On generic triality

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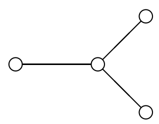
Dedicated with gratitude to Professor K. Chandrasekharan on his 80$^{\text{th}}$ birthday

1 Introduction

It is well known that the study of central simple algebras with involution is closely related to that of classical simple adjoint algebraic groups, the latter occuring as automorphism groups of these algebras (see for example [KMRT]). On the other hand there is a connection between central simple algebras with involution and algebraic groups via invariant theory. For each type of the simple groups A_n, B_n C_n and D_n there exists a generic algebra so that algebras in the corresponding class occur as specializations.The first case ever considered was by Amitsur and Procesi ([A], [P1]), who constructed a generic division algebra corresponding to the inner type of A_n. Generic central simple algebras with orthogonal or symplectic involutions were later introduced by Rowen [R1]. Procesi [P2] showed that for the groups PGL_n, PGO_n and PSp_{2n} the generic algebras occur as invariants of the groups acting on the matrix algebras over the field generated by the entries of the generic matrices. The centers of the generic algebras also occur as invariants of these groups acting on this field. As shown by Procesi [P2] for PGL_n, Berele and Saltman [BS] for PGO_n and PSp_{2n}, and more generally for any reductive group by Saltman [S1], the center can also be described as multiplicative invariant field of the corresponding Weyl group.

Let G be a simple adjoint connected algebraic group of type D_4 over a field F of characteristic different from 2. A model is the group $\mathrm{PGO}^+(q)$ of proper similitudes of a nonsingular quadratic form q of dimension 8, modulo the center. The Dynkin diagram



of G over a separable closure $\widetilde{F}$ of F admits the permutation group of three symbols S_3 as its automorphism group. It follows that S_3 acts as outer automorphisms of $G(\widetilde{F})$ and that $\mathrm{Aut}_{\widetilde{F}}\big(G(\widetilde{F})\big)$ is the semidirect product $G(\widetilde{F}) \rtimes S_3$. Following [KMRT] there is a notion of an associative trialitarian algebra T over F such that $\mathrm{Aut}(T) = G \rtimes S_3$ and $G = \mathrm{Aut}(T)^0$, the connected component of identity of $\mathrm{Aut}(T)$. Very few examples of trialitarian algebras are known. In this paper, we construct generic trialitarian algebras on the lines of Amitsur, Rowen, Procesi and Saltman. We show that these algebras occur as invariants of the group $\mathrm{PGO}_8^+ \rtimes S_3$ acting on certain matrix algebras; the base field also occurs as invariants of the same group acting on the field generated by entries of generic matrices. We further show that the generic trialitarian algebra over a field F specializes to any given trialitarian algebra defined over an extension of F.

The paper is organized as follows. In Section 2 we recall some properties of the Clifford algebra of a nonsingular quadratic form q, which plays an important rôle in our construction, and show in some detail how it is related to the Lie algebra of the orthogonal group $\mathrm{O}(q)$. In Section 3 we describe triality at the level of Lie algebras (local triality), using the Clifford algebra of the norm form of a Cayley algebra, as in [KMRT]. In Section 4 we give an explicit description of the action of S_3 on the Lie algebra of skew-symmetric (8×8)-matrices. Triality at the level of the adjoint algebraic group is the topic of Section 5. Some properties of central simple algebras with orthogonal involutions are recalled in Section 6. Trialitarian algebras are defined in Section 7. A cohomological definition of generic trialitarian algebras follows in Section 8 and it is shown that these algebras are division algebras. In the main Section 9 we give another, more explicit construction of generic trialitarian algebras, much in the spirit of the recent "Lectures on Division Algebras" [S2] of Saltman. The generic trialitarian algebra also occurs as the algebra of invariants of $\mathrm{PGO}_8^+ \rtimes S_3$ acting on generic matrices (Section 10). In Section 11 we show that the generic trialitarian algebra specializes to any trialitarian algebra. Similar constructions for generic algebras with orthogonal involution with trivial discriminant, resp. with involutions of second kind, corresponding to the groups PGO_{2n}^+, resp. PGU_n or $\mathrm{PGL}_n \rtimes \mathbb{Z}/2\mathbb{Z}$ are briefly sketched in the last Section 12.

Fields are assumed to have characteristic different from 2 and we restrict to commutative rings in which 2 is invertible. For any homomorphism of commutative rings $\phi : R \to R'$ and any R-module V, $V \otimes_\phi R'$ denotes the tensor product of V with R' over R. Unadorned tensor products are over F. For any ring A, $A^\times$ denotes the group of units of A.

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2 The Clifford algebra and the Lie algebra of a quadratic space

Let $q : V \to F$ be a quadratic form on V, with associated polar bilinear form $b_q(x,y) = q(x+y) - q(x) - q(y)$. We call the pair (V,q) a *quadratic space* if b_q is nonsingular. The *adjoint involution* σ_q on $\mathrm{End}_F(V)$ is defined through the identity

$$b_q\big(\sigma_q(f)(x), y\big) = b_q\big(x, f(y)\big)$$

for x, $y \in V$ and $f \in \mathrm{End}_F(V)$. The space

$$\begin{aligned} \mathfrak{o}(q) &= \mathrm{Skew}\big(\mathrm{End}_F(V), \sigma_q\big) \\ &= \{f \in \mathrm{End}_F(V) \mid \sigma_q(f) = -f\} \\ &= \{f \in \mathrm{End}_F(V) \mid b_q\big(x, f(y)\big) + b_q\big(f(x), y\big) = 0\} \end{aligned}$$

of skew-symmetric elements of $\mathrm{End}_F(V)$ with respect to the involution σ_q is a Lie subalgebra (of dimension $\frac{n(n-1)}{2}$) of $\mathrm{End}_F(V)$ for the Lie bracket $[f,g] = f\circ g - g\circ f$ of $\mathrm{End}_F(V)$. The Lie algebra $\mathfrak{o}(q)$ is the Lie algebra of the orthogonal group $\mathrm{O}(q)$. If $q = <1, 1, \ldots, 1>$, i.e.,

$q(\sum_i x_i e_i) = \sum_i x_i^2$ with respect to a basis $(e_1, \ldots, e_n)$, and $\mathrm{End}_F(V)$ is identified with $M_n(F)$ through the choice of the same basis, then σ_q is transpose $a \mapsto a^t$ and $\mathfrak{o}(q) = \mathrm{Skew}_n(F)$, the set of $(n \times n)$-skew-symmetric matrices with entries in F.

Let $C(V,q)$ be the Clifford algebra of the quadratic space (V,q). We recall that $C(V,q) = TV/I$ where TV is the tensor algebra of V and I is the ideal of TV generated by the elements $x \otimes x - q(x) \cdot 1$, $x \in V$. The vector space V, identified with a subspace of $C(V,q)$ through the natural map $V \to C(V,q)$, generates $C(V,q)$ as an algebra. The *even Clifford algebra* $C_0(V,q)$ is the subalgebra of $C(V,q)$ generated by products of an even number of elements of V. The properties of the Clifford algebra which we shall need are summarized in the following proposition (for a proof, see for example [SCH]):

Proposition 2.1 *Let (V,q) be a nonsingular quadratic space of even dimension $n = 2l$.*
1) The F-algebra $C(V,q)$ is central simple of dimension 2^{2l} and has a unique involution τ which is the identity on V.
2) The center Z of $C_0(V,q)$ is an étale quadratic extension of F of the form $Z = F(\sqrt{\delta})$ where $\delta = (-1)^l \det(b_q)$. The algebra $C_0(V,q)$ is central separable over Z, of rank 2^{2l-2} as Z-module. Suppose that ℓ is even. Then the involution τ restricts to an involution τ_0 of $C_0(V,q)$ which is the identity on Z. Further τ_0 is of orthogonal type if l is congruent to 0 modulo 4 and is of symplectic type if l is congruent to 2 modulo 4.

The Lie algebra $\mathfrak{o}(q)$ can be identified with a subalgebra of $\mathrm{Skew}(C_0(V,q), \tau_0)$, as follows. For x, y, $z \in V$ and the Lie product $[x,y] = xy - yx$ in $C(V,q)$, we have:

$$[[x,y],z] = 2\big(b_q(y,z)x - b_q(x,z)y\big) \in V. \qquad (1)$$

Let $[V,V] \subset C(V,q)$ be the subspace spanned by the $[x,y] = xy - yx$ for all x, $y \in V$. In view of (1) we have a linear map

$$\mathrm{ad}\colon [V,V] \to \mathrm{End}_F(V), \quad \xi \mapsto \mathrm{ad}_\xi,$$

defined by $\mathrm{ad}_\xi(z) = [\xi, z]$ for $\xi \in [V,V]$ and $z \in V$.

Lemma 2.2 *The subspace $[V,V]$ is a Lie subalgebra of* $\mathrm{Skew}\big(C_0(V,q), \tau_0\big)$*, its image under* ad *is contained in $\mathfrak{o}(q)$ and* ad *induces an isomorphism of Lie algebras:*

$$\mathrm{ad}\colon [V,V] \xrightarrow{\sim} \mathfrak{o}(q).$$

Proof See [KMRT, Lemma (45.3)].

□

Example 2.3 Let $q \xrightarrow{\sim} < 1, 1, \ldots, 1 >$ and let $(e_1, \ldots, e_n)$ be an orthonormal basis of q. Then $[e_i, e_j] = 2e_ie_j$ and $(e_ie_j,\ i < j)$, is a basis of $[V,V]$. Let $(e_{ij}, 1 \le i, j \le n)$, be the standard basis of $M_n(F)$, let $\mathcal{E}_{ij}$ be the skew-symmetric matrix $e_{ij} - e_{ji}$, $i < j$, and let f_k be the column vector with entry 1 in k-th position and zero entries elsewhere. We have

$$\mathrm{ad}_{[e_i,e_j]}(f_k) = 2\,\mathrm{ad}_{e_ie_j}(f_k) = 4(e_i\delta_{jk} - e_j\delta_{ik}) = 4\mathcal{E}_{ij}f_k$$

so that ad^{-1} maps $\mathcal{E}_{ij}$ to the element $\frac{1}{2}e_ie_j$ of $C_0(V,q)$.

We have more in dimension 8:

Lemma 2.4 *Let Z be the center of the even Clifford algebra $C_0(V,q)$. If V has dimension 8, the embedding $[V,V] \subset \operatorname{Skew}\big(C_0(V,q),\tau_0\big)$ induces a canonical isomorphism of Lie algebras over Z, $[V,V] \otimes_F Z \xrightarrow{\sim} \operatorname{Skew}\big(C_0(V,q),\tau_0\big)$. Thus* ad *induces a Z-isomorphism* $\operatorname{Skew}\big(C_0(V,q),\tau_0\big) \xrightarrow{\sim} \mathfrak{o}(q) \otimes_F Z$ *of Lie algebras.*

Proof Since $\dim_F V = 8$, the involution τ_0 of $C_0(V,q)$ is of orthogonal type (see (2.1)) and $\dim_Z \operatorname{Skew}\big(C_0(V,q),\tau_0\big) = 28$. Let $(e_1,\ldots,e_8)$ be an orthogonal basis of V. It is easy to see that the elements e_ie_j, $i<j$, are linearly independent over Z in $C_0(V,q)$. Hence the canonical map $[V,V]\otimes Z \to \operatorname{Skew}\big(C_0(V,q),\tau\big)$ induced by the product in $C_0(V,q)$ is injective. It is surjective by a dimension count.

$\square$

Similitudes of the quadratic space (V,q) are linear automorphisms of V with $q\big(f(x)\big) = m(f)q(x)$, where $m(f) \in F^\times$ is the multiplier of the similitude. They form a group $\mathrm{GO}(q)$. If $\dim_F V = n$ is even, a similitude f is *proper* if $\det(f) = m(f)^{n/2}$. Proper similitudes form a normal subgroup $\mathrm{GO}^+(q)$ of $\mathrm{GO}(q)$ of index 2. Similitudes are isometries if they have multiplier 1.

It readily follows from the definition of Clifford algebras that isometries of (V,q) induce automorphisms of $C(V,q)$. For similitudes we have:

Proposition 2.5 *Any similitude $f \in \mathrm{GO}(q)$ with multiplier $m(f)$ induces an automorphism $C(f)$ of $\big(C_0(V,q),\tau_0\big)$ such that $C(f)(xy) = m(f)^{-1}f(x)f(y)$ for x, $y \in V$. The automorphism $C(f)$ restricts to the identity on the center Z of $C_0(V,q)$ if and only if f is proper. Further, we have* $\operatorname{ad} \circ C(f) \circ \operatorname{ad}^{-1} = \operatorname{Int}(f)$.

Proof The first two claims are standard (see for example [KMRT], (13.1) and (13.2)). Using the identity in (1) we have for x, y, $z \in V$:

$$\big(\operatorname{ad} \circ C(f)\big)\big([x,y]\big)(z) = 2m(f)^{-1}\Big(\big(f(x)b_q(f(y),z\big) - f(y)b_q\big(f(x),z\big)\Big)$$

and

$$\begin{aligned}\big(\operatorname{Int}(f)\circ\operatorname{ad}\big)\big([x,y]\big)(z) &= 2f\big(xb_q(y,f^{-1}(z)) - yb_q(x,f^{-1}(z))\big)\\ &= 2\big(f(x)b_q(y,f^{-1}(z)) - f(y)b_q(x,f^{-1}(z))\big).\end{aligned}$$

The last claim follows from $m(f)^{-1}b_q\big(f(y),z\big) = b_q\big(y,f^{-1}(z)\big)$.

$\square$

For any $\nu \in F^\times$, $\nu\cdot 1_V$ is a similitude with multiplier ν^2, so that $C(\nu\cdot 1_V)$ acts trivially on $C_0(V,q)$ and $\mathrm{PGO}(q) = \mathrm{GO}(q)/F^\times$ acts on $C_0(V,q)$. We observe that the homomorphism

$$C\colon\ \mathrm{PGO}(q) \to \operatorname{Aut}_F\big(C_0(V,q),\tau_0\big) \tag{2}$$

is injective if $\dim V \geq 3$, in view of (2.5) and the fact that $\mathfrak{o}(q)$ generates $\operatorname{End}_F(V)$ as an algebra. Let $\mathrm{PGO}^+(q) = \mathrm{GO}^+(q)/F^\times$. By (2.5) the homomorphism (2) restricts to a homomorphism

$$C\colon\ \mathrm{PGO}^+(q) \to \operatorname{Aut}_Z\big(C_0(V,q),\tau_0\big).$$

3 Cayley algebras and local triality

Let $\mathfrak{C}$ be a Cayley algebra over F with conjugation $\pi : x \mapsto \overline{x}$ and norm $\mathfrak{n}\colon x \mapsto x\overline{x}$. The new multiplication $x \star y = \overline{x}\,\overline{y}$ satisfies

$$x \star (y \star x) = (x \star y) \star x = \mathfrak{n}(x)y \tag{3}$$

for x, $y \in \mathfrak{C}$. Further, the polar form $b_\mathfrak{n}$, $b_\mathfrak{n}(x,y) = \mathfrak{n}(x+y) - \mathfrak{n}(x) - \mathfrak{n}(y)$, is *associative* with respect to $\star$, in the sense that

$$b_\mathfrak{n}(x \star y, z) = b_\mathfrak{n}(x, y \star z).$$

Proposition 3.1 *For x, $y \in \mathfrak{C}$, let $r_x(y) = y \star x$ and $\ell_x(y) = x \star y$. The map $\mathfrak{C} \to \operatorname{End}_F(\mathfrak{C} \oplus \mathfrak{C})$ given by*

$$x \mapsto \begin{pmatrix} 0 & \ell_x \\ r_x & 0 \end{pmatrix}$$

induces isomorphisms $\alpha\colon \big(C(\mathfrak{C},\mathfrak{n}),\tau\big) \xrightarrow{\sim} \big(\operatorname{End}_F(\mathfrak{C} \oplus \mathfrak{C}), \sigma_{\mathfrak{n}\perp\mathfrak{n}}\big)$ and

$$\alpha_0\colon \big(C_0(\mathfrak{C},\mathfrak{n}),\tau_0\big) \xrightarrow{\sim} \big(\operatorname{End}_F(\mathfrak{C}), \sigma_\mathfrak{n}\big) \times \big(\operatorname{End}_F(\mathfrak{C}), \sigma_\mathfrak{n}\big), \tag{4}$$

of algebras with involution. Further α_0 maps $[\mathfrak{C},\mathfrak{C}] \subset C_0(\mathfrak{C},\mathfrak{n})$ into $\mathfrak{o}(\mathfrak{n}) \times \mathfrak{o}(\mathfrak{n})$.

Proof We have $r_x\big(\ell_x(y)\big) = \ell_x\big(r_x(y)\big) = \mathfrak{n}(x)\cdot y$ by (3). Thus the existence of the map α follows from the universal property of the Clifford algebra. The fact that α is compatible with involutions is equivalent to

$$b_\mathfrak{n}\big(x \star (z \star y), u\big) = b_\mathfrak{n}\big(z, y \star (u \star x)\big)$$

for all x, y, z, u in $\mathfrak{C}$. This formula follows from the associativity of $b_\mathfrak{n}$, since

$$b_\mathfrak{n}\big(x \star (z \star y), u\big) = b_\mathfrak{n}(u \star x, z \star y) = b_\mathfrak{n}\big(z, y \star (u \star x)\big).$$

Since $C(\mathfrak{C},\mathfrak{n})$ is central simple, the map α is an isomorphism by a dimension count. The fact that the image of $[\mathfrak{C},\mathfrak{C}]$ under α_0 lies in $\mathfrak{o}(\mathfrak{n}) \times \mathfrak{o}(\mathfrak{n})$ follows from the fact that α_0 is an isomorphism of algebras with involution.

□

We have an (injective) homomorphism $\alpha_0|_{[\mathfrak{C},\mathfrak{C}]} \circ \operatorname{ad}^{-1} : \mathfrak{o}(\mathfrak{n}) \to \mathfrak{o}(\mathfrak{n}) \times \mathfrak{o}(\mathfrak{n})$. For any $\lambda \in \mathfrak{o}(\mathfrak{n})$ let $\alpha_0|_{[\mathfrak{C},\mathfrak{C}]} \circ \operatorname{ad}^{-1}(\lambda) = (\lambda^+, \lambda^-)$.

Proposition 3.2 (Local triality) *For any $\lambda \in \mathfrak{o}(\mathfrak{n})$, the elements λ^+, $\lambda^- \in \mathfrak{o}(\mathfrak{n})$ satisfy*

$$\begin{aligned} \lambda^+(x \star y) &= \lambda(x) \star y + x \star \lambda^-(y), && (5) \\ \lambda^-(x \star y) &= \lambda^+(x) \star y + x \star \lambda(y), && (6) \\ \lambda(x \star y) &= \lambda^-(x) \star y + x \star \lambda^+(y) && (7) \end{aligned}$$

for all x, $y \in \mathfrak{o}(\mathfrak{n})$. Further the pair (λ^+, λ^-) is uniquely determined by any of the three relations. In particular any one of the three elements λ, λ^+ and λ^- determines the two others.

Proof Let $\xi = \mathrm{ad}^{-1}(\lambda)$. Since α_0 is an isomorphism of algebras we have

$$\alpha_0\big([\xi,x]\big) = \big[\alpha_0(\xi),\alpha_0(x)\big] = \begin{pmatrix}\lambda^+ & 0\\ 0 & \lambda^-\end{pmatrix}\begin{pmatrix}0 & \ell_x\\ r_x & 0\end{pmatrix} - \begin{pmatrix}0 & \ell_x\\ r_x & 0\end{pmatrix}\begin{pmatrix}\lambda^+ & 0\\ 0 & \lambda^-\end{pmatrix}.$$

On the other hand, $[\xi,x] = \lambda(x)$ and

$$\alpha_0\big([\xi,x]\big) = \begin{pmatrix}0 & \ell_{\lambda x}\\ r_{\lambda x} & 0\end{pmatrix}$$

so that

$$\begin{aligned}\lambda^+(x\star y) - x\star\lambda^-(y) &= \lambda(x)\star y\\ \lambda^-(y\star x) - \lambda^+(y)\star x &= y\star\lambda(x).\end{aligned}$$

This gives formulas (5) and (6). From (5) we obtain

$$b_{\mathfrak{n}}\big(\lambda^+(x\star y),z\big) = b_{\mathfrak{n}}\big(\lambda(x)\star y,z\big) + b_{\mathfrak{n}}\big(x\star\lambda^-(y),z\big).$$

Since $b_{\mathfrak{n}}(x\star y,z) = b_{\mathfrak{n}}(x,y\star z)$ and since λ^-, λ and λ^+ are in $\mathfrak{o}(\mathfrak{n})$, this implies

$$-b_{\mathfrak{n}}\big(x,y\star\lambda^+(z)\big) = -b_{\mathfrak{n}}\big(x,\lambda(y\star z)\big) + b_{\mathfrak{n}}\big(x,\lambda^-(y)\star z\big)$$

for all x, y, and z in $\mathfrak{o}(\mathfrak{n})$, hence (7). We finally check that the pair $(\lambda^+,\,\lambda^-)$ associated to λ is uniquely determined by (5). It suffices to check that the only pair of linear maps $\lambda_1,\ \lambda_2 \in \mathfrak{o}(\mathfrak{C},\mathfrak{n})$ satisfying $\lambda_1(x\star y) = x\star\lambda_2(y)$ for all $x,\ y\in\mathfrak{C}$ is the pair $(0,0)$. We have $\lambda_1(\overline{x}\,\overline{y}) = \overline{x\overline{\lambda_2(y)}}$. Then $x=1$ implies $\lambda_1(\overline{y}) = \overline{\lambda_2(y)}$, so that $\lambda_1(\overline{xy}) = \overline{x}\lambda_1(\overline{y})$ and $\lambda_1(x) = xa$ for $a=\lambda_1(1)$. This finally implies $(xy)a = x(ya)$ for all $x,\ y\in\mathfrak{C}$ and a lies in F. However $\lambda_1(x) = ax$, for $a\in F$ and $\lambda_1\in\mathfrak{o}(\mathfrak{C},\mathfrak{n})$, implies $a=0$. The proof of the uniqueness in the cases (6) and (7) is similar.

□

Let d_ρ, resp. d_{ρ^2} be the endomorphisms of $\mathfrak{o}(\mathfrak{C},\mathfrak{n})$ defined by $d_\rho(\lambda) = \lambda^+$ and $d_{\rho^2}(\lambda) = \lambda^-$ for $\lambda\in\mathfrak{o}(\mathfrak{n})$, so that

$$\alpha_0\circ\mathrm{ad}^{-1} = (d_\rho, d_{\rho^2}) \in \mathrm{End}_F(\mathfrak{C})\times\mathrm{End}_F(\mathfrak{C}).$$

Corollary 3.3 *The endomorphisms d_ρ and d_{ρ^2} are Lie algebras automorphisms of $\mathfrak{o}(\mathfrak{n})$ and satisfy the relations $(d_\rho)^2 = d_{\rho^2}$, $(d_\rho)^3 = 1$.*

Proof The claims follow from uniqueness in (3.2).

□

The conjugation π of $\mathfrak{C}$ induces an automorphism (of Lie algebras) $d_\pi : \lambda\mapsto\pi\lambda\pi$ of $\mathfrak{o}(\mathfrak{n})$ (the product $\pi\lambda\pi$ is taken in $\mathrm{End}_F(\mathfrak{C})$).

Proposition 3.4 *The relations $(d_\pi)^2 = 1$ and $d_\pi\circ d_\rho = d_{\rho^2}\circ d_\pi$ hold in $\mathrm{Aut}_F\big(\mathfrak{o}(\mathfrak{n})\big)$ and $\{d_\pi, d_\rho\}$ generate a subgroup of $\mathrm{Aut}_F\big(\mathfrak{o}(\mathfrak{n})\big)$, isomorphic to S_3.*

Proof The first relation is obvious since $x \mapsto \bar{x}$ is an involution with respect to the $\star$ product. We check the second one. Since π is an isometry of $\mathfrak{n}$, π induces an automorphism $C(\pi)$ of $C(\mathfrak{C}, \mathfrak{n})$. We have $C(\pi) = \mathrm{Int}(e)$ for $e = 1_{\mathfrak{C}} \in \mathfrak{C}$, hence $\alpha \circ C(\pi) \circ \alpha^{-1} = \mathrm{Int}\big(\alpha(e)\big)$. Plugging in the definition of α, we get for $\left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right) \in \mathrm{End}_F(\mathfrak{C} \oplus \mathfrak{C})$,

$$\big(\alpha \circ C(\pi) \circ \alpha^{-1}\big)\left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right) = \left(\begin{smallmatrix} \pi g \pi & 0 \\ 0 & \pi f \pi \end{smallmatrix}\right). \tag{8}$$

On the other hand we know that $\mathrm{ad} \circ C(\pi) \circ \mathrm{ad}^{-1} = \mathrm{Int}(\pi)$ on $\mathfrak{o}(\mathfrak{n})$ by (2.5), so that

$$\begin{aligned} \big(\alpha_0 \circ C(\pi) \circ \alpha_0^{-1}\big)\big(d_\rho(\lambda), d_{\rho^2}(\lambda)\big) &= \big(\pi d_{\rho^2}(\lambda)\pi, \pi d_\rho(\lambda)\pi\big) \\ &= \big(\alpha_0 \circ C(\pi) \circ \mathrm{ad}^{-1}\big)(\lambda) \\ &= \big(\alpha_0 \circ \mathrm{ad}^{-1} \circ \mathrm{ad} \circ C(\pi) \circ \mathrm{ad}^{-1}\big)(\lambda) \\ &= \big(\alpha_0 \circ \mathrm{ad}^{-1}\big)(\pi\lambda\pi) \\ &= \big(d_\rho(\pi\lambda\pi), d_{\rho^2}(\pi\lambda\pi)\big) \end{aligned}$$

and the second relation holds. Thus we get a homomorphism $S_3 \to \mathrm{Aut}_F\big(\mathfrak{o}(\mathfrak{n})\big)$. The fact that it is injective follows from the explicit formulas given in the next section.

$\square$

Remark 3.5 The elements λ of $\mathfrak{o}_8$ fixed under the action of S_3 are such that $\lambda(x \star y) = \lambda(x) \star y + x \star \lambda(y)$ for all $x, y \in \mathfrak{C}$. Such λ are *derivations* of $(\mathfrak{C}, \star)$. They define a Lie algebra of type G_2.

4 Triality for Skew-symmetric Matrices

Let q be the 8-dimensional quadratic form $< 1, 1, \ldots, 1 >$. The Lie algebra of skew-symmetric (8×8)-matrices is the Lie algebra $\mathfrak{o}_8 = \mathfrak{o}(q)$. In this section we give explicit formulas for the trialitarian action on $\mathfrak{o}_8$. We call the Cayley algebra with norm the identity form $< 1, 1, \ldots, 1 >$ the *standard Cayley algebra* or the *algebra of octonions* and denote it $\mathfrak{C}(8)$. There exists an orthogonal basis $(1_{\mathfrak{C}} = e_1, \ldots, e_8)$ of $\mathfrak{C}(8)$ such that the multiplication table of $\mathfrak{C}(8)$ is

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	e_2	$-e_1$	e_4	$-e_3$	$-e_6$	e_5	$-e_8$	e_7
e_3	e_3	$-e_4$	$-e_1$	e_2	$-e_7$	e_8	e_5	$-e_6$
e_4	e_4	e_3	$-e_2$	$-e_1$	$-e_8$	$-e_7$	e_6	e_5
e_5	e_5	e_6	e_7	e_8	$-e_1$	$-e_2$	$-e_3$	$-e_4$
e_6	e_6	$-e_5$	$-e_8$	e_7	e_2	$-e_1$	$-e_4$	e_3
e_7	e_7	e_8	$-e_5$	$-e_6$	e_3	e_4	$-e_1$	$-e_2$
e_8	e_8	$-e_7$	e_6	$-e_5$	e_4	$-e_3$	e_2	$-e_1$

(9)

Let x_{ij}, $1 \le i, j \le 8$, be indeterminates and let $F(x_{ij})$ be the quotient field of the polynomial ring $F[x_{ij}]$ in the indeterminates x_{ij}. The matrix $X = \sum_{i,j} x_{ij} e_{ij} \in M_8\big(F(x_{ij})\big)$ is the *generic matrix* and the matrix $\mathcal{X} = \sum_{i<j} x_{ij} \mathcal{E}_{ij} \in \mathfrak{o}_8 \otimes F(x_{ij})$, $\mathcal{E}_{ij} = e_{ij} - e_{ji}$, is the *generic skew-symmetric matrix*. We compute the image of $\mathcal{X}$ under the automorphisms d_ρ and d_π. The element $\mathcal{E}_{ij}$ corresponds to the product $\frac{1}{2} e_i e_j$ in the Clifford algebra $C(8)$, through the

map ad: $[\mathfrak{C}(8), \mathfrak{C}(8)] \to \mathfrak{o}(8)$ (see (2.3)). Thus the image of $\mathcal{E}_{ij}$ under d_ρ is the matrix of the automorphism $u \mapsto \frac{1}{2}e_i \star (u \star e_j) = \frac{1}{2}\overline{e_i} \cdot (e_j \cdot u)$ of the space $\mathfrak{C}$. Straightforward explicit calculations using the multiplication table (9) show that $\mathcal{X} = \sum_{i<j} x_{ij}\mathcal{E}_{ij}$ has as image under $\alpha_0 \circ \mathrm{ad}^{-1}$ the pair of skew-symmetric matrices $\big(d_\rho(\mathcal{X}), d_{\rho^2}(\mathcal{X})\big)$, where

$$d_\rho(\mathcal{X}) = \frac{1}{2}\left(\begin{array}{c|c|c|c|c|c|c|c}
 & \begin{smallmatrix}-x_{12}+x_{34}\\-x_{56}-x_{78}\end{smallmatrix} & \begin{smallmatrix}-x_{13}-x_{24}\\-x_{57}+x_{68}\end{smallmatrix} & \begin{smallmatrix}-x_{14}+x_{23}\\-x_{58}-x_{67}\end{smallmatrix} & \begin{smallmatrix}-x_{15}+x_{26}\\+x_{37}+x_{48}\end{smallmatrix} & \begin{smallmatrix}-x_{16}-x_{25}\\-x_{38}+x_{47}\end{smallmatrix} & \begin{smallmatrix}-x_{17}+x_{28}\\-x_{35}-x_{46}\end{smallmatrix} & \begin{smallmatrix}-x_{18}-x_{27}\\x_{36}-x_{45}\end{smallmatrix} \\ \hline
 & & \begin{smallmatrix}-x_{14}+x_{23}\\+x_{58}+x_{67}\end{smallmatrix} & \begin{smallmatrix}x_{13}+x_{24}\\-x_{57}+x_{68}\end{smallmatrix} & \begin{smallmatrix}x_{16}+x_{25}\\-x_{38}+x_{47}\end{smallmatrix} & \begin{smallmatrix}-x_{15}+x_{26}\\-x_{37}-x_{48}\end{smallmatrix} & \begin{smallmatrix}x_{18}+x_{27}\\+x_{36}-x_{45}\end{smallmatrix} & \begin{smallmatrix}-x_{17}+x_{28}\\+x_{35}+x_{46}\end{smallmatrix} \\ \hline
 & & & \begin{smallmatrix}-x_{12}+x_{34}\\+x_{56}+x_{78}\end{smallmatrix} & \begin{smallmatrix}x_{17}+x_{28}\\+x_{35}-x_{46}\end{smallmatrix} & \begin{smallmatrix}-x_{18}+x_{27}\\+x_{36}+x_{45}\end{smallmatrix} & \begin{smallmatrix}-x_{15}-x_{26}\\+x_{37}-x_{48}\end{smallmatrix} & \begin{smallmatrix}+x_{16}-x_{25}\\+x_{38}+x_{47}\end{smallmatrix} \\ \hline
 & & & & \begin{smallmatrix}x_{18}-x_{27}\\+x_{36}+x_{45}\end{smallmatrix} & \begin{smallmatrix}x_{17}+x_{28}\\-x_{35}+x_{46}\end{smallmatrix} & \begin{smallmatrix}-x_{16}+x_{25}\\+x_{38}+x_{47}\end{smallmatrix} & \begin{smallmatrix}-x_{15}-x_{26}\\-x_{37}+x_{48}\end{smallmatrix} \\ \hline
 & & & & & \begin{smallmatrix}x_{12}+x_{34}\\+x_{56}-x_{78}\end{smallmatrix} & \begin{smallmatrix}x_{13}-x_{24}\\+x_{57}+x_{68}\end{smallmatrix} & \begin{smallmatrix}x_{14}+x_{23}\\+x_{58}-x_{67}\end{smallmatrix} \\ \hline
 & & & & & & \begin{smallmatrix}x_{14}+x_{23}\\-x_{58}+x_{67}\end{smallmatrix} & \begin{smallmatrix}-x_{13}+x_{24}\\+x_{57}+x_{68}\end{smallmatrix} \\ \hline
 & & & & & & & \begin{smallmatrix}x_{12}+x_{34}\\-x_{56}+x_{78}\end{smallmatrix} \\ \hline
 & & & & & & &
\end{array}\right)$$

and

$$d_{\rho^2}(\mathcal{X}) = \frac{1}{2}\left(\begin{array}{c|c|c|c|c|c|c|c}
 & \begin{smallmatrix}-x_{12}-x_{34}\\+x_{56}+x_{78}\end{smallmatrix} & \begin{smallmatrix}-x_{13}+x_{24}\\+x_{57}-x_{68}\end{smallmatrix} & \begin{smallmatrix}-x_{14}-x_{23}\\+x_{58}+x_{67}\end{smallmatrix} & \begin{smallmatrix}-x_{15}-x_{26}\\-x_{37}-x_{48}\end{smallmatrix} & \begin{smallmatrix}-x_{16}+x_{25}\\+x_{38}-x_{47}\end{smallmatrix} & \begin{smallmatrix}-x_{17}-x_{28}\\+x_{35}+x_{46}\end{smallmatrix} & \begin{smallmatrix}-x_{18}+x_{27}\\-x_{36}+x_{45}\end{smallmatrix} \\ \hline
 & & \begin{smallmatrix}x_{14}+x_{23}\\+x_{58}+x_{67}\end{smallmatrix} & \begin{smallmatrix}-x_{13}+x_{24}\\-x_{57}+x_{68}\end{smallmatrix} & \begin{smallmatrix}-x_{16}+x_{25}\\-x_{38}+x_{47}\end{smallmatrix} & \begin{smallmatrix}x_{15}+x_{26}\\-x_{37}-x_{48}\end{smallmatrix} & \begin{smallmatrix}-x_{18}+x_{27}\\+x_{36}-x_{45}\end{smallmatrix} & \begin{smallmatrix}x_{17}+x_{28}\\+x_{35}+x_{46}\end{smallmatrix} \\ \hline
 & & & \begin{smallmatrix}x_{12}+x_{34}\\+x_{56}+x_{78}\end{smallmatrix} & \begin{smallmatrix}-x_{17}+x_{28}\\+x_{35}-x_{46}\end{smallmatrix} & \begin{smallmatrix}x_{18}+x_{27}\\+x_{36}+x_{45}\end{smallmatrix} & \begin{smallmatrix}x_{15}-x_{26}\\+x_{37}-x_{48}\end{smallmatrix} & \begin{smallmatrix}-x_{16}-x_{25}\\+x_{38}+x_{47}\end{smallmatrix} \\ \hline
 & & & & \begin{smallmatrix}-x_{18}-x_{27}\\+x_{36}+x_{45}\end{smallmatrix} & \begin{smallmatrix}-x_{17}+x_{28}\\-x_{35}+x_{46}\end{smallmatrix} & \begin{smallmatrix}x_{16}+x_{25}\\+x_{38}+x_{47}\end{smallmatrix} & \begin{smallmatrix}x_{15}-x_{26}\\-x_{37}+x_{48}\end{smallmatrix} \\ \hline
 & & & & & \begin{smallmatrix}-x_{12}+x_{34}\\+x_{56}-x_{78}\end{smallmatrix} & \begin{smallmatrix}-x_{13}-x_{24}\\+x_{57}+x_{68}\end{smallmatrix} & \begin{smallmatrix}-x_{14}+x_{23}\\+x_{58}-x_{67}\end{smallmatrix} \\ \hline
 & & & & & & \begin{smallmatrix}-x_{14}+x_{23}\\-x_{58}+x_{67}\end{smallmatrix} & \begin{smallmatrix}x_{13}+x_{24}\\+x_{57}+x_{68}\end{smallmatrix} \\ \hline
 & & & & & & & \begin{smallmatrix}-x_{12}+x_{34}\\-x_{56}+x_{78}\end{smallmatrix} \\ \hline
 & & & & & & &
\end{array}\right)$$

Since the conjugation π of $\mathfrak{C}$ is given by the matrix $P = \mathrm{diag}(1, -1, \ldots, -1)$ with respect to the basis chosen above, we have

$$d_\pi(\mathcal{X}) = P\mathcal{X}P = \sum_{1<j} -x_{1j}\mathcal{E}_{1j} + \sum_{1<i<j} x_{ij}\mathcal{E}_{ij}.$$

For any skew-symmetric matrix $\mathcal{U}$ with entries in F, we obtain formulas for $d_\alpha(\mathcal{U})$ for $\alpha = \rho$, ρ^2 and π by specializing $\mathcal{X}$ to $\mathcal{U}$. This shows that S_3 acts faithfully on $\mathfrak{o}_8$.

Proposition 4.1 *For any Cayley algebra $\mathfrak{C}$ with norm $\mathfrak{n}$ the action of S_3 on the Lie algebra $\mathfrak{o}(\mathfrak{n})$ is faithful. Further the automorphisms d_ρ and d_{ρ^2} do not extend to inner automorphisms of* $\mathrm{End}_F(\mathfrak{C})$.

Proof Since $\mathfrak{C}$ isomorphic to $\mathfrak{C}(8)$ over some field extension of F, it suffices to check the claims for $\mathfrak{C}(8)$. We already know the first assertion for $\mathfrak{C}(8)$. Let det be the determinant function in $\mathrm{End}_F\big(\mathfrak{C}(8)\big)$. If d_ρ were to extend to an inner automorphism of $\mathrm{End}_F\big(\mathfrak{C}(8)\big)$, we would have $\det\big(d_\rho(\mathcal{X})\big) = \det(\mathcal{X})$. Let x be an indeterminate. Specializing $\mathcal{X}$ to $x_{12} = x$, $x_{34} = x_{56} = x_{78} = 1$ and all other entries to 0 shows that this is not the case.

□

Remark 4.2 The relations (3.4) can be explicitly verified for skew-symmetric matrices, using the formulas for $d_\rho(\mathcal{X})$ and $d_\pi(\mathcal{X})$. The general case follows by going to some field extension of F.

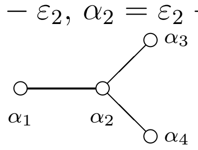
We next check that the action of S_3 on $\mathfrak{o}_8$ induces a faithful action of S_3 on the Dynkin diagram of D_4. This gives another proof that S_3 acts faithfully on $\mathfrak{o}_8$ as outer automorphisms.

Proposition 4.3 *The action defined in* (3.4) *induces permutations of the simple roots of* $\mathfrak{o}_8$.

Proof A Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{o}_8$ is generated by the four diagonal blocks $\mathcal{E}_{12}$, $\mathcal{E}_{34}$, $\mathcal{E}_{56}$ and $\mathcal{E}_{78}$ (see [H], p. 187). The automorphism d_ρ of $\mathfrak{o}_8$ restricts to an automorphism of $\mathfrak{H}$. With respect to the basis $u_1 = \mathcal{E}_{34}$, $u_2 = \mathcal{E}_{56}$, $u_3 = \mathcal{E}_{78}$ and $u_4 = \mathcal{E}_{12}$ it is given by the orthogonal matrix

$$T = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

Let $F(i)$ be a field extension of F such that $i^2 = -1$. In view of [H], p. 188, the linear forms ε_j, $j = 1, \dots, 4$, in $\mathfrak{H}^*$, defined by $\varepsilon_j(u_k) = i\delta_{jk}$, form a set of roots of $\mathfrak{o}_8 \otimes F(i)$ such that the simple roots may be taken as $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\alpha_3 = \varepsilon_3 - \varepsilon_4$ and $\alpha_4 = \varepsilon_3 + \varepsilon_4$ and the Dynkin diagram is



The action induced by d_ρ on the dual $\mathfrak{H}^*$ is given by the matrix $T^t = T^{-1}$ with respect to the basis (ε_k). On the other hand the action of S_3 on the Dynkin diagram permutes α_1, α_3, α_4 and leaves α_2 invariant. The cyclic permutation $(\alpha_1, \alpha_3, \alpha_4)$ extends uniquely to an automorphism of the dual $\mathfrak{H}^*$. This automorphism is given with respect to the basis (ε_j) by the matrix $T^t = T^{-1}$. This is the claim for the generator ρ of order 3 of S_3. The action of d_π on $\mathfrak{H}$ maps u_4 to $-u_4$ and leaves the other u's fixed. Thus its contragredient action on $\mathfrak{H}^*$ maps ε_4 to $-\varepsilon_4$ and leaves the other ε's fixed. On the level of simple roots it permutes α_3 and α_4. This concludes the proof.

□

5 Similitudes and Triality

Let $\mathfrak{C}$ be any Cayley algebra with norm $\mathfrak{n}$. Any proper similitude $f \in \mathrm{GO}^+(\mathfrak{n})$ induces an automorphism $C(f)$ of the even Clifford algebra $\big(C_0(\mathfrak{C}, \mathfrak{n}), \tau_0\big)$, which leaves the center of $C_0(\mathfrak{C}, \mathfrak{n})$ invariant (see (2.5)). Thus $\alpha_0 \circ C(f) \circ \alpha_0^{-1}$ is a pair of automorphisms of $\big(\mathrm{End}_F(\mathfrak{C}), \sigma_\mathfrak{n}\big)$. It follows from the Skolem-Noether theorem that, for any quadratic space (V, q), automorphisms of $(\mathrm{End}_F(V), \sigma_q)$ are of the form $\mathrm{Int}(f)$, where f is a similitude of q. Therefore we have

$$\alpha_0 \circ C(f) \circ \alpha_0^{-1} = \big(\mathrm{Int}(f_1), \mathrm{Int}(f_2)\big)$$

for similitudes f_1, f_2. The pair (f_1, f_2) can be normalized:

Proposition 5.1 *For any proper similitude* $f \in \mathrm{GO}^+(\mathfrak{n})$ *with multiplier* $m(f)$*, there exist proper similitudes* f_1, f_2 *such that*

$$\alpha_0 \circ C(f) \circ \alpha_0^{-1} = \big(\mathrm{Int}(f_1), \mathrm{Int}(f_2)\big),$$

i.e., the diagram

$$\begin{array}{ccc} C_0(\mathfrak{C},\mathfrak{n}) & \xrightarrow{\alpha_0} & \mathrm{End}_F(\mathfrak{C})\times \mathrm{End}_F(\mathfrak{C}) \\ {\scriptstyle C(f)}\big\downarrow & & \big\downarrow{\scriptstyle \left(\mathrm{Int}(f_1),\,\mathrm{Int}(f_2)\right)} \\ C_0(\mathfrak{C},\mathfrak{n}) & \xrightarrow{\alpha_0} & \mathrm{End}_F(\mathfrak{C})\times \mathrm{End}_F(\mathfrak{C}) \end{array} \tag{10}$$

is commutative, and

(1) $$m(f_1)^{-1}f_1(x\star y) = f(x)\star f_2(y),$$

(2) $$m(f)^{-1}f(x\star y) = f_2(x)\star f_1(y)$$

and

(3) $$m(f_2)^{-1}f_2(x\star y) = f_1(x)\star f(y).$$

The pair (f_1, f_2) is determined by f up to a factor (m, m^{-1}), $m \in F^\times$, and we have

$$m(f_1)m(f)m(f_2) = 1.$$

Furthermore, any of the formulas (1) to (3) implies the others.

Proof See the proof of Proposition (35.4) in [KMRT].

□

Passing from $\mathrm{GO}^+(\mathfrak{n})$ to $\mathrm{PGO}^+(\mathfrak{n}) = \mathrm{GO}^+(\mathfrak{n})/F^\times$, we get well defined automorphisms of $\mathrm{PGO}^+(\mathfrak{n})$, $\overline{\rho}$: $[f]\mapsto [f_1]$, ρ': $[f]\mapsto[f_2]$, and uniqueness in (5.1) implies that $\rho' = \overline{\rho}^2$, $\overline{\rho}^3 = 1$. We thus have

$$\big(\overline{\rho}(a), \overline{\rho}^2(a)\big) = \alpha_0 C(a)\alpha_0^{-1} \tag{11}$$

for any $a \in PGO^+(\mathfrak{n})$.

Let $\overline{\pi}$ be the automorphism of $\mathrm{PGO}^+(\mathfrak{n})$ given by $\overline{\pi}(x) = [\pi]x[\pi]$, where $[\pi]$ is the class in $\mathrm{PGO}(\mathfrak{n})$ of the conjugation map π of $\mathfrak{C}$. It follows from (5.1) and the identity $\pi(x\star y) = \pi(y)\star\pi(x)$ that $\overline{\pi}\,\overline{\rho} = \overline{\rho}^2\overline{\pi}$. Thus:

Corollary 5.2 (Global triality) *The set $\{\overline{\pi}, \overline{\rho}\}$ generate a subgroup of $\mathrm{Aut}_F\big(\mathrm{PGO}^+(\mathfrak{n})\big)$ isomorphic to S_3.*

Proof The fact that S_3 acts on $\mathrm{PGO}^+(\mathfrak{n})$ follows from the relations given above. The fact that the action is faithful follows from the computations in Section 4.

□

The action of S_3 on the Lie algebra $\mathfrak{o}(\mathfrak{n})$ and the action on the group $\mathrm{PGO}^+(\mathfrak{n})$ are related as follows:

Lemma 5.3 *Let $[f] \in \mathrm{PGO}^+(\mathfrak{n})$ be represented by $f \in \mathrm{GO}^+(\mathfrak{n})$ and $\overline{\rho}([f])$ (resp. $\overline{\rho}^2([f])$) be represented by f_1 (resp. f_2) as above. We have, for any element $\mathcal{U} \in \mathfrak{o}(\mathfrak{n})$,*

1) $\big(\mathrm{ad}\circ C(f)\circ \mathrm{ad}^{-1}\big)(\mathcal{U}) = f\mathcal{U}f^{-1}$,

2) $d_\rho(f\mathcal{U}f^{-1}) = f_1 d_\rho(\mathcal{U})f_1^{-1}$, $d_\rho(f_1\mathcal{U}f_1^{-1}) = f_2 d_\rho(\mathcal{U})f_2^{-1}$ *and* $d_\rho(f_2\mathcal{U}f_2^{-1}) = f d_\rho(\mathcal{U})f^{-1}$

3) $d_{\rho^2}(f\mathcal{U}f^{-1}) = f_2 d_{\rho^2}(\mathcal{U})f_2^{-1}$, $d_{\rho^2}(f_1\mathcal{U}f_1^{-1}) = f d_{\rho^2}(\mathcal{U})f^{-1}$ *and* $d_{\rho^2}(f_2\mathcal{U}f_2^{-1}) = f_1 d_{\rho^2}(\mathcal{U})f_1^{-1}$.

Furthermore, for the conjugation $\pi \in \mathrm{GO}(\mathfrak{n})$*, we have*

4) $d_\rho(\pi\mathcal{U}\pi) = \pi d_{\rho^2}(\mathcal{U})\pi$ *and* $d_{\rho^2}(\pi\mathcal{U}\pi) = \pi d_\rho(\mathcal{U})\pi$*.*

Proof Claim 1) is already in (2.5). We check the first formula of 2). By definition we have $\alpha_0 \circ \mathrm{ad}^{-1}(\mathcal{U}) = \big(d_\rho(\mathcal{U}), d_{\rho^2}(\mathcal{U})\big)$. Thus

$$\begin{aligned}
\big(d_\rho(f\mathcal{U}f^{-1}), d_{\rho^2}(f\mathcal{U}f^{-1}\big) &= \alpha_0 \circ \mathrm{ad}^{-1}(f\mathcal{U}f^{-1}) \\
&= \big(\alpha_0 \circ C(f) \circ \mathrm{ad}^{-1}\big)(\mathcal{U}) \\
&= \big(\alpha_0 \circ C(f) \circ \alpha_0^{-1}\big) \circ \big(\alpha_0 \circ \mathrm{ad}^{-1}\big)(\mathcal{U}) \\
&= \big(\alpha_0 \circ C(f) \circ \alpha_0^{-1}\big)\big(d_\rho(\mathcal{U}), d_{\rho^2}(\mathcal{U})\big) \\
&= \big(f_1 d_\rho(\mathcal{U}) f_1^{-1}), f_2 d_{\rho^2}(\mathcal{U}) f_2^{-1}\big)
\end{aligned}$$

by (5.1). The proofs of the other formulas in 2) and 3) are similar. Claim 4) is the relation $d_\pi d_\rho = d_{\rho^2} d_\pi$ in (3.4).

□

Since S_3 acts on $\mathrm{PGO}^+(\mathfrak{n})$ we may form the semidirect product $\mathrm{PGO}^+(\mathfrak{n}) \rtimes S_3$. The group $\mathrm{PGO}^+(\mathfrak{n}) \rtimes S_3$ consists of pairs (b, β), where $b \in \mathrm{PGO}^+(\mathfrak{n})$ and $\beta \in S_3$ with the product $(b, \beta) \cdot (b', \beta') = \big(b\beta(b'), \beta\beta'\big)$. For any $\beta \in S_3$ we denote the corresponding automorphism of $\mathfrak{o}(\mathfrak{n})$ by d_β.

Proposition 5.4 *1) The group* $\mathrm{PGO}^+(\mathfrak{n})$ *acts on* $\mathfrak{o}(\mathfrak{n})$ *through inner automorphisms and, for any* $b \in \mathrm{PGO}^+(\mathfrak{n})$*,* $\beta \in S_3$*,* $s \in \mathfrak{o}(\mathfrak{n})$*, we have* $\beta(b) d_\beta(s) = d_\beta(bs)$*.*
2) The group $\mathrm{PGO}^+(\mathfrak{n}) \rtimes S_3$ *acts on* $\mathfrak{o}(\mathfrak{n})$ *through the formula* $(b, \beta)(s) = b d_\beta(s)$ *and* $\mathrm{Aut}_F\big(\mathfrak{o}(\mathfrak{n})\big) = \mathrm{PGO}^+(\mathfrak{n}) \rtimes S_3$*.*

Proof Claim 1) and the first part of 2) follow from (5.3). The last part of 2) is proved for example in ([J], Theorem 5).

□

Proposition 5.5 *The differential of the action of* S_3 *on* $\mathrm{PGO}^+(\mathfrak{n})$ *in* (5.2) *is the action of* S_3 *defined on* $\mathfrak{o}(\mathfrak{n})$*.*

Proof We first recall the definition of the differential. Denote by $F[\varepsilon]$ the F-algebra of *dual numbers*, i.e., $F[\varepsilon] = F \cdot 1 \oplus F \cdot \varepsilon$ with multiplication given by $\varepsilon^2 = 0$. Let $\kappa\colon F[\varepsilon] \to F$, $\varepsilon \mapsto 0$, be the augmentation map. Let G be an algebraic group. The kernel of the induced map $G(F[\varepsilon]) \xrightarrow{G(\kappa)} G(F)$ is the Lie algebra $\mathrm{Lie}(G)$ of G. If $G \subset \mathrm{GL}_n(F)$, $\mathrm{Lie}(G) = \{a \in M_n(F) \mid 1 + a\varepsilon \in G(F[\varepsilon])\}$. Any homomorphism of algebraic groups $f\colon G \to H$ induces a commutative diagram

$$\begin{array}{ccc}
G(F[\varepsilon]) & \xrightarrow{f_{F[\varepsilon]}} & H(F[\varepsilon]) \\
{\scriptstyle G(\kappa)}\big\downarrow & & \big\downarrow{\scriptstyle H(\kappa)} \\
G(F) & \xrightarrow{f_F} & H(F)
\end{array}$$

and hence defines an F-linear map $df\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ which is the differential of f.

We only check that the differential of $\overline{\rho}$ in (11) is d_ρ as defined in (3.4). Let $[g] = [1 + a\varepsilon] \in \mathrm{PGO}^+\big(\mathfrak{n} \otimes F[\varepsilon]\big)$, so that $[a] \in \mathrm{Lie}\big(\mathrm{PGO}^+(\mathfrak{n})\big) = \mathrm{Lie}\big(\mathrm{GO}^+(\mathfrak{n})\big)/F$. By definition of the differential we have $[\overline{\rho}(g)] = [1 + d_{\overline{\rho}}(a)\varepsilon]$ and by definition of triality

$$m\big(\overline{\rho}(g)^{-1}\big)\overline{\rho}(g)(x \star y) = g(x) \star \overline{\rho}^2(g)(y).$$

Modulo scalars we have

$$\big(1 + d_{\overline{\rho}}(a)\varepsilon\big)(x \star y) = (1 + a\varepsilon)(x) \star \big(1 + d_{\overline{\rho}^2}(a)\varepsilon\big)(y)$$

or

$$d_{\overline{\rho}}(a)(x \star y) = a(x) \star y + x \star d_{\overline{\rho}^2}(a)(y)$$

hence $d_{\overline{\rho}} = d_\rho$ by (3.2) (local triality).

$\square$

6 Orthogonal Involutions

Let $\mathrm{GO}_n(F) = \{a \in \mathrm{GL}_n(F) \mid aa^t \in F^\times\}$ denote the group of similitudes of the n-dimensional identity quadratic form $< 1, \dots, 1 >$. The element $m(a) = aa^t$ is the multiplier of the similitude a. Elements of $\mathrm{GO}_n(F)$ act on $(M_n(F), t)$ through inner automorphisms, $[a](x) = axa^{-1}$, $a \in \mathrm{GO}_n(F)$, and

$$\mathrm{Aut}_F\big(M_n(F), t\big) = \mathrm{GO}_n(F)/F^\times = \mathrm{PGO}_n(F). \tag{12}$$

We assume that $n = 2l$ is even. Proper similitudes a satisfy $\det(a) = m(a)^l$ and form a subgroup $\mathrm{GO}^+_{2l}(F)$ of $\mathrm{GO}_{2l}(F)$. Going modulo the center we get the group of classes of similitudes $\mathrm{PGO}_{2l}(F)$, resp. the group $\mathrm{PGO}^+_{2l}(F)$ of classes of proper similitudes.

The group $\mathrm{PGO}^+_{2l}(F)$ can also be described as a group of automorphisms; let

$$\mathrm{Skew}_{2l}(F) = \{X \in M_{2l}(F) \mid X^t = -X\}$$

be the set of skew-symmetric $(2l \times 2l)$-matrices. The pfaffian

$$\mathrm{pf}\colon \mathrm{Skew}_{2l}(F) \to F$$

is a homogeneous polynomial function of degree l such that $\mathrm{pf}(X)^2 = \det(X)$; further, $\mathrm{pf}(AXA^t) = \det(A)\,\mathrm{pf}(X)$ for $A \in M_{2l}(F)$ and $X \in \mathrm{Skew}_{2l}(F)$. For $[A] \in \mathrm{PGO}_{2l}(F)$ we have $\mathrm{pf}(AXA^{-1}) = \mathrm{pf}(X)$ if and only if and $[A] \in \mathrm{PGO}^+_{2l}(F)$. Thus

$$\mathrm{Aut}_F\big(M_{2l}(F), t, \mathrm{pf}\big) = \mathrm{GO}^+_{2l}(F)/F^\times = \mathrm{PGO}^+_{2l}(F). \tag{13}$$

Let $\widetilde{F}$ be a separable closure of F and let $\Gamma = \mathrm{Gal}(\widetilde{F}/F)$ be the absolute Galois group of F. Let $H^1(F, G) = H^1\big(\Gamma, G(\widetilde{F})\big)$ be the pointed set of Galois cohomology of Γ with values in G. The set $H^1(F, G)$ classifies G-torsors (principal homogeneous spaces for G over F). Assume that G is the automorphism group of some tensor w over F; then $H^1(F, G)$ classifies isomorphism classes of $\widetilde{F}/F$-forms of w. Let (A, σ) be a central simple algebra of degree n over F with an F-linear involution σ_A. We say that σ is of *orthogonal type* if we have an isomorphism $(A,\sigma)\otimes\widetilde{F}\simeq(M_n(\widetilde{F}),t)$ over a separable closure $\widetilde{F}$ of F. We call such an isomorphism a *standard splitting of* (A,σ). In view of (12) the pointed set $H^1(F,\mathrm{PGO}_n)$ classifies central simple F-algebras (A,σ) of degree n with orthogonal involutions, the point being the class of $\big(M_n(F),t\big)$.

We give a similar description of $H^1(F,\mathrm{PGO}^+_{2l})$. For any algebra A with involution σ, let

$$\mathrm{Skew}(A,\sigma)=\{x\in A\mid\sigma(x)=-x\}$$

be the set of skew elements.

Proposition 6.1 *The pointed set $H^1(F,\mathrm{PGO}^+_{2l})$ classifies triples $(A,\sigma,\mathrm{Pfrd}_A)$, where A is central simple of degree $2l$, σ is an involution of orthogonal type on A and Pfrd_A is a polynomial function of degree l on $\mathrm{Skew}(A,\sigma)$ such that $\mathrm{Pfrd}_A(x)^2=\mathrm{Nrd}_A(x)$ for $x\in\mathrm{Skew}(A,\sigma)$. The point is the class of the triple $\big(M_{2l}(F),t,\mathrm{pf}\big)$. Further, the map Pfrd_A satisfies $\mathrm{Pfrd}_A(ax\sigma(a))=\mathrm{Nrd}_A(a)\,\mathrm{Pfrd}_A(x)$ for $x\in\mathrm{Skew}(A,\sigma)$ and $a\in A$.*

Proof Let $(a_s)_{s\in\Gamma}$, $a_s\in\mathrm{PGO}^+_{2l}(\widetilde{F})$, be a cocycle with values in PGO^+_{2l}. Its image as a cocycle with values in PGO_{2l} defines the central simple algebra

$$A=\{x\in M_{2l}(\widetilde{F})\mid a_s s(x)a_s^{-1}=x\text{ for all }s\in\Gamma\}$$

with an orthogonal involution σ which is the restriction of the transpose t. We claim that the pfaffian on $\mathrm{Skew}_{2l}(\widetilde{F})$ restricts to a function $\mathrm{Pfrd}_A\colon\mathrm{Skew}(A,\sigma)\to F$. It suffices to check that $s\big(\mathrm{pf}(x)\big)=\mathrm{pf}(x)$ for $x\in\mathrm{Skew}(A,\sigma)$ and $s\in\Gamma$. We have

$$\begin{aligned}\mathrm{pf}(x)&=\mathrm{pf}\big(a_s s(x)a_s^{-1}\big)=\mathrm{pf}\big(a_s s(x)a_s^t m(a_s)^{-1}\big)\\&=\mathrm{pf}\big(s(x)\big)\det(a_s)m(a_s)^{-l}=\mathrm{pf}\big(s(x)\big)\end{aligned}\tag{14}$$

since $\det(a_s)=m(a_s)^l$ for $a_s\in\mathrm{PGO}^+_{2l}(\widetilde{F})$. The claim then follows from $\mathrm{pf}\big(s(x)\big)=s\big(\mathrm{pf}(x)\big)$. The given properties of the polynomial map Pfrd_A follow from the corresponding properties of the pfaffian.

$\square$

We call a map Pfrd_A as in (6.1) a *reduced pfaffian.* An isomorphism $(A,\sigma,\mathrm{Pfrd}_A)\otimes\widetilde{F}\simeq(M_n(\widetilde{F}),t,\mathrm{pf})$ over a separable closure $\widetilde{F}$ of F is a *standard splitting* of $(A,\sigma,\mathrm{Pfrd}_A)$. We have an exact sequence of Γ-groups

$$1\longrightarrow\mathrm{PGO}^+_{2l}(\widetilde{F})\xrightarrow{\ i\ }\mathrm{PGO}_{2l}(\widetilde{F})\xrightarrow{\ d\ }\mu_2\longrightarrow 1$$

where, for a similitude a with multiplier $m(a)$, $d([a])=m(a)^{-l}\det(a)\in\mu_2$. The sequence induces an exact sequence of pointed sets in Galois cohomology:

$$\dots\longrightarrow\mu_2\longrightarrow H^1(F,\mathrm{PGO}^+_{2l})\xrightarrow{\ i_*\ }H^1(F,\mathrm{PGO}_{2l})\xrightarrow{\ d_*\ }H^1(F,\mu_2)$$

The image under i_* of the class of the triple $(A,\sigma,\mathrm{Pfrd}_A)$ is the class of the pair (A,σ). We describe the map d_*:

Proposition 6.2 *If $H^1(F,\mu_2)$ is identified with $F^\times/F^{\times2}$, then $d_*([A,\sigma])$ is the square class $[\mathrm{Nrd}_A(x)]\in F^\times/F^{\times2}$ of any skew-symmetric unit of A.*

Proof We recall the identification of $H^1(F,\mu_2)$ with $F^\times/F^{\times 2}$: any cocycle $(b_s)_{s\in\Gamma}\in Z^1(F,\mu_2)$ can be written as $b_s = s(\sqrt{b})\sqrt{b}^{-1}$ for $b\in F^\times$ and $\sqrt{b}\in\widetilde{F}^\times$. Then the square class $[b]\in F^\times/F^{\times 2}$ corresponds to the cocycle (b_s). Let $(a_s)_{s\in\Gamma}$, $a_s\in \mathrm{PGO}_{2l}(\widetilde{F})$, be a cocycle with values in PGO_{2l}. By (14) we have $d_*\big([a_s]\big) = m(a_s)^{-l}\det(a_s) = s(\mathrm{pf}(x))\,\mathrm{pf}(x)^{-1}$ for any skew-symmetric unit $x\in A$. Thus $d_*([A,\sigma]) = [\mathrm{pf}(x)^2] = [\mathrm{Nrd}_A(x)]$, as claimed.

□

Remark 6.3 The square class $[\mathrm{Nrd}_A(x)]\in F^\times/F^{\times 2}$ which, in view of (6.2), is independent of the choice of the skew-symmetric unit x, is the *discriminant* $\mathrm{disc}(\sigma)$ of the involution σ (see [KPS]). The image of the map i_* in $H^1(F,\mathrm{PGO}_{2l})$ consists of classes $[(A,\sigma)]$ such that σ has trivial discriminant. Given a class $[(A,\sigma)]$ in $H^1(F,\mathrm{PGO}_{2l})$ with $\mathrm{disc}(\sigma)=1$, there could be two possible choices of lifts in $H^1(F,\mathrm{PGO}^+_{2l})$, namely $[(A,\sigma,\pm\,\mathrm{Pfrd}_A)]$.

We briefly recall how the discriminant is related to the Clifford algebra and refer to ([KMRT], §8), for details. The map

$$C\colon\ \mathrm{PGO}_{2l}(\widetilde{F}) \to \mathrm{Aut}_{\widetilde{F}}\big(C_0(2l),\tau_0\big)$$

defined in (2) induces a map in Galois cohomology. The image $C(\xi)$ of a cocycle ξ corresponding to an algebra with orthogonal involution (A,σ) defines an algebra $C(A,\sigma)$ of (A,σ) with an involution $\underline{\sigma}$. The algebra $C(A,\sigma)$ is called the *Clifford algebra* of (A,σ). Any isomorphism $\phi\colon (A,\sigma_A)\xrightarrow{\sim}(A',\sigma_{A'})$ of algebras with orthogonal involutions induces an isomorphism $C(\phi)$ of the corresponding Clifford algebras. The algebra $C(A,\sigma)$ has a similar structure as the even Clifford algebra of a quadratic space of even dimension (see (2.1)); its center $Z(A,\sigma)$ is quadratic étale over F and $C(A,\sigma)$ is central separable over $Z(A,\sigma)$. If $(A,\sigma)=\big(\mathrm{End}_F(V),\sigma_q\big)$, then $\big(C(A,\sigma),\underline{\sigma}\big)$ is canonically isomorphic to $\big(C_0(V,q),\tau_0\big)$.

There is an embedding

$$\eta_A\colon\ \mathrm{Skew}(A,\sigma)\to\mathrm{Skew}\big(C(A,\sigma),\underline{\sigma}\big) \tag{15}$$

which is the map

$$\mathrm{ad}^{-1}:\mathfrak{o}(q)\to[V,V]\subset C_0(V,q)$$

if $(A,\sigma)=\big(\mathrm{End}_F(V),\sigma_q\big)$.

The center $Z(A,\sigma)$ of $C(A,\sigma)$ has a generator z such that $z^2=d\in F^\times$. If $(A,\sigma)=\big(\mathrm{End}_F(V),\sigma_q\big)$, we have $z^2=(-1)^l\det(b_q)$ where $\det(b_q)$ is the determinant of the matrix b of b_q with respect to some basis of V. The square class $\mathrm{disc}(q)=[\det(b_q)]\in F^\times/F^{\times 2}=H^1(F,\mu_2)$, is the *(unsigned) discriminant* of q. The isomorphism class of the center of $C_0(V,q)$ is determined uniquely by $\mathrm{disc}(q)$. The involution σ_q is given by $x\mapsto bx^tb^{-1}$ for $x\in M_{2l}(F)$ and it follows from (6.2) that $\mathrm{disc}(\sigma_q)=[\det(b)]=\mathrm{disc}(q)]$. For (A,σ) arbitrary we similarly write $z^2=(-1)^ld$ and the square class $[d]\in F^\times/F^{\times 2}$ of d is the discriminant $\mathrm{disc}(\sigma)$ of σ as defined in (6.3) (see [KMRT, (8.25)][1]).

From now on we assume that A is of degree a multiple of 4, so that l as above is even. The center $Z(A,\sigma)$ is generated by an element z such that $z^2=d$ with $[d]=\mathrm{disc}(\sigma)$. If

[1]Our definition of the discriminant of an orthogonal involution is not consistent with the definition given in [KMRT]. There it is called the *determinant* and the *discriminant* in [KMRT] is the class of $(-1)^l\cdot d$ modulo squares.

$\operatorname{disc}(\sigma) = 1$, then $Z(A,\sigma) \simeq F \times F$. The choice of an isomorphism $\gamma\colon Z(A,\sigma) \xrightarrow{\sim} F \times F$ corresponds to a choice of a sign of Pfrd_A. Thus:

Corollary 6.4 *Let n be a multiple of 4. The pointed set $H^1(F, \mathrm{PGO}_n^+)$ classifies triples (A,σ,γ) where γ is an F-algebra isomorphism $\gamma\colon Z(A,\sigma) \xrightarrow{\sim} F \times F$.*

7 Trialitarian algebras

Let (A,σ_A) be a central simple algebra of degree 8 over F, with an orthogonal involution σ_A of discriminant 1. With the notation of (6.4), the choice of an isomorphism $\gamma\colon Z(A,\sigma_A) \xrightarrow{\sim} F \times F$ leads to a decomposition

$$\big(C(A,\sigma_A),\underline{\sigma}\big) \simeq (B,\sigma_B) \times (C,\sigma_C)$$

The algebras B and C are of degree 8 and the involutions σ_B, σ_C are of orthogonal type (see (2.1)). Thus, given a cocycle $\xi \in H^1(F,\mathrm{PGO}_8^+)$, there exist algebras of degree 8 $\big((A,\sigma_A),(B,\sigma_B),(C,\sigma_C)\big)$ with orthogonal involutions together with an isomorphism

$$\alpha_A\colon \big(C(A,\sigma_A),\underline{\sigma}\big) \simeq (B,\sigma_B) \times (C,\sigma_C). \tag{16}$$

If $(A,\sigma) = \big(M_8(F),t\big)$, then $(B,\sigma_B) \simeq (M_8(F),t) \simeq (C,\sigma_C)$ and formula (4) gives an explicit isomorphism $\alpha_{M_8(F)} = \alpha_0$. The triple $\big(M_8(F),t,\alpha_0\big)$ corresponds to the point of $H^1(F,\mathrm{PGO}_8^+)$.

Lemma 7.1 *The trialitarian action on $\mathrm{PGO}_8^+(\widetilde{F})$ is compatible with the action of the Galois group $\Gamma = \mathrm{Gal}(\widetilde{F}/F)$ and leads to an action of S_3 on $H^1(F,\mathrm{PGO}_8^+)$.*

Proof Let $a \in \mathrm{PGO}_8^+(\widetilde{F}) = \mathrm{Aut}_{\widetilde{F}}\big(M_8(\widetilde{F}),t,\mathrm{pf}\big)$. The action of Γ on the group $\mathrm{PGO}_8^+(\widetilde{F})$ is given by $s(a) = (1\otimes s)a(1\otimes s)^{-1}$ for $s \in \Gamma$. The trialitarian action is defined as $\big(\overline{\rho}(a),\overline{\rho}^2(a)\big) = (\alpha_0\otimes 1_{\widetilde{F}})\circ C(a)\circ(\alpha_0\otimes 1_{\widetilde{F}})^{-1}$ (see (11)). Since α_0 is defined over F and the functor C commutes with the action of Γ, we have $s\big(\overline{\rho}(a)\big) = \overline{\rho}\big(s(a)\big)$. One can also check that $s\overline{\pi} = \overline{\pi}s$.

$\square$

Our next aim is to describe this action on $H^1(F,\mathrm{PGO}_8^+)$, in terms of the classified objects. Let $\beta_A\colon A\otimes\widetilde{F} \xrightarrow{\sim} M_8(\widetilde{F})$ be a standard splitting of $(A,\sigma_A,\mathrm{Pfrd}_A)$ and let $a_s = \beta_A\circ(1\otimes s)\circ\beta_A^{-1}\circ(1\otimes s)^{-1} \in \mathrm{PGO}_8^+(\widetilde{F})$ be the corresponding cocycle. Let

$$\begin{aligned} B &= \{y \in M_8(\widetilde{F}) \mid \overline{\rho}(a_s)(sy) = y,\ s\in\Gamma\} \quad \text{and} \\ C &= \{z \in M_8(\widetilde{F}) \mid \overline{\rho}^2(a_s)(sz) = z,\ s\in\Gamma\} \end{aligned}$$

be the central simple algebras defined by the cocycles $\overline{\rho}(a_s)$, resp. $\overline{\rho}^2(a_s)$. The involutions σ_B, resp. σ_C are the restrictions of the transpose to B, resp. C and the pfaffians Pfrd_B and Pfrd_C are the restrictions of the usual pfaffian pf. The two splittings $\beta_B\colon B\otimes\widetilde{F} \xrightarrow{\sim} M_8(\widetilde{F})$, resp. $\beta_C\colon C\otimes\widetilde{F} \xrightarrow{\sim} M_8(\widetilde{F})$ are given by multiplication in $M_8(\widetilde{F})$. The Clifford algebra of (A,σ) is

$$C(A,\sigma) = \{u \in C_0(8) \mid C(a_s)(su) = u,\ s\in\Gamma\}.$$

Lemma 7.2 *The restriction of* $\alpha_0 \otimes 1_{\widetilde{F}} \colon C_0(8) \otimes \widetilde{F} \xrightarrow{\sim} M_8(\widetilde{F}) \times M_8(\widetilde{F})$ *to* $C(A,\sigma)$ *induces an isomorphism* $\alpha_A \colon \big(C(A,\sigma_A), \underline{\sigma}\big) \xrightarrow{\sim} (B,\sigma_B) \times (C,\sigma_C)$ *such that the diagram*

$$\begin{array}{ccc} C(A,\sigma_A) \otimes \widetilde{F} & \xrightarrow{\alpha_0 \otimes 1_{\widetilde{F}}} & B \otimes \widetilde{F} \times C \otimes \widetilde{F} \\ {\scriptstyle C(\beta_A)}\big\downarrow & & \big\downarrow{\scriptstyle (\beta_B,\beta_C)} \\ C_0(8) & \xrightarrow{\alpha_0 \otimes 1_{\widetilde{F}}} & M_8(\widetilde{F}) \times M_8(\widetilde{F}) \end{array}$$

commutes.

Proof Let $(y,z) = \alpha_0(x)$ for $x \in C(A,\sigma_A)$. We have to check that $\big(\overline{\rho}(a_s), \overline{\rho}^2(a_s)\big)(sy,sz) = (y,z)$ for all $s \in \Gamma$.

Denote $\alpha_0 \otimes 1_{\widetilde{F}} = \alpha_0$; we have, since $\alpha_0 s = s\alpha_0$ and $C(a_s)sx = x$,

$$\begin{aligned} \big(\overline{\rho}(a_s), \overline{\rho}^2(a_s)\big)(sy,sz) &= \alpha_0 C(a_s)\alpha_0^{-1}(sy,\, sz) \\ &= \alpha_0 C(a_s)\alpha_0^{-1} s\big(\alpha_0(x)\big) \\ &= \alpha_0 C(a_s) sx = \alpha_0(x) = (y,z). \end{aligned}$$

□

Repeating the same procedure for (B,σ_B), resp. (C,σ_C) in (7.2) we get isomorphisms

$$\begin{aligned} \alpha_B \colon \big(C(B,\sigma_B), \underline{\sigma}\big) &\xrightarrow{\sim} (C,\sigma_C) \times (A,\sigma_A) \\ \alpha_C \colon \big(C(C,\sigma_C), \underline{\sigma}\big) &\xrightarrow{\sim} (A,\sigma_A) \times (B,\sigma_B). \end{aligned} \tag{17}$$

In particular the $\overline{\rho}$-action on $H^1(F, \mathrm{PGO}_8^+)$ corresponds to the permutation $(A,\, B,\, C) \mapsto (B,\, C,\, A)$. On can similarly check that $\overline{\pi}$ switches B and C.

The split exact sequence

$$1 \longrightarrow \mathrm{PGO}_8^+(\widetilde{F}) \longrightarrow \mathrm{PGO}_8^+(\widetilde{F}) \rtimes S_3 \longrightarrow S_3 \longrightarrow 1$$

induces an exact sequence of pointed sets in Galois cohomology

$$\longrightarrow H^1(F, \mathrm{PGO}_8^+) \longrightarrow H^1(F, \mathrm{PGO}_8^+ \rtimes S_3) \longrightarrow H^1(F, S_3). \tag{18}$$

The set $H^1(F, S_3)$ classifies cubic étale F-algebras L, i.e., F-algebras L with $L \otimes \widetilde{F} \simeq \widetilde{F} \times \widetilde{F} \times \widetilde{F}$ (see [KMRT], (29.9)). The set $H^1(F, \mathrm{PGO}_8^+)$ classifies triples (A,B,C) as in (7.2). Following ([KMRT], Chapter X), we introduce algebraic objects which are classified by $H^1(F, \mathrm{PGO}_8^+ \rtimes S_3)$.

The triple (A,B,C) gives rise to an $F \times F \times F$-algebra $T = A \times B \times C$ with an involution $\sigma_T = (\sigma_A, \sigma_B, \sigma_C)$. The triple $(\alpha_A,\, \alpha_B,\, \alpha_C)$ is an $F \times F \times F$-isomorphism of $C(T,\sigma_T)$ with the $(F \times F \times F) \times (F \times F)$-algebra $(B \times C) \times (C \times A) \times (A \times B)$; we write the cubic étale algebra $L = F \times F \times F$ as a column $(F,F,F)^t$ and its discriminant $\Delta = F \times F$ as a row (F,F), so that $L \otimes \Delta$ is represented by (3×2)-matrices: $L \otimes \Delta = \left(\begin{smallmatrix} F & F \\ F & F \\ F & F \end{smallmatrix}\right)$ and $(\alpha_A,\, \alpha_B,\, \alpha_C)$ has values in $\left(\begin{smallmatrix} B & C \\ C & A \\ A & B \end{smallmatrix}\right)$. Let (α_A^1, α_A^2), resp. (α_B^1, α_B^2), (α_C^1, α_C^2) be the two components of α_A, resp. α_B, α_C. We have $T \otimes \Delta = \left(\begin{smallmatrix} A \\ B \\ C \end{smallmatrix}\right) \otimes (F \times F) = \left(\begin{smallmatrix} A & A \\ B & B \\ C & C \end{smallmatrix}\right)$; let

$$\alpha_T = \begin{pmatrix} \alpha_C^1 & \alpha_B^2 \\ \alpha_A^1 & \alpha_C^2 \\ \alpha_B^1 & \alpha_A^2 \end{pmatrix} \colon C(T,\sigma_T) \to T \otimes \Delta, \tag{19}$$

where $C(T,\sigma_T)$ is thought as a column matrix. The map α_T is F-linear, but not L-linear. To restore L-linearity we have to twist the action of $L\otimes\Delta = \begin{pmatrix} F & F\\ F & F\\ F & F\end{pmatrix}$ on $\begin{pmatrix} A & A\\ B & B\\ C & C\end{pmatrix}$ through an automorphism of $L\otimes\Delta$; in fact $L\otimes\Delta$ is a S_3-Galois F-algebra with the action of S_3 given as

$$\widetilde{\rho}\begin{pmatrix} a_1 & a_2\\ b_1 & b_2\\ c_1 & c_2\end{pmatrix} = \begin{pmatrix} c_1 & b_2\\ a_1 & c_2\\ b_1 & a_2\end{pmatrix} \quad\text{and}\quad \widetilde{\pi}\begin{pmatrix} a_1 & a_2\\ b_1 & b_2\\ c_1 & c_2\end{pmatrix} = \begin{pmatrix} a_2 & a_1\\ b_2 & b_1\\ c_2 & c_1\end{pmatrix}. \tag{20}$$

For any $L\otimes\Delta$-module V with L and Δ as above, let ${}^{\widetilde{\rho}}V$ be the F-module V with the action of $L\otimes\Delta$ twisted through $\widetilde{\rho}$, i.e., $\xi\cdot x = \widetilde{\rho}(\xi)x$ for $\xi\in L\otimes\Delta$ and $x\in T$. For $T = A\times B\times C$ and $\Delta = F\times F$, the map (19) viewed as a map

$$\alpha_T = \begin{pmatrix} \alpha_C^1 & \alpha_B^2\\ \alpha_A^1 & \alpha_C^2\\ \alpha_B^1 & \alpha_A^2\end{pmatrix} : C(T,\sigma_T) \to {}^{\widetilde{\rho}}(T\otimes\Delta) \tag{21}$$

is L-linear. For $(A,\sigma) = \big(M_8(F), t\big)$, we have $(B,\sigma) = (C,\sigma) = \big(M_8(F), t\big)$, $T = M_8(F)\times M_8(F)\times M_8(F)$ and $(\alpha_A,\alpha_B,\alpha_C) = (\alpha_0,\alpha_0,\alpha_0)$. We denote the corresponding isomorphism α_T by $\widetilde{\alpha}$; it is uniquely determined by:

$$\widetilde{\alpha}\circ \mathrm{ad}^{-1}\begin{pmatrix} x\\ y\\ z\end{pmatrix} = \begin{pmatrix} d_\rho(z) & d_{\rho^2}(y)\\ d_\rho(x) & d_{\rho^2}(z)\\ d_\rho(y) & d_{\rho^2}(x)\end{pmatrix} \tag{22}$$

for $(x,y,z)\in\mathfrak{o}(8)\times\mathfrak{o}(8)\times\mathfrak{o}(8)$. Let

$$M(F) = M_8(F)\times M_8(F)\times M_8(F) = M_8(F\times F\times F)$$

viewed as a central separable algebra over $F\times F\times F$. We call the data

$$\big(M(F), F\times F\times F, \widetilde{\sigma} = t, \widetilde{\alpha}\big)$$

the *standard trialitarian algebra over F*. We also abbreviate the data by $M(F)$.

Let ϕ be an F-automorphism of $\big(M(F), t\big)$; ϕ restricts to an automorphism γ of the center $F\times F\times F$, i.e., ϕ is γ-semilinear. We claim that ϕ extends to a γ-semilinear automorphism $C(\phi)$ of the Clifford algebra $C\big(M(F), t\big)$. More generally we have:

Lemma 7.3 *Let L be an étale F-algebra and let (V,q) be a quadratic space over L.*
1) Let γ be an F-automorphism of L and let f be a γ-semilinear similitude of (V,q), i.e. $q(fv) = m(f)\gamma\big(q(v)\big)$ for $v\in V$, $m(f)\in L^\times$. Then f extends to a γ-semilinear automorphism of $C_0(V,q)$.
2) Any F-automorphism ϕ of $\big(\mathrm{End}_L(V),\sigma_q\big)$ restricts to an automorphism γ of the center L and is of the form $\phi(x) = fxf^{-1}$ for some γ-semilinear similitude f of V. Further it induces a γ-semilinear automorphism $C(\phi)$ of $C\big(\mathrm{End}_L(V),\sigma_q\big) = C_0(V,q)$.

Proof The proof of 1) is similar to the proof of (2.5) (see also [W]). We prove claim 2); the automorphism ϕ induces an automorphism of $\mathrm{End}_F(V)$, hence is of the form $\mathrm{Int}(f)$ with $f\in\mathrm{GL}(V)$. The fact that ϕ restricts to γ on the center implies that f is γ-semilinear.

□

Let ϕ be an F-algebra automorphism of $M(F)$. Its restriction γ to the center $F\times F\times F$ of $M(F)$ induces an automorphism $\Delta(\phi)$ of the discriminant $\Delta = F\times F$. We say that ϕ is an *automorphism* of the standard trialitarian algebra $M(F)$ if $\widetilde{\alpha}\circ C(\phi) = (\phi\otimes\Delta(\phi))\circ\widetilde{\alpha}$, i.e., the following diagram is commutative:

$$\begin{array}{ccc} C\big(M(F),t\big) & \xrightarrow{\widetilde{\alpha}} & {}^{\widetilde{\rho}}\big(M(F)\otimes\Delta\big) \\ {\scriptstyle C(\phi)}\big\downarrow & & \big\downarrow{\scriptstyle \phi\otimes\Delta(\phi)} \\ C\big(M(F),t\big) & \xrightarrow{\widetilde{\alpha}} & {}^{\widetilde{\rho}}\big(M(F)\otimes\Delta\big) \end{array} \tag{23}$$

Proposition 7.4 *We have* $\mathrm{Aut}_F(M(F))\simeq \mathrm{PGO}_8^+(F)\rtimes S_3$.

Proof Since any automorphism of $M(F)$ restricts to an automorphism of the center $F\times F\times F$ of $M(F)$, there exists a homomorphism $\theta\colon \mathrm{Aut}_F\big(M(F)\big)\to \mathrm{Aut}_F(F\times F\times F) = S_3$. We show that θ is surjective by exhibiting a section s. Let ρ_0 be the 3-cycle $(1\,2\,3)$, so that ρ_0 acts on $F\times F\times F$ as $\rho_0(x,y,z) = (z,x,y)$, and let π_0 the transposition $(2\,3)\in S_3$, so that $\pi_0(x,y,z) = (x,z,y)$. We define for U, V, and $W\in M_8(F)$: $s(\rho_0)(U,V,W) = (W,U,V)$ and $s(\pi_0)(U,V,W) = (PUP,PWP,PVP))$, where P is the matrix of the Cayley conjugation π with respect to an orthogonal basis as in Section 5. Since $s(\rho_0)s(\pi_0) = s(\pi_0)s(\rho_0^2)$, s is a section. An automorphism ϕ belonging to the kernel of θ restricts to identity on the center of $M(F)$; it is $(F\times F\times F)$-linear, hence of the form $(a,\,b,\,c)$, for a, b and $c\in \mathrm{PGO}_8^+(F)$. Further it induces the identity on Δ. It then follows from the commutativity of (23) that

$$\begin{array}{ccc} C_0(8) & \xrightarrow{\alpha_0} & M_8(F)\times M_8(F) \\ {\scriptstyle C(a)}\big\downarrow & & \big\downarrow{\scriptstyle (b,c)} \\ C_0(8) & \xrightarrow{\alpha_0} & M_8(F)\times M_8(F) \end{array}$$

commutes. By (11) we have that $b=\overline{\rho}(a)$, $c=\overline{\rho}^2(a)$ and the map $\phi\mapsto a$ gives an isomorphism of the kernel of θ with $\mathrm{PGO}_8^+(F)$. Thus $\mathrm{Aut}_F\big(M(F)\big)\simeq \mathrm{PGO}_8^+\rtimes S_3$, as claimed.

∎

Remark 7.5 An explicit describtion of the action of the semidirect product $\mathrm{PGO}_8^+(F)\rtimes S_3$ follows from the proof of (7.4): let $(b,\beta)\in \mathrm{PGO}_8^+(F)\rtimes S_3$ and let $(U,V,W)\in M(F) = M_8(F)\times M_8(F)\times M_8(F)$. Let $b=[B]$, $\overline{\rho}(b)=[B_1]$ and $\overline{\rho}^2(b)=[B_2]$ for B, B_1, $B_2\in \mathrm{GO}_8^+(F)$. We have

$$\begin{aligned} (b,1)(U,V,W) &= (BUB^{-1}, B_1VB_1^{-1}, B_2WB_2^{-1}),\\ (1,\rho_0)(U,V,W) &= (W,U,V)\\ (1,\pi_0)(U,V,W) &= (PUP,PWP,PVP) \end{aligned}$$

for ρ_0, π_0 and P as above.

Let L be a cubic étale F-algebra and let Δ be the *discriminant algebra* of L, i.e., $\Delta = F[D]$, where the class of D^2 in $F^\times/F^{\times 2}$ is the discriminant of L as an étale F-algebra. We recall that $\Delta\simeq F\times F$ if and only L is cyclic; If Δ is not isomorphic to $F\times F$ then $L\otimes\Delta$ is Galois over F with group S_3 and $L\otimes\Delta/\Delta$ is cyclic. The case $L=F\times F\times F$ was extensively used above (see (20)). Let ρ be a generator of $\mathrm{Gal}(L\otimes\Delta/\Delta)$; for any L-module V we denote ${}^{\rho}(V\otimes\Delta)$ the module $V\otimes\Delta$ with $L\otimes\Delta$-action twisted through ρ. Let T be an central separable L-algebra, with an orthogonal involution σ_T. We say that T is *pre-trialitarian* if there exists an L-isomorphism

$$\alpha_T\colon \big(C(T,\sigma_T),\underline{\sigma}\big) \xrightarrow{\sim} {}^{\rho}\big((T,\sigma_T)\otimes_F\Delta\big). \tag{24}$$

The standard trialitarian algebra $\widetilde{T}=M(F)$ is pre-trialitarian with $\alpha_T=\widetilde{\alpha}$.

An *isomorphism of pre-trialitarian* algebras $\Psi:(T,L,\sigma_T,\alpha_T)\xrightarrow{\sim}(T',L',\sigma_{T'},\alpha_{T'})$ is a pair $(\psi:T\xrightarrow{\sim}T',\phi:L\xrightarrow{\sim}L')$ of F-isomorphisms where ψ is ϕ-semilinear, such that the diagram

$$\begin{tikzcd} C(T,\sigma_T) \arrow[r, "\alpha_T"] \arrow[d, "C(\psi)"'] & {}^{\rho}\big(T\otimes\Delta(L)\big) \arrow[d, "\psi\otimes\Delta(\phi)"] \\ C(T',\sigma_{T'}) \arrow[r, "\alpha_{T'}"] & {}^{\rho}\big(T'\otimes\Delta(L')\big) \end{tikzcd} \tag{25}$$

commutes. For a field extension F'/F and any pre-trialitarian F-algebra T, $T'=T\otimes_F F'$ is pre-trialitarian over F' in an obvious way. We say that a pre-trialitarian F-algebra T is *trialitarian over F* if there exists a field extension F'/F and an isomorphism $T\otimes_F F'\xrightarrow{\sim} M(F')$, where $M(F')$ is the standard trialitarian algebra over F'.

The algebra $A\times B\times C$ with A, B and C as in (7.2) is trialitarian over F, with $L=F\times F\times F$ and α_T as defined in (21). Summarizing the previous discussion we have:

Proposition 7.6 *Trialitarian algebras over a field F are classified by the pointed set $H^1(F,\mathrm{PGO}_8^+\rtimes S_3)$, with the class of the standard trialitarian algebra $M(F)$ as the point. The map $H^1(F,\mathrm{PGO}_8^+)\to H^1(F,\mathrm{PGO}_8^+\rtimes S_3)$ of (18) associates to a central simple F-algebra (A,σ_A) of degree 8, with orthogonal involution σ_A with discriminant 1, the data $\big(A\times B\times C,F\times F\times F,(\sigma_A,\sigma_B,\sigma_C),(\alpha_A,\alpha_B,\alpha_C)\big)$ as in (7.2).*

□

Let $T=(T,L,\sigma_T,\alpha_T)$ be a trialitarian algebra and let

$$\eta_T:\mathrm{Skew}(T,\sigma_T)\to\ \mathrm{Skew}\big(C(T,\sigma_T),\underline{\sigma}\big)$$

be the canonical embedding (15). Let Z_T be the center of $C(T,\sigma_T)$. The isomorphism $\alpha_T\colon C(T,\sigma_T)\xrightarrow{\sim}{}^{\rho}(T\otimes\Delta)$ restricts to an F-isomorphism $\alpha'\colon Z_T\xrightarrow{\sim} L\otimes\Delta$ and we have an isomorphism $1\otimes\alpha'^{-1}$ of $\mathrm{Skew}(T,\sigma_T)\otimes_F\Delta=\mathrm{Skew}(T,\sigma_T)\otimes_L(L\otimes\Delta)$ with $\mathrm{Skew}(T,\sigma_T)\otimes_L Z_T$. In view of (2.4) we have $\mathrm{Skew}\big(C(T,\sigma_T),\underline{\sigma}\big)=\eta_T\big(\mathrm{Skew}(T,\sigma_T)\big)\otimes_L Z_T$ so that $\alpha_T\circ(\eta_T\otimes 1_{Z_T})$ is an isomorphism $\mathrm{Skew}(T,\sigma_T)\otimes_L Z_T\xrightarrow{\sim}{}^{\rho}\big(\mathrm{Skew}(T,\sigma_T)\otimes_F\Delta\big)$. The composition of these two isomorphisms is an isomorphism

$$\alpha_\rho\colon\ \mathrm{Skew}(T,\sigma_T)\otimes_F\Delta\xrightarrow{\sim}{}^{\rho}\big(\mathrm{Skew}(T,\sigma_T)\otimes_F\Delta\big)$$

which, in turn, can be viewed as a ρ-semilinear automorphism of the $(L\otimes\Delta)$-Lie algebra $\mathrm{Skew}\big((T,\sigma_T)\otimes_F\Delta\big)$. Let $\alpha_\pi=1_{\mathrm{Skew}(T,\sigma_T)}\otimes\iota$, where ι is conjugation on the quadratic algebra Δ.

Proposition 7.7 *The pair of automorphisms α_ρ and α_π constitutes a descent datum on the Lie algebra* $\operatorname{Skew}(T,\sigma_T)\otimes_F \Delta$ *from $L\otimes\Delta$ to F with respect to the Galois group S_3. The fixed points of the S_3-action*

$$\mathfrak{o}(T)=\{x\in \operatorname{Skew}(T,\sigma_T)\otimes_F\Delta \mid \alpha_\gamma(x)=x \quad \textit{for all } \gamma\in S_3\}$$

is a Lie algebra (of type D_4) associated with the triality T. For the standard trialitarian algebra $T=M(F)$, $\mathfrak{o}(T)=\mathfrak{o}_8$.

Proof The fact that α_ρ and α_π generate a S_3-semilinear action needs to be checked only in the standard case, by going over to a standard splitting. For the standard trialitarian algebra $T=M_8(F)\times M_8(F)\times M_8(F)$ we have $\Delta=F\times F$ and, by (20), for x_i, y_i,$z_i\in\mathfrak{o}_8(F)$

$$\alpha_\rho\begin{pmatrix}x_1 & x_2\\ y_1 & y_2\\ z_1 & z_2\end{pmatrix}=\begin{pmatrix}d_\rho(z_1) & d_{\rho^2}(y_2)\\ d_\rho(x_1) & d_{\rho^2}(z_2)\\ d_\rho(y_1) & d_{\rho^2}(x_2)\end{pmatrix}\quad\text{and}\quad \alpha_\pi\begin{pmatrix}x_1 & x_2\\ y_1 & y_2\\ z_1 & z_2\end{pmatrix}=\begin{pmatrix}x_2 & x_1\\ y_2 & y_1\\ z_2 & z_1\end{pmatrix}.$$

The first claim then follows from this explicit description of α_ρ and α_π. We leave the last claim as an exercise.

□

Remark 7.8 If L/F is cubic cyclic the discriminant Δ is split, ${}^{\rho}\big(L\otimes\Delta\big)= {}^{\rho}L\times {}^{\rho^2}L$ and we have

$$\alpha_T\colon \big(C(T,\sigma_T),\underline{\sigma}\big)\xrightarrow{\sim} {}^{\rho}(T,\sigma)\times {}^{\rho^2}(T,\sigma).$$

We say in this case that T is *cyclic trialitarian*. Cyclic trialitarian algebras are classified by the pointed set $H^1\big(F,\mathrm{PGO}_8^+\rtimes A_3\big)$. The restriction of α_T to $\operatorname{Skew}(T,\sigma_T)$ is $(\alpha_\rho,\alpha_{\rho^2})$ and we get a Galois descent datum for $\operatorname{Skew}(T,\sigma_T)$ from L to F.

8 Generic torsors

Let H be a closed subgroup of GL_n. We have an exact sequence of pointed sets

$$1\to H(F)\to \mathrm{GL}_n(F)\to \big(\mathrm{GL}_n/H\big)(F)\xrightarrow{\delta} H^1(F,H)\to H^1(F,\mathrm{GL}_n)\tag{26}$$

(see for example [KMRT]) and $H^1(F,\mathrm{GL}_n)=0$ by Hilbert 90. We recall that the set $H^1(F,H)$ classifies H-torsors. Let $F(\mathrm{GL}_n/H)$ be the function field of the homogeneous variety $\big(\mathrm{GL}_n/H\big)$ which is defined over F. Let η be the generic point of GL_n/H. In view of the exact sequence (26) (over $F(\mathrm{GL}_n/H)$), the element $\eta\in \mathrm{GL}_n/H\big(F(\mathrm{GL}_n/H)\big)$ gives rise to an H-torsor $\delta(\eta)=\xi\in H^1\big(F(\mathrm{GL}_n/H),H\big)$, called the *generic H-torsor*. Any H-torsor is a *specialization* of ξ: more precisely, let $F\subset F'$ be a field extension and let $x\in H^1(F',H)$. There exists a point ζ of the variety GL_n/H and an embedding $i_\zeta: F(\zeta)\to F'$ such that $x=i^1_\zeta\big(\delta(\zeta)\big)$, where $i^1_\zeta: H^1\big(F(\zeta),H\big)\to H^1(F',H)$ is the induced map in Galois cohomology.

Let $H=\mathrm{PGO}^+(8)\rtimes S_3$ and $H\hookrightarrow \mathrm{GL}_n$ be an embedding over F. The corresponding generic H-torsor ξ defines a trialitarian algebra $T_\xi=(T_\xi,L_\xi,\sigma_\xi,\alpha_\xi)$ over the function field $F(\mathrm{GL}_n/H)$.

Proposition 8.1 *The algebra T_ξ is a division algebra.*

Proof Let Z be the center of T_ξ. We claim that Z is a field. We may choose a specialization F_1 of $F(\mathrm{GL}_n/H)$ with a separable cubic field extension M/F_1. There is a (quasi-split) trialitarian algebra T_1 with center M (see for example [KMRT, Example (43.7)].) Specializing T to T_1 specializes the center Z of T to M. Thus Z has to be a field. Suppose that T_ξ is not a division algebra, say $T_\xi = M_2(A)$ for some central simple algebra A over Z. This implies (by specialization) that for any cubic étale algebra M_0/F_0 over any field F_0 which contains F, any trialitarian F_0-algebra with center M_0 has to be of the form $M_2(A_0)$ for some algebra A_0 over M_0, which is a specialization of A. Let (D,σ) be a central division algebra of degree 8 over some field extension F_1 of F, with an orthogonal involution σ, for example a tensor product of three generic quaternion algebras. Let Δ be the discriminant of σ (as a quadratic extension of F_1). In view of [KMRT, Proposition (43.15)], $(D,\sigma)\times\big(C(D,\sigma),\underline{\sigma}\big)$ is a trialitarian algebra over $F_1\times\Delta$. Since D is a division algebra, $(D,\sigma)\times\big(C(D,\sigma),\underline{\sigma}\big)$ is not of the form $M_2(A_0)$. This is a contradiction. Hence T_ξ has to be a division algebra.

□

9 Generic trialitarian division algebras

The aim of this section is to construct explicitly a generic trialitarian algebra. Our construction is inspired by and closely follows the presentation of generic division algebra given in Saltman [S2].

Let F be a field and let r be a positive integer. Consider the polynomial ring $S = F[x_{ijk}, y_{ijk}, z_{ijk} \mid 1\le i,j\le 8; 1\le k\le r]$. In $M_8(S)$ we have the generic matrices X_k, Y_k and Z_k whose entries are given by $(X_k)_{ij} = x_{ijk}$, $(Y_k)_{ij} = y_{ijk}$ and $(Z_k)_{ij} = z_{ijk}$. Let U^t be the transpose of the matrix U and let $\widetilde{U} = U - U^t$. We recall (Section 4) that by triality, we can associate to any skew-symmetric matrix V, skew-symmetric matrices $d_\rho(V)$ and $d_{\rho^2}(V)$. Let $D = D_1D_2D_3$ where

$$\begin{array}{rcl} D_1 &=& \mathrm{pf}\big(\widetilde{X_1} + d_\rho(\widetilde{Z_1}) + d_{\rho^2}(\widetilde{Y_1})\big),\\ D_2 &=& \mathrm{pf}\big(\widetilde{Y_1} + d_\rho(\widetilde{X_1}) + d_{\rho^2}(\widetilde{Z_1})\big),\\ D_3 &=& \mathrm{pf}\big(\widetilde{Z_1} + d_\rho(\widetilde{Y_1}) + d_{\rho^2}(\widetilde{X_1})\big). \end{array} \tag{27}$$

Let $R(F,r)$ be the F-subalgebra of $M(S) = M_8(S)\times M_8(S)\times M_8(S)$ generated by all triples

$$\begin{pmatrix} X_k\\ Y_k\\ Z_k\end{pmatrix},\quad \begin{pmatrix} X_k^t\\ Y_k^t\\ Z_k^t\end{pmatrix},\quad \begin{pmatrix} d_\rho(\widetilde{Z_k}) + d_{\rho^2}(\widetilde{Y_k})\\ d_\rho(\widetilde{X_k}) + d_{\rho^2}(\widetilde{Z_k})\\ d_\rho(\widetilde{Y_k}) + d_{\rho^2}(\widetilde{X_k})\end{pmatrix} \quad\text{and}\quad \begin{pmatrix} \big(d_\rho(\widetilde{Z_k}) - d_{\rho^2}(\widetilde{Y_k})\big)D\\ \big(d_\rho(\widetilde{X_k}) - d_{\rho^2}(\widetilde{Z_k})\big)D\\ \big(d_\rho(\widetilde{Y_k}) - d_{\rho^2}(\widetilde{X_k})\big)D\end{pmatrix}. \tag{28}$$

All these generators, in particular the last two classes, are needed to define generically the "α" map which occurs in the definition of a trialitarian algebra. Let $(T, L, \sigma_T, \alpha_T)$ be a trialitarian algebra over a field extension E of F. An F-algebra homomorphism $\psi\colon R(F,r)\to T$ is called *trialitarian* if, for a standard splitting $\phi\colon T\otimes_E E' \xrightarrow{\sim} M(E')$ of the trialitarian algebra T, there exists a homomorphism $\nu\colon S\to E'$ such that $M(\nu)\colon M(S)\to M(E')$ restricts to ψ.

Proposition 9.1 *Let $(T, L, \sigma_T, \alpha_T)$ be a trialitarian algebra over a field extension E of F. Let a_1, a_2, …, a_r be elements of T. There exists a unique trialitarian F-algebra homomorphism $\psi\colon R(F,r)\to T$ such that $\psi\big((X_k, Y_k, Z_k)\big) = a_k$ for $k = 1,\dots,r$.*

Proof We assume that the discriminant Δ of the cubic extension L is a field (and leave the split case as an exercise). Let be $E \subset \Delta \subset E'$ be a field extension such that $(T, L, \sigma_T, \alpha_T) \otimes_E E'$ is isomorphic to the standard trialitarian algebra over E', i.e., there is an isomorphism of trialitarian algebras $(T, L, \sigma_T, \alpha_T) \otimes_E E' \xrightarrow{\sim} M(E')$. We use the isomorphism to identify T with a subalgebra of $M(E')$. Let

$$a_k \otimes 1 = \big(A_k^1, A_k^2, A_k^3\big) \in M(E') = M_8(E') \times M_8(E') \times M_8(E')$$

where $(A_k^\epsilon)_{ij} = a_{ijk}^\epsilon \in E'$, $\epsilon = 1,\ 2,\ 3$. Define $\phi\colon S \to E'$ by $\phi(x_{ijk}) = a_{ijk}^1$, $\phi(y_{ijk}) = a_{ijk}^2$ and $\phi(z_{ijk}) = a_{ijk}^3$. Then ϕ induces a homomorphism of algebras $M(\phi) : M(S) \to M(E')$, where $M(\phi)\big((s_{ij}^\epsilon)\big) = \big(\phi(s_{ij}^\epsilon)\big)$ for any triple of matrices $(s_{ij}^\epsilon) \in M_8(S)$. Let ψ be the restriction of $M(\phi)$ to $R(F, r)$. We claim that $\psi\big(R(F, r)\big) \subset T$ and that ψ is uniquely determined by the condition $\psi\big((X_k, Y_k, Z_k)\big) = a_k$ for $k = 1, \dots, r$. We check that the images under ψ of the generators of $R(F, r)$ are in T and are determined by the a_k's. Since t restricts to σ_T, $\psi(X_k^t, Y_k^t, Z_k^t) = \sigma_T(a_k) \in T$ and $\psi(\widetilde{X_k}, \widetilde{Y_k}, \widetilde{Z_k}) = \widetilde{a_k} = a_k - \sigma_T(a_k) \in T$.

Let ι be the conjugation map of Δ. Since $\Delta \subset E'$ we have canonical isomorphisms $\Delta \otimes_E E' \xrightarrow{\sim} E' \times E'$, $x \otimes y \mapsto \big(xy, \iota(x)y\big)$, $x \in \Delta$ and $y \in E'$, and

$$(T \otimes_E \Delta) \otimes_E E' \xrightarrow{\sim} T \otimes_E E' \times T \otimes_E E', \quad (a \otimes x) \otimes 1 \mapsto \big(ax, a\iota(x)\big)$$

Through this canonical isomorphism we have:

$$(\alpha_T \circ \eta_T)(\widetilde{a_k}) \otimes_E 1_{E'} \mapsto \begin{pmatrix} d_\rho(\tilde{A}_k^3) & d_{\rho^2}(\tilde{A}_k^2) \\ d_\rho(\tilde{A}_k^1) & d_{\rho^2}(\tilde{A}_k^3) \\ d_\rho(\tilde{A}_k^2) & d_{\rho^2}(\tilde{A}_k^1) \end{pmatrix} \in T \otimes_E E' \times T \otimes_{F'} E'$$

thus

$$\psi \begin{pmatrix} d_\rho(\widetilde{Z_k}) & d_{\rho^2}(\widetilde{Y_k}) \\ d_\rho(\widetilde{X_k}) & d_{\rho^2}(\widetilde{Z_k}) \\ d_\rho(\widetilde{Y_k}) & d_{\rho^2}(\widetilde{X_k}) \end{pmatrix} = \alpha_T \circ \eta_T(\tilde{a}_k) \in {}^\rho\big(\mathrm{Skew}(T, \sigma_T) \otimes_E \Delta\big). \tag{29}$$

We next compute $M(\phi)(D_1, D_2, D_3)$ and $\phi(D)$. Let $\mathrm{Tr}_{T \otimes_E \Delta/T}\colon T \otimes_E \Delta \to T$ be the map induced by the trace map $\Delta \to E$ and let $\mathrm{N}_{L \otimes \Delta/\Delta}\colon L \otimes_E \Delta \to \Delta$ be induced by the norm map. Since α_T restricts to an isomorphism of centers $Z_T \xrightarrow{\sim} L \otimes_E \Delta$, $L \otimes_E \Delta \simeq L[X]/(X^2 - d)$ with $[d] = \mathrm{disc}(\sigma_T) \in L^\times/L^{\times 2}$. It follows that $\sigma_T \otimes_L 1_{L \otimes_E \Delta}$ is an involution of $T \otimes_L (L \otimes_E \Delta) = T \otimes_E \Delta$ with trivial discriminant. By (6.1) there exists a reduced pfaffian $\mathrm{Pfrd}_{T \otimes_E \Delta}\colon \mathrm{Skew}(T \otimes_E \Delta, \sigma_{T \otimes_E \Delta}) = \mathrm{Skew}(T, \sigma_T) \otimes_E \Delta \to L \otimes_E \Delta$. This pfaffian is the restriction of $\mathrm{Pfrd}_{M(E')} = (\mathrm{pf}, \mathrm{pf}, \mathrm{pf})$ to $T \otimes_E \Delta$. Since $a_1 = \psi\big((\widetilde{X_1}, \widetilde{Y_1}, \widetilde{Z_1})\big)$ and

$$\psi \begin{pmatrix} d_\rho(\widetilde{Z_k}) + d_{\rho^2}(\widetilde{Y_k}) \\ d_\rho(\widetilde{X_k}) + d_{\rho^2}(\widetilde{Z_k}) \\ d_\rho(\widetilde{Y_k}) + d_{\rho^2}(\widetilde{X_k}) \end{pmatrix} = \mathrm{Tr}_{T \otimes_E \Delta/T}\big((\alpha_T \circ \eta_T)(\tilde{a}_k)\big) \in T,$$

we have $M(\phi)(D_1, D_2, D_3) = \mathrm{Pfrd}_{T \otimes_E \Delta}\big(\tilde{a}_1 + \mathrm{Tr}_{T \otimes_E \Delta/T}\big((\alpha_T \circ \eta_T)(\tilde{a}_1)\big)\big) \in L \otimes_E \Delta$ and

$$\phi(D) = \mathrm{N}_{L \otimes_E \Delta/\Delta}\big(\mathrm{Pfrd}_{T \otimes_E \Delta}\big(\tilde{a}_1 + \mathrm{Tr}_{T \otimes_E \Delta/T}\big((\alpha_T \circ \eta_T)(\tilde{a}_1)\big)\big) \in \Delta.$$

Let $D' = \phi(D)$; since $\phi(D^2) = \mathrm{N}_{L \otimes_E \Delta/\Delta}\big(\mathrm{Nrd}_T\big(\tilde{a}_1 + \mathrm{Tr}_{T \otimes_E \Delta/T}\big((\alpha_T \circ \eta_T)(\tilde{a}_1)\big)\big) \in E$, it follows that $\iota(D') = \pm D'$. Since the discriminant Δ of σ_T is not trivial we must have $\iota(D') = -D'$ and

$$\big((\alpha_T \circ \eta_T)(\widetilde{a_k}) \otimes 1_{E'}\big)D' = \begin{pmatrix} d_\rho(\tilde{A}_k^3)D' & -d_{\rho^2}(\tilde{A}_k^2)D' \\ d_\rho(\tilde{A}_k^1)D' & -d_{\rho^2}(\tilde{A}_k^3)D' \\ d_\rho(\tilde{A}_k^2)D' & -d_{\rho^2}(\tilde{A}_k^1)D' \end{pmatrix},$$

so that

$$\psi\begin{pmatrix}(d_\rho(\widetilde{Z_k}) - d_{\rho^2}(\widetilde{Y_k}))D\\(d_\rho(\widetilde{X_k}) - d_{\rho^2}(\widetilde{Z_k}))D\\(d_\rho(\widetilde{Y_k}) - d_{\rho^2}(\widetilde{X_k}))D\end{pmatrix} = T_{T\otimes_E\Delta/T}\big((\alpha_T\circ\eta_T)(\tilde a_k)\otimes 1_{E'}D'\big)\in T.$$

Hence, as claimed, the images of the generators of $R(F,r)$ are in T and are determined by the a_k's.

□

Let $E\subset E'$ be a field extension and let

$$V_T(E') = \mathrm{Hom}_F\big(R(F,r), T\otimes_E E'\big)$$

be the set of all trialitarian F-algebra homomorphisms from $R(F,r)$ to $T\otimes_E E'$. Let $R_{L/E}(T^r)$ be the affine space T^r viewed as a variety over E. In view of (9.1) we have:

Corollary 9.2 *$V_T(E')$ can be identified with the set of E'-points of $R_{L/E}(T^r)$.*

Let $0\neq s\in R(F,r)$. The set $V_{T,s}(E')$ of all trialitarian F-algebra homomorphisms $\phi: R(F,r)\to T$ such that $\phi(s)\neq 0$ is the set of E'-points of an open subvariety $V_{T,s}$ of V_T. If E' is infinite, $V_{T,s}(E')$ is nonempty.

Proposition 9.3 *The ring $R(F,r)$ is a noncommutative domain.*

Proof Let s, t be nonzero elements of $R(F,r)$. Choose (T,L,σ_T,α_T) over some field $E\supset F$ such that T is a division algebra. Such algebras exist in view of (8.1). Since V_T is irreducible, $V_{T,s}\cap V_{T,t}$ is an open subvariety of V_T and, hence, has a E-rational point ϕ: $R(F,r)\to T$. Thus $\phi(s)$ and $\phi(t)$ are not zero and, since T is a division algebra $\phi(st)=\phi(s)\phi(t)\neq 0$, so that $st\neq 0$.

□

The center $Z(F,r)$ of $R(F,r)$ is a commutative domain (by (9.3)). Let $L(F,r)$ be its field of fractions and let $UT(F,r)$ be the ring obtained from $R(F,r)$ by inverting all the nonzero elements of $Z(F,r)$. Then $UT(F,r)$ is a (noncommutative) domain with center $L(F,r)$.

Theorem 9.4 *The ring $UT(F,r)$ is a division ring. Further it is of degree* 8 *over its center $L(F,r)$.*

Proof Since $R(F,r)\subset M(S) = M_8(S\times S\times S)$ and since $R(F,r)$ is a domain, $R(F,r)$ is a prime PI-ring. By [R2, Theorem 6.1.30] $UT(F,r)$ is central simple finite dimensional over $L(F,r)$, hence is a central division algebra over $L(F,r)$. To check that $UT(F,r)$ is of degree 8 over $L(F,r)$, we need some intermediate steps.

Proposition 9.5 *1) Assume that $r\geq 3\cdot 8^2$. There exists a basis of $M(K)$ over K consisting of triples $(X_k,Y_k,Z_k)\in R(F,r)$. In particular we have $R(F,r)K = M(K)$.*
2) Let $\mathrm{Skew}(K) = \mathrm{Skew}_8(K)\times\mathrm{Skew}_8(K)\times\mathrm{Skew}_8(K)\subset M(K)$ be the Lie algebra of skew-symmetric elements in $M(K)$. Assume that $r\geq 3\cdot 8^2$. There is a basis of $\mathrm{Skew}(K)$ over K consisting of triples $(\widetilde{X_k},\widetilde{Y_k},\widetilde{Z_k})\in R(F,r)$.

Proof For the first claim, let $\{e_{ij}\}$, $1 \leq i, j \leq 8$, be the standard basis of $M_8(K)$, so that $\big((e_{ij}, 0, 0),\ (0, e_{ij}, 0),\ (0, 0, e_{ij})\big)$ is a K-basis of $M(K)$. We have in $M(K)$:

$$(X_k, Y_k, Z_k) = \sum_{i,j} x_{ijk}(e_{ij}, 0, 0) + \sum_{i,j} y_{ijk}(0, e_{ij}, 0) + \sum_{i,j} z_{ijk}(0, 0, e_{ij})$$

for $k = 1, \ldots, r$. The matrix of the coefficients has generic entries, hence, if $r \geq 3 \cdot 8^2$, the determinant of the coefficient matrix of the first $3 \cdot 8^2$ equations is not zero in K so that the elements $\{(e_{ij}, 0, 0),\ (0, e_{ij}, 0),\ (0, 0, e_{ij})\}$ can be expressed as K-linear combinations of the triples (X_k, Y_k, Z_k), $k \leq 3 \cdot 8^2$. A similar argument gives the second claim.

□

Remark 9.6 The condition $r \geq 3 \cdot 8^2$ in (9.5) is not the best possible. Since X and X^t generate $M_8(K)$ as a K-algebra, three triples (X_k, Y_k, Z_k) and their transposes generate $M(K)$ as a K-algebra. Hence $r \geq 3$ would suffice.

Corollary 9.7 *1)* $Z(F, r) \subset S \times S \times S$.
2) If $(T, L, \sigma_T, \alpha_T)$ *is trialitarian and* $\psi\colon R(F, r) \to T$ *is an* F*-algebra homomorphism, then* $\psi\big(Z(F, r)\big) \subset L$.

Proof By (9.5) $R(F, r)$ generates $M(K)$ over K, hence $R(F, r)K = M(K)$. Since $Z(F, r)$ is central in $M(K)$, $Z(F, r)$ is contained in $K \times K \times K$. The first claim then follows from $M(S) \cap (K \times K \times K) = S \times S \times S$. Let $\phi : R(F, r) \to T$ be an F-algebra homomorphism. Let $T \otimes_E E' \xrightarrow{\sim} M(E')$ of $(T, L, \sigma_T, \alpha_T)$ be a standard splitting. We identify $(T, L, \sigma_T, \alpha_T) \otimes_E E'$ with its image in $M(E')$. Let $\psi\colon R(F, r) \to T \hookrightarrow M(E')$ be written as (ψ^1, ψ^2, ψ^3) with $\psi^i\colon R(F, r) \to M_8(E')$, $1 \leq i \leq 3$, the components of ψ. We define $\phi\colon S \to E'$ by setting $\phi(x_{ijk}) = \psi^1(X_k, Y_k, Z_k)_{ij}$, $\phi(y_{ijk}) = \psi^2(X_k, Y_k, Z_k)_{ij}$ and $\phi(z_{ijk}) = \psi^3(X_k, Y_k, Z_k)_{ij}$. The maps ψ and $M(\phi)|_{R(F,r)}$ give rise F-algebra homomorphisms $R(F, r) \to M(E')$ which coincide on (X_k, Y_k, Z_k),$1 \leq k \leq r$. By the uniqueness statement in (9.1), ψ is the restriction of $M(\phi)$ to $R(F, r)$. Since $M(\phi)(S \times S \times S) \subset L \otimes_E E'$, we have $\psi\big(Z(F, r)\big) \subset L \otimes_E E'$. The second claim then follows from $A \cap (L \otimes_E E') = L$.

□

We now prove that $UT(F, r)$ is of degree 8 over $L(F, r)$. Let $\mathcal{S}$ be the set of nonzero elements of $Z(F, r)$. Since $Z(F, r) \subset S \times S \times S$, $L(F, r)$ is contained in $K \times K \times K$. This inclusion and the inclusion $R(F, r) \subset M(S)$ induce inclusion $UT(F, r) = R(F, r)_{\mathcal{S}} \to M(K)$. We use it to identify the algebra $UT(F, r)$ with a subalgebra of $M(K)$. The induced map $UT(F, r) \otimes_{L(F,r)} (K \times K \times K) \to M(K)$ is surjective by (9.5). On the other hand $UT(F, r) \otimes_{L(F,r)} (K \times K \times K)$ is central separable over $K \times K \times K$, so $UT(F, r) \otimes_{L(F,r)} (K \times K \times K) \xrightarrow{\sim} M(K)$ and $UT(F, r)$ is of degree 8 over $L(F, r)$.

□

10 Invariants of $\mathrm{PGO}_8^+(F) \rtimes S_3$

Let G the semidirect product $\mathrm{PGO}_8^+ \rtimes S_3$ as an algebraic group over F. We keep the same notation as in the previous section. In particular we set $S = F[x_{ijk}, y_{ijk}, z_{ijk}]$. We describe an action of $G(F)$ on S and on $M(S) = M_8(S) \times M_8(S) \times M_8(S)$: for $b = [B] \in \mathrm{PGO}_8(F)$, and $U \in M_8(S)$ let $b(U) = BUB^{-1}$. Let $b_1 = \overline{\rho}(b)$, $b_2 = \overline{\rho}^2(b) \in \mathrm{PGO}_8^+(F)$, $\overline{\rho}$ as in (11). We define $\phi_b \colon S \to S$ by

$$\phi_b(x_{ijk}) = \big(b^{-1}(X_k)\big)_{ij},\ \phi_b(y_{ijk}) = \big(b_1^{-1}(Y_k)\big)_{ij} \quad \text{and} \quad \phi_b(z_{ijk}) = \big(b_2^{-1}(Z_k)\big)_{ij}. \tag{30}$$

We have, by definition of ϕ_b, $M_8(\phi_b)(X_k) = b^{-1}(X_k)$, $M_8(\phi_b)(Y_k) = b_1^{-1}(Y_k)$, $M_8(\phi_b)(Z_k) = b_2^{-1}(Z_k)$. For $\rho = (1\,2\,3)$ and $\pi = (2\,3) \in S_3$ we set

$$\begin{array}{rclcrcl} \phi_\rho(x_{ijk}) & = & z_{ijk} & & \phi_\pi(x_{ijk}) & = & (PX_kP)_{ij} \\ \phi_\rho(y_{ijk}) & = & x_{ijk} & \text{and} & \phi_\pi(y_{ijk}) & = & (PZ_kP)_{ij} \\ \phi_\rho(z_{ijk}) & = & y_{ijk} & & \phi_\pi(z_{ijk}) & = & (PY_kP)_{ij} \end{array} \tag{31}$$

where $P \in GO_8(F)$ is the matrix of the conjugation π of the standard octonions with respect to the basis chosen in Section 4.

Lemma 10.1 *The actions* (30) *and* (31) *combine to define an action $g \mapsto \phi_g$ of $G(F)$ on the ring S.*

Proof A straightforward computation!

$\square$

The action of $G(F)$ on $M_8(S) \times M_8(S) \times M_8(S)$ is as follows: let $(U, V, W) \in M_8(S) \times M_8(S) \times M_8(S)$. For $b \in \mathrm{PGO}_8^+(F)$ we set

$$\psi_b(U, V, W) = b \circ M(\phi_b) = \big(b \circ M_8(\phi_b)U,\ b_1 \circ M_8(\phi_b)V,\ b_2 \circ M_8(\phi_b)W\big), \tag{32}$$

where b, b_1 and b_2 are as in (30). It is easy to check that ψ_b restricts to the identity map on the generic matrices X_k, Y_k and Z_k. For ρ and $\pi \in S_3$ as above we set

$$\psi_\rho(U, V, W) = \rho \circ M(\phi_\rho) = \big(M_8(\phi_\rho)(V),\ M_8(\phi_\rho)(W),\ M_8(\phi_\rho)(U)\big) \tag{33}$$

and

$$\psi_\pi(U, V, W) = \pi \circ M(\phi_\pi) = \big(PM_8(\phi_\pi)(U)P,\ PM_8(\phi_\pi)(W)P,\ PM_8(\phi_\pi)(V)P\big) \tag{34}$$

Again, it is straightforward to check that:

Lemma 10.2 *The actions* (32), (33) *and* (34) *combine to give an action $g \mapsto \psi_g$ of $G(F)$ on $M_8(S) \times M_8(S) \times M_8(S)$.*

We note that the above action of $G(F)$ on $M(S)$ restricts to an action of $G(F)$ on the center $S \times S \times S$. In the next proposition we describe the action of $G(F)$ on the generators of the F-algebra $R(F, r)$:

Proposition 10.3 *1) The triples (X_k, Y_k, Z_k) and (X_k^t, Y_k^t, Z_k^t) are invariant under the action of $G(F)$ on $M(S) = M_8(S) \times M_8(S) \times M_8(S)$.*

2) For any $U \in M_8(S)$, let $\widetilde{U} = U - U^t$. Let A_3 be the alternating group with generator ρ and let $G_0(F) = \mathrm{PGO}_8^+(F) \rtimes A_3$. The triple $\big(d_\rho(\widetilde{Z_k}), d_\rho(\widetilde{X_k}), d_\rho(\widetilde{Y_k})\big)$ and the triple

$\big(d_{\rho^2}(\widetilde{Y_k}), d_{\rho^2}(\widetilde{Z_k}), d_{\rho^2}(\widetilde{X_k})\big)$ are invariant under the group $G_0(F)$ and $\big(d_\rho(\widetilde{Z_k}), d_\rho(\widetilde{X_k}), d_\rho(\widetilde{Y_k})\big)$ is mapped to $\big(d_{\rho^2}(\widetilde{Y_k}), d_{\rho^2}(\widetilde{Z_k}), d_{\rho^2}(\widetilde{X_k})\big)$ under ψ_π.

3) The triple $\big(d_\rho(\widetilde{Z_k})+d_{\rho^2}(\widetilde{Y_k}), d_\rho(\widetilde{X_k})+d_{\rho^2}(\widetilde{Z_k}), d_\rho(\widetilde{Y_k})+d_{\rho^2}(\widetilde{X_k})\big)$ is invariant under $G(F)$.

4) The triple $(D_1, D_2, D_3) \in S \times S \times S$ with

$$\begin{aligned}
D_1 &= \operatorname{pf}\big(\widetilde{X_1} + d_\rho(\widetilde{Z_1}) + d_{\rho^2}(\widetilde{Y_1})\big),\\
D_2 &= \operatorname{pf}\big(\widetilde{Y_1} + d_\rho(\widetilde{X_1}) + d_{\rho^2}(\widetilde{Z_1})\big),\\
D_3 &= \operatorname{pf}\big(\widetilde{Z_1} + d_\rho(\widetilde{Y_1}) + d_{\rho^2}(\widetilde{X_1})\big)
\end{aligned}$$

is invariant under $G_0(F)$ and changes sign under ψ_π.

5) The element $D = D_1D_2D_3 \in S$ is invariant under $G_0(F)$ and changes sign under ϕ_π.
6) The triple $\big(\big(d_\rho(\widetilde{Z_k}) - d_{\rho^2}(\widetilde{Y_k})\big)D, \big(d_\rho(\widetilde{X_k}) - d_{\rho^2}(\widetilde{Z_k})\big)D, \big(d_\rho(\widetilde{Y_k}) - d_{\rho^2}(\widetilde{X_k})\big)D\big)$ is invariant under $G(F)$.

Proof Claims 1), 2) and 3) are straightforward to verify (use (5.3)); we check that D_1 is invariant under $\mathrm{PGO}_8^+(F)$: let $b = [B] \in \mathrm{PGO}_8^+(F)$; we have $BB^t = m(B)$ and $\det(B) = m(B)^4$. In view of (5.3) we have identities $bd_\rho = d_\rho\overline{\rho}^2(b)$ and $bd_{\rho^2} = d_{\rho^2}\overline{\rho}(b)$ and the following "small miracle" occurs:

$$\begin{aligned}
\phi_b(D_1) &= \operatorname{pf}\big(M_8(\phi_b)(\widetilde{X}_1) + d_\rho\big(M_8(\phi_{\overline{\rho}^2(b)})(\widetilde{Z}_1)\big) + d_{\rho^2}\big(M_8(\phi_{\overline{\rho}(b)})(\widetilde{Y}_1)\big)\big)\\
&= \operatorname{pf}\big(b^{-1}\widetilde{X}_1 + d_\rho(\overline{\rho}^2(b)^{-1}\widetilde{Z}_1) + d_{\rho^2}(\overline{\rho}(b)^{-1}\widetilde{Y}_1)\big)\\
&= \operatorname{pf}\big(b^{-1}\widetilde{X}_1 + b^{-1}d_\rho\widetilde{Z}_1 + b^{-1}d_{\rho^2}\widetilde{Y}_1\big)\\
&= \operatorname{pf}\big(B^{-1}\big(\widetilde{X}_1 + d_{\rho^2}(\widetilde{Y}_1) + d_\rho(\widetilde{Z}_1)\big)B\big)\\
&= \operatorname{pf}\big(m(B)^{-1}B^t\big(\widetilde{X}_1 + d_{\rho^2}(\widetilde{Y}_1) + d_\rho(\widetilde{Z}_1)\big)B\big)\\
&= \det(B)m(B)^{-4}D_1 = D_1
\end{aligned}$$

The other verifications are similar. Finally 5) follows from 4) and 6) from 2), 3) and 4).

□

By (10.3) the algebra $R(F, r)$ is fixed elementwise under the action of $G(F)$. Thus $Z(F, r)$, $L(F, r)$ and $UT(F, r)$ are all fixed elementwise under this action. We shall show in (10.7) that $M(K)^{G(F)} = UT(F, r)$ and $(K \times K \times K)^{G(F)} = L(F, r)$. We start with:

Lemma 10.4 *If $a \in K^{G(F)}$, then $a = f/g$ with $f, g \in S^{G(F)}$. If $u \in M(K)^{G(F)}$, then $u = v/g$ with $v \in M(S)^{G(F)}$ and $g \in S^{G(F)}$.*

Proof Since $M(K)$ has a basis consisting of elements of $R(F, r)$, it has a $G(F)$-invariant basis. Thus the second statement follows from the first. So let $a \in K^{G(F)}$ and write $a = f_1/f_2$ where $f_1, f_2 \in S$ and have no common divisor. For any $g = (b, \beta) \in G(F)$, $\phi_g(a) = a$, so that $\phi_g(f_1)f_2 = f_1\phi_g(f_2)$. As f_1 and f_2 do not have common divisors, f_2 divides $\phi_g(f_2)$; ϕ_g being linear, we get $\phi_g(f_2) = \lambda(g)f_2$ for some $\lambda(g) \in F^\times$. Since

$$\phi_{gh}(f_2) = \phi_g\big(\lambda(h)f_2\big) = \lambda(g)\lambda(h)f_2$$

λ defines a group homomorphism $G(F) \to F^\times$. For $g \in G(F)$ we have $g^6 \in \mathrm{PGO}_8^+(F) \subset \mathrm{PGL}_8(F)$, so that $\lambda(g)^n = 1$ for $n = 48$ ([S2, Lemma 14.12]). We have $a = f_1/f_2 = f_1f_2^{n-1}/f_2^n$ with f_2^n, hence $f_1f_2^{n-1}$ fixed under $G(F)$.

□

Our next step is to show that, if $a \in K$ is invariant under $G(F)$, then it is also invariant under $G(F')$ for any field extension $F \subset F'$. This is achieved by showing that a is invariant under a "generic" element of $G(F)$. Let $F[W_{ij}, N, N']$ be the polynomial ring in $8^2 + 2$ variables W_{ij}, N and N'. Let W be the generic matrix with entries W_{ij} and let I be the ideal of $F[W_{ij}, N, N']$ defined by the relations $WW^t = N$, $\det(W) = N^4$ and $NN' = 1$. The ring $\Lambda = F[W_{ij}, N, N']/I$ is the coordinate ring of the group PGO_8^+ over F. Let $w_{ij} = W_{ij} + I$ and (abusing notations) let $W \in M_8(\Lambda)$ be the matrix $(W)_{ij} = w_{ij}$. The element $g = ([W], \gamma) \in G(\Lambda)$ with γ arbitrary in S_3 is the required "generic" element. We also need the notion of standard trialitarian algebra over a commutative ring. The given definition over a field extends to a definition over any commutative ring R in which 2 is invertible: $M(R)$ is the separable algebra $M_8(R \times R \times R)$; since standard octonions are defined over R, $\widetilde{\alpha}$ is defined over R. We further set

$$\mathrm{PGO}_n(R) = \mathrm{Aut}_R\big(M_n(R), t\big) \text{ and } \mathrm{PGO}_{2l}^+(R) = \mathrm{Aut}_R\big(M_{2l}(R), t, \mathrm{pf}\big). \tag{35}$$

Any $b \in \mathrm{PGO}_{2l}^+(R)$ induces an automorphism $C(b)$ of the Clifford algebra $C\big(M_{2l}(R), t\big)$; however, since b, viewed as an automorphism of $M_{2l}(R)$, needs not to be inner, we cannot use (2.1) to define $C(b)$. The automorphism b is inner over some faithfully flat extension R' of R (see for example [KO]). Then $C(b \otimes 1_{R'})$ restricts to an automorphism of $C\big(M_{2l}(R), t\big)$ by uniqueness. The trialitarian action of S_3 on $\mathrm{PGO}_8^+(R)$ can be defined as in (5.2). In particular we set

$$\alpha_0 C(b) \alpha_0^{-1} = \big(\overline{\rho}(b), \overline{\rho}^2(b)\big). \tag{36}$$

The definition of an automorphism of the standard trialitarian algebra over F extends to arbitrary commutative rings R as well, and we have $\mathrm{Aut}_R\big(M(R)\big) = \mathrm{PGO}_8^+(R) \rtimes S_3$. For $x \in M(R)$ and $g \in \mathrm{Aut}_R\big(M(R)\big)$ we write $g(x) = gx$. For any extension ring $\eta\colon S \subset S'$ and $b \in \mathrm{PGO}_8^+(S')$ (as defined in (35)), we define $\phi_b^\eta\colon S \to S'$ and $\psi_b^\eta\colon M(S) \to M(S')$ in a similar way as in (30) and (32):

$$\begin{aligned} \phi_b^\eta(x_{ijk}) &= \big(b^{-1}(M_8(\eta)(X_k))\big)_{ij} \\ \phi_b^\eta(y_{ijk}) &= \big(b_1^{-1}(M_8(\eta)(Y_k))\big)_{ij} \\ \phi_b^\eta(z_{ijk}) &= \big(b_2^{-1}(M_8(\eta)(Z_k))\big)_{ij} \end{aligned} \tag{37}$$

and

$$\psi_b^\eta(U, V, W) = \Big(b\big(M_8(\phi_b^\eta)U\big), b_1\big(M_8(\phi_b^\eta)V\big), b_2\big(M_8(\phi_b^\eta)W\big)\Big), \tag{38}$$

where $b_1 = \overline{\rho}(b)$, $b_2 = \overline{\rho}^2(b)$ in $\mathrm{PGO}_8^+(S')$ and $\overline{\rho}$ is as in (36). We similarly define ϕ_β^η and ψ_β^η for any $\beta \in S_3$ (see (33) and (34) for ρ and π). Let $G(S') = \mathrm{PGO}_8^+(S') \rtimes S_3$. As in (10.1) and (10.2), (37) and (38) extend to maps

$$\phi_g^\eta\colon S \to S' \quad \text{and} \quad \psi_g^\eta = g \circ M(\phi_g^\eta)\colon M(S) \to M(S') \tag{39}$$

for arbitrary $g \in G(S')$. The next proposition is a very useful result and is, as observed by Saltman, a general fact about algebraic group actions.

Proposition 10.5 *Let $x \in M(S)^{G(F)}$ and $g = \big([W], \gamma\big) \in G(\Lambda)$ generic as above. Then, under the map $\psi_g^\eta\colon M(S) \to M(S \otimes \Lambda)$ induced by the canonical embedding $\eta\colon S \to S \otimes \Lambda$, $\psi_g^\eta(x) = M(\eta)(x)$. If $\theta\colon S \to S'$ is any homomorphism of commutative rings and $g' \in G(S')$, then $\psi_{g'}^\theta(x) = M(\theta)(x)$.*

Proof The proof follows very closely the proof of [S2, Proposition 14.14]. We include it for the sake of completeness. Suppose $\psi_g^\eta(x) \neq M(\eta)(x)$. Let $s \in S \otimes \Lambda$ be a nonzero entry in the matrix triple $\psi_g^\eta(x) - M(\eta)(x) \in M(S \otimes \Lambda)$. Since s is a nonzero element in the polynomial ring $S \otimes \Lambda = \Lambda[x_{ijk}, y_{ijk}, z_{ijk}]$, there is a nonzero coefficient $\lambda \in \Lambda$ of the polynomial $s \in S \otimes \Lambda$. Since F is infinite, there is an F-algebra homomorphism $\zeta\colon \Lambda \to F$ such that $\zeta(\lambda) \neq 0$. Let $\eta'\colon \ : S \otimes \Lambda \to S$ be $1 \otimes \zeta$. Then $\eta'(s)$ is not zero since it has a nonzero coefficient $\zeta(\lambda)$. Let $a = \zeta(g) \in G(F)$. We have:

$$M(\eta') \circ M(\phi_g^\eta)(X_k, Y_k, Z_k) = M(\phi_{\zeta(g)})(X_k, Y_k, Z_k)$$

which shows that $\eta' \circ \phi_g^\eta = \phi_{\zeta(g)} = \phi_a$. We have $M(\eta')\big(M(\eta)(x)\big) = x$ since $\eta' \circ \eta = 1_S$ and

$$\begin{aligned} M(\eta')\big(\psi_g^\eta(x) - M(\eta)(x)\big) &= \eta'(g) M(\eta' \circ \phi_g^\eta)(x) - x \\ &= aM(\phi_a)(x) - x = \psi_a(x) - x \neq 0. \end{aligned}$$

However, since $a \in G(F)$ and $x \in M(S)^{G(F)}$, we have $\psi_a(x) = x$, leading to a contradiction.

Suppose that $\theta\colon S \to S'$ is any ring homomorphism and let $g' \in G(S')$. We need to show that $\psi_{g'}^\theta(x) = M(\theta)(x)$. Since S' is contained in a product of local rings, we may assume without loss of generality that S' is local. If S' is local $g' \in \mathrm{PGO}_8^+(S')$ is the class of an element $\mathrm{GO}_8^+(S')$ modulo $S'^\times$ and any element of $\mathrm{GO}_8^+(S')$ is the image of $[W] \in \mathrm{GO}_8^+(\Lambda)$ under a suitable specialization $\Lambda \to S'$. Hence there is a specialization $\zeta\colon \Lambda \to S'$ such that $\zeta(g) = g'$, where $g = ([W], \gamma)$, for some $\gamma \in S_3$. Since $\psi_g^\eta(x) = M(\eta)(x)$, where $\eta\colon S \to S \otimes \Lambda$ is as above and since $\zeta\theta\eta = \theta$, we have $\psi_{g'}^\theta(x) = M(\zeta)\big(M(\theta)\psi_g^\eta(x)\big) = M(\zeta\theta\eta)(x) = M(\theta)(x)$, as claimed.

□

The notion of trialitarian algebra extends to an arbitrary commutative ring R (where 2 is invertible). First we have an obvious definition of a pre-trialitarian algebra $(T, L, \sigma_T, \alpha_T)$ over R: L is étale over R, T central separable over L, the Clifford algebra of (T, σ_T) is central separable over its center $Z(T, \sigma)$, which is quadratic étale over R. Isomorphisms of pre-trialitarian algebras are defined as in the field case (see (25)). Let $\theta\colon R \to R'$ be a homomorphism of commutative rings and let T be a pre-trialitarian algebra over R. Then $T \otimes_\theta R'$ is pre-trialitarian over R'. We say that T is a *trialitarian algebra over* R if there exists a faithfully flat morphism $\theta\colon R \to S$ and an S-isomorphism of $T \otimes_\theta S$ with the standard trialitarian algebra $M(S)$. Let $(T, L, \sigma_T, \alpha_T)$ resp. $(T', L', \sigma_{T'}, \alpha_{T'})$ be trialitarian over R, resp. R'. A *morphism* $\Theta\colon (T, L, \sigma_T, \alpha_T) \to (T', L', \sigma_{T'}, \alpha_{T'})$ *of trialitarian algebras* is a homomorphism $\Theta\colon T \to T'$ of rings which restricts to a homomorphism $\theta\colon R \to R'$ such that Θ induces an isomorphism $(T, L, \sigma_T, \alpha_T) \otimes_\theta R' \xrightarrow{\sim} (T', L', \sigma_{T'}, \alpha_{T'})$ of trialitarian algebras.

Proposition 10.6 *Let E be a commutative F-algebra. Suppose $\theta_i\colon M(S) \to M(E)$, $i = 1, 2$, are morphisms of trialitarian algebras such that θ_1 and θ_2 are equal on $R(F, r)$. Then θ_1 and θ_2 are equal on $M(S)^{G(F)}$.*

Proof Since any commutative ring embeds in a product of local commutative rings we may assume that E is local. Let e_{ij}^ϵ, $\epsilon = 1, 2, 3$, and let $1 \leq i, j \leq 8$, be the standard matrix units of $M(S) = M_8(S) \times M_8(S) \times M_8(S)$; let f_{ij}^ϵ be the standard units in $M(E)$. By definition of a morphism, $\theta_\ell(e^\epsilon_{ij}) = f^\epsilon_{\ell ij}$, $\ell = 1, 2$, form a complete set of matrix units of $M(E)$. Let $g_1\colon M(E) \to M(E)$ be defined as the composite

$$M(E) \xrightarrow{\ i\ } M(S) \otimes_{\theta_1} E \xrightarrow{\ \widetilde{\theta}_1\ } M(E)$$

where i maps the standard units f^ϵ_{ij} to $e^\epsilon_{ij} \otimes 1$ and $\widetilde{\theta}_1$ is induced by θ_1. Being the composition of two morphisms, g_1 is an automorphism of the trialitarian algebra $M(E)$ with the property that $g_1(f^\epsilon_{ij}) = f^\epsilon_{1ij}$. The elements $\theta'_1 = g_1^{-1}\theta_1$ and $\theta'_2 = g_1^{-1}\theta_2$ satisfy the conditions of the proposition. If the claim is true for (θ'_1, θ'_2), the claim is also true for (θ_1, θ_2), since g_1 is the identity on $M(S)^{G(F)}$ by (10.5). Thus, replacing θ_1 by $g_1^{-1}\theta_1$, we may assume that $\theta_1(e^\epsilon_{ij}) = f^\epsilon_{ij}$, so that $\theta_1 = M(\eta_1)$ for some ring homomorphism $\eta_1\colon S \to E$. Similarly we have

$$\theta_2(e^\epsilon_{ij}) = f^\epsilon_{2ij} = g^{-1}(f^\epsilon_{ij})$$

for some $g \in G(E)$ and $g\theta_2 = M(\eta_2)$ for $\eta_2\colon S \to E$. Let $S' = S \otimes_F E$ and let $\eta'_\ell\colon S' \to E$, $\ell = 1, 2$ be defined as $\eta'_\ell(s \otimes t) = \eta_\ell(s)t$, i.e., η'_ℓ is the identity on E. We note that $M(\eta'_\ell)(\alpha) = \alpha$ for any $\alpha \in M(E)$. We define $\theta'_\ell\colon M(S') = M(S) \otimes E \to M(E)$ as $\theta'_\ell(\alpha \otimes e) = \theta_\ell(\alpha)e$ for $e \in E$. We claim that $g\theta'_2 = M(\eta'_2)$ on $M(S')$. This is clear on $M(S)$, since $g\theta_2 = M(\eta_2)$ on $M(S)$, and is obvious on E, since all the maps are E-linear. Since $(X_k, Y_k, Z_k) \in R(F, r)$ and $M(\eta_1) = \theta_1$, we have

$$\begin{aligned} M(\eta'_1)\big((X_k, Y_k, Z_k)\big) &= \theta'_1\big((X_k, Y_k, Z_k)\big) = \theta'_2\big((X_k, Y_k, Z_k)\big) \\ &= g^{-1}M(\eta'_2)\big((X_k, Y_k, Z_k)\big). \end{aligned}$$

We have a commutative diagram

$$\begin{array}{ccc} S \otimes M(E) = M(S \otimes E) & \xrightarrow{M(\eta'_2) = \eta'_2 \otimes 1_{M(E)}} & M(E) = E \otimes_E M(E) \\ {\scriptstyle 1_S \otimes g}\big\downarrow & & \big\downarrow{\scriptstyle 1_E \otimes g} \\ S \otimes M(E) = M(S \otimes E) & \xrightarrow{M(\eta'_2) = \eta'_2 \otimes 1_{M(E)}} & M(E) = E \otimes_E M(E) \end{array}$$

It follows that that g^{-1} and $M(\eta'_2)$ commute. Now, since $\psi_g\big((X_k, Y_k, Z_k)\big) = (X_k, Y_k, Z_k)$, we have $g^{-1}\big((X_k, Y_k, Z_k)\big) = M(\phi_g)\big((X_k, Y_k, Z_k)\big)$. It follows that $M(\eta'_1)\big((X_k, Y_k, Z_k)\big) = M(\eta'_2\phi_g)\big((X_k, Y_k, Z_k)\big)$ hence $\eta'_1 = \eta'_2\phi_g$. We then have for any $\beta \in M(S')$:

$$\begin{aligned} \theta'_1(\beta) &= M(\eta'_1)(\beta) = M(\eta'_2)M(\phi_g)(\beta) \\ &= M(\eta'_2)g^{-1}gM(\phi_g)(\beta) = g^{-1}M(\eta'_2)\psi_g(\beta) = \theta'_2\big(\psi_g(\beta)\big), \end{aligned}$$

so that $\theta'_1 = \theta'_2\psi_g$. Choose $\alpha \in M(S)^{G(F)}$. By (10.5) $\psi_g(\alpha) = \alpha$ and $\theta_1(\alpha) = \theta'_1(\alpha) = \theta'_2(\alpha) = \theta_2(\alpha)$, hence the claim.

□

We are now ready for the description of $UT(F, r)$ and $L(F, r)$ as invariant rings:

Theorem 10.7 *Let $G(F) = \mathrm{PGO}^+_8(F) \rtimes S_3$. We have $UT(F, r) = M(K)^{G(F)}$ and $L(F, r) = (K \times K \times K)^{G(F)}$.*

Proof It suffices to prove the first claim, since the second follows by restricting the action to the center. We know already that $UT(F,r) \subset M(K)^{G(F)}$. Set $UT = UT(F,r)$, $L = L(F,r)$ and $K^3 = K \times K \times K$. Define $\eta_\ell \colon K^3 \to K^3 \otimes_L K^3$, $\ell = 1, 2$, by $\eta_1(x) = x \otimes 1$ and $\eta_2(x) = 1 \otimes x$. Since L is a field, it is clear that for $x \in K^3$, $x \in L$ if and only if $\eta_1(x) = \eta_2(x)$. We set $\theta_\ell = M_8(\eta_\ell) \colon M_8(K^3) \to M_8(K^3 \otimes_L K^3)$, $\ell = 1, 2$. We have $M_8(K^3) = M(K)$ and $M_8(K^3 \otimes_L K^3) = M(K) \otimes_L K^3$. We identify $M(K)$ with $UT(F,r) \otimes_L K^3$ and $M_8(K^3 \otimes_L K^3)$ with $UT(F,r) \otimes_L (K^3 \otimes_L K^3)$. Then the θ_ℓ are maps

$$\theta_l : UT(F,r) \otimes_L K^3 \to UT(F,r) \otimes_L (K^3 \otimes_L K^3)$$

and, for $\alpha \in M(K)$, $\alpha \in UT(F,r)$ if and only if $\theta_1(\alpha) = \theta_2(\alpha)$. Let $\alpha \in M(K)^{G(F)}$ and write (by (10.4)) $\alpha = \beta/h$ with $\beta \in M(S)^{G(F)}$ and $h \in S^{G(F)}$. Thus it suffices to show that $M(S)^{G(F)} \subset UT(F,r)$. Since the θ_ℓ are equal on $UT(F,r)$ they are equal on $R(F,r)$. By (10.6) they are equal on $M(S)^{G(F)}$, hence $M(S)^{G(F)} \subset UT(F,r)$.

□

We still need to define a trialitarian structure on the central division algebra $UT(F,r)$ over $L(F,r)$, in particular we need a base field $k(F,r)$ over which $L(F,r)$ is cubic separable field extension. In view of (10.7) an obvious candidate is the field $k(F,r) = K^{G(F)}$. We start with an intermediate result:

Lemma 10.8 *1) Let $D \in S$ be as in ((10.3), 5); then $D \notin k(F,r)$ and $D^2 \in k(F,r)$, so that $k(F,r)[D] \subset K$ is a quadratic separable extension of $k(F,r)$.*
2) Let $L(F,r)[D] = L(F,r) \otimes_{k(F,r)} k(F,r)[D] \subset K^3$. Let $\widetilde{\rho} \colon K^3 \to K^3$ be the cyclic permutation $(x,y,z) \mapsto (z,x,y)$. Then $\widetilde{\rho}\big(L(F,r)[D]\big) \subset L(F,r)[D]$ and $L(F,r)[D]/k(F,r)[D]$ is a cubic extension, which is cyclic, with the restriction of $\widetilde{\rho}$ to $L(F,r)[D]$ as a generator of the Galois group.

Proof The first claim follows from ((10.3), 5). We prove 2): Let $\rho = (1\,2\,3) \in S_3$. Since ψ_ρ leaves D fixed (see (10.3), 5)), $L(F,r)[D]$ is invariant under $G_0(F) = \mathrm{PGO}_8^+(F) \rtimes A_3$. Since $D \notin L(F,r)$, $[L(F,r)[D] : L(F,r)] = 2$. It then follows from $\big(G(F) : G_0(F)\big) = 2$ that $L(F,r)[D] = (K \times K \times K)^{G_0(F)}$. Let $(x,y,z) \in L(F,r)[D] \subset K \times K \times K$. We show that $\widetilde{\rho}\big((x,y,z)\big)$ is invariant under $G_0(F)$. Let $g = g_1\rho'$, $g_1 \in \mathrm{PGO}_8^+(F)$ and $\rho' = (1\,2\,3) \in A_3$. Since $L(F,r)[D]$ is invariant under $G_0(F)$, we have

$$\begin{aligned}(x,y,z) &= \psi_{g_1\rho'}(x,y,z) = \psi_{g_1}\big(\rho'(\phi_{\rho'}x, \phi_{\rho'}y, \phi_{\rho'}z)\big) = \psi_{g_1}(\phi_{\rho'}z, \phi_{\rho'}x, \phi_{\rho'}y) \\ &= \big(\phi_{g_1}\phi_{\rho'}(z), \phi_{g_1}\phi_{\rho'}(x), \phi_{g_1}\phi_{\rho'}(y)\big)\end{aligned}$$

so that $x = \phi_{g_1}\phi_{\rho'}(z)$, $y = \phi_{g_1}\phi_{\rho'}(x)$ and $z = \phi_{g_1}\phi_{\rho'}(y)$. On the other hand we have

$$\psi_{g_1\rho'}\big(\widetilde{\rho}(x,y,z)\big) = \big(\phi_{g_1}\phi_{\rho'}(y), \phi_{g_1}\phi_{\rho'}(z), \phi_{g_1}\phi_{\rho'}(x)\big) = (z,x,y) = \widetilde{\rho}(x,y,z).$$

The same holds for $g_1\rho'^{-1}$ so that $\widetilde{\rho}(x,y,z) \in L(F,r)[D]$ and $\widetilde{\rho}\big(L(F,r)[D]\big) \subset \big(L(F,r)[D]\big)$. Replacing $\widetilde{\rho}$ by $\widetilde{\rho}^{-1}$ we see that $\widetilde{\rho}$ is an automorphism of $L(F,r)[D]$. We claim that $k(F,r)[D]$ is the fixed field of $L(F,r)[D]$ under $\widetilde{\rho}$. The field $k(F,r)[D]$ is clearly invariant under $\widetilde{\rho}$. For the other inclusion, we recall first that $L(F,r)[D] = (K \times K \times K)^{G_0(F)}$. We show that $k(F,r)[D] = K^{G_0(F)}$: clearly $k(F,r)[D] \subset K^{G_0(F)}$ and (by definition) $k(F,r) = K^{G(F)}$. Since

$D \notin k(F,r)$ and $D^2 \in k(F,r)$ we have $[k(F,r)[D] : k(F,r)] = 2$, so $k(F,r)[D] = K^{G_0(F)}$. Now, if $x \in L(F,r)[D]$ is invariant under $\widetilde{\rho}$, then $x \in L(F,r)[D] \cap K \subset K^{G_0(F)} = k(F,r)[D]$.

□

Theorem 10.9 *The extension $L(F,r)/k(F,r)$ is of degree 3 and has discriminant D^2.*

Proof The first claim follows from $[k(F,r)[D] : k(F,r)] = [L(F,r)[D] : L(F,r)] = 2$ and $[L(F,r)[D] : k(F,r)[D]] = 3$. The last claim holds since by adjoining D, $L(F,r)[D]$ over $k(F,r)[D]$ becomes cyclic.

□

Theorem 10.10 *Let $\Delta(F,r) = k(F,r)[D]$. There is an orthogonal involution σ_{UT} on $UT(F,r)$ and an isomorphism*

$$\alpha_{UT} \colon C\big(UT(F,r), \sigma_{UT}\big) \xrightarrow{\sim} {}^{\widetilde{\rho}}\big(UT(F,r) \otimes_{k(F,r)} \Delta(F,r)\big)$$

such that $\big(UT(F,r), L(F,r), \sigma_{UT}, \alpha_{UT}\big)$ is a trialitarian algebra over $k(F,r)$.

Proof Let $M(K)$ be the standard trialitarian structure over K, with isomorphism

$$\widetilde{\alpha} \colon C\big(M(K), t\big) \xrightarrow{\sim} {}^{\rho}M(K) \times {}^{\rho^2}M(K).$$

Let $\eta \colon \operatorname{Skew}(K) \to C\big(M(K), t\big)$ be the canonical embedding (15). The isomorphism $\widetilde{\alpha}$ is uniquely determined by the condition

$$\widetilde{\alpha} \circ \eta\big((a,b,c)\big) = \Big(\big(d_\rho(c), d\rho(a), d_\rho(b)\big), \big(d_{\rho^2}(b), d_{\rho^2}(c), d_{\rho^2}(a)\big)\Big) \tag{40}$$

for any $(a,b,c) \in \operatorname{Skew}\big(M(K), t\big)$. We set $UT = UT(F,r)$. Since UT is a subalgebra of $M(K)$ which generates $M(K)$ over K we have $UT \otimes_{k(F,r)} K \xrightarrow{\sim} M(K)$. The transpose t on $M(K)$ maps $R(F,r)$ to itself and hence restricts to an involution σ_{UT} on UT and $C\big(UT, \sigma_{UT}\big)$ is a subalgebra of the Clifford algebra $C\big(M(K), t\big)$; we set $\alpha_{UT} = \alpha|_{UT}$. To check that α_{UT} defines a trialitarian structure on UT, it suffices to verify that $\alpha_{UT} \circ \eta\big(\operatorname{Skew}(UT, \sigma_{UT})\big) \subset {}^{\widetilde{\rho}}(UT \otimes_{k(F,r)} \Delta)$. Since $\operatorname{Skew}(UT, \sigma_{UT})$ is (linearly) generated over $L(F,r)$ by the (X_k, Y_k, Z_k) (see (9.5)), it suffices to check that the images of the (X_k, Y_k, Z_k) belong to ${}^{\widetilde{\rho}}\big(UT \otimes_{k(F,r)} \Delta\big)$. The embedding $UT \subset M(K)$ induces an embedding

$${}^{\widetilde{\rho}}\big(UT \otimes_{k(F,r)} \Delta(F,r)\big) \subset {}^{\rho}\big(M(K) \otimes_K \Delta(F,r)\big).$$

We may choose D^{-1} as a generator of $\Delta(F,r)$, instead of D. Since $D \in K$, we have an identification $M(K) \otimes_{k(F,r)} \Delta(F,r) \xrightarrow{\sim} M(K) \times M(K)$ given by $a \otimes 1 + b \otimes D^{-1} \mapsto (a + bD^{-1}, a - bD^{-1})$, so that for a_1, $a_2 \in M(K)$:

$$\frac{a_1 + a_2}{2} + \frac{(a_1 - a_2)}{2} D \otimes D^{-1} \mapsto (a_1, a_2). \tag{41}$$

With

$$a_1 = \big(d_\rho(\widetilde{Z_k}), d_\rho(\widetilde{X_k}), d_\rho(\widetilde{Y_k})\big)^t \quad \text{and} \quad a_2 = \big(d_{\rho^2}(\widetilde{Y_k}), d_{\rho^2}(\widetilde{Z_k}), d_{\rho^2}(\widetilde{X_k})\big)^t$$

in (41), (40) implies that the triple $(\widetilde{X_k}, \widetilde{Y_k}, \widetilde{Z_k})$ maps to

$$\frac{1}{2}\begin{pmatrix} d_\rho(\widetilde{Z_k}) + d_{\rho^2}(\widetilde{Y_k}) \\ d_\rho(\widetilde{X_k}) + d_{\rho^2}(\widetilde{Z_k}) \\ d_\rho(\widetilde{Y_k}) + d_{\rho^2}(\widetilde{X_k} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \big(d_\rho(\widetilde{Y_k}) - d_{\rho^2}(\widetilde{Z_k})\big)D \\ \big(d_\rho(\widetilde{Z_k}) - d_{\rho^2}(\widetilde{X_k})\big)D \\ \big(d_\rho(\widetilde{X_k}) - d_{\rho^2}(\widetilde{Y_k})\big)D \end{pmatrix}$$

This element is in ${}^\rho(UT \otimes_{k(F,r)} \Delta)$, since it is the sum of elements in $R(F, r)$, hence the claim.

□

We call $UT(F, r)$ the *generic trialitarian algebra*.

Remark 10.11 The association $F \mapsto UT(F, r)$ is functorial: any injection $F \to F'$ of fields yields in a natural way a morphism $UT(F, r) \to UT(F', r)$ of trialitarian algebras.

Remark 10.12 The field $k(F, r)$ has transcendental dimension $3r - 1 + 36$. We plan to give some properties of $k(F, r)$ in a coming paper.

11 Specialization

Let $T = (T, L, \sigma_T, \alpha_T)$ be a trialitarian algebra over a field k and $T' = (T', L', \sigma_{T'}, \alpha_{T'})$ be a trialitarian algebra over a field k'. We say that T *specializes* to T' if there is a domain $R \subset k$ and a trialitarian algebra $(B, L_B, \sigma_B, \alpha_B)$ over R such that the following conditions are satisfied:
1) k is the field of fractions of R and there is an isomorphism of trialitarian algebras $B \otimes_R k \simeq T$;
2) there is a homomorphism $\phi \colon R \to k'$ inducing an isomorphism $B \otimes_\phi k' \xrightarrow{\sim} T'$ of trialitarian algebras over k'.

In the rest of the section we give a proof of the following

Theorem 11.1 *For any trialitarian algebra $(T, L, \sigma_T, \alpha_T)$ over any field extension $k \supset F$, there exists a specialization of the generic trialitarian algebra $(UT(F, r), L(F, r), \sigma_{UT}, \alpha_{UT})$ to $(T, L, \sigma_T, \alpha_T)$.*

The proof of the theorem is in several steps. The first step is to lift the trialitarian algebra $(UT(F, r), L(F, r), \sigma_{UT}, \alpha_{UT})$ to a trialitarian algebra over a commutative ring whose field of fractions is $k(F, r)$. We set $V_s = V \otimes_R R[1/s]$ for a module V over a domain R and a nonzero element $s \in R$.

Lemma 11.2 *Let (T, L, σ, α) be a trialitarian algebra over a field k. Let $B \hookrightarrow T$, $S \hookrightarrow L$, $R \hookrightarrow k$, $\delta \hookrightarrow \Delta$ be subrings such that $B \supset S \supset R$, $\delta \supset R$ and such that k is the quotient field of R; further assume that B is central separable over S of degree 8, S is cubic étale over R and δ is quadratic étale over R. Then there exists $s \in R$, $s \neq 0$, such that σ restricts to an involution of B_s, ρ on $L \otimes \delta$ restricts to an automorphism of $S_s \otimes \delta_s$ and α restricts to an isomorphism $C(B_s, \sigma) \xrightarrow{\sim} {}^\rho\big((B_s, \sigma) \otimes_{R_s} \delta_s\big)$; i.e., $(B_s, S_s, \sigma, \alpha)$ is a trialitarian algebra over R_s with the natural map $(B_s, S_s, \sigma, \alpha) \otimes_{R_s} k \xrightarrow{\sim} (T, L, \sigma, \alpha)$ an isomorphism of trialitatrian algebras over k.*

Proof The lemma is immediate, observing that B, S and δ are finitely generated as R-modules (in particular $Sk = L$, $\delta k = \Delta$ and $Bk = T$).

$\square$

From now on we use the same notations $R(F,r)$, $Z(F,r)$, $UT(F,r)$, $L(F,r)$ and $k(F,r)$ as in Section 9.

Proposition 11.3 *There exists an element $s \in Z(F,r)$, $s \neq 0$, and F-algebras $E \subset Z(F,r)_s$, $\delta \subset Z(F,r)_s[D]$, $\delta \supset E$, such that:*

1) $R(F,r)_s$ is Azumaya over $Z(F,r)_s$,
2) $Z(F,r)_s$ is cubic étale over E,
3) δ is quadratic étale over E,
4) $k(F,r)$ is the quotient field of E.

Proof The ring $R(F,r)$ is a prime PI-ring (see the proof of (9.4)). By [R2, 6.3.27] there exists $s \in Z(F,r)$ such that $R(F,r)_s$ is central separable over $Z(F,r)_s$. Since $R(F,r)_s$ is central separable over $Z(F,r)_s$ and is finitely generated as an F-algebra, $Z(F,r)_s$ is a finitely generated F-algebra. Let D be as in (27); since $D^2 \in k(F,r) \hookrightarrow L(F,r)$ we may assume, after inverting a further element of $Z(F,r)$, that $D^2 \in Z(F,r)_s$. Then the ring $Z(F,r)_s[D]$ is a finitely generated F-algebra with quotient field $L(F,r)[D]$. Let $\widetilde{\rho}$ be the automorphism of $L(F,r)[D]$ as in (10.8) and let $\widetilde{\pi}$ be the automorphism sending D to $-D$ and fixing $L(F,r)$. Then $\widetilde{\rho}$ and $\widetilde{\pi}$ generate a subgroup $\widetilde{S}_3$ of $\mathrm{Aut}_{k(F,r)}\big(L(F,r)[D]\big)$ isomorphic to S_3 and with fixed field $k(F,r)$. Because of the finite generation of $Z(F,r)_s$ over F, one may invert a further element of $Z(F,r)_s$ and assume that $\widetilde{S}_3$ restricts to an action on $Z(F,r)_s[D]$. We set $E = \big(Z(F,r)_s[D]\big)^{\widetilde{S}_3}$, $\delta = \big(Z(F,r)_s\big)^{\widetilde{\rho}}$. Then $E \subset Z(F,r)_s[D]^{\widetilde{\pi}} = Z(F,r)_s$. We claim that $k(F,r)$ is the quotient field of E. Since $\big(Z(F,r)_s[D]\big)^{\widetilde{S}_3} = E$, the norm map $L(F,r) \to k(F,r)$ restricts to a norm map $N\colon Z(F,r)_s[D] \to E$. Let $y \in k(F,r)$, $y = a/b$, a, $b \in Z(F,r)_s[D]$; then $y = a(Nb)b^{-1}(Nb)^{-1}$, with $Nb \in E$. The element $a(Nb)b^{-1} = a \cdot \prod_{\sigma \in \widetilde{S}_3,\, \sigma \neq 1} \sigma(b)$ belongs to $Z(F,r)_s[D]$ and, since y is invariant under $\widetilde{S}_3$, $yNb = aNbb^{-1}$ is invariant under $\widetilde{S}_3$. Hence $aNbb^{-1}$ belongs to E. Thus $k(F,r)$ is the quotient field of E. Since $Z(F,r)_s[D]^{\widetilde{S}_3} = E$, $Z(F,r)_s[D]^{\widetilde{\rho}} = \delta$; δ and $Z(F,r)_s$ are finitely generated F-algebras which are integral over E. Hence they are finite as E-modules and we may choose $s' \in E$ such that $Z(F,r)_{ss'}$ and $\delta_{s'}$ are étale over $E_{s'}$. Replacing $E_{s'}$ by E, the proposition follows from (11.2).

$\square$

We thus have proved the following

Theorem 11.4 *There exists a nonzero element $s \in Z(F,r)$ such that the trialitarian algebra $\big(UT(F,r), L(F,r), \sigma_{UT}, \alpha_{UT}\big)$ restricts to a trialitarian algebra $\big(R(F,r)_s, Z(F,r)_s, \overline{\sigma}, \overline{\alpha}\big)$ over $E \subset k(F,r)$, the quotient field of E being $k(F,r)$, such that the natural map $R(F,r)_s \otimes_E k(F,r) \to UT(F,r)$ is an isomorphism of trialitarian algebras over $k(F,r)$.*

We next show that for any field extension $F \subset k$ and any trialitarian algebra (T, L, σ, α) over k, there is a specialization of $\big(R(F,r)_s, Z(F,r)_s, \overline{\sigma}, \overline{\alpha}\big)$ to (T, L, σ, α).

Let $N\colon K \times K \times K \to K$ be the norm, i.e., $N(x,y,z) = xyz$ for x, y and $z \in K$. Let $s \in Z(F,r)$ be as in (11.4). Since $Z(F,r)_s \subset (S \times S \times S)_s$, we have $E \subset S_{Ns}$. We further have $s_0 \in S$ such that $R(F,r) \cdot S_{s_0} = M(S_{s_0})$ (see the proof of (9.5)); in particular $R(F,r)_s \cdot S_{Ns\cdot s_0} = M(S_{Ns\cdot s_0})$ and the natural map $\theta\colon R(F,r)_s \otimes_E S_{Ns\cdot s_0} \to M(S_{Ns\cdot s_0})$ is an isomorphism of $S_{Ns\cdot s_0}$-algebras. Since the restriction of the standard triality on $M(K)$ to $M(S_{Ns\cdot s_0})$ is the standard triality on this algebra and since its restriction to $R(F,r)_s$ is the restriction of the triality on $UT(F,r)$ to $R(F,r)_s$, θ is an isomorphism of trialitarian algebras. Thus we have proved:

Proposition 11.5 *The natural map $\theta\colon R(F,r)_s \otimes_E S_{Ns\cdot s_0} \to M(S_{Ns\cdot s_0})$ is an isomorphism of trialitarian algebras.*

Let (T, L, σ, α) be a trialitarian algebra over $k \supset F$. We recall (see (9.2)) that $X = \mathrm{Hom}_F\big(R(F,r), T\big)$ is an affine space $R_{L/F}(T^r)$. Let $k \subset k'$ be such that there is an isomorphism $T \otimes_k k' \simeq M(k')$ of trialitarian algebras. We regard T as a subalgebra of $M(k')$ through this isomorphism. We view any $\psi \in X$ as a map $R(F,r) \to T \hookrightarrow M(k')$. Let $\phi_\psi\colon S \to k'$ be such that $M(\phi_\psi)$ restricted to $R(F,r)$ is ψ. We recall that

$$\phi_\psi(x_{ijk}) = a_{ijk}, \quad \phi_\psi(y_{ijk}) = b_{ijk} \quad \text{and} \quad \phi_\psi(z_{ijk}) = c_{ijk}$$

if

$$\psi\big((X_k, Y_k, Z_k)\big) = \big((a_{ijk}), (b_{ijk}), (c_{ijk})\big) \in T \hookrightarrow M(k'), \quad 1 \le i, j \le 8.$$

Further, given an r-tuple $\underline{a} = (a_1, a_2, \dots, a_r)$ of elements of $T \hookrightarrow M(k')$, let $\psi_{\underline{a}}\colon R(F,r) \to T$ be the restriction of $M(\phi_{\underline{a}})$, where

$$\phi_{\underline{a}}(x_{ijk}) = (A_k^1)_{ij}, \quad \phi_{\underline{a}}(y_{ijk}) = (A_k^2)_{ij}, \quad \phi_{\underline{a}}(z_{ijk}) = (A_k^3)_{ij},$$

for $a_k = (A_k^1, A_k^2, A_k^3) \in M(k')$, $1 \le k \le r$. We already remarked that the following subvarieties of X are open:

$$\begin{aligned} U_s &= \{\psi \in X \mid \psi(s) \in T^\times\}, \quad s \in Z(F,r) \\ V &= \{\psi_{\underline{a}} \mid (a_1, \dots, a_r), \text{ with } (a_1, \dots, a_\ell), \ell = 3 \cdot 8^2 \le r, \text{ a } k\text{-basis of } T\}. \end{aligned}$$

Let $s_0 \in S$ be as in (11.5) and let $W = \{\psi \in X \mid \phi_\psi(s_0) \neq 0\}$. Clearly $X = \mathrm{Hom}_F\big(R(F,r), T\big)$ is identified with a subvariety of $\mathrm{Hom}_F(S, k')$ under the map $\psi \mapsto \phi_\psi$ and the set $\{\phi \in \mathrm{Hom}_F(S, k'),\ \phi(s_0) \neq 0\}$ is open in $\mathrm{Hom}_F(S, k')$. Hence W is open in X. Let s be as in (11.3). The intersection of the three open subsets U_s, V and W of X is not empty (since F is infinite), hence contains a k-rational point $\psi_{\underline{a}}$. The element $\underline{a} = (a_1, a_2, \dots, a_r)$ is such that $(a_1, a_2, \dots, a_\ell)$, $\ell = 3 \cdot 8^2 \le r$, is a basis of T over k, $\psi_{\underline{a}}\big((X_k, Y_k, Z_k)\big) = a_k$, $1 \le k \le r$, $\phi_{\psi_{\underline{a}}}(s_0) \neq 0$ in k' and $\psi_{\underline{a}}(s)$ is a unit of T. We abbreviate $\psi_{\underline{a}} = \psi$ and $\phi_{\psi_{\underline{a}}} = \phi$. The homomorphism $\phi\colon S \to k'$ has a nonzero value on $N(s) \cdot s_0$, hence yields a homomorphism $\phi\colon S_{Ns\cdot s_0} \to k'$. Since $E \subset S_{Ns} \cap R(F,r)$, ϕ restricts to $\phi\colon E \to k$, because $M(\phi)\big(R(F,r)\big) = \psi\big(R(F,r)\big)$ and $T \cap k' = k$. The homomorphism $\psi\colon R(F,r) \to T$ extends to $\psi\colon R(F,r)_s \to T$ since $\psi(s)$ is a unit of T. Since ψ is the restriction of $M(\phi)\colon M(S_s) \to M(k')$, $\psi\big(Z(F,r)_s\big) \subset L$ and $\psi(E) \subset k$ (see (11.3)). We claim that the map $\widetilde{\psi}\colon R(F,r)_s \otimes_\phi k \to T$ induced by ψ is an isomorphism of trialitarian algebras. First, $\widetilde{\psi}$ is an isomorphism of k-algebras since the k-basis $(a_1, \dots, a_\ell)$ is caught in the image of $\widetilde{\psi}$. To verify that it is an isomorphism of trialities, it suffices to check this after base change to k'. The composite

$$\begin{aligned} R(F,r)_s \otimes_\phi k \otimes_k k' &\xrightarrow{\sim} R(F,r)_s \otimes_\phi k' \simeq \\ R(F,r)_s \otimes_E S_{Ns\cdot s_0} \otimes_\phi k' &\xrightarrow{\sim} M(S_{Ns\cdot s_0}) \otimes_\phi k' \simeq M(k') \end{aligned}$$

is precisely $\widetilde{\psi} \otimes 1$. The isomorphism $R(F,r)_s \otimes_E S_{Ns\cdot s_0} \xrightarrow{\sim} M(S_{Ns\cdot s_0})$ is an isomorphism of trialities. Hence each arrow in the above composition is an isomorphism of trialities and so is $\widetilde{\psi} \otimes 1$. Thus we have proved:

Theorem 11.6 *Let (T, L, σ, α) be a trialitarian algebra over $k \supset F$. With the notation as above, there is an homomorphism $\phi\colon E \to k$ and an homomorphism $\psi\colon R(F,r)_s \to T$ of F-algebras such that $\psi|_E = \phi$ and $\widetilde{\psi}\colon R(F,r)_s \otimes_\phi k \to T$ is an isomorphism of trialitarian algebras. In particular the trialitarian algebra $\big(UT(F,r), L(F,r), \sigma_{UT}, \alpha_{UT}\big)$ admits a specialization.*

Remark 11.7 Let Δ be the discriminant of the cubic extension $L(F,r)$ (as a quadratic algebra). Then $L(F,r)\otimes\Delta$ is cubic cyclic over Δ, $UT(F,r)\otimes\Delta = M(K)^{\mathrm{PGO}_8^+ \rtimes A_3}$ is a generic cyclic trialitarian algebra and admits a specialization to any cyclic trialitarian algebra.

12 Generic Orthogonal and Unitary Involutions

Following the pattern used in constructing the generic trialitarian algebra $UT(F,r)$, it is straightforward to construct a generic algebra with an involution of orthogonal type and trivial discriminant, as well as a generic algebra with a unitary involution (i.e., of the second kind). This is well-known, but we did not find an explicit description in the literature. We first recall the orthogonal case with arbitrary discriminant, which is in Rowen [R1]. Let $S = F[x_{ijk}, y_{ijk}]$, $1 \le i,j \le n$, $k = 1,\dots,r$, $r \ge n^2$, with field of fractions K. Let X_k, $k = 1,\dots,r$, be the generic matrix with entries $(X_k)_{ij} = x_{ijk}$; Let $R(F,n,r)$ be the F-subalgebra of $M_n(K)$ generated by the generic matrices X_k and their transposes X_k^t. Let $Z(F,n,r)$ be the center of $R(F,n,r)$. The rings $R(F,n,r)$ and $Z(F,n,r)$ are domains. Let $k(F,n,r)$ be the quotient field of $Z(F,n,r)$ and $U(F,n,r)$ be the central localization of $R(F,n,r)$, i.e., the localization with respect to all nonzero elements of $Z(F,n,r)$. The algebra $U(F,n,r)$ is a central simple algebra over $k(F,n,r)$. The transpose on $M_n(K)$ restricts to an involution, denoted σ, of $R(F,n,r)$, hence of $U(F,n,r)$. The involution σ restricts to the identity on the center $k(F,n,r)$ and is of orthogonal type. The pair $\big(U(F,n,r),\sigma\big)$ is the generic central simple algebra with orthogonal involution considered in [R1]. On the same lines as in the proof of (10.7) we get Procesi's result [P2]:

$$U(F,n,r) = M_n(K)^{\mathrm{PGO}_n(F)} \quad \text{and} \quad k(F,n,r) = K^{\mathrm{PGO}_n(F)}$$

We assume from now on that $n = 2l$ is even. The matrix $X_1 - X_1^t$ is skew-symmetric; let $D = \mathrm{pf}(X_1 - X_1^t)$. The element $D^2 = \det(X_1 - X_1^t)$ belongs to $k(F,2l,r)$; further $D \notin k(F,2l,r)$, otherwise $\mathrm{disc}(\sigma) = 1$, but σ specializes to involutions which do not have trivial discriminant. Thus $\Delta(F,2l,r) = k(F,2l,r)[D] \subset K$ is a quadratic field extension of $k(F,2l)$. Let $\widetilde{U}(F,2l,r) = U(F,2l,r) \otimes_{k(F,2l,r)} \Delta(F,2l,r)$ and $\widetilde{\sigma} = \sigma \otimes 1_{\Delta(F,2l,r)}$; then $\big(\widetilde{U}(F,2l,r),\widetilde{\sigma}\big)$ is the generic central simple algebra of degree $2l$ with involution of orthogonal type with trivial discriminant; this generic algebra admits a specialization to any central simple algebra of degree $2l$ with involution of orthogonal type with trivial discriminant. The algebra $\widetilde{U}(F,2l,r)$ is the invariant ring of $M_{2l}(K)$ under the action of $\mathrm{PGO}_{2l}^+(F)$ and $\Delta(F,2l,r)$ is the field of invariants for the same action restricted to K.

We finally consider the case of unitary involutions. Let, as above, $S = F[x_{ijk}, y_{ijk}]$, $1 \leq i,j \leq n$, $k = 1,\dots,r$, with field of fractions K. Let $R^u(F,n,r)$ be the F-subalgebra of $M_n(K) \times M_n(K)$ generated by the pairs (X_k, Y_k), (Y_k^t, X_k^t). Let $Z^u(F,n,r)$ be the center of $R^u(F,n,r)$. The rings $R^u(F,n,r)$ and $Z^u(F,n,r)$ are domains. Let $K(F,n,r)$ be the quotient field of $Z^u(F,n,r)$ and let $U^u(F,n,r)$ be the central localization of $R^u(F,n,r)$ with respect to all nonzero elements of $Z^u(F,n,r)$. The algebra $U^u(F,n,r)$ is a central division algebra over $Z(F,n,r)$. The involution $\tau\colon (x,y) \mapsto (y^t, x^t)$ of $M_n(K) \times M_n(K)$ is of second kind and restricts to an involution (also denoted τ) of $R^u(F,n,r)$, hence of $U^u(F,n,r)$. It further restricts to an automorphism ι of the center $Z^u(F,n,r)$ of order 2. Let $k(F,n,r)$ be the elements of $K(F,n,r)$ fixed under ι. Then $\big(U^u(F,n,r),\tau\big)$ is a generic division algebra with unitary involution, in the sense that it admits a specialization to any central simple algebra with unitary involution over a field k which contains F. Let further $G(F) = \mathrm{PGL}_n(F) \rtimes \mathbb{Z}/2\mathbb{Z}$, where the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{PGL}_n(F)$ is through $x \mapsto (x^t)^{-1}$ for $x \in \mathrm{GL}_n(F)$. The group $G(F)$ acts on S and on $M_n(S) \times M_n(S)$, hence on K and $M_n(K) \times M_n(K)$ and $\big(M_n(K) \times M_n(K)\big)^{\mathrm{PGL}_n(F)\rtimes\mathbb{Z}/2\mathbb{Z}} = U^u(F,n,r)$, $(K \times K)^{\mathrm{PGL}_n(F)\rtimes\mathbb{Z}/2\mathbb{Z}} = Z(F,n,r)$ and $K^{\mathrm{PGL}_n(F)\rtimes\mathbb{Z}/2\mathbb{Z}} = k(F,n,r)$.

Remark 12.1 To construct the generic algebra with unitary involution, we may also start with a half-split datum: let $F_1 = F(\sqrt{u})$ be a generic quadratic extension, i.e., u is an indeterminate, and let $\iota\colon F_1 \to F_1$ be the F-linear map with $\iota(\sqrt{u}) = -\sqrt{u}$. Let $K_1 = K \otimes_F F_1$, where $K = F(x_{ijk}, y_{ijk})$, and let $R(F_1,n,r)$ be the F_1-subalgebra of $M_n(K_1)$ generated by generic matrices X_k and their transposes X_k^t. The unitary involution τ of $M_n(K_1)$ which restricts to $1_K \otimes \iota$ on the center and maps X_k to X_k^t restricts to a unitary involution of $R(F_1,n,r)$. We let it as an exercise to show that the central localization $U^*(F,n,r)$ of $R(F_1,n,r)$ is isomorphic to $U^u(F,n,r)$. Let $K^*(F,n.r)$ be the center of $U^*(F,n,r)$. The group $\mathrm{PGU}_n(F) = \{x \in \mathrm{PGL}_n(F_1) \mid x\tau(x) \in F_1^\times\}$; acts on $M_n(K_1)$ and $M_n(K_1)^{\mathrm{PGU}_n(F)} = U^*(F,n,r)$, $\quad K_1^{\mathrm{PGU}_n(F)} = K^*(F,n,r)$.

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