COUNTING PRIMES IRRATIONALLY

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During his lecture for the ALGANT week, Henryk Iwaniec observed[1] that besides using the asymptotic $\zeta(s) \sim 1/(s-1)$ as $s \to 1$ and the Euler product

$$\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}$$

to prove that there are infinitely many primes, he could have said that

$$\prod_p (1 - p^{-2})^{-1} = \zeta(2) = \frac{\pi^2}{6} \notin \mathbf{Q},$$

to get the same conclusion.

It is well-known that the asymptotic behavior as $s \to 1$ yields the stronger results

$$\pi(x) \gg x^{1-\varepsilon} \text{ for all } \varepsilon > 0,$$

for $x \geqslant 2$, the implied constant depending on ε, and

$$\sum_{p \leqslant x} \frac{1}{p} \sim \log \log x$$

as $x \to +\infty$.

Does the irrationality observation give a quantitative lower bound for the number of primes $p \leqslant x$? There is indeed a way to do this, but the result is very mediocre... and needs much stronger results as input!

Proposition 1. *We have*

$$\pi(x) \gg \log \log x$$

for $x \geqslant 2$.

Proof. Consider the partial product

$$z(x) = \prod_{p \leqslant x} \frac{1}{1 - 1/p^2} = \prod_{p \leqslant x} \frac{p^2}{p^2 - 1} \in \mathbf{Q}.$$

Clearly

$$|\zeta(2) - z(x)| \leqslant \sum_{n > x} n^{-2} \ll x^{-1}$$

for $x \geqslant 2$. On the other hand, it is known[2] that $\beta = 5.441243\ldots$ is an irrationality measure for $\zeta(2)$, i.e., there exists a constant $c > 0$ such that

$$\left| \zeta(2) - \frac{p}{q} \right| \geqslant \frac{c}{q^\beta}$$

for any coprime integers p, q, $q \geqslant 1$. (The value of β is unimportant).

Writing

$$z(x) = \frac{p(x)}{q(x)}$$

[1] This had never been noticed or mentioned previously to the author, but may of course have been known to others.

[2] G. Rhin and C. Viola, *On a permutation group related to* $\zeta(2)$, Acta Arith. 77 (1996), no. 1, 23–56.

in lowest terms, we claim that

$$\log q(x) \ll \tilde{\pi}(x) \log x \tag{1}$$

where $\tilde{\pi}(x) = \pi(x) - \pi(Cx^{1/5.5})$ for some constant C to be determined. Indeed, consider a prime $\ell \leqslant Cx^{1/5.5}$. According to Heath-Brown's version of Linnik's Theorem on the least prime in an arithmetic progression[3], if C is small enough, there exists primes $p \leqslant C^{5.5}x \leqslant x$ such that $p \equiv 1 \pmod{\ell}$, and $q \leqslant C^{5.5}x \leqslant x$ such that $q \equiv -1 \pmod{\ell}$. Then $\ell^2 \mid (p^2-1)(q^2-1)$, so that the factor ℓ^2 in the "apparent" numerator of $z(x)$ is cancelled out in the denominator. So the denominator is smaller than the apparent one by the product of all those primes. Hence, as claimed, we have

$$\log q(x) \leqslant \sum_{Cx^{1/5.5} \leqslant p \leqslant x} \log(p^2-1) \leqslant 2\tilde{\pi}(x) \log x.$$

So we have, for some constant C'

$$\frac{c}{e^{\beta C' \tilde{\pi}(x) \log x}} \leqslant \frac{c}{q(x)^\beta} \leqslant |\zeta(2) - z(x)| \ll \frac{1}{x}$$

which gives

$$\tilde{\pi}(x) \gg 1.$$

In other words, between $Cx^{1/5.5}$ and x, if x is large enough, there is at least one prime. It is clear that this implies the stated bound. □

Obviously this "proof" is quite ridiculous (considering the difficulty of Linnik's Theorem well beyond the Prime Number Theorem). What is somewhat interesting as a problem is the following: given an integer $k \geqslant 2$, what is the asymptotic behavior of the denominator of the rational number

$$\prod_{p \leqslant x} \frac{1}{1-p^{-k}} = \prod_{p \leqslant x} \frac{p^k}{p^k - 1}.$$

Let us write $q_k(x)$ this denominator. Slightly more precisely than what was stated before, we have:

Proposition 2. *We have for k fixed and $x \geqslant 2$*

$$(1 + o(1))\pi(x) \log x \leqslant \log q_2(x) \leqslant 2\pi(x) \log x$$

as $x \to +\infty$.

Proof. The upper bound is obvious. For a lower bound, notice that if ℓ is an odd prime such that $2\ell - 1 > x$, it is not possible that the factor ℓ^2 in the numerator cancels out at all, since any prime p for which $p^2 - 1 \equiv 0 \pmod{\ell}$ must be $\geqslant 2\ell - 1 > x$.

Hence the numerator $d_2(x)$ satisfies

$$\log d_2(x) \geqslant 2 \sum_{(x+1)/2 < \ell \leqslant x} \log \ell \geqslant 2(\log x)\Big(\pi(x) - \pi\Big(\frac{x+1}{2}\Big)\Big) \geqslant (1 + o(1))\pi(x) \log(x)$$

by the Prime Number Theorem. Since $d_2(x)/q_2(x) \to \zeta(2)$, we have

$$\log d_2(x) \sim \log q_2(x)$$

as $x \to +\infty$, and we get the result. □

[3] Linnik's version suffices... or would, if it did not use much more precise information on primes than what we are "proving"!