TRIALITARIAN AUTOMORPHISMS OF LIE ALGEBRAS

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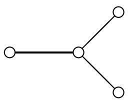
Abstract. Over an algebraically closed field of characteristic zero simple Lie algebras admit outer automorphisms of order 3 if and only if they are of type D_4. Moreover there are two conjugacy classes of such automorphisms. Among orthogonal Lie algebras over arbitrary fields of characteristic zero, only orthogonal Lie algebras relative to quadratic norm forms of Cayley algebras admit outer automorphisms of order 3. We give a complete list of conjugacy classes of outer automorphisms of order 3 for orthogonal Lie algebras over arbitrary fields of characteristic zero. For the norm form of a given Cayley algebra, one class is associated with the Cayley algebra and the others with central simple algebras of degree 3 with involution of second kind such that the cohomological invariant of the involution is the norm form.

1. Introduction

Simple Lie algebras over algebraically closed fields of characteristic zero are classified by their Dynkin diagrams, and the group of automorphisms of the Lie algebra modulo the subgroup of inner automorphisms is isomorphic to the group of symmetries of the corresponding Dynkin diagram. In most cases this group of symmetries has at most two elements. The case of the Lie algebra of skew-symmetric 8×8-matrices is exceptional. The Dynkin diagram is of type D_4:



and has the permutation group S_3 as a group of automorphisms. The existence of peculiar automorphisms of order 3 for Lie algebras of type D_4 is one aspect of the phenomenon known as triality.

The automorphisms of the Dynkin diagram can easily be extended to automorphisms of the Lie algebra using the root system. Thus over an algebraically closed field, the classes of automorphisms modulo inner automorphisms are explicitly known.

A complete list of conjugacy classes of outer automorphisms of order 3 over an algebraically closed field of characteristic zero can be deduced from the classification of automorphisms of finite order of simple Lie algebras. Such a classification is given for example in [WG], [N], [K69], [K85], [H] or [JZ]. Besides the conjugacy class of the automorphism constructed with help of the root system, whose fixed point algebra is of type G_2, there is one more conjugacy class in the full group of automorphisms, whose fixed point algebra is a simple Lie algebra of type A_2.

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In this work we consider outer automorphisms of order 3 of orthogonal Lie algebras over arbitrary fields of characteristic 0. The orthogonal Lie algebra relative to the quadratic norm form of a Cayley algebra always admits such automorphisms. This is known as the "local triality principle" (see for example [VBS], [J64], [SV] or Example 7). The following converse also holds: if an orthogonal Lie algebra of type D_4 admits an outer automorphism of order 3, then it is the orthogonal Lie algebra relative to the quadratic norm form of a Cayley algebra. Thus (local) triality and octonions are mutually "responsible" (Tits, [T]) for existence. By descent, conjugacy classes of order 3 outer automorphisms must have as fixed point algebras Lie algebras of type G_2 or A_2. We show that conjugacy classes of the orthogonal Lie algebra relative to a given norm form are classified by the set consisting of the isomorphism class of the corresponding Cayley algebra (type G_2) and the isomorphism classes of central simple algebras of degree 3 with specific involutions of second kind over the quadratic extension generated by a cubic root of unity (type A_2).We describe an explicit rational construction of a set of representatives of conjugacy classes of outer automorphisms of order 3. For example, over $\mathbb{R}$ there are two conjugacy classes for both the identity norm form and the hyperbolic norm form.

In Section 2 we reproduce the construction of outer automorphisms based on a Cartan decomposition of the Lie algebra. Symmetric compositions and their Lie algebras of derivations are discussed in Sections 3 and 4. We then recall in Section 5 the construction of outer automorphisms via symmetric compositions given in [KMRT]. In the last section we give a proof that local triality for an orthogonal Lie algebra $\mathfrak{g}$ implies that $\mathfrak{g}$ is the orthogonal Lie algebra of a norm of a Cayley algebra and we show that the classification of symmetric compositions leads to a classification of conjugacy classes of outer automorphisms of order three of the Lie algebra. In the Appendix (in collaboration with Larissa Cadorin), we compare the definitions of triality given in Sections 2 and 5.

If not explicitly mentioned, we assume in the whole paper that fields have characteristic 0. However it should be enough to assume characteristic different from 2 and 3.

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2. Orthogonal Lie algebras

Let V be a finite dimensional vector space over a field F and let $q : V \to F$ be a quadratic form on V, with *associated polar form* $b_q(x, y) = q(x + y) - q(x) - q(y)$. We call the pair (V, q) a *quadratic space* if b_q is nonsingular. The *adjoint involution* σ_q on $\mathrm{End}_F(V)$ is defined through the formula $b_q\big(\sigma_q(f)(x), y\big) = b_q\big(x, f(y)\big)$ for x, $y \in V$ and $f \in \mathrm{End}_F(V)$. The space

$$\begin{aligned}\mathfrak{o}(V, q) \;=\; \mathfrak{o}(q) \;&=\; \mathrm{Skew}\big(\mathrm{End}_F(V), \sigma_q\big)\\ &=\; \{f \in \mathrm{End}_F(V) \mid \sigma_q(f) = -f\}\\ &=\; \{f \in \mathrm{End}_F(V) \mid b_q\big(x, f(y)\big) + b_q\big(f(x), y\big) = 0\}\end{aligned}$$

of skew-symmetric elements of $\operatorname{End}_F(V)$ with respect to the involution σ_q is a Lie subalgebra (of dimension $\frac{n(n-1)}{2}$) of $\operatorname{End}_F(V)$ for the Lie bracket $[f,g]=f\circ g-g\circ f$ of $\operatorname{End}_F(V)$. The Lie algebra $\mathfrak{o}(q)$ is called the *orthogonal Lie algebra relative to q*.

Example 1. Let V=F^{2m}, $m\geq 4$, and let b_{2m} be the symmetric bilinear form on V given by the matrix

$$s=\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$$

where I_m is the m-dimensional identity matrix. The quadratic space (F^{2m},b_{2m}) is the $2m$-dimensional *hyperbolic space* H_{2m}. A basis $(e_1,\dots,e_m,f_1,\dots,f_m)$ with

$$b_{2m}(e_i,e_j)=0,\ b_{2m}(f_i,f_j)=0,\ b_{2m}(e_i,f_i)=1,\ i,\ j=1,\dots,m$$

is called *hyperbolic*. The corresponding orthogonal Lie algebra $\mathfrak{o}_{2m}$ can be identified with the set of matrices

$$\begin{pmatrix} X & Y \\ Z & -X^t \end{pmatrix}$$

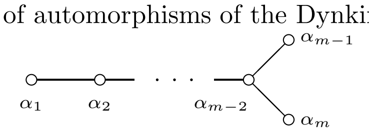
where X, Y, Z are $(m\times m)$-matrices, Y and Z are skew-symmetric and X^t is the transpose of X. Orthogonal Lie algebras relative to quadratic spaces of even dimension $2m$ are forms of $\mathfrak{o}_{2m}$, i.e., isomorphic to $\mathfrak{o}_{2m}$ over an algebraic closure $\overline{F}$ of F.

A *similitude* of quadratic spaces (V,q) and (V',q') is an isometry of (V,q) with $(V',\lambda q')$ for some $\lambda\in F^\times=F\setminus\{0\}$. Any similitude s induces an isomorphism $\operatorname{Int}(s):\big(\operatorname{End}_F(V),\sigma_q\big)\xrightarrow{\sim}\big(\operatorname{End}_F(V'),\sigma_{q'}\big)$ of algebras with involutions and a Lie algebra isomorphism $\mathfrak{o}(s):\mathfrak{o}(q)\xrightarrow{\sim}\mathfrak{o}(q')$. Similitudes of the quadratic space (V,q) are linear automorphisms f of V with $q\big(f(x)\big)=\mu(f)q(x)$, where $\mu(f)\in F^\times$ is the *multiplier* of the similitude. They form a group denoted by $\operatorname{GO}(q)$. If $\dim_F V=2m$ is even, a similitude f is *proper* if $\det(f)=\mu(f)^m$. Proper similitudes form a normal subgroup $\operatorname{GO}^+(q)$ of $\operatorname{GO}(q)$ of index 2. Elements of $\operatorname{GO}(q)$ act by inner conjugation as automorphisms of $\big(\operatorname{End}_F(V),\sigma_q\big)$ and the group of automorphisms of the algebra with involution $\big(\operatorname{End}_F(V),\sigma_q\big)$ is isomorphic to the projective group $\operatorname{PGO}(q)=\operatorname{GO}(q)/F^\times$. The group $\operatorname{PGO}^+(q)=\operatorname{GO}^+(q)/F^\times$ also acts on $\mathfrak{o}(q)$ by conjugations and the corresponding automorphisms are called *inner automorphisms* of $\mathfrak{o}(q)$. Inner automorphisms can be identified over an algebraically closed field with invariant automorphisms of the Lie algebra $\mathfrak{o}(q)$ (see [J62, Chap. IX]). Inner automorphisms form a normal subgroup $\operatorname{Int}\big(\mathfrak{o}(q)\big)$ of the full automorphism group $\operatorname{Aut}\big(\mathfrak{o}(q)\big)$ of $\mathfrak{o}(q)$. Automorphisms which are not inner automorphisms are called *outer automorphisms*. The definitions of inner and outer automorphisms are invariant under base field extensions and are independent of the realization of the Lie algebra in the form $\mathfrak{o}(q)$.

It is a classical result (see for example [J62, Chap. IX] or [SP]) that, over an algebraically closed field of characteristic zero, the factor group

$$\operatorname{Out}\big(\mathfrak{o}(q)\big)=\operatorname{Aut}\big(\mathfrak{o}(q)\big)/\operatorname{Int}\big(\mathfrak{o}(q)\big)$$

is isomorphic to the group of automorphisms of the Dynkin diagram of type D_m:



If $m \neq 4$, the Dynkin diagram admits only one nontrivial automorphism, of order 2 (which corresponds to the automorphism of the Lie algebra given by conjugation with a similitude which is not proper). The case of type D_4 is exceptional, in the sense that the group of automorphisms of the Dynkin diagram is S_3, the group of permutations of 3 objects. Thus we have an exact sequence

$$1 \to \mathrm{Int}\big(\mathfrak{o}_8\big) \to \mathrm{Aut}\big(\mathfrak{o}_8\big) \to S_3 \to 1 \tag{2}$$

Proposition 3. *Let (V,q) and (V',q') be quadratic spaces of even dimension $2m$, $m \geq 4$.*
1) Any similitude $s : (V,q) \xrightarrow{\sim} (V',q')$ induces an isomorphism

$$\mathrm{Int}(s) : \big(\mathrm{End}_F(V), \sigma_q\big) \xrightarrow{\sim} \big(\mathrm{End}_F(V'), \sigma_{q'}\big)$$

of algebras with involution and conversely any isomorphism

$$\big(\mathrm{End}_F(V), \sigma_q\big) \xrightarrow{\sim} \big(\mathrm{End}_F(V'), \sigma_{q'}\big)$$

of algebras with involution is of the form $\mathrm{Int}(s)$ for a similitude $s : (V,q) \xrightarrow{\sim} (V',q')$.
2) The quadratic spaces (V,q) and (V',q') are similar if and only if the Lie algebras $\mathfrak{o}(q)$ and $\mathfrak{o}(q')$ are isomorphic.

Proof. The first claim follows from the fact that any isomorphism of algebras $\mathrm{End}_F(V) \xrightarrow{\sim} \mathrm{End}_F(V')$ is induced by an isomorphism $V \xrightarrow{\sim} V'$. Claim 2) for $m \neq 4$ is in [J62] and in [J64] for $m = 4$. □ □

Following [N] (see also [WG]) we construct in this section outer automorphisms of order 3 of the Lie algebra $\mathfrak{o}_8$ using root systems. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{o}_8$, for example

$$\mathfrak{h} = \{\left(\begin{smallmatrix} D & 0 \\ 0 & -D \end{smallmatrix}\right) \mid D = \mathrm{diag}(\lambda_1, \dots, \lambda_4),\ \lambda_i \in F\}.$$

Let $(E_{ij},\ i,j = 1,\dots,n)$ be the standard basis of the algebra $M_n(F)$ of $(n \times n)$-matrices with entries in F. The matrices

$$u_j = E_{jj} - E_{j+4,j+4},\ j = 1,\dots,4, \tag{4}$$

form a basis of $\mathfrak{h}$. We have a Cartan decomposition of $\mathfrak{o}_8$

$$\mathfrak{o}_8 = \mathfrak{h} \oplus (\oplus_\alpha \mathfrak{g}_\alpha),$$

where the one-dimensional root spaces $\mathfrak{g}_\alpha$ are generated by the 24 matrices

$$\begin{array}{rcll} G^X_{ij} & = & E_{i,j} - E_{j+4,i+4} & 1 \leqslant i,j \leqslant 4, i \neq j, \\ G^Y_{ij} & = & E_{i,j+4} - E_{j,i+4} & 1 \leqslant i < j \leqslant 4, \\ G^Z_{ij} & = & E_{i+4,j} - E_{j+4,i} & 1 \leqslant i < j \leqslant 4. \end{array}$$

Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{H}$ and let $\varepsilon_i \in \mathfrak{h}^*,\ i = 1,\dots,4$, be the dual basis of the basis (u_j) of $\mathfrak{h}$. The simple roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3,\ \alpha_3 = \varepsilon_3 - \varepsilon_4 \text{ and } \alpha_4 = \varepsilon_3 + \varepsilon_4.$$

The rule $\alpha_1 \mapsto \alpha_4,\ \alpha_4 \mapsto \alpha_3,\ \alpha_3 \mapsto \alpha_1,\ \alpha_2 \mapsto \alpha_2$ defines an automorphism ρ of the Dynkin diagram D_4. Let $\overline{\rho}$ be its (unique) extension to $\mathfrak{h}^*$. The automorphism $\overline{\rho}$ is given by the matrix

$$R = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix} \tag{5}$$

with respect to the basis (ε_i). Similarly, let π be the automorphism of the Dynkin diagram given by $\alpha_3 \mapsto \alpha_4,\ \alpha_4 \mapsto \alpha_3,\ \alpha_1 \mapsto \alpha_1,\ \alpha_2 \mapsto \alpha_2$. Its extension $\overline{\pi}$ to $\mathfrak{h}^*$ has the matrix

$$P = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

One checks that $R^3 = 1$, $P^2 = 1$ and $PR = R^2P$. Thus $\{R, P\}$ generates a group isomorphic to S_3. Let $H \in \mathfrak{h}$ and $e_\alpha \neq 0 \in \mathfrak{g}_\alpha$. Any automorphism ϕ of the root system induces an automorphism f_ϕ of $\mathfrak{o}_8$:

$$f_\phi\big(H + \sum_\alpha s_\alpha e_\alpha\big) = (\phi^*)^{-1}(H) + \sum_\alpha s_\alpha e_{\phi(\alpha)}$$

where ϕ^* is the dual of ϕ and the sum is taken over all roots. In particular ρ and π induce automorphisms f_ρ and f_π. Let ω be a primitive cubic root of 1. Let (v_i) be the dual basis of (α_i), so that $v_2 = u_1 + u_2$. The map

$$f_2\big(H + \sum_\alpha s_\alpha e_\alpha\big) = H + \sum_\alpha \omega^{\alpha(v_2)} s_\alpha e_\alpha$$

is an automorphism of order 3 of $\mathfrak{o}_8$ and is invariant, since it is the identity on the Cartan algebra (see [J62]). Hence it is an inner automorphism. Moreover f_ρ and f_2 commute. Let $f'_\rho = f_\rho \circ f_2$. We have (see [N] or [WG]):

Theorem 6. *Assume that F is algebraically closed.*
1) Every outer automorphism of order 3 of $\mathfrak{o}_8$ is conjugate in $\operatorname{Aut}(\mathfrak{o}_8)$ to f_ρ or f'_ρ and in $\operatorname{Int}(\mathfrak{o}_8)$ to $f_\rho^{\pm 1}$ or to $f'^{\pm 1}_\rho$.
2) The set $\{f_\rho, f_\pi\}$ generates a group isomorphic to S_3.
3) The fixed point algebra of f_ρ is a Lie algebra of type G_2 and that of f'_ρ of type A_2.
4) Fixed point subalgebras under outer automorphisms of order 3 are isomorphic if and only if the automorphisms are conjugate in $\operatorname{Aut}(\mathfrak{o}_8)$.

Example 7. Let $\mathcal{C}$ be a Cayley algebra with norm n and multiplication $x, y \mapsto xy$. The classical local triality principle (see for example [VBS], [SV] or [J64]) states that for every $s \in \mathfrak{o}(n) \subset \operatorname{End}_F(\mathcal{C})$ there are unique $t, u \in \mathfrak{o}(n)$ such that

$$t(x)y + xs(y) = u(xy) \quad \text{for all } x, y \in \mathcal{C}.$$

Moreover the map $f : s \mapsto t$ is an automorphism of order 3 and the map $g : s \mapsto u$ is an automorphism of order 2 of $\mathfrak{o}(n)$; they generate a group isomorphic to S_3. The fixed point Lie algebra $\operatorname{Fix}(f) = \{s \in \mathfrak{o}(n) \mid f(s) = s\}$ is the Lie algebra of derivations of $\mathcal{C}$ which is a Lie algebra of type G_2. Thus by Theorem 6 f is conjugate to f_ρ over an algebraic closure of F.

3. Symmetric compositions

Let S be a finite dimensional F-vector space with a bilinear multiplication $(x, y) \mapsto x \star y$. We say that a quadratic form n on S is *multiplicative* if

$$n(x \star y) = n(x)n(y) \tag{8}$$

for all x, $y \in S$. A triple $(S, \star, n)$ with a nonsingular multiplicative quadratic form n is called a *composition algebra* and n is the *norm* of S. By a classical result of Hurwitz (see for example [KMRT, (33.17)]), a complete list of composition algebras with identity is given by the base field F, quadratic separable algebras, quaternion algebras and Cayley algebras over F. Composition algebras with identity are usually called *Hurwitz algebras*. If a composition algebra does not have an identity, one can define a new multiplication such that the corresponding algebra is a composition algebra with identity with the same norm (see Kaplansky [KA] or [KMRT, (33.27)]). Thus:

Proposition 9. *Composition algebras only occur in dimension* 1, 2, 4 *and* 8, *and their norms are norms of Hurwitz algebras.*

Let $\alpha_1, \dots, \alpha_n \in F^\times$. We denote by $\langle \alpha_1, \dots, \alpha_n \rangle$ the diagonal quadratic form

$$\langle \alpha_1, \dots, \alpha_n \rangle = \alpha_1 x_1^2 + \cdots + \alpha_n x_n^2$$

and by $\langle\!\langle \alpha_1, \dots, \alpha_k \rangle\!\rangle$ the *k-fold Pfister quadratic form*

$$\langle\!\langle \alpha_1, \dots, \alpha_k \rangle\!\rangle = \langle 1, \alpha_1 \rangle \otimes_F \cdots \otimes_F \langle 1, \alpha_k \rangle.$$

Norm forms of composition algebras are k-Pfister forms for $k = 1$, 2 and 3.
We recall that a symmetric bilinear form b on an algebra $(S, \star)$ is called *associative or invariant* with respect to the multiplication $\star$ of S if

$$b(x \star y, z) = b(x, y \star z)$$

holds for all x, y, $z \in S$. If the polar form b_n of a composition algebra S is associative, we say that $(S, \star, n)$ is a *symmetric composition*. Symmetric compositions have the following nice property (see [OO] or [KMRT, (34.1)]:

Proposition 10. *Let $(S, \star)$ be an algebra and let be n be a nonsingular multiplicative quadratic form on S. The following conditions are equivalent:*

(1) *$(S, \star, n)$ is a symmetric composition.*
(2) *The norm form n satisfies the relations $x \star (y \star x) = n(x)y = (x \star y) \star x$ for x, $y \in S$.*

If $(S, \star)$ is a symmetric composition, the opposite algebra $(S^{\mathrm{op}}, \star)$ is also a symmetric composition.

Example 11. (Symmetric compositions of type G_2, [KMRT]) or [EM])
Let $\mathcal{C}$ be a Hurwitz algebra with multiplication $(x, y) \mapsto xy$ and norm n. The algebra $\mathcal{C}$ admits a conjugation $\pi_0 : x \mapsto \overline{x}$ which satisfies:

$$\overline{1} = 1,\ \overline{\overline{x}} = x,\ \ \overline{xy} = \overline{y}\,\overline{x} \ \text{ and } \ b_n(xy, z) = b_n(x, z\overline{y}) \ \text{ for all } x, y, z \in \mathcal{C}. \tag{12}$$

The form n is also multiplicative for the multiplication

$$(x, y) \mapsto x \star y = \overline{x}\,\overline{y}$$

and it follows from (12) that n is associative for the $\star$-multiplication. We call $(\mathcal{C}, \star, n)$ a *para-Hurwitz algebra*. In dimension 8 para-Hurwitz algebras are called *para-Cayley algebras*, resp. *symmetric compositions of type G_2*.

Example 13. (Symmetric compositions of type A_2, [KMRT]) or [EM])
Assume that the base field contains a primitive cubic root ω of 1. Let A be a central simple algebra of dimension 9 over F (for example $A = M_3(F)$). Let

$$P_{A,a}(X) = X^3 - T_A(a)X^2 + S_A(a)X^2 - N_A(a)$$

be the reduced characteristic polynomial of A and let A^0 be the 8-dimensional space of reduced trace zero elements

$$A^0 = \{x \in A \mid T_A(x) = 0\}.$$

We define a multiplication $\star$ on A^0 by

$$x \star y = \mu xy + (1-\mu)yx - \tfrac{1}{3}T_A(xy)1_A \tag{14}$$

where $\mu = \frac{1-\omega}{3}$. The algebra $(A^0, \star)$ is a symmetric composition algebra with norm $n(x) = -\frac{1}{3}S_A(x) = \frac{1}{6}T_A(x^2)$ (see for example [KMRT, (34.19)]).

If F does not contain a primitive cubic root of unity ω, one considers central simple algebras B of dimension 9 over the quadratic field extension $K = F(\omega) = F(\sqrt{-3})$, with an involution τ of second kind, i.e., $\tau|_K \neq 1_K$. Let $\mathrm{Sym}(B,\tau)$ be the set of symmetric elements in B and let

$$\mathrm{Sym}(B,\tau)^0 = \{x \in \mathrm{Sym}(B,\tau) \mid T_B(x) = 0\}.$$

The multiplication defined on B^0 by (14) induces a product on $\mathrm{Sym}(B,\tau)^0$ and with the quadratic form $n(x) = \frac{1}{6}T_B(x^2)$, the space $\big(\mathrm{Sym}(B,\tau)^0, \star, n\big)$ is a symmetric composition. This type of composition was first considered by Okubo [O] for matrix algebras and by Faulkner [F] for cubic alternative algebras. They are called *Okubo compositions* or *symmetric compositions of type* A_2, more precisely *of type* ${}^1A_2$, resp. *of type* ${}^2A_2$ depending if F contains a primitive cubic root of 1 or not.

It follows from the classification of symmetric compositions (see [EM] or [KMRT, (34.37)]) that in dimension 1, 2 and 4 Para-Hurwitz and Okubo compositions are essentially the same. In dimension 8 we have ([EM] or [KMRT, (34.37)]):

Theorem 15. *Let F be a field of characteristic different from 2 and 3. A symmetric composition $(S, \star, n)$ of dimension 8 is either isomorphic to:*
1) A para-Cayley algebra $(\mathcal{C}, \star)$ for a Cayley algebra $\mathcal{C}$,
2) A composition algebra of the form $(A^0, \star)$ for a central simple F-algebra A of degree 3 if F contains a primitive cubic root of unity,
3) A composition algebra of the form $\big(\mathrm{Sym}(B,\tau)^0, \star, n\big)$ for a central simple $F(\omega)$-algebra B of degree 3 with an involution τ of the second kind if F does not contain a primitive cubic root of unity ω.

If the quadratic extension K is allowed to be split, i.e., $K \simeq F \times F$, then types ${}^1A_2$ and ${}^2A_2$ need not to be distinguished: we replace A of type ${}^1A_2$ by $B = H(A) = A \times A^{\mathrm{op}}$ and τ is the twist of the two factors. We say also in this case that $H(A)$ is central simple over K/F and from now on symmetric compositions of type A_2 will be represented as $\mathrm{Sym}(B,\tau)^0$. The algebras $\mathcal{C}$, A or (B,τ) in Theorem 15 are uniquely determined up to isomorphisms by the corresponding symmetric compositions. More precisely,

Proposition 16. *1) Any isomorphism $\mathcal{C}_1 \simeq \mathcal{C}_2$ of Cayley algebras is an isomorphism of the associated para-Cayley algebras and conversely.*
2) Let (B_i, τ_i), $i = 1, 2$, be central simple of degree 3 with involutions of second kind over $K = F[x]/(x^2+3)$. Any K-isomorphism $(B_1,\tau_1) \simeq (B_2,\tau_2)$ of algebras with involution induces by restriction an isomorphism $\mathrm{Sym}(B_1,\tau_1)^0 \simeq \mathrm{Sym}(B_2,\tau_2)^0$ of the associated Okubo algebras and conversely.
3) Any F-isomorphism $(B_1,\tau_1) \simeq (B_2,\tau_2)$ of algebras with involution induces by restriction an isomorphism or an anti-isomorphism $\mathrm{Sym}(B_1,\tau_1)^0 \simeq \mathrm{Sym}(B_2,\tau_2)^0$

of the associated Okubo algebras and conversely.

Proof. 1) follows from the fact that isomorphisms of Cayley algebras are compatible with conjugations. We refer to [KMRT, (34.35)] for 2). Claim 3) follows from the fact that if $\phi : (B_1, \tau_1) \simeq (B_2, \tau_2)$ is an F-isomorphism which is not K-linear, then $\tau_2 \circ \phi = \phi \circ \tau_1$ is a K-anti-isomorphism, hence induces an anti-isomorphism of the associated Okubo algebras. □ □

The conjugation π_0 of a Cayley algebra $(\mathcal{C}, n)$ is a F-linear involution for the associated para-Cayley algebra $(\mathcal{C}, \star)$:

$$\pi_0(x \star y) = \pi_0(y) \star \pi_0(x),\ \pi_0^2 = 1_{\mathcal{C}} \tag{17}$$

for $x,\ y \in \mathcal{C}$. The map π_0 is an isometry of n, but is improper since $\det(\pi_0) = -1$. For symmetric composition algebras of type A_2 we have:

Proposition 18. *Let (B, τ) be a central simple $F(\omega)$-algebra B of degree 3 with an involution τ of the second kind. The following properties are equivalent:*
1) The symmetric composition $S = \big(\mathrm{Sym}(B, \tau)^0, \star, n\big)$ admits an F-linear involution π_0,
2) S is isomorphic to its opposite algebra S^{op},
3) B is split.

Proof. 1) clearly implies 2). If ϕ is an F-linear anti-automorphism of $(S, \star)$, then ϕ defines also a K-anti-automorphism of B since the multiplication of B restricted to B^0 is given by

$$xy = (1 + \omega)x \star y - \omega\, y \star x + b_n(x, y) \cdot 1$$

(see [KMRT, §34] It follows that B has order two in the Brauer group of K. Since B, being of degree 3, has exponent 1 or 3, B is split. If B is split and $x \mapsto \overline{x}$, $x \in M_3(K)$, is element wise conjugation, then $\overline{\tau} : x \mapsto \overline{\tau(x)}$ is an F-linear involution of B, hence 1). □ □

4. Derivations of symmetric compositions

For any F-algebra A (i.e., a F-vector space with a bilinear multiplication $(x, y) \mapsto xy$), the set of F-derivations

$$\mathrm{Der}(A) = \{d \in \mathrm{End}_F(A) \mid d(xy) = xd(y) + d(x)y\}$$

is a Lie subalgebra of $\mathrm{End}_F(A)$. Let $(\mathcal{C}, n)$ be a Cayley algebra with norm n and let

$$(\mathcal{C}, n) = F \cdot 1 \perp (\mathcal{C}^0, n^0)$$

where $(\mathcal{C}^0, n^0)$ is the quadratic subspace of trace 0 elements. The Lie algebra $\mathrm{Der}(\mathcal{C})$ is a subalgebra of $\mathfrak{o}(n)$ (see [J39]) and the 8-dimensional representation $\mathrm{Der}(\mathcal{C}) \subset \mathrm{End}_F(\mathcal{C})$ decomposes as the sum of a 1-dimensional trivial representation and the 7-dimensional standard representation

$$\gamma : \mathrm{Der}(\mathcal{C}) \subset \mathfrak{o}(n^0) \subset \mathrm{End}_F(\mathcal{C}^0) \tag{19}$$

which is known to be absolutely irreducible (see [J39]).

Proposition 20. *Let* $\phi : \mathrm{Der}(\mathcal{C}) \to \mathfrak{o}(q)$ *be a faithful orthogonal representation for some 7-dimensional quadratic space* (V, q)*. There exists a similitude* $s : (V, q) \xrightarrow{\sim} (\mathcal{C}^0, n^0)$ *such that* $\mathrm{Int}(s) \circ \phi = \gamma$*.*

Proof. The claim is true over an algebraically closed field (see [J39]). Let $t : (V, q) \xrightarrow{\sim} (\mathcal{C}^0, n^0)$ be a similitude over an algebraic closure $\overline{F}$ of F such that $\mathrm{Int}(t) \circ \phi = \gamma$. Since $\gamma\big(\mathrm{Der}(\mathcal{C})\big)$ generates $\mathrm{End}_F(\mathcal{C}^0)$ (see [J39]), $\mathrm{Int}(t)$ is unique, hence by descent restricts to an isomorphism $\psi : \big(\mathrm{End}_F(V), \sigma_q\big) \xrightarrow{\sim} \big(\mathrm{End}_F(\mathcal{C}^0), \sigma_{n^0}\big)$ of algebras with involution. By Proposition 3 we have $\psi = \mathrm{Int}(s)$ for a similitude $s : (V, q) \xrightarrow{\sim} (\mathcal{C}^0, n^0)$. □ □

Corollary 21. *Let* $(\mathcal{C}, n)$ *and* $(\mathcal{C}', n')$ *be Cayley algebras. If the Lie algebra* $\mathrm{Der}(\mathcal{C})$ *is isomorphic to a subalgebra of* $\mathfrak{o}(n')$*, then* $\mathcal{C}$ *and* $\mathcal{C}'$ *are isomorphic.*

Proof. For reason of dimension the representation $\mathrm{Der}(\mathcal{C}) \subset \mathfrak{o}(n') \subset \mathrm{End}_F(\mathcal{C}')$ decomposes into a 1-dimensional representation V_1 and a 7-dimensional irreducible representation V_2. We claim that the decomposition $\mathcal{C}' = V_1 \oplus V_2$ is orthogonal with respect to the norm n'. Let b' be the polar of n' and let $x_i \in V_i$, $i = 1, 2$. The 1-dimensional representation is the trivial representation, hence

$$b'\big(d(x_2), x_1\big) = -b'\big(x_2, d(x_1)\big) = 0.$$

Thus $\mathrm{Der}(\mathcal{C})(V_2)$ is orthogonal to V_1 and since $\mathrm{Der}(\mathcal{C})$ generates $\mathrm{End}_F(V_2)$ as an F-algebra, we have an orthogonal decomposition

$$(\mathcal{C}', n') = (V_1, \langle\lambda\rangle) \perp (V_2, q)$$

for some quadratic form q on V_2. By Proposition 20 the form q is isometric to $\langle\mu\rangle \cdot n^0$ for some $\mu \in F^\times$, thus we get $n' \simeq \langle\lambda\rangle \perp \langle\mu\rangle \cdot n^0$. Since $\det(n')$ is congruent to 1 modulo squares in F, $\langle\lambda\rangle \simeq \langle\mu\rangle$ and

$$n' \simeq \langle\mu\rangle \perp \langle\mu\rangle \cdot n^0 \simeq \langle\mu\rangle \cdot n.$$

Thus n and n' are similar, which implies that $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic (see for example [KMRT, (33.19)]). □ □

Corollary 22. *Let* $(\mathcal{C}, n)$ *be a Cayley algebra, let* $\gamma : \mathrm{Der}(\mathcal{C}) \to \mathfrak{o}(n)$ *be the standard embedding and let* $\mathfrak{g}$ *be a Lie subalgebra of* $\mathfrak{o}(n)$ *isomorphic to* $\mathrm{Der}(\mathcal{C})$*. Any isomorphism* $\phi : \gamma\big(\mathrm{Der}(\mathcal{C})\big) \xrightarrow{\sim} \mathfrak{g}$ *can be extended to an inner automorphism of* $\mathfrak{o}(n)$*.*

Proof. The isomorphism ϕ induces a 8-dimensional representation

$$\tilde{\phi} : \mathrm{Der}(\mathcal{C}) \xrightarrow{\sim} \mathfrak{g} \subset \mathfrak{o}(n) \subset \mathrm{End}_F(\mathcal{C})$$

which decomposes as a direct sum of a 1-dimensional and a 7-dimensional representation

$$(\mathcal{C}, n) = (F \cdot e, \langle\lambda\rangle) \perp (V, q).$$

By Proposition 20 there is an isometry $s : (V, q) \xrightarrow{\sim} \langle\mu\rangle \cdot (\mathcal{C}^0, n^0)$, $\mu \in K^\times$, such that $\mathrm{Int}(s) \circ \tilde{\phi} = \gamma$. Since $\det(n)$ is congruent to 1 modulo squares, the element e can be chosen such that $n(e) = \mu^{-1}$ and $e \mapsto 1$ extends s to a similitude of $(\mathcal{C}, n)$, also denoted s. If s is improper, we may replace s by $\pi_0 \circ s$, where π_0 is the conjugation of $\mathcal{C}$, since $\mathrm{Int}(\pi_0)$ is the identity on $\mathrm{Der}(\mathcal{C})$ ([S, III.8, (3.75)]). □ □

Proposition 23. *1) Let $(S,\star)$ be the Para-Cayley algebra associated with a Cayley algebra $\mathcal{C}$. Then $\operatorname{Der}(S,\star)=\operatorname{Der}(\mathcal{C})$, so that $\operatorname{Der}(S,\star)$ is a simple Lie algebra of type G_2.*
2) Let $(\mathcal{C}_i,n_i)$ be Cayley algebras with norms n_i. Any isomorphism $\operatorname{Der}(\mathcal{C}_1)\simeq\operatorname{Der}(\mathcal{C}_2)$ of Lie algebras is induced by a unique isomorphism $\mathcal{C}_1\simeq\mathcal{C}_2$ of Cayley algebras.

Proof. We refer to [J39] for a proof of 2). To prove 1), let first d be a derivation of $\mathcal{C}$, $d(xy)=xd(y)+d(x)y$. In view of [S, III.8, (3.75)], we have $d(\overline{x})=\overline{d(x)}$ for $x\in\mathcal{C}$. It follows that

$$d(x\star y)=d(\overline{x}\,\overline{y})=\overline{x}\,d(\overline{y})+d(\overline{x})\,\overline{y}=x\star d(y)+d(x)\star y$$

and d is also a derivation of the associated para-Cayley algebra. Conversely, if d is a derivation of the para-Cayley algebra, one first check that $d(1)=0$, which implies that $d(x)=-d(\overline{x})$. Using that the conjugation $\pi_0: x\mapsto\overline{x}$ is an involution of the para-Cayley algebra, we get

$$d(x\star y)=d(y)\star\overline{x}+\overline{y}\star d(x).$$

With $x=1$ this implies $d(\overline{y})=\overline{d(y)}$ and one concludes that d is also a derivation of the Cayley algebra. □ □

Let K/F be a quadratic extension and let (B,τ) be a central simple algebra over K/F with an involution τ of second kind. The quadratic form $Q: B\to F,\ Q(x)=T_B\big(x\tau(x)\big)$ is nonsingular, the involution τ is an isometry of Q and we have

$$b_Q(xy,z)=b_Q\big(x,z\tau(y)\big)$$

for x, y and $z\in B$. Let $\operatorname{Skew}(B,\tau)^0=[\operatorname{Skew}(B,\tau),\operatorname{Skew}(B,\tau)]$ be the Lie algebra of skew-symmetric elements of B of reduced trace zero. The quadratic form $Q: B\to F,\ Q(x)=T_B\big(x\tau(x)\big)$ restricts to a nonsingular quadratic form on $\operatorname{Skew}(B,\tau)^0$ denoted by Q_- and to a nonsingular quadratic form on $\operatorname{Sym}(B,\tau)^0$ denoted by Q_+.

Proposition 24. *The map* $\operatorname{ad}:\operatorname{Skew}(B,\tau)^0\to\operatorname{End}_F\big(\operatorname{Skew}(B,\tau)^0\big),\ \operatorname{ad}_y(x)=[x,y]$, *induces an orthogonal representation*

$$\operatorname{Skew}(B,\tau)^0\to\mathfrak{o}(Q_-)$$

which is absolutely irreducible.

Proof. The map ad induces an isomorphism

$$\operatorname{Skew}(B,\tau)^0\xrightarrow{\sim}\operatorname{Der}\big(\operatorname{Skew}(B,\tau)^0\big)\tag{25}$$

of Lie algebras since $\operatorname{Skew}(B,\tau)^0$ is simple and any derivation of $\operatorname{Skew}(B,\tau)^0$ is inner. Using again that any derivation of $\operatorname{Skew}(B,\tau)^0$ is inner, it is easy to check that $\operatorname{Der}\big(\operatorname{Skew}(B,\tau)^0\big)\subset\mathfrak{o}(Q_-)$. Thus the adjoint representation is orthogonal. The fact that it is absolutely irreducible is for example in [J62]. □ □

Proposition 26. *Let (B,τ) be a central simple algebra of rank 9 over K/F with an involution of second kind τ.*
1) Let (V,q) be a 8-dimensional quadratic space and let $\phi:\operatorname{Skew}(B,\tau)^0\to\mathfrak{o}(q)$ be a faithful orthogonal representation. There exists a similitude $s:(V,q)\xrightarrow{\sim}\big(\operatorname{Skew}(B,\tau)^0,Q_-\big)$ such that $\operatorname{Int}(s)\circ\phi=\operatorname{ad}$.

2) Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{o}(Q_-)$ isomorphic to $\mathrm{Der}\big(\mathrm{Skew}(B,\tau)^0\big)$. Any isomorphism of Lie algebras $\phi : \mathrm{Der}\big(\mathrm{Skew}(B,\tau)^0\big) \xrightarrow{\sim} \mathfrak{g}$ can be extended to an automorphism of $\mathfrak{o}(Q_-)$ of the form $\mathrm{Int}(s)$, for a similitude s of Q_-.

Proof. 1) Over an algebraic closure $\overline{F}$ of F, the adjoint representation is up to isomorphism the unique irreducible 8-dimensional orthogonal representation of $\mathrm{Skew}(B,\tau)^0$. Thus the map ϕ can be extended to an isomorphism $\psi = \mathrm{Int}(t)$ of $\big(\mathrm{End}_F(V),\sigma_q\big)\otimes_F \overline{F}$ onto $\big(\mathrm{End}_F\big(\mathrm{Skew}(B,\tau)^0\big),\sigma_{Q_-}\big)\otimes_F \overline{F}$, which is uniquely determined by the the condition $\mathrm{Int}(t)\circ\phi = \mathrm{ad}$, since the image of the map ad in $\mathrm{End}_{\overline{F}}\big(\mathrm{Skew}(B,\tau)^0\big)$ generates $\mathrm{End}_{\overline{F}}\big(\mathrm{Skew}(B,\tau)^0\big)$. By descent ψ restricts to an isomorphism of $\big(\mathrm{End}_F(V),\sigma_q\big)$ onto $\big(\mathrm{End}_F\big(\mathrm{Skew}(B,\tau)^0\big),\sigma_{Q_-}\big)$, hence is induced by a similitude of (V,q) with $\big(\mathrm{Skew}(B,\tau)^0,Q_-\big)$ by Proposition 3. Claim 2) follows from claim 1). □ □

Proposition 27. *Let $K = F[x]/(x^2+3)$ and let (B,τ) be central simple of rank 9 over K/F, with an involution of second kind τ. Let $(S,\star,n) = \mathrm{Sym}(B,\tau)^0$ be the symmetric composition of type A_2 associated with (B,τ). Let u be a skew-symmetric non-zero element of K.*
1) The map $\ell_u : x \mapsto ux$ induces a similitude

$$\big(\mathrm{Sym}(B,\tau)^0,n\big) \xrightarrow{\sim} \big(\mathrm{Skew}(B,\tau)^0,Q_-\big) \tag{28}$$

and an isomorphism

$$\mathrm{Int}(\ell_u) : \mathrm{Der}(S,\star) \xrightarrow{\sim} \mathrm{Der}\big(\mathrm{Skew}(B,\tau)\big) \tag{29}$$

of Lie algebras. Hence the Lie algebra $\mathrm{Der}(S,\star)$ is a simple Lie algebra of type A_2.
2) Let (B_i,τ_i) be central simple algebras with involutions of second kind over K/F, $i=1,2$. Any isomorphism of Lie algebras $\mathrm{Skew}(B_1,\tau_1)^0 \xrightarrow{\sim} \mathrm{Skew}(B_2,\tau_2)^0$ can be uniquely extended to an F-isomorphism $(B_1,\tau_1)\xrightarrow{\sim}(B_2,\tau_2)$ and to a K-isomorphism or a K-anti-isomorphism $(B_1,\tau_1)\xrightarrow{\sim}(B_2,\tau_2)$.

Proof. The first claim of 1) is clear since $n = \frac{1}{6}Q_+$ and since Q_+ and Q_- are similar. The second claim follows by a computation using (14) and (17). The fact that $\mathrm{Der}(S,\star,n)$ is a Lie algebra of type A_2 is a consequence of (29) and (25). For a proof of 2), see for example [J62]. □ □

Corollary 30. *Let (B,τ) be a central simple algebra of degree 3 over a quadratic extension K of F, with an involution of second kind. The Lie algebra $\mathrm{Skew}(B,\tau)^0$ is a subalgebra of the orthogonal Lie algebra of a 3-Pfister form n if and only if $K = F[x]/(x^2+3)$.*

Proof. Assume that $\mathrm{Skew}(B,\tau)^0$ is a subalgebra of the orthogonal Lie algebra of a 3-Pfister form n. In view of Proposition 26 the form Q_- is similar to n. It is also similar to $\big(\mathrm{Sym}(B,\tau)^0,Q_+\big)$ and, if $K = F[x]/(x^2-\alpha)$,

$$\big(\mathrm{Sym}(B,\tau)^0,Q_+\big) \simeq \langle 2,6\rangle \perp \langle 2\rangle\cdot\langle\langle -\alpha\rangle\rangle\cdot q_0, \tag{31}$$

where q_0 is a quadratic space over F of dimension 3 and determinant 1 (see [HKRT] or[KMRT, § 19]). Thus the determinant of Q_- is congruent to -3α modulo squares. Since the determinant of a 3-Pfister form is congruent to 1, α must be congruent to -3. The converse follows from Proposition 24. □ □

By Proposition 9, the norm form n of a symmetric composition of type A_2 is a 3-Pfister form. More precisely:

Proposition 32. *The norm of a symmetric composition of type A_2 is of the form $\langle\langle 3, b, c\rangle\rangle$ for some $b,\ c \in F^\times$. If F contains a primitive cubic root of 1, n is hyperbolic.*

Proof. By Corollary 30 the element α in (31) is congruent to -3 and, if we write q_0 in 31 as $\langle b, c, bc\rangle$, we get

$$n \simeq \langle \tfrac{1}{6}\rangle \cdot Q_+ \simeq \langle 1, \tfrac{1}{3}\rangle \perp \langle 1, \tfrac{1}{3}\rangle \cdot \langle b, c, bc\rangle \simeq \langle\langle 3, b, c\rangle\rangle$$

as claimed. □ □

Remark 33. The isomorphism class of the 3-Pfister form given by the norm of the symmetric composition $\mathrm{Sym}(B, \tau)^0$ defines a cohomological invariant which classifies involutions of second kind on B. For details see [HKRT] or [KMRT, (19.6)], where this invariant is denoted by $\pi(\tau)$. Those involutions for which the invariant is trivial (i.e., the form is hyperbolic) are called *distinguished*. Any central simple algebra of degree 3 with involution of the second kind admits a distinguished involution (see [HKRT] or [KMRT, (19.30)]). If B is a matrix algebra and τ is the hermitian transpose composed with conjugation by the hermitian matrix $\langle \delta_1, \delta_2, \delta_3\rangle$, $\delta_i \in F^\times$, then $\pi(\tau) \simeq \langle\langle 3, \delta_1\delta_2, \delta_1\delta_3\rangle\rangle$.

Example 34. If $F = \mathbb{R}$ and $B = M_3(\mathbb{C})$, there are two types of involutions of second kind on B: the hermitian transpose, for which the cohomological invariant is the identity form and the hermitian transpose conjugated with $\langle -1, 1, 1\rangle$, which is distinguished. Thus there are 4 classes of symmetric compositions over $\mathbb{R}$.

Proposition 35. *Two symmetric compositions of dimension 8 are isomorphic or anti-isomorphic if and only if their Lie algebras of derivations are isomorphic.*

Proof. Derivations of S are obviously also derivations of S^{op}. Conversely, in view of Theorem 15 of symmetric compositions, it suffices to check the claim for symmetric compositions S of type G_2 and A_2. The claim then follows from Corollary 21 and Proposition 27. □ □

5. The Clifford algebra of a symmetric composition

Let (V, q) be a quadratic space and let $C(V, q)$ be the *Clifford algebra* of the quadratic space (V, q). We recall that $C(V, q) = TV/I$ where TV is the tensor algebra of V and I is the ideal of TV generated by the elements $x \otimes x - q(x) \cdot 1$, $x \in V$. The space V, identified with a subspace of $C(V, q)$ through the natural map $V \to C(V, q)$, generates $C(V, q)$ as an algebra. The *even Clifford algebra* $C_0(V, q)$ is the subalgebra of $C(V, q)$ generated by products of an even number of elements of V. The Clifford algebra is characterized by a universal property: any F-linear map α of (V, q) to an associative F-algebra A such that $\alpha(x)^2 = q(x)$ for all $x \in V$ factorises over $C(V, q)$. There is a unique F-linear involution τ on $C(V, q)$ which is the identity on V. The structure of the Clifford algebra is described in the following proposition (see for example [SCH] for a proof):

Proposition 36. *Let (V, q) be a nonsingular quadratic space of even dimension $n = 2m$.*
1) The F-algebra $C(V, q)$ is central simple of dimension 2^n.
2) The center Z of the even Clifford algebra is an étale quadratic extension of F of the form $Z = F(\sqrt{\delta})$ where $\delta = (-1)^m \det(b_q)$. The algebra $C_0(V, q)$ is central separable over Z of rank 2^{2m-1}. Suppose that m is even. Then the involution τ

restricts to an involution τ_0 of $C_0(V,q)$ which is the identity on Z. Further τ_0 is of orthogonal type if m is congruent to 0 modulo 4 and is of symplectic type if m is congruent to 2 modulo 4.

The Lie algebra $\mathfrak{o}(q)$ can be identified with a Lie subalgebra of $\operatorname{Skew}\big(C_0(V,q),\tau_0\big)$, as follows. For x, y, $z \in V$ and the Lie product $[x,y] = xy - yx$ in $C(V,q)$ we have:

$$[[x,y],z] = 2\big(b_q(y,z)x - b_q(x,z)y\big) \in V. \tag{37}$$

Let $[V,V]$ be the subspace of $C_0(V,q)$ spanned by the $[x,y] = xy - yx$ for all x, $y \in V$. In view of (37) we have a linear map

$$\operatorname{ad}\colon [V,V] \to \operatorname{End}_F(V), \quad \xi \mapsto \operatorname{ad}_\xi,$$

defined by $\operatorname{ad}_\xi(z) = [\xi, z]$ for $\xi \in [V,V]$ and $z \in V$.

Lemma 38. *The subspace $[V,V]$ of $C_0(q)$ is a Lie subalgebra of $\operatorname{Skew}\big(C_0(V,q),\tau_0\big)$ and* ad *induces an isomorphism of Lie algebras:*

$$\operatorname{ad}\colon [V,V] \xrightarrow{\sim} \mathfrak{o}(q).$$

Proof. See [KMRT, Lemma (45.3)]. □ □

It readily follows from the definition of Clifford algebras that isometries of (V,q) induce automorphisms of $C(V,q)$. For similitudes we have:

Proposition 39. *Any similitude $f \in \mathrm{GO}(q)$ with multiplier $m(f)$ induces an automorphism $C(f)$ of $\big(C_0(V,q),\tau_0\big)$ such that $C(f)(xy) = m(f)^{-1}f(x)f(y)$ for x, $y \in V$. The automorphism $C(f)$ restricts to the identity on the center Z of $C_0(V,q)$ if and only if f is proper. Further the automorphism $\operatorname{ad} \circ C(f) \circ \operatorname{ad}^{-1}$ of $\mathfrak{o}(q)$ is equal to the restriction of $\operatorname{Int}(f)$ to $\mathfrak{o}(q)$ in $\operatorname{End}_F(V)$.*

Proof. The first two claims are standard (see for example [KMRT]). Using (37) we have for x, y, $z \in V$:

$$\big(\operatorname{ad} \circ C(f)\big)\big([x,y]\big)(z) = 2\mu(f)^{-1}\big(\big(f(x)b_q(f(y),z\big) - f(y)b_q\big(f(x),z\big)\big)$$

and

$$\begin{aligned}\big(\operatorname{Int}(f) \circ \operatorname{ad}\big)\big([x,y]\big)(z) &= 2f\big(xb_q(y,f^{-1}(z)) - yb_q(x,f^{-1}(z))\big)\\ &= 2\big(f(x)b_q(y,f^{-1}(z)) - f(y)b_q(x,f^{-1}(z))\big).\end{aligned}$$

The last claim follows from $\mu(f)^{-1}b_q\big(f(y),z\big) = b_q\big(y,f^{-1}(z)\big)$. □ □

Let $(S,\star,n)$ be a symmetric composition of dimension 8. Let $C(S,n)$ be the Clifford algebra and $C_0(S,n)$ the even Clifford algebra of (S,n). Let τ be the involution of $C(S,n)$ which is the identity on S. Let $r_x(y) = y \star x$ and $\ell_x(y) = x \star y$ for $x, y \in S$.

Proposition 40. ([KMRT, 35.1]) *The map $S \to \operatorname{End}_F(S \oplus S)$ given by*

$$x \mapsto \begin{pmatrix} 0 & \ell_x \\ r_x & 0 \end{pmatrix}$$

induces isomorphisms

$$\alpha_S : \big(C(S,n),\tau\big) \xrightarrow{\sim} \big(\operatorname{End}_F(S \oplus S), \sigma_{n\perp n}\big)$$

and

$$\alpha_S : \big(C_0(S,n),\tau\big) \xrightarrow{\sim} \big(\operatorname{End}_F(S),\sigma_n\big) \times \big(\operatorname{End}_F(S),\sigma_n\big), \tag{41}$$

of algebras with involution. Further α_S maps $[S,S] \subset C_0(S,n)$ to $\mathfrak{o}(n) \times \mathfrak{o}(n)$.

Proof. We have $r_x \circ \ell_x(y) = \ell_x \circ r_x(y) = n(x) \cdot y$ by Proposition 10. Thus the existence of the map α_S follows from the universal property of the Clifford algebra. We refer to [KMRT, (35.1)] for a proof of the other claims. □ □

We get an (injective) homomorphism

$$\alpha_S \circ \mathrm{ad}^{-1} : \mathfrak{o}(n) \to \mathfrak{o}(n) \times \mathfrak{o}(n) \subset \mathrm{End}_F(S) \times \mathrm{End}_F(S). \tag{42}$$

For any $\lambda \in \mathfrak{o}(n)$, let

$$\alpha_S \circ \mathrm{ad}^{-1}(\lambda) = (\lambda^+, \lambda^-).$$

Theorem 43. *For any $\lambda \in \mathfrak{o}(n)$, the elements λ^+, $\lambda^- \in \mathfrak{o}(n)$ satisfy*

$$\begin{aligned} \lambda^+(x \star y) &= \lambda(x) \star y + x \star \lambda^-(y), \\ \lambda^-(x \star y) &= \lambda^+(x) \star y + x \star \lambda(y), \\ \lambda(x \star y) &= \lambda^-(x) \star y + x \star \lambda^+(y) \end{aligned}$$

for all x, $y \in \mathfrak{o}(n)$. Any of the above three relations determines the two other. Further the pair (λ^+, λ^-) is uniquely determined by any of the three relations.

Proof. See [KMRT, §45]. □ □

Let ρ_S, resp. ρ'_S be the endomorphisms of $\mathfrak{o}(n)$ defined by $\rho_S(\lambda) = \lambda^+$ and $\rho'_S(\lambda) = \lambda^-$ for $\lambda \in \mathfrak{o}(n)$, so that $\alpha_S \circ \mathrm{ad}^{-1} = (\rho_S, \rho'_S) \in \mathrm{End}_F\big(\mathfrak{o}(n)\big) \times \mathrm{End}_F\big(\mathfrak{o}(n)\big)$. Theorem 43 leads to a description of local triality which is more symmetric than the description given in Example 7:

Theorem 44. *The endomorphisms ρ_S and ρ'_S are outer automorphisms of $\mathfrak{o}(n)$ and satisfy the relations $\rho_S^2 = \rho'_S$, $\rho_S^3 = 1$. Hence ρ_S generates a cyclic subgroup of order 3 of $\mathrm{Aut}_F\big(\mathfrak{o}(n)\big)$. The Lie algebra of fixed points under ρ_S is the Lie algebra of derivations of $(S, \star)$. Moreover, if S^{op} is the opposite symmetric composition, then $\rho_{S^{\mathrm{op}}} = \rho_S^2$, so that ρ_S and $\rho_{S^{\mathrm{op}}}$ generate the same cyclic subgroup of order 3 of $\mathrm{Aut}_F\big(\mathfrak{o}(n)\big)$.*

Proof. The fact that $\rho_S^2 = \rho'_S$ and that ρ_S generates a cyclic subgroup of order 3 of $\mathrm{Aut}_F\big(\mathfrak{o}(n)\big)$ follows from Theorem 43. The automorphism ρ_S is not inner since it permutes the three 8-dimensional representations of $\mathfrak{o}(n)$ (the vector and the two spin-representations). Another proof follows from the direct computations given in the Appendix. The last claim readily follows from the definition of the isomorphism α_S. □ □

In view of Corollary 21 and Proposition 27 the fixed point Lie algebra of ρ_S is a simple Lie algebra of type G_2 if S is of type G_2 and of type A_2 if S is of type A_2. Thus the automorphism ρ_S is conjugate over an algebraic closure of F (with respect to the full group of automorphisms of $\mathfrak{o}(n)$) to the automorphism f_ρ of Theorem 6 if S is a Cayley algebra and to f'_ρ of Theorem 6 in the other case. If the symmetric composition S is a para-Cayley algebra or is of type A_2 for a split algebra, the involution π_0 of S induces an automorphism (of Lie algebras) $\pi : \lambda \mapsto \pi_0 \lambda \pi_0$ of $\mathfrak{o}(n)$ (the product $\pi_0 \lambda \pi_0$ is taken in $\mathrm{End}_F(S)$). Since π_0 is an isometry of (S, n), π_0 induces an automorphism $C(\pi_0)$ of the Clifford algebra $C(S, n)$.

Lemma 45. *For $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \in \mathrm{End}_F(S \oplus S)$, we have*

$$\big(\alpha_S \circ C(\pi_0) \circ \alpha_S^{-1}\big) \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} \pi(g) & 0 \\ 0 & \pi(f) \end{pmatrix}.$$

Proof. Since $C_0(S,n)$ is generated by all products uv, $u, v \in S$ (the product is taken in $C(S,n)$), it suffices to check the claim for $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} = \alpha_S(uv) \in \operatorname{End}_F(S \oplus S)$ which is a straightforward computation. □ □

Theorem 46. *If S is of type G_2 or is of type A_2 for a split algebra, the relations $\pi^2 = 1$ and $\pi \circ \rho_S = \rho_S^2 \circ \pi$ hold in $\operatorname{Aut}_F\big(\mathfrak{o}(n)\big)$ and $\{\pi, \rho_S\}$ generate a subgroup of $\operatorname{Aut}_F\big(\mathfrak{o}(n)\big)$ isomorphic to S_3.*

Proof. The first relation is obvious. We check the second one. Since π_0 is an isometry of (S,n), π_0 induces an automorphism $C(\pi_0)$ of the Clifford algebra $C(S,n)$. We know that $\operatorname{ad} \circ C(\pi_0) \circ \operatorname{ad}^{-1} = \pi = \operatorname{Int}(\pi_0)$ on $\mathfrak{o}(n)$ by Proposition 39; further $(\alpha_S \circ \operatorname{ad}^{-1})(\lambda) = \big(\rho_S(\lambda), \rho_S^2(\lambda)\big)$, so that

$$\begin{aligned}
\big(\pi(\rho_S^2(\lambda)), \pi(\rho_S(\lambda))\big) &= \big(\alpha_S \circ C(\pi_0) \circ \alpha_S^{-1}\big)\big(\rho_S(\lambda), \rho_S^2(\lambda)\big) \\
&= \big(\alpha_S \circ C(\pi_0) \circ \operatorname{ad}^{-1}\big)(\lambda) \\
&= \big(\alpha_S \circ \operatorname{ad}^{-1} \circ \operatorname{ad} \circ C(\pi_0) \circ \operatorname{ad}^{-1}\big)(\lambda) \\
&= \big(\alpha_S \circ \operatorname{ad}^{-1}\big)(\pi(\lambda)) \\
&= \big(\rho_S(\pi(\lambda)), \rho_S^2(\pi(\lambda))\big).
\end{aligned}$$

The second relation then follows from Theorem 44. Thus we get get a homomorphism $S_3 \to \operatorname{Aut}_F\big(\mathfrak{o}(n)\big)$. It is obviously injective since $\rho_S \neq 1$. □ □

Remark 47. Let $(S, \star, n)$ be a symmetric composition and let (ρ_1, ρ_2) be the two automorphisms of order three of the Lie algebra $\mathfrak{o}(n)$ constructed in Theorem 44. If $(S, \diamond, n)$ is another symmetric composition with the same underlying quadratic space, we get a different pair of automorphisms (ϕ_1, ϕ_2) of order three of the Lie algebra $\mathfrak{o}(n)$ and it follows from the exact sequence (2) that, at least over an algebraic closure of F, (and after renumbering ϕ_1, ϕ_2 if necessary) there exists $f, g \in \mathrm{GO}^+(n)$ such that $(\phi_1, \phi_2) = \operatorname{Int}(f,g)(\rho_1, \rho_2)$. This also follows directly from Theorem 43 and is valid over F: let

$$\alpha_S : \big(C_0(S,n), \tau\big) \xrightarrow{\sim} \big(\operatorname{End}_F(S), \sigma_n\big) \times \big(\operatorname{End}_F(S), \sigma_n\big),$$

be the isomorphism defined in (41) and let α'_S be the corresponding isomorphism defined via $\diamond$. Then $\alpha'_S \circ \alpha_S^{-1}$ is an automorphism of $\big(\operatorname{End}_F(S), \sigma_n\big) \times \big(\operatorname{End}_F(S), \sigma_n\big)$ and we may assume that $\alpha'_S \circ \alpha_S^{-1} = \operatorname{Int}(f,g)$ with $(f,g) \in \mathrm{GO}^+(n)$. Since $(\rho_1, \rho_2) = \alpha_S \circ \operatorname{ad}^{-1}$ and $(\phi_1, \phi_2) = \alpha'_S \circ \operatorname{ad}^{-1}$ it follows that $(\alpha'_S \circ \operatorname{ad}^{-1}) \circ (\operatorname{ad} \circ \alpha_S^{-1} = \operatorname{Int}(f,g)$ and $(\phi_1, \phi_2) = \operatorname{Int}(f,g)(\rho_1, \rho_2)$.

6. Automorphisms of order 3 and composition algebras

A Lie algebra $\mathfrak{g}$ of type D_4 over F is a form of $\mathfrak{o}_8$, i.e., there exists an isomorphism

$$\beta : \mathfrak{g} \otimes_F \overline{F} \xrightarrow{\sim} \overline{\mathfrak{o}}_8 = \mathfrak{o}_8 \otimes_F \overline{F},$$

where $\overline{F}$ is an algebraic closure of F. The following result is well known. We include a proof via symmetric compositions for completeness.

Theorem 48. *An orthogonal Lie algebra $\mathfrak{g}$ of type D_4 over F admits an outer automorphism of order 3 if and only if $\mathfrak{g}$ is isomorphic to the orthogonal Lie algebra of the norm form n of a Cayley algebra $\mathcal{C}$ over F.*

Proof. The fact that the orthogonal Lie algebra of the norm form n of a Cayley algebra admits an outer automorphism of order 3 follows for example from Theorem 44. The proof of the converse relies on the following simple, but useful principle, see [KMRT, (2.25)].

Lemma 49. *Let Ω and C be F-algebras, and let $U \subset \Omega$ be a vector subspace which generates Ω as an F-algebra. Let $f : U \to C$ be a F-linear map. If there exists a field extension K of F such that $f \otimes_F K : U \otimes_F K \to C \otimes_F K$ extends to a K-algebra homomorphism $\hat{f}_K : \Omega \otimes_F K \to C \otimes_F K$, then f extends to an F-algebra homomorphism $\hat{f} : \Omega \to C$.*

Proof. The restriction of $\hat{f}_K$ to Ω is an F-algebra homomorphism $\Omega \to C \otimes_F K$. Since the restriction of $\hat{f}_K$ to U is f and U generates Ω, the image of this homomorphism is in fact contained in C. □ □

We return to the proof of Theorem 48. Let $\overline{F}$ be an algebraic closure of F and let $(\overline{V}, \overline{q}) = (V, q) \otimes_F \overline{F}$. Then $(\overline{V}, \overline{q})$ is isometric to the norm of the split Cayley algebra $(\overline{\mathcal{C}}, \overline{n})$ since both quadratic forms are hyperbolic over $\overline{F}$. Let

$$\beta : (\overline{V}, \overline{q}) \xrightarrow{\sim} (\overline{\mathcal{C}}, \overline{n}) \tag{50}$$

be an isometry. We have an induced commutative diagram

$$\begin{tikzcd}
\mathfrak{o}(\overline{q}) \arrow[r, "\mathrm{ad}^{-1}"] \arrow[d, "\mathfrak{o}(\beta)"'] & {[\overline{V}, \overline{V}]} \arrow[r, "i"] \arrow[d, "C_0(\beta)"] & C_0(\overline{V}, \overline{q}) \arrow[d, "C_0(\beta)"] & \mathrm{End}_{\overline{F}}(\overline{V}) \times \mathrm{End}_{\overline{F}}(\overline{V}) \arrow[d, "\mathrm{Int}\beta \times \mathrm{Int}\beta"] \\
\mathfrak{o}(\overline{n}) \arrow[r, "\mathrm{ad}^{-1}"] & {[\overline{\mathcal{C}}, \overline{\mathcal{C}}]} \arrow[r, "i"] & C_0(\overline{\mathcal{C}}, \overline{n}) \arrow[r, "\alpha_{\overline{\mathcal{C}}}"] & \mathrm{End}_{\overline{F}}(\overline{\mathcal{C}}) \times \mathrm{End}_{\overline{F}}(\overline{\mathcal{C}})
\end{tikzcd}$$

Let (f_1', f_2') be the two components of the map $\mathfrak{o}(\overline{q})$ to $\mathfrak{o}(\overline{q}) \times \mathfrak{o}(\overline{q})$ given by the composition $(\mathrm{Int}\beta \times \mathrm{Int}\beta)^{-1} \circ \alpha_{\overline{\mathcal{C}}} \circ C_0(\beta) \circ i \circ \mathrm{ad}^{-1}$. By Theorem 43 (f_1', f_2') is a pair of outer automorphisms of $\mathfrak{o}(\overline{q})$. Let f_1 be an outer automorphism of order 3 of $\mathfrak{o}(q)$ and let $f_2 = f_1^2$. In view of Remark 47 (and renumbering f_1 and f_2 if necessary), there exist inner automorphisms $\mathrm{Int}h_1, \mathrm{Int}h_2$ of $\mathfrak{o}(\overline{q})$ such that

$$f_1 = \mathrm{Int}h_1 \circ f_1', \ f_2 = \mathrm{Int}h_2 \circ f_2'.$$

The composition

$$(\mathrm{Int}h_1 \times \mathrm{Int}h_2) \circ (\mathrm{Int}\beta \times \mathrm{Int}\beta)^{-1} \circ \alpha_{\overline{\mathcal{C}}} \circ C_0(\beta)$$

maps the algebra $C_0(V, q)$ to $\mathrm{End}_{\overline{F}}(\overline{V}) \times \mathrm{End}_{\overline{F}}(\overline{V})$ and its restriction to $[V, V]$ has as image the image of (f_1, f_2) which is contained in $\mathfrak{o}(q) \times \mathfrak{o}(q) \subset \mathrm{End}_F(V) \times \mathrm{End}_F(V)$. Since $\mathfrak{o}(q) \xrightarrow{\sim} [V, V]$ generates $C_0(V, q)$ as an algebra, it follows from Lemma 49 that $(f_1, f_2) \circ \mathrm{ad}^{-1}$ extends to isomorphism $\gamma : C_0(V, q) \xrightarrow{\sim} \mathrm{End}_F(V) \times \mathrm{End}_F(V)$. Replacing q by a multiple λq if necessary, we may assume that there is an element $e \in V$ such that $q(e) = 1$. Let

$$V = F \cdot e \perp U.$$

For any $u \in U$, let $\gamma(eu) = (\phi_u, \psi_u) \in \mathrm{End}_F(V) \times \mathrm{End}_F(V)$. Since $eu + ue = 0$ in $C_0(q)$, we have $2eu = [e, u] \in [V, V]$ and $(\phi_u, \psi_u) \in \mathfrak{o}(q) \times \mathfrak{o}(q)$, moreover $(\phi_u, \psi_u)^2 = \gamma(eu)^2 = -q(u) \cdot (1, 1)$ and

$$b_q\big(\phi_u(y), \phi_u(z)\big) = -b_q\big(\phi_u^2(y), z\big) = q(u) b_q(y, z) \tag{51}$$

for all $u \in U$, $y, z \in V$. Let $x = \lambda e + u \in V$. It follows from (51) that the bilinear product on V given by

$$x \diamond y = \lambda y + \phi_u(y),\ y \in V.$$

satisfies

$$b_q(x \diamond y, x \diamond y) = q(x) b_q(y, z),$$

hence defines a composition algebra on (V, q). By Proposition 9 q is the norm form of a Cayley algebra. □ □

A result corresponding to Theorem 48 for projective groups of similitudes is proved in [WO] with a quite settled geometric argument.

Theorem 52. *Let $\mathfrak{o}(n)$ be the orthogonal Lie algebra relative to a 3-Pfister form n.*
1) The cyclic subgroups $G(f)$ and $G(g)$ of $\mathrm{Aut}\big(\mathfrak{o}(n)\big)$ generated by two outer automorphisms f, g of $\mathfrak{o}(n)$ of order 3 are conjugate and only if they have isomorphic fixed point subalgebras.
2) Each conjugacy class of a cyclic subgroup $G(f)$ of $\mathrm{Aut}\big(\mathfrak{o}(n)\big)$, generated by an outer automorphism f of order 3, is the class of a subgroup $G(f_S)$ for an automorphism f_S induced by some symmetric composition $(S, \star)$. The Lie algebra $\mathrm{Der}(S, \star)$ is the fixed point algebra of the group $G(f_S)$. Up to isomorphism the conjugacy class of $G(f_S)$ is uniquely determined by the pair $\{S, S^{\mathrm{op}}\}$.

Proof. 1) If $G(f)$ and $G(g)$ are conjugate and for example $g = \phi \circ f \circ \phi^{-1}$ with $\phi \in \mathrm{Aut}\big(\mathfrak{o}(n)\big)$, then ϕ restricts to an isomorphism $\mathrm{Fix}\big(G(f)\big) \xrightarrow{\sim} \mathrm{Fix}\big(G(g)\big)$. Conversely, let $\mathfrak{g} = \mathrm{Fix}\big(G(f)\big)$ and $\mathfrak{h} = \mathrm{Fix}\big(G(g)\big)$ be isomorphic Lie algebras. By Theorem 6, $\mathfrak{g}$ and $\mathfrak{h}$ are both of type G_2 or of type A_2. If they are both of type G_2, they are isomorphic to derivation algebras of Cayley algebras $\mathcal{C}_1$ and $\mathcal{C}_2$. In view of Corollary 21 we may assume that $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$, where $\mathcal{C}$ is the (up to isomorphism) unique Cayley algebra with norm n. Moreover we can assume that $\mathrm{Der}(\mathcal{C})$ is canonically embedded in $\mathfrak{o}(n)$. Let $\phi_1 : \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}(\mathcal{C})$ and $\phi_2 : \mathfrak{h} \xrightarrow{\sim} \mathrm{Der}(\mathcal{C})$ be isomorphisms. In view of Corollary 22, the isomorphisms ϕ_1 and ϕ_2 can be extended to inner automorphisms s_i of $\mathfrak{o}(n)$. Conjugating f by s_1^{-1} and g by s_2^{-1} we are reduced to the case where f and g have the same fixed point algebra $\mathrm{Der}(\mathcal{C})$. It then follows from Remark 47 that $f \circ g^{-1}$ or $f^2 \circ g^{-1}$ is an inner automorphism $s = \mathrm{Int}(t)$, $t \in \mathrm{GO}^+(n)$. Let $\mathcal{C} = \mathcal{C}^0 \perp F{\cdot}1$. The image of $\mathrm{Der}(\mathcal{C})$ in $\mathfrak{o}(n) \subset \mathrm{End}_F(\mathcal{C})$ lies in $\mathrm{End}_F(\mathcal{C}^0)$ and t commutes with this image. Thus t is of the form $\mu{\cdot}1_{\mathcal{C}^0} + \lambda{\cdot}1_F$, $\lambda,\ \mu \in F^\times$. The multiplier of t is equal to λ^2 and to μ^2, so that $\lambda = \pm\mu$. Since t is proper, the case $\lambda = -\mu$ cannot occur and $s = \mathrm{Int}(t)$ is the identity on $\mathfrak{o}(n)$. A similar proof, using Proposition 26 and Corollary 30, holds if both fixed point algebras are of type A_2. Observe that in this case, we only get conjugation up to automorphisms, not necessarily inner automorphisms.
2) Let f be an outer automorphism of order 3 of $\mathfrak{o}(n)$ and let $\mathrm{Fix}(f) = \mathfrak{g}$. Then $\mathfrak{g}$ is of type G_2 or A_2. Assume first that $\mathfrak{g} \simeq \mathrm{Der}(\mathcal{C})$ for a Cayley algebra $\mathcal{C}$. Let $f_{\mathcal{C}}$ be an outer automorphism of $\mathfrak{o}(n)$ induced through the para-Cayley algebra $\mathcal{C}$ (see Theorem 44). In view of 1) $G(f)$ and $G(f_{\mathcal{C}})$ are conjugate. If $\mathfrak{g}$ is of type A_2 we know by descent that that $\mathfrak{g} \simeq \mathrm{Skew}(B, \tau)^0$ for B central simple of degree 3 over a quadratic extension K/F. In view of Corollary 30, K must be isomorphic to $F[x]/(x^2 + 3)$. Let $(S, \star, n)$ be the symmetric composition associated with (B, τ). The fixed point algebra of the corresponding automorphism f_S of order 3 of $\mathfrak{o}(n)$ is isomorphic to $\mathrm{Fix}(f)$. Hence by 1) the cyclic subgroups $G(f)$ and $G(f_S)$ are in the same conjugacy class. Unicity follows from Proposition 35. □ □

In view of the classification of symmetric compositions we have the following classification of outer automorphisms:

Corollary 53. *Let n be a 3-Pfister form over a field F of characteristic zero and let $\mathrm{Conj}(n)$ be the set of conjugacy classes of order 3 outer automorphisms of the orthogonal algebra $\mathfrak{o}(n)$.*
1) If F does not contain a primitive cubic root of 1, then $\mathrm{Conj}(n)$ is in bijection with the set consisting of the isomorphism class of a Cayley algebra with norm n and all $F(\sqrt{-3})$-isomorphism classes of pairs consisting of a central simple algebras of degree 3 over $F(\sqrt{-3})/F$ and an involution of second kind such that the cohomological invariant of the involution is the class of n.
2) If F contains a primitive cubic root of 1 and n is hyperbolic, then $\mathrm{Conj}(n)$ is in bijection with the set consisting of the isomorphism class of the split Cayley algebra and the isomorphism classes of central simple algebras of degree 3 over F.
3) If F contains a primitive cubic root of 1 and n is anisotropic, then $\mathrm{Conj}(n)$ reduces to the isomorphism class of a Cayley algebra with norm n.

Proof. 1) If the composition algebra S admits an involution σ_S, then by Theorem 46 f_S and $f_S^2 = f_{S^{\mathrm{op}}}$ are conjugate in $\mathrm{Aut}\big(\mathfrak{o}(n)\big)$. Thus in this case the conjugacy class is determined by S. The left cases are where the central simple algebra B of degree 3 is not split. Then B and B^{op} are not isomorphic so that S and S^{op} are not isomorphic either. The claim then follows from the fact that the form n determines the isomorphism type of the involution of second kind on B. Claims 2) and 3) follow from 1) and the fact that n is hyperbolic if F contains a primitive cubic root of unity (see Proposition 32). □ □

Remark 54. As observed in [HKRT], given an algebra B of degree 3 with an involution of second kind and given a norm form n, B does not necessarily admits an involution of second kind with invariant n. However, since any central simple algebra B of degree 3 with an involution of second kind admits a distinguished involution, there is at least one class of outer automorphisms of order 3 for each isomorphism class of central simple algebras B of degree 3 admitting an involution of second kind over the extension $F[x]/(x^2+3)$.

Example 55. Over $\mathbb{R}$, there are two 3-Pfister forms, the hyperbolic form and the identity form, hence two isomorphism classes of orthogonal Lie algebras admitting outer automorphisms of order 3. For each one there is a conjugacy class of outer automorphisms of order 3 given by the corresponding Cayley algebra. Moreover we have two classes of involutions of second kind on $M_3(\mathbb{C})$ (see Example 34). For the hermitian transpose τ, the invariant is the identity form. The other class of involutions is given by the distinguished involution $\mathrm{Int}(u) \circ \tau$, where $u = \mathrm{diag}(-1, 1, 1)$ (see Example 34). Thus we have two classes of outer automorphisms of order 3 for each of the two 3-Pfister forms.

7. Appendix (in collaboration with Larissa Cadorin)

Using adequate multiplication tables of symmetric compositions, one can use Theorem 43 to give explicit descriptions of trialitarian actions defined through symmetric compositions. Computations of this kind were done in [KPS] and [C]. We first consider the case of a split para-Cayley algebra $\mathcal{C}_s$. Let $Q_s = M_2(F)$ be the split quaternion algebra, with basis $v_1 = 1,\ v_2,\ v_3,\ v_4 = v_2v_3$ and relations $v_2^2 = v_3^2 = 1$ and $v_2v_3 + v_3v_2 = 0$. Let $x \mapsto \overline{x}$ be the conjugation in Q_s. We set $C_s = Q_s + v_5Q_s$ and define the multiplication in $\mathcal{C}_s$ by

$$(a + v_5b)(a' + v_5b') = (aa' + b'\bar{b}) + v_5(\bar{a}b' + a'b) \text{ for } a, a', b, b' \in Q_s.$$

The norm n of $\mathcal{C}_s$ with respect to the basis $(v_1, \ldots, v_4, v_5, v_6 = v_5v_2, v_7 = v_5v_3, v_8 = v_5v_7)$ is the diagonal form $< 1, -1, \ldots, 1, -1 >$. It is more convenient to work with the basis

$$\begin{array}{llll} e_1 = \frac{1}{2}(v_4 - v_8), & e_2 = \frac{1}{2}(-v_2 + v_6), & e_3 = \frac{1}{2}(-v_3 + v_7), & e_4 = \frac{1}{2}(v_1 + v_5), \\ f_1 = \frac{1}{2}(v_4 + v_8), & f_2 = \frac{1}{2}(v_2 + v_6), & f_3 = \frac{1}{2}(v_3 + v_7), & f_4 = \frac{1}{2}(v_1 - v_5) \end{array}$$

which is hyperbolic for n. The multiplication table for the corresponding para-Cayley algebra is given in Table 1.

$\star$	e_1	f_1	e_2	f_2	e_3	f_3	e_4	f_4
e_1	0	$-f_4$	f_3	0	$-f_2$	0	0	$-e_1$
f_1	$-e_4$	0	0	e_3	0	$-e_2$	$-f_1$	0
e_2	$-f_3$	0	0	$-f_4$	f_1	0	0	$-e_2$
f_2	0	$-e_3$	$-e_4$	0	0	e_1	$-f_2$	0
e_3	f_2	0	$-f_1$	0	0	$-f_4$	0	$-e_3$
f_3	0	e_2	0	$-e_1$	$-e_4$	0	$-f_3$	0
e_4	$-e_1$	0	$-e_2$	0	$-e_3$	0	f_4	0
f_4	0	$-f_1$	0	$-f_2$	0	$-f_3$	0	e_4

TABLE 1

We have $1 = e_4 + f_4$ and conjugation in $\mathcal{C}_s$ is given by $\pi(e_i) = -e_i$, $\pi(f_i) = -f_i$ for $i = 1, \ldots, 3$ and $\pi(e_4) = f_4$. The following relations hold in the Clifford algebra $C(\mathcal{C}_s, n)$:

(56) $$e_ie_j + e_je_i = f_if_j + f_jf_i = 0 \text{ and } e_if_j + f_je_i = \delta_{ij}$$

for all $i, j = 1, \ldots, 4$. A computation shows that the isomorphism $\mathrm{ad}^{-1} : \mathfrak{o}_8 \xrightarrow{\sim} [\mathcal{C}_s, \mathcal{C}_s] \subset C_0(\mathcal{C}_s, n)$ of Lemma 38 is given by

$$\begin{array}{rclcll} G_{ij}^Y & = & E_{i,j+4} - E_{j,i+4} & \mapsto & e_ie_j & 1 \leqslant i < j \leqslant 4 \\ G_{ij}^Z & = & E_{i+4,j} - E_{j+4,i+4} & \mapsto & f_if_j & 1 \leqslant i < j \leqslant 4 \\ G_{ij}^X & = & E_{i,j} - E_{j+4,i+4} & \mapsto & e_if_j & 1 \leqslant i, j \leqslant 4, i \neq j \\ G_{ii}^X & = & E_{i,i} - E_{i+4,i+4} & \mapsto & e_if_i - \frac{1}{2} & 1 \leqslant i \leqslant 4. \end{array}$$

The image of a generic skew-symmetric matrix

$$\mathcal{X} \;=\; \textstyle\sum_{1 \leqslant i,j \leqslant 4} x_{ij}G_{ij}^X \;+\; \sum_{1 \leqslant i<j \leqslant 4} y_{ij}G_{ij}^Y \;+\; \sum_{1 \leqslant i<j \leqslant 4} z_{ij}G_{ij}^Z$$

(57) $$= \left(\begin{array}{cccc|cccc} x_{11} & x_{12} & x_{13} & x_{14} & 0 & y_{12} & y_{13} & y_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} & -y_{12} & 0 & y_{23} & y_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} & -y_{13} & -y_{23} & 0 & y_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} & -y_{14} & -y_{24} & -y_{34} & 0 \\ \hline 0 & z_{12} & z_{13} & z_{14} & -x_{11} & -x_{21} & -x_{31} & -x_{41} \\ -z_{12} & 0 & z_{23} & z_{24} & -x_{12} & -x_{22} & -x_{32} & -x_{42} \\ -z_{13} & -z_{23} & 0 & z_{34} & -x_{13} & -x_{23} & -x_{33} & -x_{43} \\ -z_{14} & -z_{24} & -z_{34} & 0 & -x_{14} & -x_{24} & -x_{34} & -x_{44} \end{array}\right)$$

under the map $\alpha_{\mathcal{C}_s} \circ \operatorname{ad}^{-1}$ of (42) is

$$\begin{aligned}&\alpha_S \circ \operatorname{ad}^{-1}(\mathcal{X}) = \\ &\textstyle\sum_{\substack{1\leqslant i,j\leqslant 4\\ i\neq j}} x_{ij}\begin{pmatrix} L_{e_i}R_{f_j} & 0 \\ 0 & R_{e_i}L_{f_j}\end{pmatrix} + \sum_{i=1}^4 x_{ii}\left[\begin{pmatrix} L_{e_i}R_{f_i} & 0 \\ 0 & R_{e_i}L_{f_i}\end{pmatrix} - \frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}\right] \\ &+ \textstyle\sum_{1\leqslant i<j\leqslant 4} y_{ij}\begin{pmatrix} L_{e_i}R_{e_j} & 0 \\ 0 & R_{e_i}L_{e_j}\end{pmatrix} + \sum_{1\leqslant i<j\leqslant 4} z_{ij}\begin{pmatrix} L_{f_i}R_{f_j} & 0 \\ 0 & R_{f_i}L_{f_j}\end{pmatrix}.\end{aligned}$$

Let $\alpha_{\mathcal{C}_s} \circ \operatorname{ad}^{-1}(\mathcal{C}_s) = (\rho(\mathcal{X}), \rho^2(\mathcal{X}))$. Using multiplication table 1, it is straightforward to show that $\rho(\mathcal{X})$ is given by

$$\left(\begin{array}{cccc|cccc} +--+ & x_{12} & x_{13} & -y_{14} & 0 & -x_{43} & x_{42} & -z_{23} \\ x_{21} & -+-+ & x_{23} & -y_{24} & x_{43} & 0 & -x_{41} & z_{13} \\ x_{31} & x_{32} & --++ & -y_{34} & -x_{42} & x_{41} & 0 & -z_{12} \\ z_{14} & z_{24} & z_{34} & ---- & z_{23} & -z_{13} & z_{12} & 0 \\ \hline 0 & x_{34} & -x_{24} & -y_{23} & -++- & -x_{21} & -x_{31} & -z_{14} \\ -x_{34} & 0 & x_{14} & y_{13} & -x_{12} & +-+- & -x_{32} & -z_{24} \\ x_{24} & -x_{14} & 0 & -y_{12} & -x_{13} & -x_{23} & ++-- & -z_{34} \\ y_{23} & -y_{13} & y_{12} & 0 & y_{14} & y_{24} & y_{34} & ++++ \end{array}\right)$$

and $\rho^2(\mathcal{X})$ by

$$\left(\begin{array}{cccc|cccc} +--- & x_{12} & x_{13} & z_{23} & 0 & -z_{34} & z_{24} & -x_{14} \\ x_{21} & -+-- & x_{23} & -z_{13} & z_{34} & 0 & -z_{14} & -x_{24} \\ x_{31} & x_{32} & --+- & z_{12} & -z_{24} & z_{14} & 0 & -x_{34} \\ -y_{23} & y_{13} & -y_{12} & +++- & x_{14} & x_{24} & x_{34} & 0 \\ \hline 0 & -y_{34} & y_{24} & x_{41} & -+++ & -x_{21} & -x_{31} & y_{23} \\ y_{34} & 0 & -y_{14} & x_{42} & -x_{12} & +-++ & -x_{32} & -y_{13} \\ -y_{24} & y_{14} & 0 & x_{43} & -x_{13} & -x_{23} & ++-+ & y_{12} \\ -x_{41} & -x_{42} & -x_{43} & 0 & -z_{23} & z_{13} & -z_{12} & ---+ \end{array}\right)$$

where $+/-$ are the signs of $\frac{1}{2}(x_{11} + x_{22} + x_{33} + x_{44})$.

The automorphism ρ restricts to an automorphism of the Cartan subalgebra $\mathfrak{h}$ which we used already in Section 2; the action is given by the matrix

$$X = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

with respect to the basis $u_j = E_{jj} - E_{j+4,j+4}$, $j = 1, \dots, 4$ of $\mathfrak{h}$. We recall (see (5)) that the matrix P of the automorphism f_ρ induced on $\mathfrak{h}$ by the corresponding automorphism of the Dynkin diagram is

$$R = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}.$$

Obviously we have $X = YRY^{-1}$ with $Y = \operatorname{diag}(-1, 1, 1, 1)$. This shows explicitly that both actions ρ and f_ρ are conjugate.

Similar computations can be made for all actions constructed with symmetric compositions. We sketch the case of the split algebra of type A_2. We represent $M_3(F)$ as a cyclic algebra with generators x, y and relations

$$yx = \omega xy, \quad x^3 = 1, \quad y^3 = 1$$

where ω is a primitive cubic root of 1. The monomials $x^i y^j$, $(i, j) \neq (0, 0) > 0, i, j \leq 2$ have trace zero and form a basis of $M_3(F)^0$. The basis

$$\begin{array}{llll} e_1 = -x & e_2 = -y & e_3 = -\omega xy & e_4 = -\omega^2 x^2 y \\ f_1 = -x^2 & f_2 = -y^2 & f_3 = -\omega x^2 y^2 & f_4 = -\omega^2 y x^2 \end{array}$$

is hyperbolic and the multiplication $\star$ has the following nice multiplication table (see [E]):

$\star$	e_1	f_1	e_2	f_2	e_3	f_3	e_4	f_4
e_1	$-f_1$	0	0	f_4	0	f_2	0	f_3
f_1	0	$-e_1$	e_4	0	e_2	0	e_3	0
e_2	e_3	0	$-f_2$	0	f_4	0	0	e_1
f_2	0	f_3	0	$-e_2$	0	e_4	f_1	0
e_3	e_4	0	0	e_1	$-f_3$	0	f_2	0
f_3	0	f_4	f_1	0	0	$-e_3$	0	e_2
e_4	e_2	0	f_3	0	0	e_1	$-f_4$	0
f_4	0	f_2	0	e_3	f_1	0	0	$-e_4$

TABLE 2

Since the norm form is hyperbolic we choose the same Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{o}_8$ as above and similar computations give a trialitarian action ρ'. The image of the generic skew-symmetric matrix (57) under ρ' is

$$\left(\begin{array}{cccc|cccc} -+++ & y_{23} & y_{34} & -y_{24} & 0 & x_{41} & x_{21} & x_{31} \\ -z_{23} & ---+ & x_{42} & x_{43} & -x_{41} & 0 & -z_{13} & z_{12} \\ -z_{34} & x_{24} & -+-- & x_{23} & -x_{21} & z_{13} & 0 & -z_{14} \\ z_{24} & x_{34} & x_{32} & --+- & -x_{31} & -z_{12} & z_{14} & 0 \\ \hline 0 & -x_{14} & -x_{12} & -x_{13} & +--- & z_{23} & -z_{34} & -z_{24} \\ x_{14} & 0 & -y_{13} & y_{12} & -y_{23} & +++- & -x_{24} & -x_{34} \\ x_{12} & y_{13} & 0 & -y_{14} & -y_{34} & -x_{42} & +-++ & -x_{32} \\ x_{13} & -y_{12} & y_{14} & 0 & y_{24} & -x_{43} & -x_{23} & ++-+ \end{array}\right),$$

where $+/-$ are as above the signs of $\frac{1}{2}(x_{11}+x_{22}+x_{33}+x_{44})$. Here again the action restricts to an automorphism of the Cartan algebra, given by the matrix

$$X' = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & +1 & -1 \end{pmatrix}.$$

Since $\mathrm{tr}(X') = -2$ and $\mathrm{tr}(X) = 1$, X and X' are clearly not conjugate.

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