BEZOUT CURVES IN THE PLANE

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Update (February 2019): *The main part of this note goes back to* 2000–2001*, but at the end of the introduction is the content of an email from W. Sawin that sketches a proof of the main question from the original text: a Bezout curve is indeed algebraic.*

1. Introduction

Among various categories of curves (defined by different regularity conditions, or rings of functions), algebraic curves (over algebraically closed fields) are rather strongly distinguished by their intersection-theoretic properties. The basic result in that direction is Bezout's Theorem, a version of which reads (see e.g [H,] for a proof, in the more general case of hypersurfaces):

Theorem 1. *Let C and D be projective algebraic curves of degree c and d (respectively) over an algebraically closed field k, with no common irreducible component. Then the intersection $C \cap D$ is finite and has cd points "counted with multiplicity": more precisely, there exists an intersection multiplicity $i(C, D; x) \geqslant 1$ defined for every $x \in C \cap D$, and*

$$\sum_x i(C, D; x) = cd.$$

In addition, the intersection $C \cap D$ is "generically transverse", meaning that all the multiplicities are $= 1$, and Bezout's Theorem proves that in this case the intersection $C \cap D$ is a finite set containing cd points.

In this paper, we investigate whether a converse to Bezout's theorem holds: that is, is it true that algebraic curves, as sets in the plane, are *characterized* by their intersection with other algebraic curves satisfying Bezout's Theorem? We will show that this is true, at least in some cases.

All results of algebraic geometry used in this paper can be found in the first two chapters of Hartshorne's book [H]. Some knowledge of number theory (automorphic forms, of all things!) is useful to understand the original motivation of this study, the "application" in Section 3, but that Section is independent of the rest of the paper.

First, recall how Bezout's Theorem can be made more precise by specifying the meaning of the word "generically" used above, using elementary scheme-theoretic language ([H, II]).

Fix an algebraically closed field k. There is little loss in assuming $k = \mathbf{C}$ (until Section 3). Since we work over an algebraically closed field, we often identify reduced algebraic sets with their k-valued points.

In this paper, a *plane curve* over k is a purely 1-dimensional closed subscheme of $\mathbf{P}^2$ (not necessarily reduced or irreducible, but projective by definition). Such a scheme is always of the form $\mathrm{Proj}(k[X, Y, Z]/(f))$ for some homogeneous polynomial $f \in k[X, Y, Z]$, uniquely determined up to scalars ([H, I.1.13]), hence curves of a given degree $d \geqslant 1$ are naturally parameterized by an irreducible algebraic variety $\mathcal{H}_d$ over k, namely the projective space $\mathbf{P}_k^{N-1}$ associated with the $N = (d + 1)(d + 2)/2$ coefficients of a homogeneous polynomial $P \in k[X, Y, Z]$ of degree d (which gives the equation for the curve).

We also consider subsets $S \subset \mathbf{P}^2(k)$ of the set of closed points of $\mathbf{P}^2$, and say that such an S *is* a curve when $S = C(k)$ for some plane curve C, and C is unambiguously defined if we ask that it be reduced. Similarly the Zariski closure of a subset $S \subset \mathbf{P}^2(k)$ is the smallest reduced closed subscheme $\overline{S}$ with $S \subset \overline{S}(k)$.

Now a more precise form of Bezout's Theorem is: let C/k be a reduced curve of degree c. Then for any $d \geqslant 1$, there exists a Zariski open subset $\mathcal{U}_d \subset \mathcal{H}_d$ such that for all curves $D \in \mathcal{U}_d$, and all $x \in C \cap D$, $i(C, D; x) = 1$. Hence, for D outside a closed subset of $\mathcal{H}_d$, the intersection $C \cap D$ is a finite set with cd elements.

We now make the following definition.

Definition. Let k be an algebraically closed field. A *Bezout curve* of degree $n \geqslant 1$ in $\mathbf{P}^2$ (over k) is a subset $\mathcal{C} \subset \mathbf{P}^2(k)$ satisfying the following property: for any integer $d \geqslant 1$, there exists a Zariski open subset $\mathcal{U}_d$ of the space $\mathcal{H}_d$ of curves of degree d in $\mathbf{P}^2$ such that for any $C \in \mathcal{U}_d$, the intersection $\mathcal{C} \cap C = \mathcal{C} \cap C(k)$ is finite and contains nd (distinct) points.

Bezout's Theorem as we described it now says that any reduced plane curve C of degree c in $\mathbf{P}^2$ *is* a Bezout curve of degree c.

Remark. The exceptional set $\mathcal{H}_d \setminus \mathcal{U}_d$ is not necessarily unique if we want it to be indeed closed: this might not be true for the set of all curves not satisfying the intersection property. The definition only says what it says.

Because adding a finite set of points doesn't alter the property of being a Bezout curve, we introduce some more terminology: let $C \subset \mathbf{P}^2$ be a (closed) algebraic set in the plane. The *dimension* d of C is the maximal dimension of an irreducible component of C, and the *essential part* of C is the union of its irreducible components of dimension d. Similarly, the *essential closure* of a set $\mathcal{C} \subset \mathbf{P}^2$ is the essential part C_0 of its Zariski closure C, and its *essential part* is $\mathcal{C}_0 = \mathcal{C} \cap C_0$. So for instance, if $\mathcal{C}$ has Zariski closure of dimension 1, and $\mathcal{C}' = \mathcal{C} \cup F$, where F is a finite set of points, the essential closure of $\mathcal{C}$ and $\mathcal{C}'$ coincide.

Definition. Let $\mathcal{C}$ be a Bezout curve in the plane. Then $\mathcal{C}$ is said to be *algebraic* if and only if $\mathcal{C} = U(k) \cup F$, where U is a plane algebraic curve and F is a finite set of points. In other words, the essential closure C of $\mathcal{C}$ is of dimension 1 and $\mathcal{C}$ is Zariski-open in C.

Question. Under what conditions is it true that a Bezout curve is algebraic? When this is the case, is it possible to restrict the definition to testing the intersection condition for curves of degree $\leqslant N$ for some N, or even of a fixed degree (for example, of degree 1, so that only the intersection with lines is considered)?

We will prove a result in that direction in the next section, together with a modest corollary.

Proposition 2. *Let $\mathcal{C}$ be a Bezout curve of degree $c \geqslant 1$ in $\mathbf{P}^2$.*

(1) *The essential closure of $\mathcal{C}$ is a curve in $\mathbf{P}^2$.*

(2) *If either $c = 1$ or the essential closure of $\mathcal{C}$ is irreducible, then $\mathcal{C}$ is algebraic, i.e. $\mathcal{C} = U(k) \cup F$ where U is a Zariski open subset of an irreducible plane curve C of degree c, and F is a finite set of points.*

Remark. Although quite different in emphasis, one can compare our results with the subject of "n-point sets", i.e. subsets X of the real plane having the property that for *any* line L, the intersection $L \cap X$ contains n (distinct) points. Mazurkiewicz proved that "n-point sets" exist for $n \geqslant 2$, using the axiom of choice; it is not known if this can be avoided (see e.g. [M]). Other (maybe) unlikely places where Bezout's Theorem appear are in some of Gromov's work [G, 3.1/2.29] and in the study of Zariski Geometries [HZ].[1]

Section 3 explains a restatement (over finite fields), which was the original motivation of the question above.

Acknowledgements. I wish to thank E. Nelson for finding the example at the end of Section 2, and P. Michel, J. Ellenberg and E. Ullmo for discussions and/or encouragements.

We now include the email from W. Sawin, February 2019, lightly edited:

[1]The author doesn't claim a deep knowledge of all those subjects.

"Let $\mathcal{C}$ be a Bezout curve of degree c, contained inside a union C of irreducible curves C_i of degrees d_i, and let $d = \sum_i d_i$.

Using the Riemann–Roch Theorem, we can find natural numbers e_i associated to the components C_i of the curve such that any line bundle on C of degree e_i on C_i has a section that is everywhere nonzero. (Without the intersections between the different curves, and singularities of the individual curves, we could just take $e_i = \mathrm{genus}(C_i)$, but the intersections may require us to increase these e_i, but only a bounded amount).

We then take $nd_i - e_i$ general points in $\mathcal{C} \cap C_i$, for each i, and consider the line bundle on C of sections of $\mathcal{O}(n)$ vanishing at all these points, which by construction has degree e_i on C_i. We take a general section of this line bundle and obtain a general section σ of $\mathcal{O}(n)$ that passes through $dn - \sum_i e_i$ points of $\mathcal{C}$. (The section is general because the points are independent general among all points lying on the curve, and a general section vanishes at independent general points). We then extend σ to a general section of $\mathcal{O}(n)$ on $\mathbf{P}^2$. Because this passes through at least $dn - \sum_i e_i$ points, we must have $cn \geqslant dn - \sum_i e_i$, so $c \geqslant d$, and hence $c = d$, and then we are done by Lemma 6."

2. Irreducible Bezout curves are algebraic

We will prove Proposition 2. To get an idea of what happens, it is best to read first Lemma 1 below, which treats the case $c = 1$ separately, and shows quite clearly that the "algebraic structure" on $\mathcal{C}$ is "imposed" from the definition which, after all, does involve algebraic sets, the Zariski-open $\mathcal{U}_d \subset \mathcal{H}_d$.

We will need some very basic geometry of the space $\mathcal{H}_d$ of curves of a given degree d. Recall that $\mathcal{H}_d \simeq \mathbf{P}^{N-1}$, where

$$N = N(d) = \frac{(d+1)(d+2)}{2}. \tag{1}$$

For any point $x \in \mathbf{P}^2(k)$, we define

$$\mathcal{U}_x = \{C \in \mathcal{H}_d \mid x \notin C(k)\};$$

clearly $\mathcal{U}_x$ is an open set in $\mathcal{H}_d$. Indeed, the complement $\mathcal{Z}_x = \mathcal{H}_d \backslash \mathcal{U}_x$ is a hyperplane in $\mathbf{P}^{N-1}$.

Let $\underline{x} = (x_i)$ be a n-tuple of points in $\mathbf{P}^2(k)$ for some $n \geqslant 1$; it will be convenient to denote $|\underline{x}| = n$, because various "tupleities" will appear simultaneously. We define $\check{\underline{x}}$ to be the set

$$\check{\underline{x}} = \{C \in \mathcal{H}_d \mid x_i \in C(k) \text{ for } 1 \leqslant i \leqslant |\underline{x}|\}$$

(all curves of degree d passing through all points in $\underline{x}$). So $\mathcal{Z}_x = \check{x}$.

For $|\underline{x}| \leqslant N$, by linear algebra, it follows that $\check{\underline{x}}$ is a linear subspace of $\mathcal{H}_d$ of dimension

$$\dim \check{\underline{x}} \geqslant N - 1 - |\underline{x}|, \tag{2}$$

and "generically" there is equality. In particular, for $|\underline{x}| = N - 1$, this says that there exists a curve of degree d passing through $N - 1$ given points in $\mathbf{P}^2(k)$ (and if they are "in general position", only one).

Moreover, being a linear subspace of the projective space $\mathcal{H}_d$, $\check{\underline{x}}$ is irreducible.

(Note that we omit the degree d from the notation $\check{\underline{x}}$; if will always be clear from the context with what value of d we are dealing, and different values will not be mixed).

We now start the proof of the proposition by showing that we have enough points to play with.

Lemma 1. *Let $\mathcal{C}$ be a Bezout curve in the plane, $\mathcal{E} \subset \mathcal{H}_1$ an exceptional set for the intersection property with lines. There are only finitely many $x \in \mathcal{C}$ for which $\check{x} \cap \mathcal{E}$ is Zariski-dense in $\check{x}$, where $\check{x} \subset \mathcal{H}_1$ is the set of lines containing x.*

Proof. This is clear, since $\dim \mathcal{E} \leqslant 1$ so that $\check{x} \cap \mathcal{E}$ is either an irreducible component of $\mathcal{E}$ (which can happen only for finitely many x), or a finite set. ☐

Lemma 2. *Let $\mathcal{C}$ be a Bezout curve in the plane. Then $\mathcal{C}$ is infinite. More precisely, the cardinality of $\mathcal{C}$ is the cardinality of k.*

Proof. Again we consider $d = 1$, i.e. the intersection of $\mathcal{C}$ with lines. If $F \subset \mathbf{P}^2(k)$ is a finite set, none of the lines in the open subset

$$\mathcal{U} = \bigcap_{x \in F} \mathcal{U}_x$$

intersects F, so F cannot be a Bezout curve.

To prove the more precise statement, let $x \in \mathcal{C}$ be any point such that $\check{x} \cap \mathcal{E}$ is finite, which exists now by Lemma 1. The non-exceptional lines in $\check{x}$ are parameterized by an open subset U of $\mathbf{P}^1(k)$, and choosing one intersection point $x_L \in \mathcal{C} \cap L$ for $L \in U(k)$, we obtain an injection $U(k) \to \mathcal{C}$, so the cardinality of $\mathcal{C}$ is $\geqslant$ that of k. This is also the cardinality of $\mathbf{P}^2(k)$, so there must be equality. □

Lemma 3. *Let $\mathcal{C}$ and $\mathcal{D}$ be two subsets of $\mathbf{P}^2(k)$. If $\mathcal{C}$ and $\mathcal{D}$ differ by a finite set, then $\mathcal{C}$ is a Bezout curve of degree c if and only if $\mathcal{D}$ is a Bezout curve of degree c.*

Proof. This is immediate. □

The next lemma is the crucial step: it will imply that a Bezout curve is not Zariski dense in the plane. The idea is that unless $\mathcal{C}$ is so special that it lies inside some curve, we can find many points in general position on $\mathcal{C}$, and control somewhat the exceptional curves passing through these points. Actually, the lemma has little to do with Bezout curves (the property defining $\mathcal{E}_d$ is hardly used).

Lemma 4. *Let $\mathcal{C}$ be a Bezout curve of degree c in the plane. Let $d \geqslant 1$ be an integer, and $\mathcal{E} \subset \mathcal{H}_d$ the exceptional set for the intersection property with curves of degree d. Let $N = N(d)$. Then one of the following a. or b. is true:*
a. We have $\mathcal{C} \subset C(k) \cup F$, where C is a plane curve, all irreducible components of which are of degree $\leqslant d$, and F is finite;
b. There exists an $(N-2)$-tuple $\underline{x} = (x_i)$ such that:
  1. For all i, $x_i \in \mathcal{C}$.
  2. The intersection $\check{\underline{x}} \cap \mathcal{E}$ is finite (possibly empty).

Proof. Here we work, of course, with the intersection of $\mathcal{C}$ with curves of degree d only. We assume that Property a. is not true, and prove b. To do this, we will prove by induction on $n \leqslant N-2$ the following property $\mathcal{P}(n)$:

$\mathcal{P}(n)$: *There exists $\underline{x}$ with $|\underline{x}| = n$, such that*
  (i) *For all $i \leqslant n$, $x_i \in \mathcal{C}$.*
  (ii) *We have* $\dim \check{\underline{x}} = N - 1 - |x|$, (i.e. equality in (2)).
  (iii) *We have* $\dim(\check{\underline{x}} \cap \mathcal{E}) \leqslant N - 1 - (|\underline{x}| + 1)$.

Observe that for $n = N-2$, (iii) of Property $\mathcal{P}(N-2)$ is the conclusion b. that we want to obtain.

To prove $\mathcal{P}(1)$, let x be any point of $\mathcal{C}$. Then we always have $\dim \check{x} = N - 2$, whereas $\dim \mathcal{E} \leqslant N - 2$ by assumption. Since $\check{x}$ is irreducible, we have

$$\dim(\check{x} \cap \mathcal{E}) \leqslant N - 3 \tag{3}$$

unless $\check{x}$ is an irreducible component of $\mathcal{E}$. This can only happen for finitely many $x \in \mathbf{P}^2(k)$ (because $\check{x}$ determines x) while by Lemma 2, we know that $\mathcal{C}$ is infinite. So $x \in \mathcal{C}$ exists such that (3) holds, which is the starting point $\mathcal{P}(1)$ of the induction.

Now assume $\mathcal{P}(n)$ is valid, and consider a n-tuple $\underline{x}$ given by this assumption. For any $x_{n+1} \in \mathcal{C}$, let $\underline{y} = (\underline{x}, x_{n+1})$. We claim that for all but finitely many such $\underline{y}$, there is equality in (2), i.e.

$$\dim \check{\underline{y}} = N - 1 - |\underline{y}|. \tag{4}$$

Indeed, we certainly have $\check{\underline{y}} \subset \check{\underline{x}}$, so by (ii) of $\mathcal{P}(n)$ (and irreducibility), (4) holds unless $\check{\underline{y}} = \check{\underline{x}}$. But if this is true for all but finitely many $\underline{y} = (\underline{x}, x_{n+1})$ with $x_{n+1} \in \mathcal{C}$, it means that

$$\mathcal{C} \setminus F \subset \bigcap_{C \in \underline{x}} C(k) \subset C_0(k)$$

where F is the finite set of x_{n+1} excluded and C_0 is any curve in $\check{\underline{x}}$ that we choose. In particular C_0 is of degree d, hence its irreducible components are of degree $\leqslant d$, so this inclusion contradicts the assumption that $\mathcal{C}$ does not satisfy a.

For any $\underline{y}$ satisfying (4) we have

$$\dim \check{\underline{y}} = N - 1 - |\underline{y}| = N - 1 - (|\underline{x}| + 1).$$

and on the other hand by (iii) of $\mathcal{P}(k)$, we have

$$\dim(\check{\underline{x}} \cap \mathcal{E}) \leqslant N - 1 - (|\underline{x}| + 1) = \dim \check{\underline{y}}.$$

so

$$\dim(\check{\underline{y}} \cap \mathcal{E}) = \dim(\check{\underline{y}} \cap (\check{\underline{x}} \cap \mathcal{E})) < \dim \check{\underline{y}} \tag{5}$$

unless $\check{\underline{y}}$ is an irreducible component of $\check{\underline{x}} \cap \mathcal{E}$. If there is one $\underline{y}$ for which this is not true, it follows that $\mathcal{P}(n+1)$ holds, since this proves (iii).

So assume we cannot find such a $\underline{y}$. Let $\underline{y}_1, \ldots, \underline{y}_m$ be such that $\check{\underline{y}}_j$ are all the distinct irreducible components of $\check{\underline{x}} \cap \mathcal{E}$ (of this form), and let

$$C_j = \bigcap_{C \in \check{\underline{y}}_j} C.$$

Every $\underline{y} = (\underline{x}, x_{n+1})$ with x_{n+1} not in $\underline{x}$ is such that

$$\check{\underline{y}} = \check{\underline{y}}_j \text{ for some } j,$$

so x_{n+1} is in every curve of degree n passing through $\underline{x}$ and y_j, i.e. $x_{n+1} \in C_j$. Hence we find that

$$\mathcal{C} \setminus \{x_i \mid 1 \leqslant i \leqslant n\} \subset \bigcup_{1 \leqslant j \leqslant m} C_j(k)$$

contradicting again the assumption that a. does not hold since the C_j and their irreducible components must also be of degree $\leqslant d$. ☐

The next corollary gives part (1) of Proposition 2.

Corollary 3. *Let $\mathcal{C}$ be a Bezout curve of degree $c \geqslant 2$ in the plane, and let C be the essential closure of $\mathcal{C}$. Then C is a curve, all irreducible components of which are of degree $< 2c$.*

Proof. Let $d = 2c - 1$, so $c < d < 2c$. We apply Lemma 4 to $\mathcal{C}$ and this d. If a. of Lemma 4 holds, we have

$$\mathcal{C} \subset C(k) \cup F$$

for some algebraic curve C, with irreducible components of degree $\leqslant d < 2c$, as desired. So if we can exclude Property b., the Lemma will be proved.

Assume therefore that b. holds and let $\underline{x}$ be an $(N-2)$-tuple as given there with $\check{\underline{x}} \cap \mathcal{E}$ finite.

For any $y \in \mathbf{P}^2(k)$, let $\underline{y} = (\underline{x}, y)$. Then $\check{\underline{y}}$ is non-empty, hence there exists a curve C_y of degree d containing the x_i's and y. Since $\mathbf{P}^2$ is not the union of finitely many curves, there must exist infinitely many different C_y. However, because

$$N(d) - 2 = \frac{(d+1)(d+2)}{2} - 2 > cd \tag{6}$$

for $d = 2c - 1$ and $c \geqslant 2$ (for $c = 2$, $N(d) - 2 = 8$), the distinct points x_i, $1 \leqslant i \leqslant N - 2$ which are in every C_y show that *all* curves C_y are exceptional, hence there can only be finitely many of them. This is a contradiction, so b. can not hold. ☐

The case of Bezout lines ($c = 1$) is very simple to settle separately by the same kind of arguments. This is the first statement in (2) of Proposition 2.

Lemma 5. *Let $\mathcal{L}$ be a Bezout line in the plane. Then L is algebraic.*

Proof. Let $\mathcal{E} \subset \mathcal{H}_1$ be an exceptional set for the intersection with lines. We have $\dim \mathcal{E} \leqslant 1$, so there are only finitely many $x \in \mathcal{L}$ with $\check{x} \subset \mathcal{E}$: pick x not among those, so that $\check{x} \cap \mathcal{E}$ is of dimension 0, i.e. finite. As before, for any $y \in \mathcal{L}$, $y \neq x$, the line joining x and y contains two points of $\mathcal{L}$, hence is exceptional, and there can be only finitely many, which shows that $\mathcal{L}$ is included in finitely many lines L_i.

Since $\mathcal{L}$ is infinite, there exists one L_i, say L_1, with $L_1 \cap \mathcal{L}$ infinite. Now for any point x in $\mathcal{L}$ not in L_1, and any $y \in L_1 \cap \mathcal{L}$, the infinitely many lines joining x and y are exceptional: this implies that there are only finitely many x in $\mathcal{L}$ outside L_1. Similarly, if x is in L_1 and not in $\mathcal{L}$, $\check{x} \cap \mathcal{E}$ is obviously infinite (it contains all lines joining x and not passing through the finitely many points of $\mathcal{L}$ outside L_1), hence again there are only finitely many such x.

Thus $\mathcal{L}$ is an open subset of L_1 union finitely many points, as was to be proved. □

Now we finish the proof of Proposition 2. Let $\mathcal{C}_0$ be the essential part of $\mathcal{C}$. Notice that $\mathcal{C} \setminus \mathcal{C}_0$ is finite, as well as $\mathcal{C} \setminus C$ (since $\mathcal{C}$ is not dense, its closure has only components of dimension 0 and 1). What we need to prove can be rephrased as:

Proposition 4. *Let $\mathcal{C}$ be a Bezout curve of degree c in the plane such that its essential closure C is irreducible. Then $\mathcal{C}_0$ is an open subset of C.*

Proof. We proceed by induction on $c \geqslant 1$. For $c = 1$, this is Lemma 5. Now assume this holds for all Bezout curves of degree $< c$. By Lemma 3, we may assume $\mathcal{C} = \mathcal{C}_0$.

Let C be the essential closure of $\mathcal{C}$. By Corollary 3, it is an algebraic curve, of degree d say, and since C was assumed to be irreducible, we have $d < 2c$.

We claim that $d = c$. Indeed, by intersecting with a generic line, we see that $c \leqslant d$. Now assume that $d - c > 0$. Because C is an algebraic curve, it is also a Bezout curve of degree d, and it follows immediately that the complement $\mathcal{C}' = C(k) \setminus \mathcal{C}$ is a Bezout curve of degree $d - c > 0$.

Clearly the essential closure of $\mathcal{C}'$ is equal to C (otherwise $\mathcal{C}'$ would be finite), so we can apply the induction hypothesis to $\mathcal{C}'$ which is of degree $d - c < c$. This shows that $\mathcal{C}'$ is open in C, so $\mathcal{C}$ is finite, which is a contradiction.

Then the next lemma shows that $\mathcal{C}$ is open in C. □

Lemma 6. *Let $\mathcal{C} \subset C(k)$ be a Bezout curve of degree c inside a plane curve of degree c. Then $\mathcal{C}$ is open in $C(k)$.*

Proof. For every point $x \in \mathbf{P}^2(k)$, let $\check{x}$ be the set of lines passing through x: it is a line in $\mathcal{H}_1$. Let $\mathcal{E} \subset \mathcal{H}_1$ be an exceptional set for the intersection of $\mathcal{C}$ with lines, and $\mathcal{F} \subset \mathcal{H}_1$ an exceptional set for the intersection of C with lines.

For any $x \in C(k) \setminus \mathcal{C}$ and any line $L \in \check{x}$, we have

$$|C \cap L| > |\mathcal{C} \cap L|. \tag{7}$$

By Lemma 1 applied to $\mathcal{D} = C$ (which *is* a Bezout curve of degree d), there are only finitely many $x \in C(k)$ for which $\check{x} \cap \mathcal{F}$ is Zariski dense in $\check{x}$. If x is not one of those and is not in $\mathcal{C}$, (7) shows that $\check{x} \cap \mathcal{E}$ contains all lines $L \in \check{x}$ which are *non-exceptional* for C, so is Zariski-dense in $\check{x}$. By Lemma 1 for $\mathcal{C}$ now, we see that there are only finitely many such x, and so $C(k) \setminus \mathcal{C}$ is finite, which proves the Proposition. □

We mention one simple corollary of Proposition 2, which manages to bypass the irreducibility assumption. Here we take $k = \mathbf{C}$ and use also the "usual" complex topology on $\mathbf{P}^2(\mathbf{C})$ (referred to as the $\mathbf{C}$-topology to distinguish from the Zariski topology). The question is which compact, connected, 2-manifolds embedded in $\mathbf{P}^2$ are Bezout curves.

Corollary 5. *Let $V \hookrightarrow \mathbf{P}^2(\mathbf{C})$ be a compact connected 2-manifold*[2] *embedded in the complex projective plane. Assume that V is a Bezout curve of degree d. Then V is a smooth algebraic curve in $\mathbf{P}^2(\mathbf{C})$.*

Lemma 7. *Let C_1 and C_2 be two algebraic curves in $\mathbf{P}^2(\mathbf{C})$ with no common irreducible component, $x \in C_1 \cap C_2$ an intersection point. Let $V \subset \mathbf{P}^2(\mathbf{C})$ be a 2-manifold such that $V \subset C_1 \cup C_2$ and $x \in V$. Then there is a neighborhood U of x in $\mathbf{P}^2(\mathbf{C})$ (in the $\mathbf{C}$-topology) such that $U \cap V \subset C_1$ or $U \cap V \subset C_2$.*

Proof. Locally in the $\mathbf{C}$-topology around x (say in the open set U), $C_1 \cup C_2$ is homeomorphic to a finite union of planes (i.e. $\mathbf{R}^2$) in $\mathbf{R}^4$ meeting at x only. Since $C_1 \cap C_2$ is discrete, such a topological space is (locally) disconnected by removing x. Let $V_i = V \cap C_i \cap U$, so that $V \cap U = V_1 \cup V_2$. If $V_i \neq \emptyset$ for $i = 1$, $i = 2$, it follows that V is also (locally) disconnected by removing x; but a topological 2-manifold can not be disconnected in this manner. □

Proof. By Corollary 3, the Zariski closure of V is an algebraic curve, say C. We claim that C is irreducible, so that we can apply Proposition 2 to prove that $V = C$ (since V is compact connected, there can be no missing point). Since V is a topological manifold, it follows that C (hence V) is smooth.

To prove the claim, assume $C = C_1 \cup C_2$ with C_i algebraic curves with no common irreducible component. Since V is connected, $C_1 \cap C_2$ is non-empty, and also $C_1 \cap C_2 \cap V$. For any $x \in C_1 \cap C_2 \cap V$, applying the lemma shows that a neighborhood of x in V (for its manifold topology) is entirely inside C_1 or C_2. This means that from V we can never "cross" from one C_i to another, and clearly contradicts the fact that V is connected and Zariski-dense in both C_1 and C_2. □

Remark. What prevents us from proving that all Bezout curves are algebraic is the following: we know that $\mathcal{C}_0 \subset C_1 \cup \ldots \cup C_k$ where the C_j are the irreducible components of the essential closure of $\mathcal{C}$, and each C_j has degree $< 2c$. It seems natural to consider the intersections $\mathcal{C}_j = \mathcal{C} \cap C_j$. *If* those were Bezout curves, the result would follow, but it seems hard to establish this a priori.

Another approach would be to try to prove directly that the essential closure is of degree c, and then apply Lemma 6.

The following example, due to E. Nelson, shows that one cannot simply hope that an open set of a curve always arises by selecting one intersection point with each line.

Example. Consider the curve $C : y = x^3$ (in affine coordinates) over $\mathbf{C}$. It contains the point $P = (1, 1)$. For each line L in $\mathbf{P}^2(\mathbf{C})$, let $x_L \in C \cap L$ be chosen such that the (euclidean) distance from P to x_L is the greatest among elements of $C \cap L$. Let $\mathcal{C}$ be the set of all points of the form x_L. Then $\mathcal{C} \subset C$ has the property that it intersects every line in one point. However, there is an open neighborhood U of P (in the euclidean metric) such that $\mathcal{C} \cap U = \emptyset$. Indeed, assume $x \in C$ is very close to P. To be "chosen" by a line L, that line must intersect C in points closer to P. By if y is one of those other points, the line $\overline{xy}$ is close to the tangent line to C at P, and the third intersection z point is close to $(-2, -8)$; therefore by definition $x_L = z$.

Note that the set $\mathcal{C}$ does not satisfy the Bezout property for the intersection with lines: although we select one point $x_L \in L \cap C$ for each L, other points in $L \cap C$ are "selected" by intersecting with other lines.

If it turns out that there are non-algebraic Bezout curves, they still have some structure: indeed, by the above, such a $\mathcal{C}$, of degree c, would come automatically with an algebraic curve C (its essential closure) of degree $d > c$, and a complementary Bezout curve $\mathcal{C}' = C(k) \setminus \mathcal{C}$ of degree $d - c$.

Finally, it is easy to formulate a definition of Bezout curves on a (fixed) smooth projective algebraic surface S over k; instead of the degree, it would be determined by an element c of the Picard group of S, with the condition that generically, it meets curves in a given class c' of

[2] I.e. of dimension 2 over $\mathbf{R}$.

the Picard group in (c,c') points, where $(\cdot,\cdot)$ is the intersection pairing on $\mathrm{Pic}(S)$ (cf.[H, V-1]). Then one may ask if/when a Bezout curve on S is necessarily algebraic. The simplest case, apart from $S=\mathbf{P}^2_k$, would be $S=\mathbf{P}^1_k\times\mathbf{P}^1_k$, with $\mathrm{Pic}(S)=\mathbf{Z}^2$ (the bidegree).

3. An application: a "converse theorem"

We now consider curves defined over a finite field. We will show that Proposition 2 has a nice interpretation in this case in terms of a kind of "converse theorem" to see, using functional equations of zeta functions, when a subset of the plane "is" an algebraic curve. This is an example of another general kind of problems that might be of interest.

Recall that if $X/\mathbf{F}_q$ is a scheme of finite type over a finite field with q elements, its zeta function is the formal power series

$$Z(X)=\exp\Bigl(\sum_{n\geqslant 1}\frac{|X(\mathbf{F}_{q^n})|}{n}T^n\Bigr)$$

which admits an Euler product expansion

$$Z(X)=\prod_{x\in|X|}\frac{1}{1-T^{\deg(x)}} \tag{8}$$

where $|X|^\circ$ is the set of closed points of X and $\deg(x)$ is the degree of the residue field extension $[k(x):\mathbf{F}_q]$ at $x\in|X|^\circ$.[3]

Schmidt for curves and Dwork in general have proved that $Z(X)$ is actually a rational function and that if X is irreducible of dimension n it satisfies a functional equation

$$Z\Bigl(X,\frac{1}{q^nT}\Bigr)=\pm q^{ne/2}t^eZ(X,T)$$

for some sign $\pm$ and some integer $e\in\mathbf{Z}$ (see e.g. [H, App. C] for a short survey).

Notice that the definition of $Z(X)$ only depends on the number of points of the finite sets $X(\mathbf{F}_{q^n})$ for $n\geqslant 1$. Now we can ask two questions:

Question. (1) For every $n\geqslant 1$, assume given a finite set $X_n\subset\mathbf{P}^m(\mathbf{F}_{q^n})$. Can we find a way of determining when there exists a subscheme X of $\mathbf{P}^m/\mathbf{F}_q$ such that $X(\mathbf{F}_{q^n})=X_n$?

(2) For every $n\geqslant 1$, assume given an integer $N_n\geqslant 0$. Can we find a way of determining when there exists a subscheme X such that $|X(\mathbf{F}_{q^n})|=N_n$?

In both cases, the data which is given suffices to define a zeta function

$$Z((X_n)_n)=\exp\Bigl(\sum_{n\geqslant 1}\frac{|X_n|}{n}T^n\Bigr). \tag{9}$$

(in the first case, with N_n instead of $|X_n|$ in the second) so necessary conditions arise from the theory described above. In particular, in the first case, we can characterize the existence of an Euler product as in (8). Let G be the Galois group $\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$, topologically generated by the Frobenius automorphism $\sigma: x\mapsto x^q$.

Lemma 8. *Let $\underline{X}=(X_n)$ be a sequence of finite sets $X_n\subset\mathbf{P}^m(\mathbf{F}_{q^n})$, $\mathcal{X}=\bigcup X_n$. Let $|X|^\circ$ be the quotient of $\mathcal{X}$ by the equivalence relation induced by the action of G on $\mathbf{P}^m(\overline{\mathbf{F}}_q)$. Then $Z(\underline{X})$ has an Euler product expansion over $|X|$, i.e.*

$$Z(\underline{X})=\prod_{x\in|X|^\circ}\frac{1}{1-T^{\deg(x)}} \tag{10}$$

if and only if $\mathcal{X}$ is invariant under the action of G on $\mathbf{P}^m(\overline{\mathbf{F}}_q)$ and for $n\geqslant 1$

$$X_n=\mathcal{X}^{\sigma^n}. \tag{11}$$

[3] Because we work with a non-algebraically closed field here, notation has to be more rigorous that in the first section.

Proof. Let $\tilde{X}$ be the union of Galois-orbits of points of $\mathcal{X}$, so that $|X|^\circ = \tilde{X}/G$, and let $\tilde{X}_n = \tilde{X}^{\sigma^n} = \tilde{X} \cap \mathbf{P}^m(\mathbf{F}_{q^n})$ be the (finite) set of $\mathbf{F}_{q^n}$-rational points. Obviously

$$X_n \subset \tilde{X}_n. \tag{12}$$

Moreover, applying the operator $T d\log$ to $Z(\underline{X})$ and the Euler product in (10), we obtain

$$\sum_{n \geqslant 1} |X_n| T^n$$

and

$$\sum_{n \geqslant 1} T^n \Bigl(\sum_{d | n} d |\{x \in |X| \mid \deg(x) = d\}| \Bigr)$$

respectively, so that the Euler product expansion holds if and only if

$$|X_n| = \sum_{d|n} d |\{x \in |X|^\circ \mid \deg(x) = d\}|$$

for all $n \geqslant 1$. The right-hand side is equal to $|\tilde{X}_n|$, since the Galois orbit of a point x in $\mathbf{P}^m(\overline{\mathbf{F}}_q)$ with $\deg(x) = d$ has d points. Thus, because of the inclusion (12), the Euler product is equivalent to $X_n = \tilde{X}_n$ for all n.

If this is true, then by definition of $\tilde{X}$ and $\tilde{X}_n$, the two conditions stated are satisfied. Conversely, if those conditions hold, we have $\mathcal{X} = \tilde{X}$, and taking invariants under σ^n gives $\tilde{X}_n = X_n$. ☐

The condition (11) can be rephrased as: for all n, $d \geqslant 1$, we have $X_n = X_{nd}^{\sigma_n}$. The lemma is probably best seen as saying that if this is true and X_n is Galois-stable, then $Z(\underline{X})$ has an Euler product expansion over the Galois orbits of $\mathcal{X}$.

The questions above are reminiscent of the *converse theorems* in the theory of automorphic forms, due to Hecke and Weil in the classical case and much generalized by Jacquet-Langlands, Jacquet-Piatetski Shapiro-Shalika, Cogdell-Piatetski Shapiro, and others (see e.g. [I, Ch. 7] for a proof of Weil's "classical" converse theorem). Those results characterize automorphic forms, among a larger class of objects, by asking for functional equations for certain L-functions. What takes here the place of the modular form is the algebraic variety X. So it is interesting to ask whether some answers to the questions above can be found using the same main idea, which is to ask for functional equations for twists of the object under study. Also, the second question is similar to the problem of finding those Weil numbers of weight 1 which correspond to curves (see e.g. [TV]).

We will apply the theory of Bezout curves to derive such a result for plane curves. Whether this has more than anecdotal value remains to be seen.

The situation we consider is the special case of Question 1, when $m = 2$. We write C_n instead of X_n and let $\underline{C} = (C_n)_{n \geqslant 1}$, and

$$\mathcal{C} = \bigcup_{n \geqslant 1} C_n \subset \mathbf{P}^2(\overline{\mathbf{F}}_q).$$

We write $Z(\mathcal{C})$ for the zeta function, abusing notation slightly. Our "twists" will be given by intersecting $\mathcal{C}$ with algebraic curves,[4] and asking for functional equations for (almost all) the intersections. Let D be any algebraic curve over $\overline{\mathbf{F}}_q$. It is defined over some minimal finite extension $\mathbf{F}_{q^n}$, and the intersection $\mathcal{C} \cap D$ can be put into the framework above, over $\mathbf{F}_{q^n}$.

The following Proposition is a tautology.

[4]The idea of studying an algebraic variety V by intersecting it with other algebraic varieties can be seen in action in many circumstances. In particular the use of hyperplane sections $V \cap H$ is very fruitful: one may mention Bertini's Theorem and the theory of Lefschetz pencils, which is of great importance for instance in Deligne's proof of the Riemann Hypothesis for varieties over a finite field.

Proposition 6. *Let* $\underline{C} = (C_n)$, $\mathcal{C}$ *be as above. Assume that*

1. $\mathcal{C}$ *is Galois invariant.*

2. There is an integer $c \geqslant 1$ *such that, for every* $d \geqslant 1$, *there exists an open subset* $\mathcal{U}_d \subset \mathcal{H}_d$ *of the space of curves of degree* d *over* $\overline{\mathbf{F}}_q$ *with the property that for any* $D \in \mathcal{U}_d$, *the intersection* $\mathcal{C} \cap D$ *is finite and its zeta function* $Z(\mathcal{C} \cap D)$ *(computed over the minimal defining field* $\mathbf{F}_{q^n}$ *of* D*) satisfies the functional equation*

$$Z\Big(\mathcal{C} \cap D, \frac{1}{T}\Big) = \pm T^{cd} Z(\mathcal{C} \cap D, T). \tag{13}$$

Then, $\mathcal{C}$ *is a Bezout curve of degree* c.

Remark. To justify somehow the interpretation of the intersections as twists, expand formally a zeta function $Z(\underline{X})$ over the semi-group M of effective ($\mathbf{F}_q$-rational) 0-cycles in $\mathbf{P}^m/\mathbf{F}_q$ (i.e. the formal finite $\mathbf{N}$-linear combinations of closed points of $\mathbf{P}^m/\mathbf{F}_q$), to get a formal power series expansion

$$Z(\underline{X}) = \sum_{D \in M} \alpha(D) T^{\deg(D)}$$

where $\alpha(D) = 1$ if D "is" a 0-cycle on X (a divisor if X is a curve) and $\alpha(D) = 0$ otherwise. Similarly for some $\underline{Y}$

$$Z(\underline{Y}) = \sum_{D \in M} \beta(D) T^{\deg(D)}.$$

Then for the intersection (say $\underline{Z}$) we have

$$Z(\underline{Z}) = \sum_{D \in M} \alpha(D)\beta(D) T^{\deg(D)}$$

which resembles, at least in a naïve way, the usual twists in analytic number theory (cf. Rankin-Selberg convolution).

Proof. Let $D/\overline{\mathbf{F}}_q$ be a curve, defined over the minimal field $\mathbf{F}_{q^n}$. The intersection points $\mathcal{C} \cap D$ are necessarily defined over extensions $\mathbf{F}_{q^{n\delta}}$ of $\mathbf{F}_{q^n}$. The first assumption implies that the zeta function $Z(\mathcal{C} \cap D)$ has an Euler product: indeed it suffices to check the assumptions of Lemma 8, but they are true for both factors, hence obviously for the intersection.

For $D \in \mathcal{U}_d$, because $\mathcal{C} \cap D$ is finite, the Euler product means that $Z(\mathcal{C} \cap D)$ is a rational function

$$Z(\mathcal{C} \cap D) = \prod_{x \in |\mathcal{C} \cap D|^\circ} \frac{1}{1 - T^{\deg(x)}}$$

(with a finite product). Then we have

$$\begin{aligned} Z(\mathcal{C} \cap D, 1/T) &= \prod_{x \in |\mathcal{C} \cap D|^\circ} \frac{1}{1 - T^{-\deg(x)}} \\ &= \prod_{x \in |\mathcal{C} \cap D|^\circ} \frac{T^{\deg(x)}}{T^{\deg(x)} - 1} \\ &= (-1)^E T^E Z(\mathcal{C} \cap D, T) \end{aligned}$$

with

$$E = \sum_{x \in |\mathcal{C} \cap D|^\circ} \deg(x).$$

By the Galois invariance (compare the proof of Lemma 8), we have

$$E = |\mathcal{C} \cap D|$$

and comparing with the functional equation (13) which is assumed to hold, it follows that

$$|\mathcal{C} \cap D| = cd.$$

Hence $\mathcal{C}$ is a Bezout curve of degree c in $\mathbf{P}^2(\overline{\mathbf{F}}_q)$. □

Corollary 7. *Let $\underline{C} = (C_n)$ be as in the previous proposition. Assume that the essential closure of $\mathcal{C}$ is irreducible. Then up to finitely many points, $\mathcal{C}$ is the set of closed points of a plane curve $C/\mathbf{F}_q$ of degree c.*

Proof. Applying the previous Proposition and Proposition 2, $\mathcal{C}$ is of the form

$$\mathcal{C} = C(\overline{\mathbf{F}}_q) \cup F$$

for some algebraic curve $C/\overline{\mathbf{F}}_q$ of degree c and some finite set F.

It remains to show that C is actually defined over $\mathbf{F}_q$. But let C^σ be the transform of C by the Frobenius; we must show that $C^\sigma \simeq C$. By the invariance of $\mathcal{C}$, we have

$$C(\overline{\mathbf{F}}_q) \cup F = \mathcal{C} = C^\sigma(\overline{\mathbf{F}}_q) \cup F^\sigma$$

so $C^\sigma(\overline{\mathbf{F}}_q)$ and $C(\overline{\mathbf{F}}_q)$ differ by finitely many points. In particular (for example) they have the same function fields, so they are isomorphic, and C is defined over $\mathbf{F}_q$. $\square$

Remark. Subjectively speaking, Proposition 6 suffers from the assumption that $\mathcal{C} \cap D$ be finite for generic D; it may be true that assuming the Euler product, the rationality of $Z(\mathcal{C} \cap D)$ for D generic, and the functional equations (13) *implies* that the intersection is generically finite, but I have not been able to prove this.

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