Geometry and probability of exponential sums

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But in applications, this is not usually the case, and we can hope to prove

$$|S| \leq \frac{N}{\theta(N)}$$

where $\theta(N) > 1$ is "large".

First example

For $n \geq 1$, define

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$$\Big|\sum_{1\leq n\leq N} \lambda(n)\Big| \leq C\sqrt{N}(\log N)^2,$$
but this is *equivalent* to the Riemann Hypothesis for the Riemann zeta function.

Digression: why is that so?

Using the Euler product

$$\sum_{n\geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

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for all $N\geq 2$, the right-hand side is holomorphic for $\mathrm{Re}(s)>1/2$. So $\zeta(s)\neq 0$ for $\mathrm{Re}(s)>1/2$.

Next examples

Let p be a prime number. For integers a, b, not divisible by p, define

$$K(a,b;p)=\sum_{1\le x\le p-1} e\Big(\frac{ax+b\bar{x}}{p}\Big),\quad B(a;p)=\sum_{0\le x\le p-1} e\Big(\frac{ax+x^3}{p}\Big),$$

where $e(z)=e^{2i\pi z}$, and $x\bar{x}\equiv 1 \pmod{p}$ if p does not divide x. (E.g., for $p=11$, $\bar{3}=4$).

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These are called *Kloosterman sums* and *Birch sums* respectively. They are classical examples of exponential sums over finite fields.

Question: How large can $|K(a, b; p)|$ or $|B(a; p)|$ be, in terms of p?

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Geometry:

- Many properties of sums like $K(a, b; p)$ and $B(a; p)$ turn out to be best studied using methods from algebraic geometry;

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Probability:

- Heuristic reasoning about these sums is often phrased in probabilistic terms;
- And they satisfy probabilistic limit theorems that justify these heuristics.

History

Kloosterman sums were first written down by Poincaré around 1911 as coefficients in Fourier expansions of Poincaré series. They are discrete analogues of Bessel functions.

Nous allons maintenant grouper ensemble les termes qui correspondent aux diverses valeurs de δ non congrues entre elles suivant le module γ. Si nous appelons $\omega(\gamma)$ la somme de ces termes, le coefficient de q^j dans $\omega(\gamma)$ sera

$$\mu_j \mathrm{J}(m, \mathrm{G}) \sum \mathrm{E}.$$

Il faut donc calculer $\sum \mathrm{E}$, c'est-à-dire

$$\sum e^{\frac{2i\pi}{\gamma}(j\delta - p\alpha)}.$$

Les entiers j, p et γ sont donnés; mais on donne à α toutes les valeurs entières premières avec γ et incongrues entre elles par rapport au module γ, et à δ les valeurs correspondantes, de telle façon que

$$\alpha\delta \equiv 1 \pmod{\gamma}.$$

Je me bornerai à constater que $\sum \mathrm{E}$ n'est pas nul en général. Il reste à sommer

Kloosterman re-defined them in 1925 and used them to establish the solubility of equations

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = n$$

for fixed positive integers $(a_1, \ldots, a_4)$ and $x_i \in \mathbf{Z}$ and suitable $n \geq 1$.

2. 4. The sum $S(u, v; \lambda, \Lambda; q)$.

We shall show afterwards, that the approximation for large values of q of the sum occurring on the right hand side of the formula of lemma 3*, can be reduced to the calculation for large values of q of the sum

$$S(u, v; \lambda, \Lambda; q) = \sum_{p \equiv \lambda \pmod{\Lambda}}' \exp\left(\frac{2\pi i u p}{q} + \frac{2\pi i v p'}{q}\right).$$

But before performing the reduction, we shall first consider this sum S. The object of this section is the proof of lemma 4. The lemmas 4 b—4 e are special cases of lemma 4, from which the general lemma 4 will be deduced.

“Square-root cancellation” philosophy

The standard heuristic for guessing the size of a sum

$$S = \sum_{n \leq N} \alpha_n, \qquad |\alpha_n| \leq 1,$$

is that if the arguments of the complex numbers α_n vary “randomly”, then the sum should have size about $\sqrt{N}$.

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The problem is to show that this heuristic applies to deterministic sums, like Kloosterman sums, or to the Möbius function.

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H. Weyl introduced a general technique for exponential sums that leads to

$$|B(a; p)| \leq C_\varepsilon p^{7/8+\varepsilon}$$

for any $\varepsilon > 0$.

The Weil bounds

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The geometric idea is to relate the Kloosterman sums to the algebraic curve with equation

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where (x, y) belong to an algebraic closure of the finite field $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$. The geometry of algebraic curves is key to the proof. Later, Stepanov found a proof which is elementary; as interpreted by Bombieri, the key point is the Riemann-Roch theorem.

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Theorem (Deligne; Katz)

As $p \to +\infty$, the normalized Kloosterman sums $K(a, b; p)/p^{1/2}$ for $1 \leq a, b \leq p - 1$ become equidistributed with respect to the measure

$$\mu_{ST} = \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}dx$$

on $[-2, 2]$. The same holds for Birch sums $B(a; p)/p^{1/2}$.

What does this mean?

(1) For any continuous function $f : [-2, 2] \longrightarrow \mathbf{C}$, we have

$$\lim_{p\to+\infty} \frac{1}{(p-1)^2} \sum_{1\leq a,b\leq p-1} f\Big(\frac{K(a,b;p)}{\sqrt{p}}\Big) = \int_{-2}^{2} f(x) d\mu_{ST}(x).$$

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(2) Or equivalently: the sequences of random variables

$$(a, b) \mapsto \frac{K(a, b; p)}{\sqrt{p}}$$

on $\{1 \leq a, b \leq p-1\}$ with uniform probability measure converges weakly to μ_{ST}.

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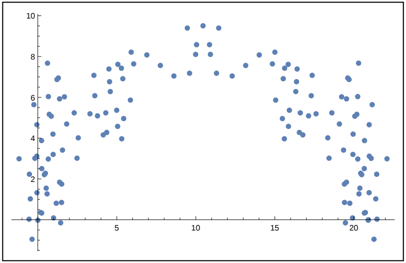
$$\sum_{1 \leq x \leq j} e\Big(\frac{ax + b\bar{x}}{p}\Big)$$

for $0 \leq j \leq p-1$,



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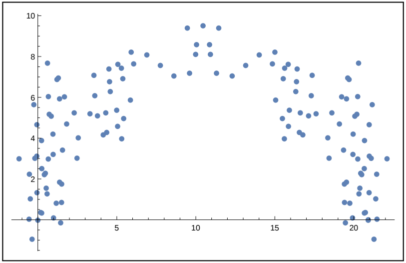

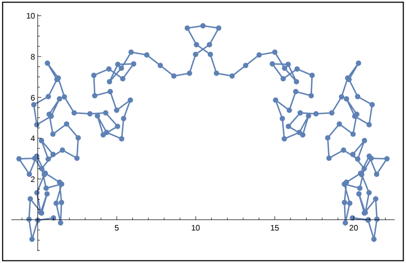

and join these points by line segments, to obtain a polygonal curve in the plane.

History

D.H. Lehmer and J.H. Loxton (1970's–1980's) looked at and studied similar graphs for more regular exponential sums, especially quadratic Gauss sums

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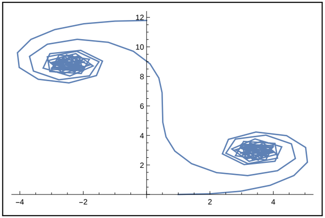
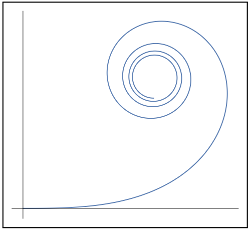
$$\sum_{0 \leq x \leq j} e\left(\frac{x^2}{p}\right).$$

These behave more regularly, staying close to Cornu spirals

$$\int_0^j e^{2i\pi x^2/p} dx$$

up to j about $p/2$.





“Absolutely chaotic”

Loxton mentions in a paper the case of Kloosterman sums:

> The other extreme may be exemplified by the incomplete Kloosterman sum
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Is he right, or wrong?

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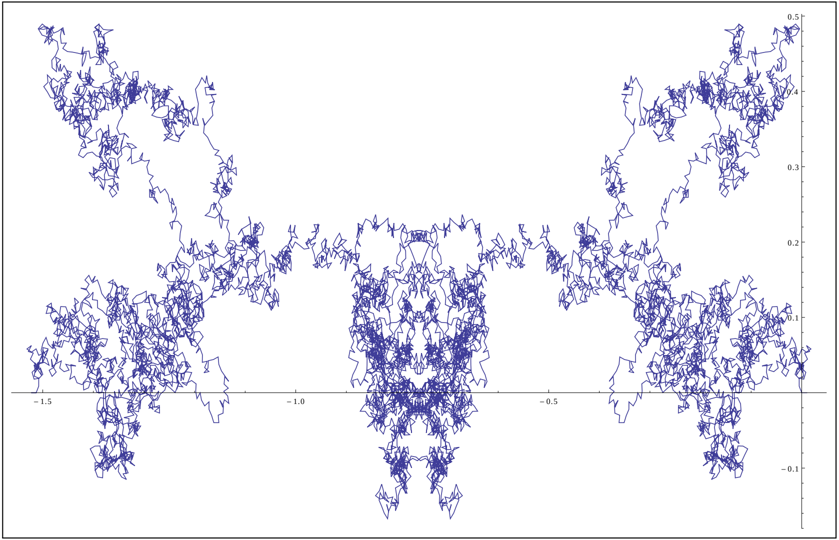

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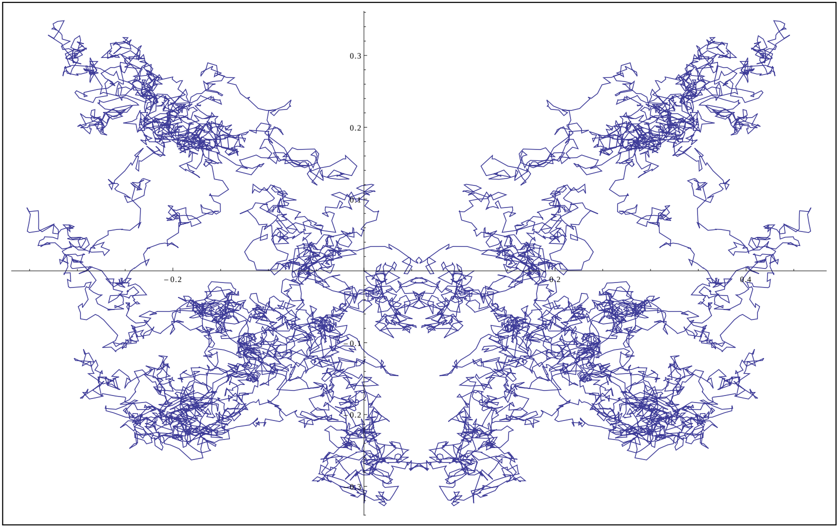

A probabilistic limit theorem

For p prime and $1 \leq a, b \leq p-1$, we define a continuous map

$$\mathcal{K}\ell_p(a,b) \,:\, [0,1] \longrightarrow \mathbf{C}$$

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For each p, we view $(\mathcal{K}\ell_p(\cdot,\cdot)(t))_{t\in[0,1]}$ as a stochastic process, defined on the finite probability space

$$\Omega_p = \{1 \leq a, b \leq p-1\}$$

with uniform probability.
We can also view this as a $C([0,1])$-valued random variable on this space.

Theorem (K.–Sawin, 2014)

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- *This limiting process is the random Fourier series*

$$V(t) = \sum_{h \in \mathbf{Z}} \frac{e^{2i\pi ht} - 1}{2i\pi h} X_h,$$

where (X_h) *is a sequence of independent random variables, identically distributed according to* μ_{ST}.

(Note: the term $h = 0$ should be interpreted as tX_0).

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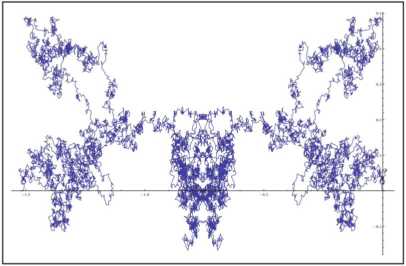
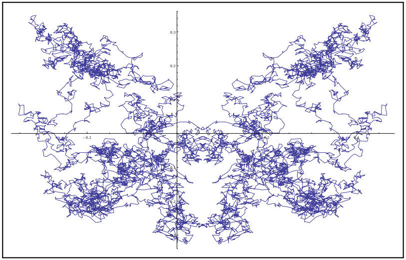
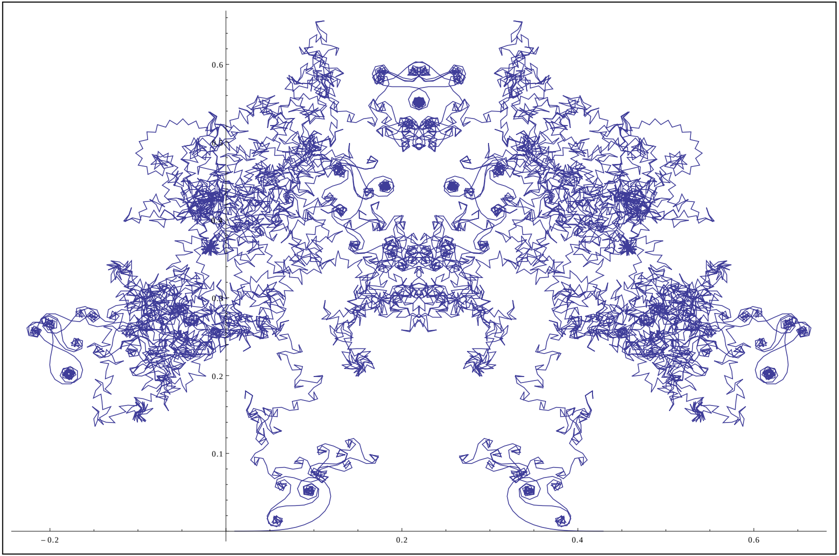
$$V(t) = \sum_{h\in\mathbf{Z}} \frac{e^{2i\pi ht}-1}{2i\pi h} X_h,$$

where (X_h) is a sequence of independent random variables, identically distributed according to μ_{ST}.

(Note: the term $h=0$ should be interpreted as tX_0).

Different look...



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The different appearance between these graphs and those of Kloosterman sums is only at *smaller scales* than those that are retained in the limit.

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  converge in law to $(V(t_1), \ldots, V(t_k))$.
- Step 2: tightness / weak-compactness in $C([0,1])$: by Kolmogorov's criterion, it is enough to prove that

  $$\mathbf{E}(|\mathcal{K}\ell_p(t) - \mathcal{K}\ell_p(s)|^{\alpha}) \leq C|t-s|^{1+\delta}$$

  for $0 \leq s, t \leq 1$ and $C \geq 0$, $\alpha > 0$ and $\delta > 0$ independent of (p, t, s).

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- Then unwind...

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- If $0 \leq \gamma \leq 1/2 - \varepsilon_1$: use equidistribution as for Step 1;
- If γ is close to $1/2$: take $\alpha = 4$, and apply Kloosterman's method!

First application

Using some relatively basic probability in Banach spaces, we get a limiting distribution μ for

$$\max_{1\leq j\leq p-1} \frac{1}{\sqrt{p}}\Big|\sum_{1\leq x\leq j} e\Big(\frac{ax+b\bar{x}}{p}\Big)\Big|$$

and doubly-exponential tail bounds

$$c^{-1}\exp(-\exp(ct)) \leq \mu([t,+\infty[) \leq c\exp(-\exp(c^{-1}t)).$$

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- Others?

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- What are further properties of $V(t)$ that would have nice consequences for Kloosterman sums?