ON THE COMPLEXITY OF DUNFIELD-THURSTON RANDOM 3-MANIFOLDS

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1. Introduction

This short note is a follow-up to (the second part of) Section 7.6 of my book [3]. There, earlier results of N. Dunfield and W. Thurston [1] on a certain type of random 3-manifolds (compact, connected) were improved and were made quantitative, using large-sieve inequalities for random walks on discrete groups. Roughly speaking, these results expressed the fact that "typical" 3-manifolds could be expected to have finite, but large, first homology with integer coefficients $H_1(M, \mathbf{Z})$. Here, we will first obtain a stronger form of the basic result of [3], by a simple refinement of the underlying arithmetic argument. Then, we will obtain a more satisfactory understanding of the nature of the result by relating the asymptotic parameter defining Dunfield-Thurston manifolds (the length of the underlying random walk) with more classical and intrinsic invariants of the manifolds themselves. This is of interest because otherwise it is by no means clear how, exactly, the results can be interpreted as probable (heuristic) properties of a fixed 3-manifold.

We first recall the definition of the Dunfield-Thurston manifolds, and the result obtained in [3, Prop. 7.19]. Their construction is based on a classical topological description of compact 3-manifolds, due to Heegaard. First, fix an integer $g \geqslant 2$. Let Γ_g denote the mapping class group of a closed surface Σ_g of genus g, and let S be a fixed finite set of generators of Γ_g, such that $S = S^{-1}$ (i.e., a symmetric generating set). Then consider a random walk (X_k) on Γ_g defined by

$$X_0 = 1, \qquad X_{k+1} = X_k \xi_{k+1} \text{ for } k \geqslant 0,$$

where (ξ_k) is a sequence of independent S-valued random variables with uniform distribution

$$\boldsymbol{P}(\xi_k = s) = \frac{1}{|S|}, \qquad \text{for all } s \in S$$

(other distributions for S are allowable, but we use the simplest). Now the associated random manifolds M_k (which depend on the underlying variable ω in the probability space on which the random walk is defined) are obtained from two copies of a handlebody H_g of genus g with boundary $\partial H_g = \Sigma_g$ by identifying their common boundary Σ_g using the mapping class $X_k \in \Gamma_g$.

We proved the following (which, in qualitative form, was proved in [1, Th. 8.4, Cor. 8.5]):

Proposition 1. *With notation as before, we have*

$$\boldsymbol{P}\Big(\textit{The order of } H_1(M_k, \mathbf{Z})_{tors} \textit{ is } < k^{\alpha \log\log k}\Big) \ll \frac{1}{\log k},$$

and for some constant $\alpha > 0$, we have

$$\boldsymbol{E}\Big(\textit{Order of } H_1(M_k, \mathbf{Z})_{tors}\Big) \gg k^{\alpha \log\log k}$$

where $H_1(M_k, \mathbf{Z})_{tors}$ is the torsion subgroup of $H_1(M_k, \mathbf{Z})$.

This is strongly improved:

Proposition 2. *Let ψ be any positive increasing function defined for integers $k \geqslant 1$ such that $\lim \psi(k) = +\infty$ as $k \to +\infty$. Then we have*

$$\lim_{k\to+\infty} \boldsymbol{P}\Big(\text{The order of } H_1(M_k, \mathbf{Z})_{tors} \text{ is } < \exp\Big(\frac{k}{\psi(k)}\Big)\Big) = 0. \tag{1}$$

Moreover, there exists a constant $\alpha > 0$ such that we have

$$\boldsymbol{E}\Big(\text{Order of } H_1(M_k, \mathbf{Z})_{tors}\Big) \gg \exp(\alpha k) \tag{2}$$

for k large enough.

This result is still expressed in terms of the length of the random walk. In the next result, we can replace this by a topological invariant of M_k instead, which gives a more satisfactory interpretation.

Corollary 3. *Let ψ be any positive increasing function defined for integers $k \geqslant 1$ such that $\lim \psi(k) = +\infty$ as $k \to +\infty$. Then we have*

$$\lim_{k\to+\infty} \boldsymbol{P}\Big(\text{The complexity } c(M_k) \text{ is } < \frac{k}{\psi(k)}\Big) = 0,$$

and also

$$\boldsymbol{E}(c(M_k)) \gg k$$

for $k \geqslant 1$.

The complexity $c(M)$ of a 3-manifold which is used here is defined by Matveev (see [5, §2]); it is a non-negative integer which has the property that (up to issues of irreducibility) there are only finitely many closed 3-manifolds M with a given value $c(M) = c$ (see Sequence `A12885` in the Online Encyclopedia of Integer Sequences). In many cases, $c(M)$ is simply the minimal number of tetrahedra in a triangulation of M (see [5, Rem. 2.1.7]).

Remark 4. In [1, Conjecture 2.11], Dunfield and Thurston conjecture that M_k should be hyperbolic with large probability as $k \to +\infty$ and that "the expected volume of M_k grows linearly in k". This suggests another geometric interpretation of the parameter k as related to the volume. Indeed J. Maher [4] (using the geometrization conjecture of Thurston, recently proved using Perelman's methods) has proved the first part: M_k is hyperbolic (i.e., can be given the structure of a compact Riemannian manifold with constant negative curvature -1) with probability tending to 1 as $k \to +\infty$. Moreover, Maher states that work in progress of Souto and Brock should prove the second part, but it is not stated if this would yield an asymptotic comparison (with probability going to 1) of k and $\mathrm{Vol}(M_k)$, or a weaker estimate like

$$c_1^k \ll \boldsymbol{E}(\mathrm{Vol}(M_k)) \ll c_2^k$$

for some constants $c_1 > 1$, $c_2 > 1$.

If the former is correct, then Corollary 3 would also hold with complexity replaced by volume.

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2. Proofs

Proof of Proposition 2. The basic tool is the following inequality, which is a consequence of the large sieve for random walks on $Sp(2g, \mathbf{Z})$, and which is proved in [3, p. 142, line -1]: for any real numbers $1 < M < L$, and any choice of a finite set $\mathcal{L}$ of primes such that

$$M < \ell \leqslant 2L$$

for $\ell \in \mathcal{L}$, we have

$$\boldsymbol{E}\Big(\Big(\sum_{\ell\in\mathcal{L}} \delta_\ell(M_k) - V(\mathcal{L})\Big)^2\Big) \leqslant (1 + CL^A \exp(-ck))V(\mathcal{L})$$

where $\delta_\ell(M_k)$ is 1 if $H_1(M_k, \mathbf{Z}/\ell\mathbf{Z}) \neq 0$ and 0 otherwise, c, C and A are constants depending only on g and S, and

$$V(\mathcal{L}) = \sum_{\ell\in\mathcal{L}} \Big(1 - \prod_{1\leqslant j\leqslant g} \frac{1}{1+\ell^{-j}}\Big) = \sum_{\ell\in\mathcal{L}} \frac{1}{\ell} + O(1)$$

(see [3, p. 142, eq. (7.24)] for the last step).

Moreover, we also proved in [3, Prop. 7.19 (1)] that if B_k is the event $\{H_1(M_k, \mathbf{Q}) \neq 0\}$, we have

$$\boldsymbol{P}(B_k) \ll \exp(-\delta k) \tag{3}$$

for some $\delta > 0$ and $k \geqslant 2$.

From this, we deduce first

$$\boldsymbol{P}(H_1(M_k, \mathbf{Z}/\ell\mathbf{Z}) = 0 \text{ for at least } \tfrac{1}{2}V(\mathcal{L}) \text{ primes}) \leqslant 4(1 + CL^A \exp(-ck))V(\mathcal{L})^{-1},$$

by positivity, and then using the dyadic localization of the primes involved,[1] we get

$$\boldsymbol{P}(|H_1(M_k, \mathbf{Z})_{tors}| \geqslant M^{\frac{1}{2}V(\mathcal{L})}) \geqslant 1 - 4(1 + CL^A \exp(-ck))V(\mathcal{L})^{-1} - \boldsymbol{P}(B_k).$$

Let now $f(k)$ be a positive increasing function defined for $k \geqslant 1$; we take L and M as follows:

$$L = C^{-1/A} \exp(ck/A), \qquad \log M = \frac{\log L}{f(k)},$$

assuming that $M > 2$, which is certainly the case for all k large enough (depending on the choice of f, and on c, C and A). Note that (again for k large enough) we have

$$\log L \geqslant \frac{ck}{2A}.$$

Then, we select $\mathcal{L}$ to be all primes between M and L. Then we have

$$V(\mathcal{L}) = \sum_{M<\ell\leqslant L} \frac{1}{\ell} + O(1) = \log\log L - \log\log M + O(1) = \log f(k) + O(1) \geqslant \tfrac{1}{2}\log f(k)$$

for k sufficiently large (again in terms of (g, c, C, A)).

Thus, for these k, the basic inequality translates to

$$\boldsymbol{P}(|H_1(M_k, \mathbf{Z})_{tors}| \geqslant M^{\frac{1}{2}V(\mathcal{L})}) \geqslant 1 - \frac{8}{\log f(k)} - \boldsymbol{P}(B_k) \to 1,$$

by (3). Now, we also have

$$M^{\frac{1}{2}V(\mathcal{L})} \geqslant \exp(\tfrac{1}{4}\log f(k)\log M) = \exp\Big(\log L \times \frac{\log f(k)}{4f(k)}\Big) \geqslant \exp\Big(\frac{k}{\phi(k)}\Big),$$

with

$$\phi(k) = \frac{8A}{c}\frac{f(k)}{\log f(k)}.$$

Now for an arbitrary positive increasing function $\psi(k)$, we can select $f(k)$ so that $\psi(k) \geqslant \phi(k)$ for k large enough; thus (1) follows.

The proof of (2) is very similar; the only trick is to use positivity to write

$$\boldsymbol{E}(|H_1(M_k, \mathbf{Z})_{tors}|) \geqslant M\boldsymbol{P}(|H_1(M_k, \mathbf{Z})_{tors}| \geqslant M)$$

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[1] We are being wasteful here since rarefaction of primes implies a better lower bound for the product of the first $\frac{1}{2}V(\mathcal{L})$ primes larger than M, but this is not important here since $\psi(k)$ is arbitrary.

(for $M \geqslant 0$) and to check that one can take M of size $\exp(\alpha k)$ for some $\alpha > 0$ so that

$$\boldsymbol{P}(|H_1(M_k, \mathbf{Z})_{tors}| \geqslant M) \gg 1$$

for k large enough. To do this, one proceeds with $\mathcal{L}$ chosen as above, but now with M as large as possible so that

$$V(\mathcal{L}) > 0,$$

which ensures that at least one prime $\geqslant M$ divides $|H_1(M_k, \mathbf{Z})_{tors}|$ (of course, one restricts to the event B_k). Clearly, taking

$$L = C^{-1/A} \exp(ck/A), \qquad \log M = \frac{1}{\Delta} \log L$$

with $\Delta \geqslant 3$ is enough, provided k is sufficiently large. Then since

$$V(\mathcal{L}) = \log \Delta + O(1) \geqslant \tfrac{1}{2} \log \Delta$$

for k large enough, we find

$$\boldsymbol{P}(|H_1(M_k, \mathbf{Z})_{tors}| \geqslant M) \geqslant 1 - \boldsymbol{P}(B_k) - \frac{16}{\log \Delta}.$$

For k large enough and Δ sufficiently large, this gives

$$\boldsymbol{P}(|H_1(M_k, \mathbf{Z})_{tors}| \geqslant M) \gg 1, \tag{4}$$

and since

$$M = \exp(\tfrac{1}{\Delta} \log L) \geqslant \exp\Bigl(\frac{ck}{2\Delta A}\Bigr),$$

we get (2). □

Corollary 3 is an immediate consequence of a general bound linking the complexity and the size of the torsion of $H_1(M, \mathbf{Z})$: we have

$$c(M_k) \geqslant 2 \log_5(|H_1(M_k, \mathbf{Z})|_{tors}) + \dim H_1(M_k, \mathbf{Q}) - 1$$

(a result of Matveev and Pervova, see [5, Th. 2.6.2]; $\log_5 x$ is the logarithm in base 5), if the manifold M_k is irreducible (which means that any embedded sphere $\mathbf{S}^2$ bounds a 3-ball in M_k). However, a result of J. Maher (see [4, Th. 1.1] and the remarks after the statement that Heegaard splitting distance greater than 2 implies irreducibility) implies that

$$\lim_{k \to +\infty} \boldsymbol{P}(M_k \text{ is reducible}) = 0,$$

hence we see that Corollary 3 is immediate from Proposition 2 and from (4).

Remark 5. In the database used in [2], containing 10986 distinct hyperbolic 3-manifolds, the maximal size of the torsion subgroup of $H_1(M, \mathbf{Z})$ is 423; the number of prime factors doesn't exceed 5.

References

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