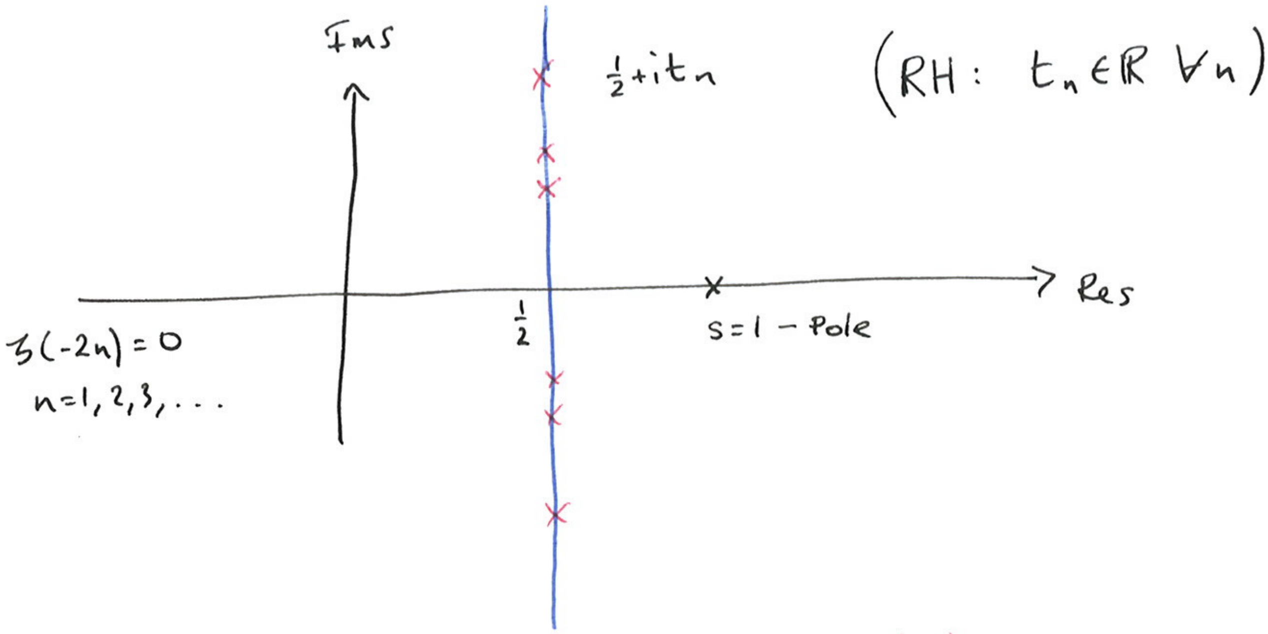
$\mathbb{R}$MT and the zeta-function:

some background and some foreground

Zeros

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad \mathrm{Re}\, s > 1$$



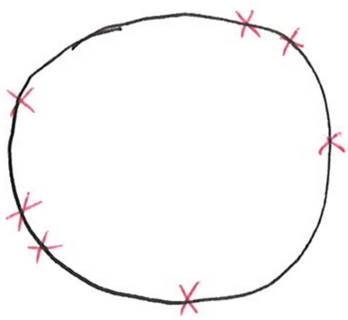
unfolded zeros: $\quad w_n = t_n \frac{1}{2\pi} \log \frac{|t_n|}{2\pi}$

Eigenvalues

A – $N \times N$ unitary matrix

eigenvalues – $e^{i\theta_n}$

unfolded eigenvalues – $\phi_n = \theta_n \frac{1}{2\pi} N$



Montgomery's Conjecture

(Montgomery 1973)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\le n,m\le N} f(w_n - w_m) \overset{?}{=} \int_{-\infty}^{\infty} f(x) \overbrace{\left(\delta(x) + 1 - \frac{\sin^2 \pi x}{\pi^2 x^2}\right)}^{R_2(x)} dx \qquad (M)$$

$$= \lim_{N\to\infty} \int_{U(N)} \frac{1}{N} \sum_{n,m} f(\phi_n - \phi_m)\, d\mu(A)$$

(Dyson 1963)

- extension to n-point correlations (Hejhal, Rudnick-Sarnak)

explicit formula $t_n \longleftrightarrow$ primes

So $\sum_{n,m} \longleftrightarrow \sum_{p,q}$

zeros

primes

evaluate $p=q$ terms using the Prime Number Theorem

(heuristically) evaluate $p \neq q$ terms using Hardy-Littlewood conjecture:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \Lambda(n)\Lambda(n+h) \overset{?}{=} \begin{cases} 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{q>2 \\ q\mid h}} \left(\frac{q-1}{q-2}\right) & h \text{ even} \\ 0 & h \text{ odd} \end{cases} \qquad (HL)$$

$$\left[\ \Lambda(n) = \begin{cases} \log p & n = p^k \\ 0 & \text{otherwise} \end{cases}\ \right]$$

- smoothed form of (HL) $\Longleftrightarrow$ (M)

  (Montgomery 1973, Goldston + Montgomery 1987)

- extension to n-point correlations

  (Bogomolny + K 1995, 1996)

- (HL) $\Rightarrow$ (M) + lower order terms (LOT):

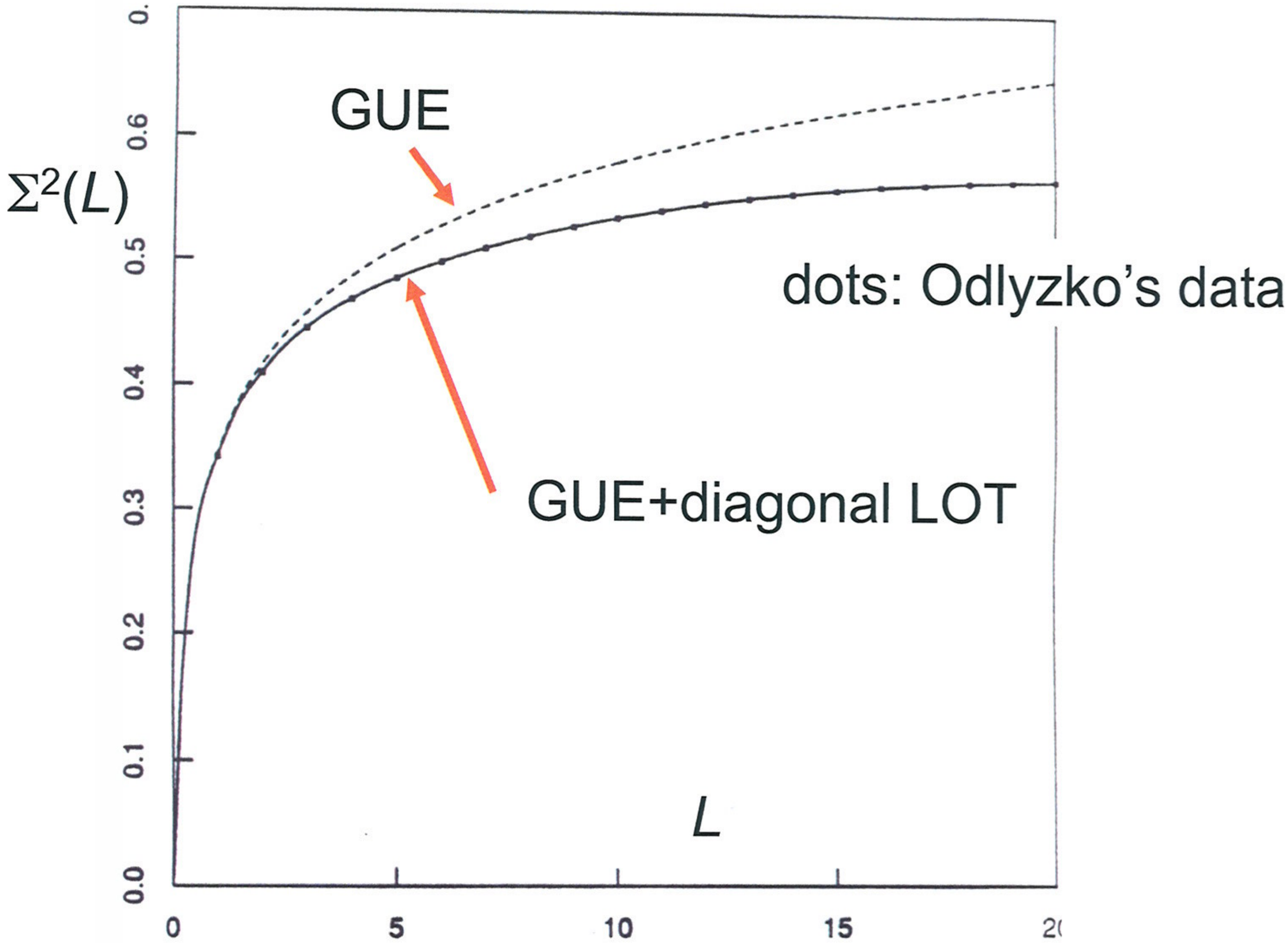
  (Bogomolny + K 1996)

$$\boxed{\begin{aligned} R_2(x;T) \simeq\ & \delta(x) + 1 - \frac{\sin^2 \pi x}{\pi^2 x^2} + \\ & + \frac{2}{\log^2 \frac{T}{2\pi}} \left[\frac{1}{z^2} - \frac{d^2}{dz^2} \mathrm{Re} \log \zeta(1-iz) - \mathrm{Re} \sum_{p=2}^{\infty} \frac{\log^2 p}{(p^{1+iz}-1)^2} \right]_{z=\frac{2\pi x}{\log \frac{T}{2\pi}}} \\ & + \frac{2}{\log^2 \frac{T}{2\pi}} \left[-\frac{\cos(2\pi x)}{z^2} + |\zeta(1+iz)|^2 \mathrm{Re}\, e^{2\pi i x} \prod_p \left(1 - \frac{(p^{iz}-1)^2}{(p-1)^2}\right) \right]_{z=\frac{2\pi x}{\log \frac{T}{2\pi}}} \\ & \qquad\qquad\qquad \text{(M+LOT)} \end{aligned}}$$

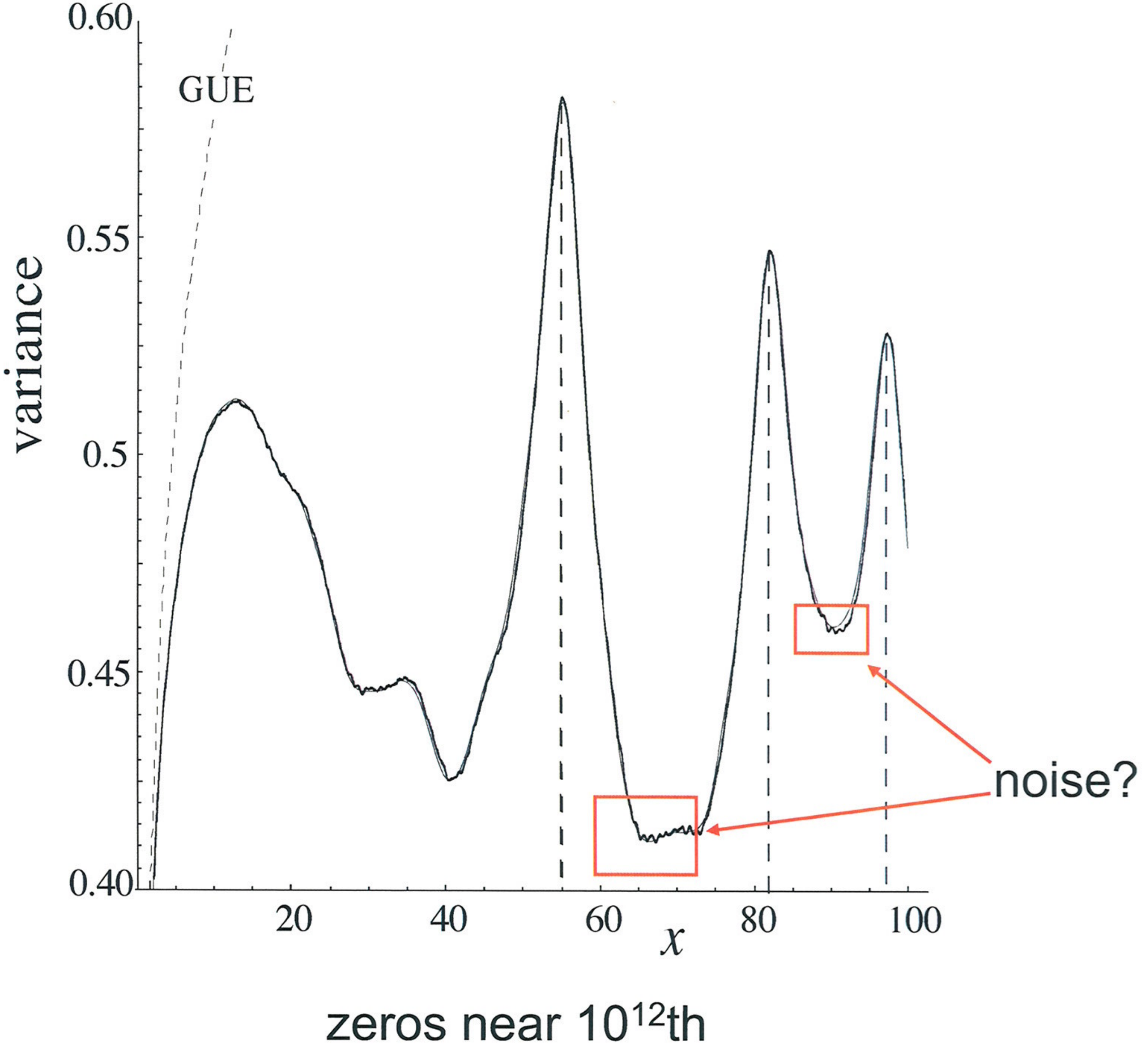
- (M+LOT) follows from several other approaches that do not rely on (HL): the universal RMT form of $R_2(x)$ (Bogomolny + K 1996); symmetrized, truncated Euler product (Bogomolny + K 1996); ratios conjecture (Conrey + Snaith 2006, 2008).

- (M + LOT) $\Rightarrow$ (HL) (Keating 2008)

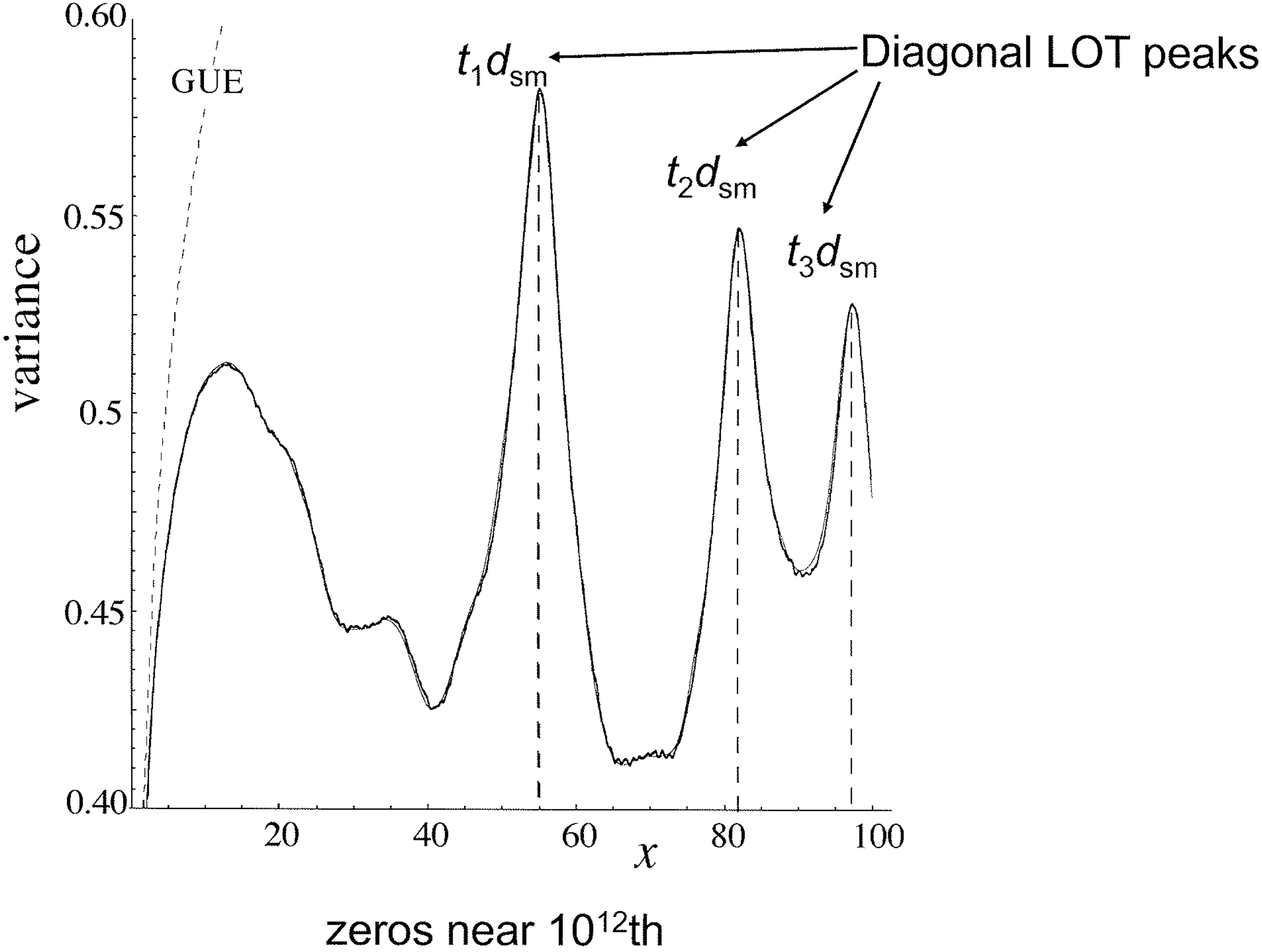
- extension to (families of) L-functions deviations from random matrix theory require contributions from ***lower order terms (LOTs)***, which are non-universal



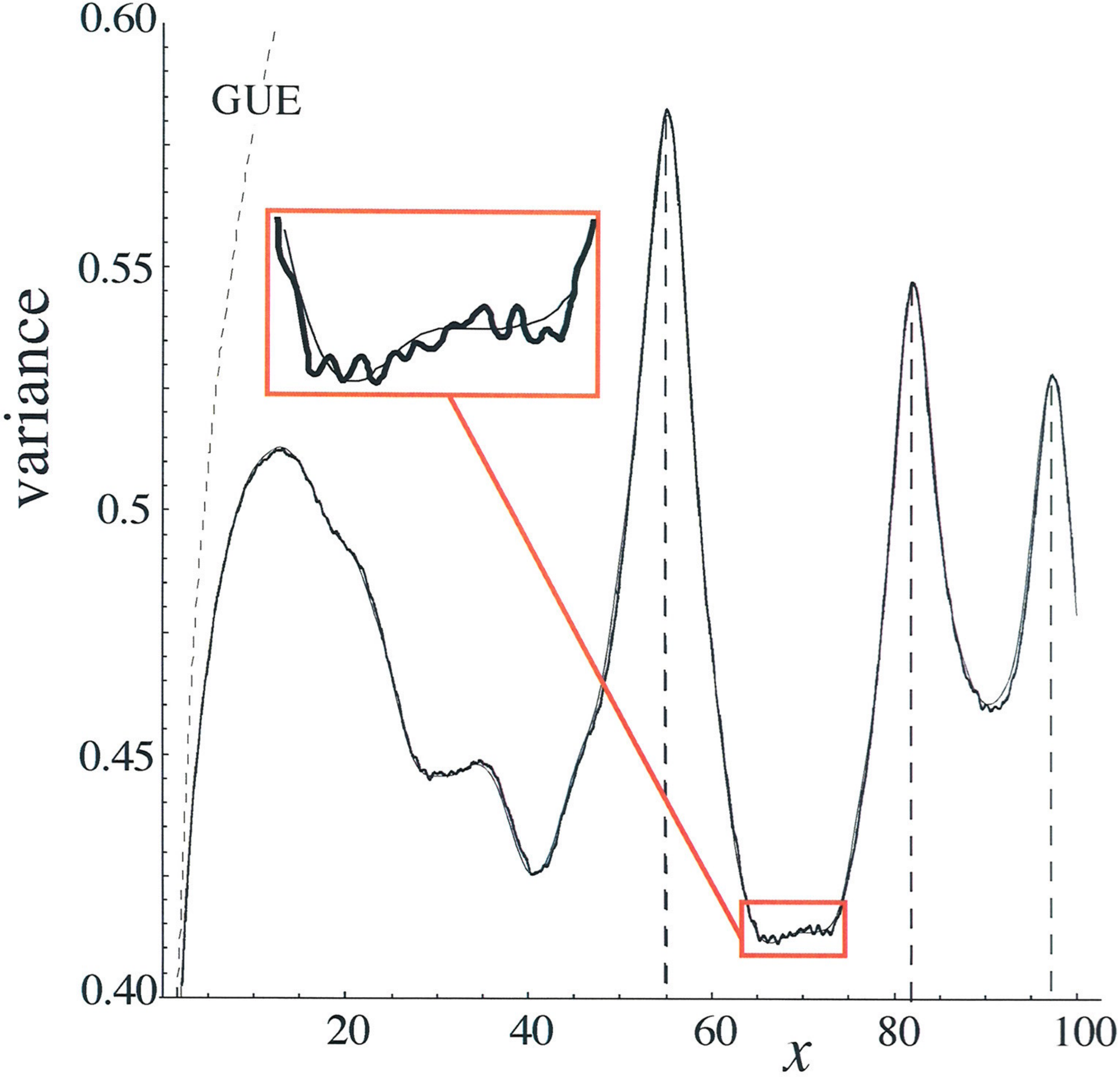
zeros near 10^{20}th



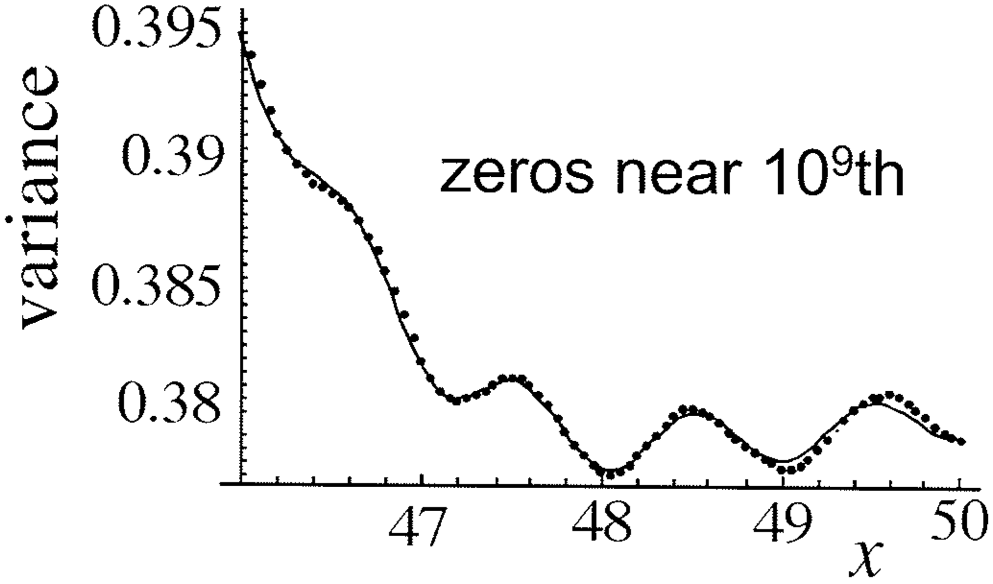
0.60
0.55
0.5
0.45
0.40
variance
GUE
noise?
20
40
60
80
100
x
zeros near 10^12th



Diagonal LOT peaks
$t_1 d_{sm}$
$t_2 d_{sm}$
$t_3 d_{sm}$
GUE
variance
0.60
0.55
0.5
0.45
0.40
20
40
60
80
100
x
zeros near 10^{12}th



0.60
0.55
0.5
0.45
0.40
GUE
variance
20
40
60
80
100
x



R_{GUE}+ *diagonal LOT* + *off-diagonal LOT*

Moments of $\zeta(\frac{1}{2}+it)$ and RMT

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad \mathrm{Re}\, s > 1$$

$$I(k;T) \equiv \frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt$$

Moment conjecture : leading order asymptotics
(K-Snaith 2000)

$$I(k;T) \underset{T\to\infty}{\sim} a(k) \left\langle |\det(I - Ae^{-i\theta})|^{2k} \right\rangle_{A \in CUE_N} \Big|_{N = \log\frac{T}{2\pi}}$$

$$a(k) = \prod_p \left[\left(1-\frac{1}{p}\right)^{k^2} {}_2F_1\left(k,k;1;\frac{1}{p}\right)\right]$$

$$\Rightarrow I(k;T) \underset{T\to\infty}{\sim} a(k) \underbrace{\prod_{j=0}^{k-1} \frac{j!}{(j+k)!}}_{= \frac{G^2(1+k)}{G(1+2k)}} \left(\log\frac{T}{2\pi}\right)^{k^2}$$

- invert to calculate value distribution of $|\zeta(\frac{1}{2}+it)|$
- generalization to families of L-functions

Hybrid Products

$$\zeta(\tfrac{1}{2}+it) \text{ "="} \prod_p \left(1-\frac{1}{p^{1/2+it}}\right)^{-1}$$

$$= \quad \cdots \prod_n \left(1-\frac{t}{t_n}\right) \qquad (\zeta(\tfrac{1}{2}+it_n)=0)$$

$$\text{"="} \quad \cdots \prod_{p<X} \left(1-\frac{1}{p^{1/2+it}}\right)^{-1} \prod_{|t-t_n|\le \frac{1}{\log X}} \left(1-\frac{t}{t_n}\right)$$

Theorem (Gonek, Hughes, K 2007)

Let $s=\sigma+it$, $\sigma\geq 0$, $|t|\geq 2$, let $X\geq 2$ and let K be any fixed positive integer. Let $f(x)$ be a nonnegative C^∞-function of mass one supported on $[0,1]$ and set $u(x)=Xf(X\log\frac{x}{e}+1)/x$. Set $U(z)=\int_0^\infty u(x)\,E_1(z\log x)\,dx$, where $E_1(z)=\int_z^\infty e^{-w}\frac{dw}{w}$.

Then

$$\zeta(s)= P_X(s)\,Z_X(s)\left(1+O\left(\frac{X^{K+2}}{(|s|\log X)^K}\right)+O(X^{-\sigma}\log X)\right),$$

where $$P_X(s) = \exp\left(\sum_{n\le X}\frac{\Lambda(n)}{n^s\log n}\right)$$

and $$Z_X(s) = \exp\left(-\sum_n U((s-\tfrac{1}{2}-it_n)\log X)\right).$$

$\Rightarrow$ moment conjecture assuming independence of P_X and Z_X

- generalization to L-functions (Bui-K 2007, 2008)

Moment Conjecture: Lower Order Terms

(Conrey, Farmer, K, Rubinstein, Snaith 2005, 2008)

$$\left\langle |\det(I - Ae^{-i\theta})|^{2k} \right\rangle_{A \in CUE_N} = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \prod_{i=1}^{k} (N+i+j)$$

(K - Snaith 2000)

$$= \sum_{n=0}^{k^2} \alpha_n N^{k^2-n}$$

conjecture: $$I(k;T) = \frac{1}{T}\int_0^T P_k\left(\log \frac{t}{2\pi}\right) dt + o(1)$$

where

$$P_k(x) = \sum_{n=0}^{k^2} c_n(k)\, x^{k^2-n}$$

$$= \frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \oint \cdots \oint \frac{G(z_1,\dots,z_{2k})\,\Delta^2(z_1,\dots,z_{2k})}{\prod_{i=1}^{2k} z_i^{2k}} \times e^{\frac{x}{2}\sum_{i=1}^{k} z_i - z_{i+k}}\, dz_1\, dz_2 \dots dz_{2k}$$

where

$$G(z_1,\dots,z_{2k}) = A_k(z_1,\dots,z_{2k}) \prod_{i=1}^{k}\prod_{j=1}^{k} \zeta(1+z_i - z_{j+k})$$

with

$$A_k(z_1,\dots,z_{2k}) = \prod_p \sum_{j=1}^{k} \prod_{i\neq j} \frac{\prod_{m=1}^{k}\left(1 - p^{-1+z_{i+k}-z_m}\right)}{1 - p^{z_{i+k}-z_{j+k}}}$$

$$P_2(x) =$$
$$\frac{1}{2\pi^2}x^4 + \frac{8}{\pi^4}\left(\gamma\pi^2 - 3\zeta'(2)\right)x^3 +$$
$$\frac{6}{\pi^6}\left(-48\gamma\zeta'(2)\pi^2 - 12\zeta''(2)\pi^2 + 7\gamma^2\pi^4 + 144\zeta'(2)^2 - 2\,\gamma_1\pi^4\right)x^2 +$$
$$\frac{12}{\pi^8}\Big(6\gamma^3\pi^6 - 84\gamma^2\zeta'(2)\pi^4 + 24\gamma_1\zeta'(2)\pi^4 - 1728\zeta'(2)^3 + 576\gamma\zeta'(2)^2\pi^2 +$$
$$288\zeta'(2)\zeta''(2)\pi^2 - 8\zeta'''(2)\pi^4 - 10\gamma_1\gamma\pi^6 - \gamma_2\pi^6 - 48\gamma\zeta''(2)\pi^4\Big)x +$$
$$\frac{4}{\pi^{10}}\Big(-12\zeta''''(2)\pi^6 + 36\gamma_2\zeta'(2)\pi^6 + 9\gamma^4\pi^8 + 21\gamma_1^2\pi^8 + 432\zeta''(2)^2\pi^4 +$$
$$3456\gamma\zeta'(2)\zeta''(2)\pi^4 + 3024\gamma^2\zeta'(2)^2\pi^4 - 36\gamma^2\gamma_1\pi^8 - 252\gamma^2\zeta''(2)\pi^6 +$$
$$3\gamma\gamma_2\pi^8 + 72\gamma_1\zeta''(2)\pi^6 + 360\gamma_1\gamma\zeta'(2)\pi^6 - 216\gamma^3\zeta'(2)\pi^6 - 864\gamma_1\zeta'(2)^2\pi^4 +$$
$$5\gamma_3\pi^8 + 576\zeta'(2)\zeta'''(2)\pi^4 - 20736\gamma\zeta'(2)^3\pi^2 - 15552\zeta''(2)\zeta'(2)^2\pi^2 -$$
$$96\gamma\zeta'''(2)\pi^6 + 62208\zeta'(2)^4\Big)$$
$$= .0506605918211688857219397316048638\,x^4 +$$
$$.69886988487897996984709628427658502\,x^3 +$$
$$2.425962198846682004756575310160663\,x^2 +$$
$$3.227907964901254764380689851274668\,x +$$
$$1.312424385961669226168440066229978$$

CFKRS 2005

46

$$
\begin{aligned}
P_2(x) = {} & 0.05066059182116888572193973160486638\, x^4 \\
& + 0.69886988487897996984709628427658502\, x^3 \\
& + 2.4259621988466820047565753101606663\, x^2 \\
& + 3.2279079649012547643806898512746668\, x \\
& + 1.3124243859616692261684400662299978
\end{aligned}
$$

$$
\begin{aligned}
P_3(x) = {} & 0.000005708527034652788398376841445252313\, x^9 \\
& + 0.000405021330884114403312153320259844\, x^8 \\
& + 0.011072455215246998350410400826667\, x^7 \\
& + 0.148400730801502726808514015187744\, x^6 \\
& + 1.0459251779054883439385323798059\, x^5 \\
& + 3.9843850948235347247479640734429\, x^4 \\
& + 8.6073191457812067561483476362 9\, x^3 \\
& + 10.274330830703446134183009522\, x^2 \\
& + 6.5939130206497581046571339 2\, x \\
& + 0.91651550763789305901785 43.
\end{aligned}
$$

CFKRS 2005

4c

interval	conjecture	reality	ratio
[0, 50000]	7236872972.7	7231005642.3	.999189
[50000, 100000]	15696470555.3	15723919113.6	1.001749
[100000, 150000]	21568672884.1	21536840937.9	.998524
[150000, 200000]	26381397608.2	26246250354.1	.994877
[200000, 250000]	30556177136.5	30692229217.8	1.004453
[250000, 300000]	34290291841.0	34414329738.9	1.003617
[300000, 350000]	37695829854.3	37683495193.0	.999673
[350000, 400000]	40843941365.7	40566252008.5	.993201
[400000, 450000]	43783216365.2	43907511751.1	1.002839
[450000, 500000]	46548617846.7	46531247056.9	.999627
[500000, 550000]	49166313161.9	49136264678.2	.999389
[550000, 600000]	51656498739.2	51744796875.0	1.001709
[600000, 650000]	54035153255.1	53962410634.2	.998654
[650000, 700000]	56315178564.8	56541799179.3	1.004024
[700000, 750000]	58507171421.6	58365383245.2	.997577
[750000, 800000]	60619962488.2	60870809317.1	1.004138
[800000, 850000]	62661003164.6	62765220708.6	1.001663
[850000, 900000]	64636649728.0	64227164326.1	.993665
[900000, 950000]	66552376294.2	65994874052.2	.991623
[950000, 1000000]	68412937271.4	68961125079.8	1.008013
[1000000, 1050000]	70222493232.7	70233393177.0	1.000155
[1050000, 1100000]	71984709805.4	72919426905.7	1.012985
[1100000, 1150000]	73702836332.4	72567024812.4	.984589
[1150000, 1200000]	75379769148.4	76267763314.7	1.011780
[1200000, 1250000]	77018102997.5	76750297112.6	.996523
[1250000, 1300000]	78620173202.6	78315210623.9	.996121
[1300000, 1350000]	80188090542.5	80320710380.9	1.001654
[1350000, 1400000]	81723770322.2	80767881132.6	.988303
[1400000, 1450000]	83228956776.3	83782957374.3	1.006656
[0, 2350000]	3317437762612.4	3317496016044.9	1.000017

Sixth moment of ζ versus Conjecture. The 'reality' column, i.e. integrals involving ζ, were computed using Mathematica.

CFKRS 2005

Theorem (CFKRS 2008)

$$\boxed{c_r(k) = a(k) \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \sum_{|\alpha|+|\beta|=r} 2^{1-\delta_{\alpha,\beta}}\, b_k(\alpha;\beta)\, N_k(\alpha;\beta)}$$

where $a(k) = \prod_p \left(1-\frac{1}{p}\right)^{k^2} {}_2F_1(k,k;1;1/p)$

$$\frac{1}{a(k)} A_k(z_1,\ldots,z_{2k}) = \sum_{\alpha,\beta} b_k(\alpha;\beta)\, z_1^{\alpha_1} \ldots z_k^{\alpha_k} z_{k+1}^{\beta_1} \ldots z_{2k}^{\beta_k}$$

$$|\alpha| = \sum_{i=1}^{k} \alpha_i \quad , \quad |\beta| = \sum_{i=1}^{k} \beta_i$$

$$N_k(\alpha;\beta) = \frac{1}{2^{k^2-r}} \left(\prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \right)^{-1} \sum_{\substack{\text{distinct} \\ \text{perms} \\ \sigma,\tau \text{ of} \\ \alpha,\beta}} M_k(\sigma(\alpha),\tau(\beta))$$

with

$$M_k(\sigma(\alpha),\tau(\beta)) = (-1)^{\Sigma\beta_i} \begin{vmatrix} \Gamma(2k-\alpha_{\sigma_1})^{-1} & \Gamma(2k-1-\alpha_{\sigma_1})^{-1} & \cdots & \Gamma(1-\alpha_{\sigma_1})^{-1} \\ \Gamma(2k-1-\alpha_{\sigma_2})^{-1} & \Gamma(2k-2-\alpha_{\sigma_2})^{-1} & \cdots & \Gamma(-\alpha_{\sigma_2})^{-1} \\ \vdots & \vdots & & \vdots \\ (-1)^k\Gamma(k+1-\beta_{\tau_k})^{-1} & (-1)^{k+1}\Gamma(k-\beta_{\tau_k})^{-1} & \cdots & (-1)^{3k-1}\Gamma(2-k-\beta_{\tau_k})^{-1} \end{vmatrix}$$

$\nearrow$

$2k \times 2k$ determinant

Theorem ! (CFKRS 2008)

$$\boxed{N_k(\alpha;\beta) \text{ is a polynomial in } k \text{ of degree } \le 2(|\alpha|+|\beta|)}$$

$\therefore$ $N_k(\alpha;\beta)$ can be determined from $2(|\alpha|+|\beta|+1)$ values of k.

eg $N_k(1;) = k^2$, $N_k(2;) = 0$, $N_k(1,1;) = \frac{1}{2}k^2(k^2-1)$

$N_k(1;1) = -k^2(k^2-1)$

Theorem (CFKRS 2008)

The Taylor coefficients $b_k(\alpha;\beta)$ are polynomials in k, the Taylor coefficients of $s\zeta(1+s)$ and the Taylor coefficients of $\log A_k(z_1,\dots,z_{2k})$

so

$$c_1(k) = a(k)\frac{g^2(1+k)}{g(1+2k)}\, 2k^2(\gamma k + B_k(1;))$$

$$c_2(k) = a(k)\frac{g^2(1+k)}{g(1+2k)}\, k^2(k^2-1) \times \Big(2(B_k(1;)+\gamma k)^2$$

$$-\gamma^2 - 2\gamma_1 + B_k(1,1;) - B_k(1;1)\Big).$$

etc

where

$$B_k(1;) = \sum_p \left(\frac{k \log p}{p-1} - \frac{k \log p \ {}_2F_1(k+1, k+1; 2; p^{-1})}{p \ {}_2F_1(k,k;1;p^{-1})} \right)$$

$$B_k(1,1;) = -\sum_p \left(\frac{k^2 \log^2 p \ {}_2F_1(k+1, k+1; 2; p^{-1})^2}{p^2 \ {}_2F_1(k,k;1;p^{-1})^2} \right.$$

$$\left. - \frac{\log^2 p \ \binom{k+1}{2} \ {}_2F_1(k+2, k+2; 3; p^{-1})}{p^2 \ {}_2F_1(k,k;1;p^{-1})} \right)$$

$$B_k(1;1) = \sum_p \frac{p \log^2 p}{(p-1)^2} + \left(\frac{\log^2 p \ k^2 \ {}_2F_1(k+1,k+1;2;p^{-1})^2}{p^2 \ {}_2F_1(k,k;1;p^{-1})^2} \right.$$

$$\left. - \frac{\log^2 p \ {}_2F_1(k+1,k+1;1;p^{-1})}{p \ {}_2F_1(k,k;1;p^{-1})} \right)$$

etc.

and $\quad s\zeta(1+s) = 1 + \gamma s - \gamma_1 s^2 + \frac{\gamma_2}{2!} s^3 + \ldots$

- generalization to L-functions (CFKRS 2005, 2008)

Numerical Tests

Compare

i) $\displaystyle\int_C^D |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt$

ii) $\displaystyle\int_C^D P_k\left(\log\tfrac{t}{2\pi}\right)dt$

iii) when $k \notin \mathbb{N}$, $\displaystyle\int_C^D \sum_{r=0}^{R} c_r(k)\left(\log\tfrac{t}{2\pi}\right)^{k^2-r}dt$

Table 1

This table compares the conjectured value (5.2) to actual data (5.1) for intervals $[50\,000n, 50\,000(n+1)]$, $n = 0, 1, \ldots, 16$, and $k = 3, 4, 5$. The fit is to two or three decimal places, consistent with the remainder stated in (1.1)

n	conj. $k=3$	data $k=3$	conj. $k=4$	data $k=4$	conj. $k=5$	data $k=5$
0	7.23687×10^{9}	7.23101×10^{9}	1.89527×10^{12}	1.88501×10^{12}	6.00428×10^{14}	5.91051×10^{14}
1	1.56965×10^{10}	1.57239×10^{10}	5.67575×10^{12}	5.70833×10^{12}	2.45298×10^{15}	2.47886×10^{15}
2	2.15687×10^{10}	2.15368×10^{10}	9.17127×10^{12}	9.12987×10^{12}	4.68619×10^{15}	4.64908×10^{15}
3	2.63814×10^{10}	2.62463×10^{10}	1.24573×10^{13}	1.23432×10^{13}	7.10198×10^{15}	7.04187×10^{15}
4	3.05562×10^{10}	3.06922×10^{10}	1.55847×10^{13}	1.5683×10^{13}	9.63318×10^{15}	9.6445×10^{15}
5	3.42903×10^{10}	3.44143×10^{10}	1.8585×10^{13}	1.87265×10^{13}	1.22457×10^{16}	1.24349×10^{16}
6	3.76958×10^{10}	3.76835×10^{10}	2.14798×10^{13}	2.15861×10^{13}	1.4919×10^{16}	1.51619×10^{16}
7	4.08439×10^{10}	4.05663×10^{10}	2.42845×10^{13}	2.37201×10^{13}	1.76398×10^{16}	1.66972×10^{16}
8	4.37832×10^{10}	4.39075×10^{10}	2.70108×10^{13}	2.724×10^{13}	2.03988×10^{16}	2.06017×10^{16}
9	4.65486×10^{10}	4.65312×10^{10}	2.96679×10^{13}	2.94271×10^{13}	2.3189×10^{16}	2.26023×10^{16}
10	4.91663×10^{10}	4.91363×10^{10}	3.22631×10^{13}	3.24807×10^{13}	2.60051×10^{16}	2.69184×10^{16}
11	5.16565×10^{10}	5.17448×10^{10}	3.48022×10^{13}	3.47606×10^{13}	2.88433×10^{16}	2.87018×10^{16}
12	5.40352×10^{10}	5.39624×10^{10}	3.72905×10^{13}	3.73482×10^{13}	3.17002×10^{16}	3.18035×10^{16}
13	5.63152×10^{10}	5.65418×10^{10}	3.97319×10^{13}	4.00187×10^{13}	3.45733×10^{16}	3.48184×10^{16}
14	5.85072×10^{10}	5.83654×10^{10}	4.21303×10^{13}	4.1917×10^{13}	3.74603×10^{16}	3.70813×10^{16}
15	6.062×10^{10}	6.08708×10^{10}	4.44887×10^{13}	4.48257×10^{13}	4.03594×10^{16}	4.08236×10^{16}
16	6.2661×10^{10}	6.27652×10^{10}	4.68097×10^{13}	4.69566×10^{13}	4.32693×10^{16}	4.3287×10^{16}

Table 2

Conjecture vs. data for $k = 6, 7$, same intervals as the previous table

n	conj. $k=6$	data $k=6$	conj. $k=7$	data $k=7$
0	2.15456×10^{17}	2.08527×10^{17}	8.45652×10^{19}	7.99015×10^{19}
1	1.18835×10^{18}	1.20686×10^{18}	6.24627×10^{20}	6.3773×10^{20}
2	2.69034×10^{18}	2.66481×10^{18}	1.67709×10^{21}	1.66563×10^{21}
3	4.56155×10^{18}	4.56713×10^{18}	3.18661×10^{21}	3.25679×10^{21}
4	6.72399×10^{18}	6.61933×10^{18}	5.1125×10^{21}	4.87831×10^{21}
5	9.12928×10^{18}	9.3828×10^{18}	7.42365×10^{21}	7.74635×10^{21}
6	1.17439×10^{19}	1.21474×10^{19}	1.00952×10^{22}	1.06992×10^{22}
7	1.45431×10^{19}	1.31266×10^{19}	1.31065×10^{22}	1.11053×10^{22}
8	1.75076×10^{19}	1.75386×10^{19}	1.64403×10^{22}	1.61306×10^{22}
9	2.06221×10^{19}	1.95439×10^{19}	2.00815×10^{22}	1.83038×10^{22}
10	2.3874×10^{19}	2.61353×10^{19}	2.40171×10^{22}	2.8627×10^{22}
11	2.72527×10^{19}	2.70986×10^{19}	2.82354×10^{22}	2.82074×10^{22}
12	3.07492×10^{19}	3.06639×10^{19}	3.2726×10^{22}	3.20372×10^{22}
13	3.43557×10^{19}	3.43848×10^{19}	3.74797×10^{22}	3.70176×10^{22}
14	3.80656×10^{19}	3.7414×10^{19}	4.24878×10^{22}	4.13975×10^{22}
15	4.18729×10^{19}	4.25286×10^{19}	4.77427×10^{22}	4.86676×10^{22}
16	4.57724×10^{19}	4.53193×10^{19}	5.32373×10^{22}	5.1628×10^{22}
17	4.97592×10^{19}	4.98651×10^{19}	5.89648×10^{22}	6.0058×10^{22}

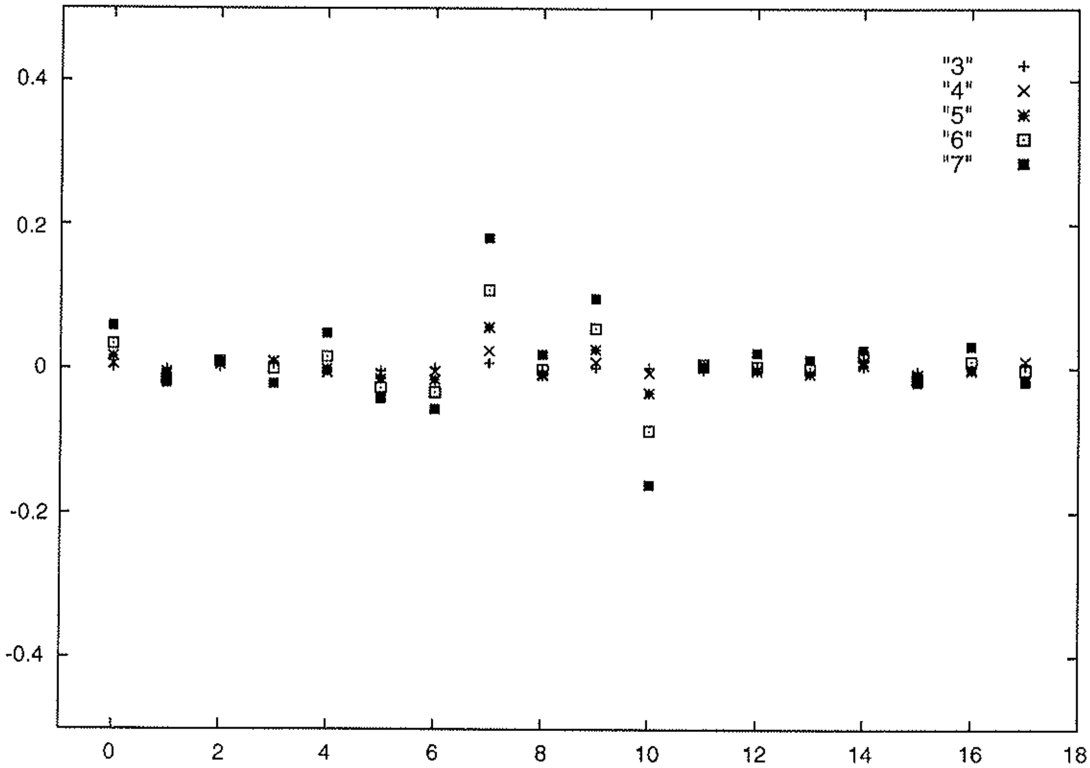


Fig. 1. The horizontal axis is n, and the vertical axis depicts (conjecture − data)/data for the values in Tables 1 and 2. For any finite interval, as $k \to \infty$, the main contribution to the $2k$th moment comes from the largest value of $|\zeta(1/2+it)|^{2k}$ on that interval. This explains the feature that, for a fixed interval, the actual moment tends to progressively deviate from the conjectured value as k increases.

Table 3
The coefficients $c_R(k)$, and conjecture v.s. data for $k = 0.5$ for three intervals. The bottom row gives (5.1) for the interval $[100, D]$, with $D = 1000$, $10\,000$, and $100\,000$. For each D, we compare this to the value of (5.3), $R = 0, 1, \ldots, 7$

R	$c_R(0.5)$	(5.3), $D = 1000$	(5.3), $D = 10\,000$	(5.3), $D = 100\,000$
0	1.1299287453321533	1463.83	17768.4	193494
1	0.19628236755422853	1523.55	18258.1	197413
2	0.03248602185728907	1525.93	18271.4	197491
3	−0.5289095729314908	1516.37	18234.3	197335
4	3.2346669444094671	1531.24	18275.5	197459
5	−21.381296730027876	1505.43	18222.2	197343
6	166.38844209028643	1559.87	18310.8	197488
7	−1529.2695739774642	1419.97	18120.1	197237
(5.1)		1521.27	18257.1	197425

Table 4
Conjecture vs. data for $k = 3.2$

R	$c_R(3.2)$	(5.3), $D = 1000$	(5.3), $D = 10\,000$	(5.3), $D = 100\,000$
0	$0.37531596173465401 \times 10^{-6}$	1968.83	1.16353×10^6	2.19960×10^8
1	$0.34462154217944847 \times 10^{-4}$	40049.5	1.65169×10^7	2.41753×10^9
2	$0.12662390083082525 \times 10^{-2}$	336190	9.78885×10^7	1.12289×10^{10}
3	$0.23963666452208821 \times 10^{-1}$	1.52891×10^6	3.2097×10^8	2.94868×10^{10}
4	0.2526426167678357	4.22213×10^6	6.6336×10^8	5.06417×10^{10}
5	1.5214668466274718	7.72205×10^6	9.6529×10^8	6.47041×10^{10}
6	5.3060442651520751	1.03793×10^7	1.12055×10^9	7.01449×10^{10}
7	11.121264784324178	1.16045×10^7	1.16894×10^9	7.14177×10^{10}
(5.1)		1.15305×10^7	1.16746×10^9	7.16886×10^{10}

Table 5
Conjecture vs. data for $k = 1.8$. For this value of k, and the range we examined, $R = 4$ or 5 give the best approximation

R	$c_R(1.8)$	(5.3), $D = 1000$	(5.3), $D = 10\,000$	(5.3), $D = 100\,000$
0	0.13885991555298723	15298.6	604203	1.58922×10^7
1	1.2590684761107478	46198.5	1.42746×10^6	3.20931×10^7
2	2.4174835075472416	59612.2	1.66821×10^6	3.56224×10^7
3	2.546894763686222	62863.9	1.70753×10^6	3.60492×10^7
4	−2.21426710514627	62199.9	1.7021×10^6	3.60059×10^7
5	3.223904454789757	62432.5	1.70339×10^6	3.60134×10^7
6	46.42674651960987	63260.1	1.70659×10^6	3.60268×10^7
7	−840.1304443557953	59448.3	1.69608×10^6	3.59953×10^7
(5.1)		61744.5	1.70134×10^6	3.60129×10^7

Table 6
Conjecture vs. data for $k = 0.5 + i$. The data here is not as convincing as for the other values of k, but, nonetheless, the early terms do give a reasonable approximation, and we believe the fit would improve with more substantial data

R	$c_R(0.5 + i)$	(5.3), $D = 1000$	(5.3), $D = 10\,000$	(5.3), $D = 100\,000$
0	$1.3117481341987813 + 1.211708767666727i$	$-308.872 + 439.126i$	$-3698.00 + 2357.78i$	$-34129.1 + 8908.71i$
1	$-3.0693034820213132 + 2.309977688777579i$	$-508.454 + 246.589i$	$-4331.26 + 957.574i$	$-35042.9 - 44.9533i$
2	$23.861826126198446 - 5.4045694962616631i$	$-335.035 + 646.347i$	$-4243.1 + 2618.59i$	$-36981.1 + 6688.15i$
3	$-111.54278536885322 - 35.79807241977336i$	$-285.625 + 118.290i$	$-3667.307 + 1304.32i$	$-34054.1 + 3538.21i$
4	$828.16689710582718 + 437.514818042632i$	$-546.679 + 1199.98i$	$-4747.02 + 3257.57i$	$-37543.5 + 6786.35i$
5	$-5808.11341189128 - 8339.592888954564i$	$1514.537 - 1342.42i$	$-738.290 - 0.1097i$	$-30111.2 + 3451.52i$
6	$15613.29091863494 + 101218.4464636376i$	$-6736.01 + 2796.92i$	$-12358.2 + 3830.64i$	$-45377.2 + 5624.57i$
7	$188541.27977634034 - 1175857.723687032i$	$23708.43 - 2789.22i$	$24043.14 + 702.364i$	$-4932.63 + 6133.48i$
(5.1)		$-340.843 + 383.859i$	$-3946.25 + 1883.17i$	$-35140 + 4830.47i$

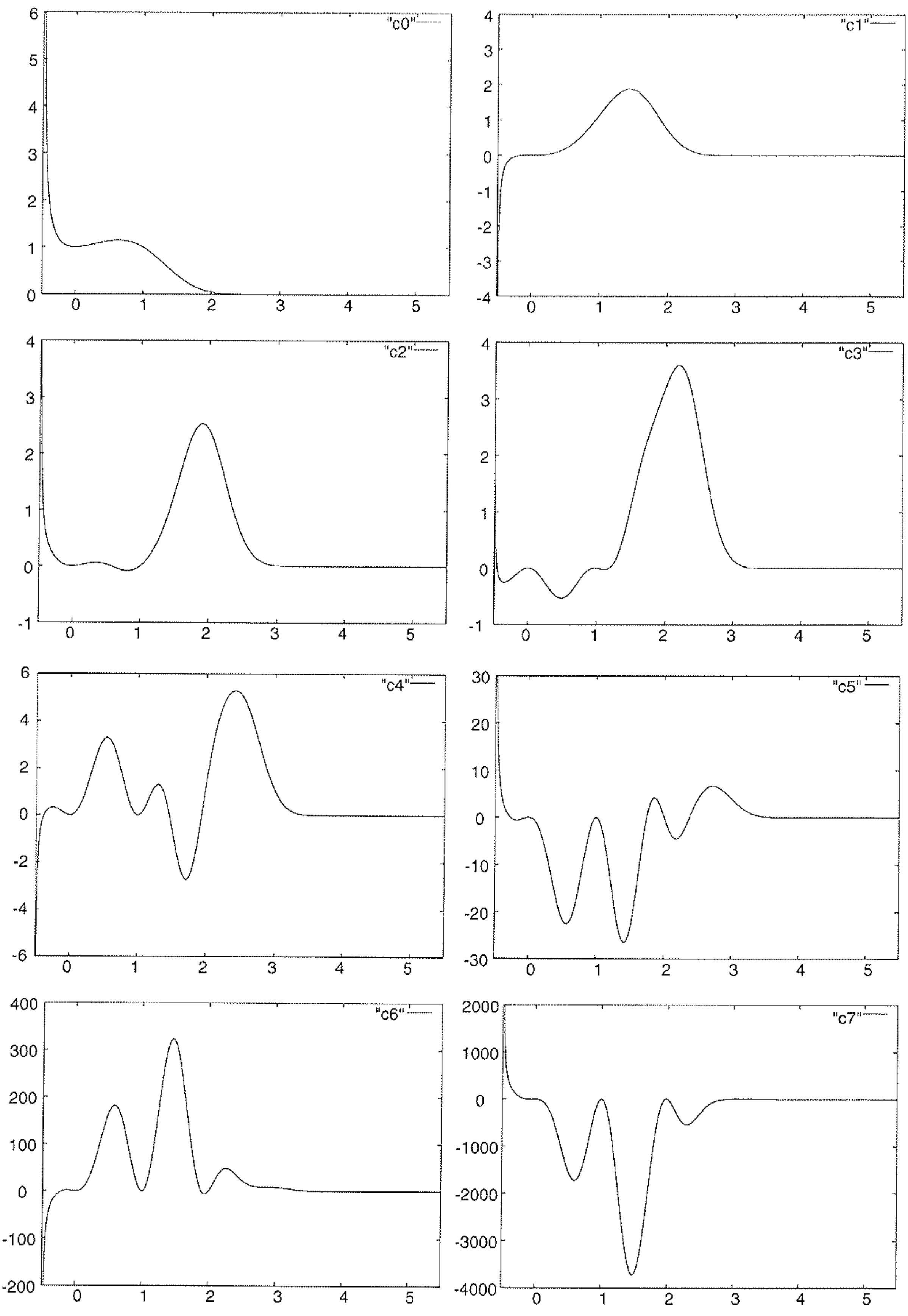


Fig. 2. Graphs of $c_r(k)$ with $-1/2 < k < 11/2$, for $r = 0, \ldots, 7$.

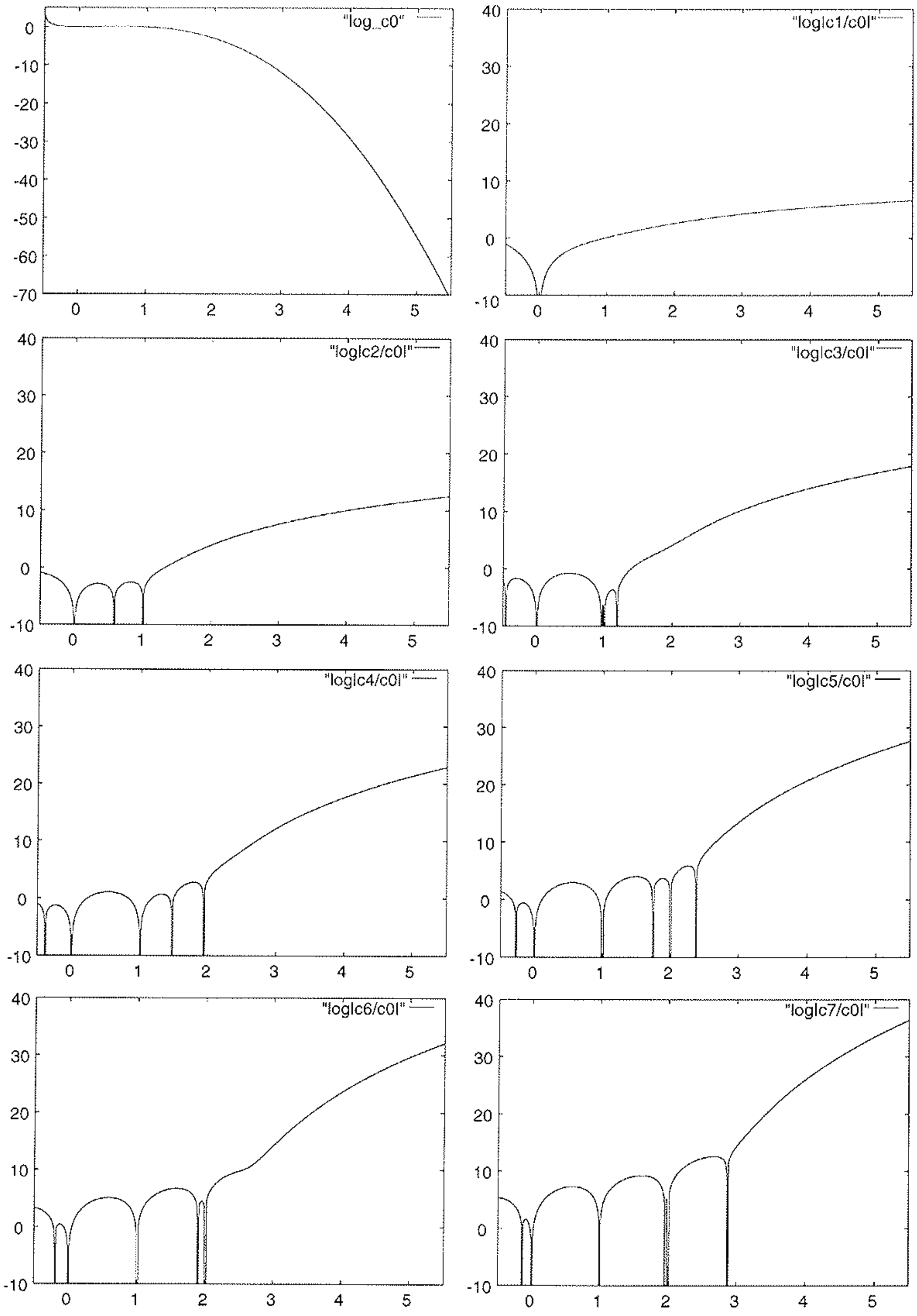


Fig. 3. The first figure depicts the graph of $\log(|c_0(k)|)$, while the next seven depict $\log(|c_r(k)/c_0(k)|)$, for $r = 1, \ldots, 7$. The asymptotic behavior of $\log(c_0(k))$ as $k \to \infty$ is implied by [CGo,KeS] and is, to leading order, $-k^2 \log(k)$. The cusps occur at zeros of $c_r(k)$, some of which are accounted for by the fact that, for non-negative $k \in \mathbb{Z}$, $P_k(x)$ is a polynomial of degree k^2 so that $c_r(k) = 0$ if $r > k^2$.