

Counting the ζ zeroes in distinct intervals

Zurich, October 31, 2008

Paul Bourgade, Enst, Université Paris 6

Random matrices, L-functions and primes

Gaussian fluctuations for the number of eigenvalues in distinct arcs of the circle (Wieand, Diaconis-Evans). Analogous result for ζ ?

Conjecture for the lim sup of $|\zeta|$.

The Keating-Snaith central
limit theorem

$$u_n \sim \mu_{U(n)}$$

$$\frac{\log \det(\text{Id} - u_n)}{\sqrt{\frac{1}{2} \log n}} \xrightarrow{\text{law}} \mathcal{N}_1 + i\mathcal{N}_2$$

as $n \rightarrow \infty$.

Selberg's central limit theorem
 ω uniform on $(0, 1)$

$$\frac{\log \zeta\left(\frac{1}{2} + i\omega t\right)}{\sqrt{\frac{1}{2} \log \log t}} \xrightarrow{\text{law}} \mathcal{N}_1 + i\mathcal{N}_2$$

as $t \rightarrow \infty$.

\mathcal{N}_1 and \mathcal{N}_2 independent standard real normal variables

The Keating-Snaith central
limit theorem

$$u_n \sim \mu U(n)$$

$$\frac{\log \det(\text{Id} - u_n)}{\sqrt{\frac{1}{2} \log n}} \xrightarrow{\text{law}} \mathcal{N}_1 + i\mathcal{N}_2$$

as $n \rightarrow \infty$.

Selberg's central limit theorem
 ω uniform on $(0, 1)$

$$\frac{\log \zeta\left(\frac{1}{2} + i\omega t\right)}{\sqrt{\frac{1}{2} \log \log t}} \xrightarrow{\text{law}} \mathcal{N}_1 + i\mathcal{N}_2$$

as $t \rightarrow \infty$.

\mathcal{N}_1 and \mathcal{N}_2 independent standard real normal variables

$$\epsilon_n \rightarrow 0, \epsilon_n \gg 1/n, u_n \sim \mu_{U(n)}$$

$$\frac{\log \det(\text{Id} - e^{-\epsilon_n} u_n)}{\sqrt{-\frac{1}{2} \log \epsilon_n}} \xrightarrow{\text{law}} \mathcal{N}_1 + i\mathcal{N}_2$$

Normalization $\sqrt{\frac{1}{2} \log n}$
 $\epsilon_n \ll 1/n$

$$\epsilon_t \rightarrow 0, \epsilon_t \gg 1/\log t, \omega$$

uniform on $(0, 1)$

$$\frac{\log \zeta\left(\frac{1}{2} + i\epsilon_t + i\omega T\right)}{\sqrt{-\frac{1}{2} \log \epsilon_t}} \xrightarrow{\text{law}} \mathcal{N}_1 + i\mathcal{N}_2$$

Normalization $\sqrt{\frac{1}{2} \log \log t}$ if
 $\epsilon_t \ll 1/\log t$

$$\epsilon_n \rightarrow 0, \epsilon_n \gg 1/n, u_n \sim \mu_{U(n)}$$

$$\frac{\log \det(\text{Id} - e^{-\epsilon_n} u_n)}{\sqrt{-\frac{1}{2} \log \epsilon_n}} \xrightarrow{\text{law}} \mathcal{N}_1 + i\mathcal{N}_2$$

Normalization $\sqrt{\frac{1}{2} \log n}$
 $\epsilon_n \ll 1/n$

$$\epsilon_t \rightarrow 0, \epsilon_t \gg 1/\log t, \omega$$

uniform on $(0, 1)$

$$\frac{\log \zeta\left(\frac{1}{2} + i\epsilon_t + i\omega T\right)}{\sqrt{-\frac{1}{2} \log \epsilon_t}} \xrightarrow{\text{law}} \mathcal{N}_1 + i\mathcal{N}_2$$

Normalization $\sqrt{\frac{1}{2} \log \log t}$ if
 $\epsilon_t \ll 1/\log t$

$N_n(\alpha, \beta)$: number of eigenvalues $e^{i\theta}$ of u_n with $\alpha < \theta < \beta$,
 $\delta_n(\alpha, \beta) = N_n(\alpha, \beta) - \mathbb{E}_{\mu_{U(n)}}(N_n(\alpha, \beta))$,

$$\delta_n(\alpha, \beta) = \frac{1}{\pi} (\Im \log Z(u_n, e^{i\beta}) - \Im \log Z(u_n, e^{i\alpha}))$$

$N(t)$ for the number of zeroes z of ζ with $0 < \Im z \leq t$

$$N(t) = \frac{t}{2\pi} \log \frac{t}{2\pi e} + \frac{1}{\pi} \Im \log \zeta(1/2 + it) + \frac{7}{8} + O\left(\frac{1}{t}\right)$$

$$\Delta(t_1, t_2) = (N(t_2) - N(t_1)) - \left(\frac{t_2}{2\pi} \log \frac{t_2}{2\pi e} - \frac{t_1}{2\pi} \log \frac{t_1}{2\pi e} \right),$$

$N_n(\alpha, \beta)$: number of eigenvalues $e^{i\theta}$ of u_n with $\alpha < \theta < \beta$,
 $\delta_n(\alpha, \beta) = N_n(\alpha, \beta) - \mathbb{E}_{\mu_{U(n)}}(N_n(\alpha, \beta))$,

$$\delta_n(\alpha, \beta) = \frac{1}{\pi} (\Im \log Z(u_n, e^{i\beta}) - \Im \log Z(u_n, e^{i\alpha}))$$

$N(t)$ for the number of zeroes z of ζ with $0 < \Im z \leq t$

$$N(t) = \frac{t}{2\pi} \log \frac{t}{2\pi e} + \frac{1}{\pi} \Im \log \zeta(1/2 + it) + \frac{7}{8} + O\left(\frac{1}{t}\right)$$

$$\Delta(t_1, t_2) = (N(t_2) - N(t_1)) - \left(\frac{t_2}{2\pi} \log \frac{t_2}{2\pi e} - \frac{t_1}{2\pi} \log \frac{t_1}{2\pi e} \right),$$

Wieand, 1998 As $n \rightarrow \infty$, the finite-dimensional distributions of the process

$$\frac{\delta_n(\alpha, \beta)}{\frac{1}{\pi} \sqrt{\log n}}, \quad 0 \leq \alpha < \beta < 2\pi,$$

converge to those of a Gaussian process

$\{\delta(\alpha, \beta), 0 \leq \alpha < \beta < 2\pi\}$ with covariance function

$$\mathbb{E}(\delta(\alpha, \beta)\delta(\alpha', \beta')) = \begin{cases} 1 & \text{if } \alpha = \alpha' \text{ and } \beta = \beta' \\ 1/2 & \text{if } \alpha = \alpha' \text{ and } \beta \neq \beta' \\ 1/2 & \text{if } \alpha \neq \alpha' \text{ and } \beta = \beta' \\ -1/2 & \text{if } \beta = \alpha' \\ 0 & \text{elsewhere} \end{cases} .$$

Analogue on ζ . ω uniform on $(0, 1)$. As $t \rightarrow \infty$, the finite-dimensional distributions of the process

$$\frac{\Delta(\omega t + \alpha, \omega t + \beta)}{\frac{1}{\pi} \sqrt{\log \log t}}, \quad 0 \leq \alpha < \beta < 2\pi,$$

converge to those of a Gaussian process

$\{\delta(\alpha, \beta), 0 \leq \alpha < \beta < 2\pi\}$ with covariance function

$$\mathbb{E}(\delta(\alpha, \beta)\delta(\alpha', \beta')) = \begin{cases} 1 & \text{if } \alpha = \alpha' \text{ and } \beta = \beta' \\ 1/2 & \text{if } \alpha = \alpha' \text{ and } \beta \neq \beta' \\ 1/2 & \text{if } \alpha \neq \alpha' \text{ and } \beta = \beta' \\ -1/2 & \text{if } \beta = \alpha' \\ 0 & \text{elsewhere} \end{cases} .$$

For $\alpha_1 < \dots < \alpha_\ell$, convergence of

$$\frac{1}{\sqrt{\log \log t}} (\log \zeta(1/2 + i\omega t + i\alpha_1), \dots, \log \zeta(1/2 + i\omega t + i\alpha_\ell))$$

to a vector of independent Gaussian random variables. Finite Dirichlet series are sufficient thanks to Selberg :

$$\frac{1}{t} \int_0^t \left| \log \zeta(1/2 + is) - \sum_{p \leq t} \frac{p^{-is}}{\sqrt{p}} \right|^2 ds < c.$$

Landau

$$\frac{\zeta'}{\zeta}(s) = - \sum_{n \leq x} \frac{\Lambda(n)}{n^s} + \frac{x^{1-s}}{1-s} - \sum_{\rho} \frac{x^{\rho-s}}{\rho-s} + \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n+s},$$

Selberg

$$\begin{aligned} \frac{\zeta'}{\zeta}(s) = & - \sum_{n \leq x^2} \frac{\Lambda_x(n)}{n^s} + \frac{x^{2(1-s)} - x^{1-s}}{(1-s)^2 \log x} + \frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(\rho-s)^2} \\ & + \frac{1}{\log x} \sum_{n=1}^{\infty} \frac{x^{-2n-s} - x^{-2(2n+s)}}{(2n+s)^2} \end{aligned}$$

where

$$\Lambda_x(n) = \begin{cases} \Lambda(n) & \text{for } 1 \leq n \leq x, \\ \Lambda(n) \frac{\log \frac{x^2}{n}}{\log n} & \text{for } x \leq n \leq x^2, \end{cases}$$

Landau

$$\frac{\zeta'}{\zeta}(s) = - \sum_{n \leq x} \frac{\Lambda(n)}{n^s} + \frac{x^{1-s}}{1-s} - \sum_{\rho} \frac{x^{\rho-s}}{\rho-s} + \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n+s},$$

Selberg

$$\begin{aligned} \frac{\zeta'}{\zeta}(s) = & - \sum_{n \leq x^2} \frac{\Lambda_x(n)}{n^s} + \frac{x^{2(1-s)} - x^{1-s}}{(1-s)^2 \log x} + \frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(\rho-s)^2} \\ & + \frac{1}{\log x} \sum_{n=1}^{\infty} \frac{x^{-2n-s} - x^{-2(2n+s)}}{(2n+s)^2} \end{aligned}$$

where

$$\Lambda_x(n) = \begin{cases} \Lambda(n) & \text{for } 1 \leq n \leq x, \\ \Lambda(n) \frac{\log \frac{x^2}{n}}{\log n} & \text{for } x \leq n \leq x^2, \end{cases}$$

Complex numbers a_{pt} ($p \in \mathcal{P}$, $t \in \mathbb{R}^+$), $\sup_p |a_{pt}| \rightarrow 0$,

$\sum_p |a_{pt}|^2 \rightarrow \sigma^2$ as $t \rightarrow \infty$.

Existence of (m_t) with $\log m_t / \log t \rightarrow 0$ and

$$\sum_{p > m_t} |a_{pt}|^2 \left(1 + \frac{p}{t}\right) \xrightarrow{t \rightarrow \infty} 0.$$

Then, if ω is a uniform random variable on $(0, 1)$,

$$\sum_{p \in \mathcal{P}} a_{pt} p^{-i\omega t} \xrightarrow{\text{law}} \sigma Y$$

as $t \rightarrow \infty$, Y being a standard complex normal variable.

Analogue of this result, which implies the fluctuations observed by Wieand.

Diaconis-Evans, 2001. Complex constants $\{a_{nj} \mid n \in \mathbb{N}, j \in \mathbb{N}\}$, $\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} |a_{nj}|^2 (j \wedge n) = \sigma^2$. Existence of positive integers $\{m_n \mid n \in \mathbb{N}\}$ such that $\lim_{n \rightarrow \infty} m_n/n = 0$ and

$$\lim_{n \rightarrow \infty} \sum_{j=m_n+1}^{\infty} |a_{nj}|^2 (j \wedge n) = 0,$$

Then, if $u_n \sim \mu_{U(n)}$, as $n \rightarrow \infty$

$$\sum_{j=1}^{\infty} a_{nj} \operatorname{Tr}(u_n^j) \xrightarrow{\text{law}} \sigma Y.$$

$$\limsup_{t \rightarrow \infty} \frac{\log |\zeta(1/2 + it)|}{\sqrt{\log t / \log \log t}} \geq c$$

Montgomery, 1977 : on RH, $c = 1/20$

Soundararajan, 2007 : $c = 1$

Farmer, Gonek, Hughes, 2007. Analogy with a convergence in probability for matrices of increasing size :

$$\limsup_{t \rightarrow \infty} \frac{\log |\zeta(1/2 + it)|}{\sqrt{\log t \log \log t}} = \frac{1}{\sqrt{2}}$$

$$\limsup_{t \rightarrow \infty} \frac{\log |\zeta(1/2 + it)|}{\sqrt{\log t / \log \log t}} \geq c$$

Montgomery, 1977 : on RH, $c = 1/20$

Soundararajan, 2007 : $c = 1$

Farmer, Gonek, Hughes, 2007. Analogy with a convergence in probability for matrices of increasing size :

$$\limsup_{t \rightarrow \infty} \frac{\log |\zeta(1/2 + it)|}{\sqrt{\log t \log \log t}} = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{\log \log n}} (\log \zeta(1/2 + i\omega n + i), \dots, \log \zeta(1/2 + i\omega n + ni))$$

Resnick-Tomkins. (X_i) i.i.d. random variables and $F(x) = \mathbb{P}(X_1 < x)$. Let $\mu_n = F^{-1}(1 - 1/n)$. Then

$$\max(X_1, \dots, X_n) / \mu_n \rightarrow 1$$

almost surely when $n \rightarrow \infty$ iff, for all $\delta > 1$, $\int_1^\infty \frac{dF(x)}{1-F(\delta x)} < \infty$.
 For i.i.d. standard Gaussians,

$$\frac{\max(X_1, \dots, X_n)}{\sqrt{2 \log n}} \rightarrow 1$$

almost surely. This supports the conjecture

$$\limsup_{t \rightarrow \infty} \frac{\log |\zeta(1/2 + it)|}{\sqrt{\log t \log \log t}} \geq 1.$$

$$\frac{1}{\sqrt{\log \log n}} (\log \zeta(1/2 + i\omega n + i), \dots, \log \zeta(1/2 + i\omega n + ni))$$

Resnick-Tomkins. (X_i) i.i.d. random variables and $F(x) = \mathbb{P}(X_1 < x)$. Let $\mu_n = F^{-1}(1 - 1/n)$. Then

$$\max(X_1, \dots, X_n) / \mu_n \rightarrow 1$$

almost surely when $n \rightarrow \infty$ iff, for all $\delta > 1$, $\int_1^\infty \frac{dF(x)}{1-F(\delta x)} < \infty$.
 For i.i.d. standard Gaussians,

$$\frac{\max(X_1, \dots, X_n)}{\sqrt{2 \log n}} \rightarrow 1$$

almost surely. This supports the conjecture

$$\limsup_{t \rightarrow \infty} \frac{\log |\zeta(1/2 + it)|}{\sqrt{\log t \log \log t}} \geq 1.$$

Shrinking intervals $(\alpha/K_t, \beta/K_t)$:

- ▶ $K_t / \log t \rightarrow 0$: Gaussian fluctuations.
- ▶ $K_t = \lambda \log t$: Montgomery-Odlyzko law, the number of zeroes may converge to a discrete random variable (Gram law).

Correlations in the mesoscopic regime : $\delta \in (0, 1)$,

$$\frac{1}{\sqrt{\frac{1}{2} \log \log t}} \left(\log \left| \zeta \left(\frac{1}{2} + i\omega t \right) \right|, \log \left| \zeta \left(\frac{1}{2} + i\omega t + \frac{i}{(\log t)^\delta} \right) \right| \right)$$

converges in law to

$$(\mathcal{N}_1, \delta \mathcal{N}_1 + \sqrt{1 - \delta^2} \mathcal{N}_2).$$