteristic  $p$ , let  $d = p^n + 1$  and note that  $Fr_{p^n} \circ \zeta_d = \zeta_d^{-1} \circ Fr_{p^n}$ . Similarly,

$$(Fr_{p^n} \circ \zeta_d^i) \circ \zeta_d = \zeta_d^{-1} \circ (Fr_{p^n} \circ \zeta_d^i)$$

for all  $i$ . This gives many independent endomorphisms in  $\text{End}_{k-av}(J_{C_d})^{anti-\mu_d}$ . Tracing through the geometry leads to many independent points in  $E(K_d)$ .

Assume  $k = \mathbb{C}$  for simplicity. Let  $\mathcal{C} = \mathcal{D} = \mathbb{P}^1$ . Let  $f(x) = x(x - a)/(x - 1)$  and  $g(y) = y(y - a)/(y - 1)$ . Here  $a \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ .

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$X = E$  is again an elliptic curve. Let

$$S = \mathbb{P}^1 \setminus \{0, 1, \infty, -1, 1/2, 2, \zeta_6, \bar{\zeta}_6\}$$

For  $a \in S$  one finds  $c_1 = 1$ ,  $c_2 = 4$  and

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The term  $-d$  cancels out the obvious endomorphisms  $\zeta_d^i : J_{\mathcal{C}_d} \rightarrow J_{\mathcal{D}_d}$ . To get rank we need some extra endomorphisms, i.e., CM.

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A-O: for fixed  $d$  only finitely many good  $a$ . But what about if  $d$  varies?