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On Quadratic Quaternion Forms and Triality

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Contents

Riassunto

Le forme quadratiche sono state generalizzate in diversi modi: da una parte abbiamo la teoria delle forme parametriche di Bak, dall'altra ci sono le paia quadratiche descritte in [14]. In primo luogo spieghiamo il rapporto fra le due generalizzazioni, analizzando in particolare le forme sui quaternioni.
In secondo luogo studiamo l'algebra di Clifford e la trialità nel caso delle forme sui quaternioni.

Abstract

There are different generalizations of the theory of quadratic forms: on one hand we have the theory of parameter forms due to Bak, on the other hand there is the notion of the quadratic pair, given in [14]. First we explain the relation between the two generalizations, focusing our attention on the case of forms over quaternions. Then we study the Clifford algebra and the triality for forms over quaternions.

Introduction

The aim of this work is to study quadratic forms over quaternions and the effect of triality on them, without excluding fields of characteristic 2.
There exist different generalizations of the classical theory of quadratic forms. In a characteristic free way, a quadratic form can be viewed as the class of a bilinear form modulo alternating bilinear forms. This interpretation was extended to sesquilinear forms (see [22], [21]) and culminated in the theory of parameter forms, due to Bak (see [2]). The first case beyond classical quadratic forms is the one of forms over quaternions, which was already systematically studied by Seip-Hornix (see [20]). Another generalization was recently given in [14], based on the notion of a quadratic pair for a central simple algebra. An important tool for the study of generalized quadratic forms (and of their isometry groups) is the Clifford algebra. We compare the different definitions of quadratic forms, and of their corresponding Clifford algebras, in general and in the special case of forms over quaternions.
Triality is associated with algebraic groups of type D_4. Groups of type D_4 are exceptional in the sense that their Dynkin diagram admits an automorphism of order 3: in all other cases there is at most an automorphism of order 2. For quadratic forms triality occurs in dimension 8. Accordingly, we have triality for 4-dimensional forms over quaternions.
In the first chapter we define the basic objects and constructions from the theory of central simple algebras needed for our work. Then we explain how quadratic pairs on a central simple algebra may be viewed as generalized quadratic forms.
The second chapter is devoted to quadratic quaternion forms. We show that the different notions of forms given by Seip-Hornix, Wall and in [14] are equivalent. This allows to translate the results of Seip-Hornix in the setting of quadratic pairs or parameter forms. As an application we get an exact sequence of Lewis valid in any characteristic.
In the third chapter we give the construction of the (generalized even) Clifford algebra. In particular we prove that the different definitions given in the literature are in fact equivalent. We conclude this chapter with some remarks on the discriminant.
The last chapter contains results on triality: we use triality for semilinear similitudes to give a Clifford algebra version of triality for quadratic quaternion forms. Then we describe a geometric triality which occurs for isotropic forms. The final sections describe the spinor group and the effect of triality

for quadratic quaternion forms on the Chevalley algebra.

Chapter 1

Central Simple Algebras

1.1 Central Simple Algebras

If not explicitly stated, we consider algebras over a field F which are finite dimensional and which have an identity 1. For any subset X of an algebra A, the *centralizer* of X in A is defined as $C_A(X) := \{a \in A \mid ax = xa \ \forall x \in X\}$. In particular, we call $Z(A) := C_A(A)$ the *center* of A. The F-algebra A is *central* if its center is F and *simple* if $A \neq \{0\}$ and the only two-sided ideals of A are $\{0\}$ and A. The algebra A is a *division* algebra if $A \neq \{0\}$ and every non-zero element of A is invertible. For every F-algebra A, we define the *opposite algebra* A^{op} by $A^{op} = \{a^{op} \mid a \in A\}$, endowed with the same vector space structure as A and with the new multiplication $a^{op}b^{op} := (ba)^{op}$ for $a, b \in A$. For any algebra A over a field F and any field extension K/F, we write A_K for the K-algebra obtained from A by extending scalars to K, i.e. $A_K = A \otimes_F K$. We summarize without proofs some basic results from the theory of central simple algebras. The proofs may be found in [19],[12], [8], [17].

Theorem 1.1.1. *(Wedderburn) For an algebra A over a field F, the following conditions are equivalent:*
(1) A is central simple.
(2) The canonical map $A \otimes_F A^{op} \longrightarrow \mathrm{End}_F(A)$ which associates to $a \otimes b^{op}$ the linear map $x \mapsto axb$ is an isomorphism.
(3) There is a field K containing F such that A_K is isomorphic to a matrix algebra over K, i.e., $A_K \cong M_n(K)$ for some n.

(4) If Ω is an algebraically closed field containing F,

$$A_\Omega \cong M_n(\Omega) \text{ for some } n.$$

(5) There is a finite dimensional central division algebra D over F and an integer r such that $A \cong M_r(D)$.
Moreover, if these conditions hold, all the simple left (or right) A-modules are isomorphic, and the division algebra D is uniquely determined up to an algebra isomorphism as $D = \mathrm{End}_A(M)$ for any simple left A-module M.

The fields K for which condition (3) holds are called *splitting fields* of A. Accordingly, the algebra A is called *split* if it is isomorphic to a matrix algebra $M_n(F)$. Since the dimension of an algebra does not change under a scalar extension, it follows from the above theorem that the dimension of a central simple algebra is a square: $\dim_F(A) = n^2$ if $A_K \cong M_n(K)$ for some extension K/F. The integer n is called the *degree* of A and is denoted by $\deg(A)$. In view of the uniqueness of the division algebra D in the preceding theorem, we introduce the following definition. Two central simple algebras A and B over F are called *similar* ($A \sim B$) if they belong to isomorphic division algebras: $A \cong M_n(D)$, $B \cong M_k(D)$. Similarity is an equivalence relation and the set of similarity classes is denoted by $Br(F)$. It can be identified with the set of isomorphism classes of finite dimensional central division algebras. The tensor product of central simple algebras defines a commutative and associative product on $Br(F)$. The class of the matrix algebra $M_n(F)$ is a neutral element for this product and every element has an inverse, namely the similarity class of the opposite algebra. Therefore $Br(F)$ is a group which is called the *Brauer group* of F. If A is a central simple algebra, $[A]$ will denote the similarity class of A. If A is an F-algebra and $a \in A^\times$, the map $\mathrm{Int}(a) : A \longrightarrow A$ defined by $\mathrm{Int}(a)(x) = axa^{-1}$ is called an *inner automorphism.*

Theorem 1.1.2. *(Skolem-Noether) Let A be a central simple F-algebra and let $B \subset A$ be a simple subalgebra. Every F-algebra homomorphism $\rho : B \longrightarrow A$ extends to an inner automorphism of A: there exists $a \in A^\times$ such that $\rho(b) = \mathrm{Int}(a)(b) = aba^{-1}$ for all $b \in B$. In particular, every F-algebra automorphism of A is inner.*

Theorem 1.1.3. *(Double Centralizer) Let A be a central simple F-algebra and let $B \subset A$ be a simple subalgebra with center $K \supset F$. The centralizer*

$C_A(B)$ is a simple subalgebra of A with center K which satisfies

$$\dim_F A = \dim_F B \cdot \dim_F C_A(B) \quad \textit{and} \quad C_A(C_A(B)) = B.$$

If $K = F$, then the multiplication in A defines a canonical isomorphism $A = B \otimes C_A(B)$.

Let Ω denote an algebraic closure of F. Under scalar extension to Ω, every central simple F-algebra A of degree n becomes isomorphic to $M_n(\Omega)$. We may therefore fix an F-algebra embedding $A \hookrightarrow M_n(\Omega)$ and view every element $a \in A$ as a matrix in $M_n(\Omega)$. Its characteristic polynomial has coefficients in F and is independent of the embedding of A in $M_n(\Omega)$; it is called the *reduced characteristic polynomial* of A and is denoted

$$\mathrm{Prd}_{A,a}(X) = X^n - s_1(a)X^{n-1} + s_2(a)X^{n-2} - \cdots + (-1)^n s_n(a).$$

The *reduced trace* and *reduced norm* of a are denoted $\mathrm{Trd}_A(a)$ and $\mathrm{Nrd}_A(a)$:

$$\mathrm{Trd}_A(a) = s_1(a), \quad \mathrm{Nrd}_A(a) = s_n(a).$$

We also write

$$\mathrm{Srd}_A(a) = s_2(a).$$

1.2 Involutions

Let F be a field. Suppose that A is a central simple F-algebra. An *involution* on A is a map $\sigma : A \longrightarrow A$ subject to the following conditions:
1. $\sigma(x+y) = \sigma(x) + \sigma(y)$ for all $x, y \in A$.
2. $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in A$.
3. $\sigma^2(x) = x$ for $x \in A$.

We are interested in involutions which satisfy the extra condition $\sigma \mid_F = \mathrm{id}_F$. Such involutions are called *involutions of the first kind* (an involution σ with $\sigma \mid_F \neq \mathrm{id}_F$ is called *involution of the second kind*). Let $A = \mathrm{End}_F(V)$ be a split central simple algebra. If $b : V \times V \longrightarrow F$ is a *nonsingular* bilinear form (i.e. $b(v, w) = 0 \ \forall w \in V$ implies $v = 0$), the *adjoint anti-automorphism* $\sigma_b : A \longrightarrow A$ is defined by the equation

$$b(v, f(w)) = b(\sigma_b(f)(v), w) \text{ for } v, w \in V.$$

Following [14], we state the next result:

Theorem 1.2.1. *The map which associates to each nonsingular bilinear form b on V its adjoint anti-automorphism σ_b induces a one-to-one correspondence between equivalence classes of nonsingular bilinear forms on V modulo multiplication by a factor in $F^\times$ and linear anti-automorphism of* $\mathrm{End}_F(V)$. *Under this correspondence, F-linear involutions on* $\mathrm{End}_F(V)$ *(i.e., anti-automorphisms of period* 2*) correspond to nonsingular bilinear forms which are either symmetric or skew-symmetric.*

Hence, if A is an arbitrary central simple F-algebra with involution σ and if we extend scalars to a splitting field K, we obtain a split algebra $\mathrm{End}_K(V)$ with involution. The involution on this split algebra is, as above, the adjoint involution with respect to a bilinear form b. The property of b being symmetric or skew-symmetric or alternating (i.e. $b(v,v)=0$ for all $v\in V$) depends only on the involution and not on the choice of K nor of b. Therefore it makes sense to call an involution *symplectic* if for any splitting field we get an involution adjoint to some nonsingular alternating form. Otherwise it is called *orthogonal.* At this point we define the spaces of *symmetric*, *skew-symmetric*, *symmetrized* and *alternating* elements in a central simple F-algebra A with involution of the first kind as follows:
$\mathrm{Sym}(A,\sigma)=\{a\in A\mid \sigma(a)=a\}$,
$\mathrm{Skew}(A,\sigma)=\{a\in A\mid \sigma(a)=-a\}$,
$\mathrm{Symd}(A,\sigma)=\{a+\sigma(a)\mid a\in A\}$,
$\mathrm{Alt}(A,\sigma)=\{a-\sigma(a)\mid a\in A\}$.
If $\mathrm{char}F\neq 2$, then $\mathrm{Symd}(A,\sigma)=\mathrm{Sym}(A,\sigma)$, $\mathrm{Alt}(A,\sigma)=\mathrm{Skew}(A,\sigma)$ and $A=\mathrm{Sym}(A,\sigma)\oplus\mathrm{Skew}(A,\sigma)$ since every element $a\in A$ decomposes as $a=\frac{1}{2}(a+\sigma(a))+\frac{1}{2}(a-\sigma(a))$. If $\mathrm{char}F=2$, then $\mathrm{Symd}(A,\sigma)=\mathrm{Alt}(A,\sigma)\subset \mathrm{Skew}(A,\sigma)=\mathrm{Sym}(A,\sigma)$, and 1.2.3 below shows that this inclusion is strict. We quote two propositions that can be found in [14].

Proposition 1.2.2. *Let A be a central simple algebra of degree n over a field F with an involution of the first kind σ. Then we have* $\dim_F\mathrm{Sym}(A,\sigma)+\dim_F\mathrm{Alt}(A,\sigma)=n^2$. *Moreover,* $\mathrm{Alt}(A,\sigma)$ *is the orthogonal complement of* $\mathrm{Sym}(A,\sigma)$ *in A for the bilinear form T_A on A induced by the reduced trace:*

$$\mathrm{Alt}(A,\sigma)=\{a\in A\mid \mathrm{Trd}_A(as)=0\ for\ s\in\mathrm{Sym}(A,\sigma)\}.$$

Similarly, $\dim_F\mathrm{Skew}(A,\sigma)+\dim_F\mathrm{Symd}(A,\sigma)=n^2$, *and* $\mathrm{Symd}(A,\sigma)$ *is the orthogonal complement of* $\mathrm{Skew}(A,\sigma)$ *in A for the bilinear form T_A.*

Proposition 1.2.3. *Let (A, σ) be a central simple F-algebra of degree n with involution of the first kind.*
(1) *Suppose that* $\mathrm{char}F \neq 2$, *hence* $\mathrm{Symd}(A, \sigma) = \mathrm{Sym}(A, \sigma)$ *and* $\mathrm{Alt}(A, \sigma) = \mathrm{Skew}(A, \sigma)$. *If σ is of orthogonal type, then*

$$\dim_F \mathrm{Sym}(A, \sigma) = \frac{n(n+1)}{2} \quad \text{and} \quad \dim_F \mathrm{Skew}(A, \sigma) = \frac{n(n-1)}{2}.$$

If σ is of symplectic type, then

$$\dim_F \mathrm{Sym}(A, \sigma) = \frac{n(n-1)}{2} \quad \text{and} \quad \dim_F \mathrm{Skew}(A, \sigma) = \frac{n(n+1)}{2}.$$

Moreover, in this case n is necessarily even.
(2) *Suppose that* $\mathrm{char}F = 2$, *hence* $\mathrm{Sym}(A, \sigma) = \mathrm{Skew}(A, \sigma)$ *and* $\mathrm{Alt}(A, \sigma) = \mathrm{Symd}(A, \sigma)$; *then*

$$\dim_F \mathrm{Sym}(A, \sigma) = \frac{n(n+1)}{2} \quad \text{and} \quad \dim_F \mathrm{Alt}(A, \sigma) = \frac{n(n-1)}{2}.$$

The involution σ is of symplectic type if and only if $\mathrm{Trd}_A(\mathrm{Sym}(A, \sigma)) = \{0\}$, *which holds if and only if* $1 \in \mathrm{Alt}(A, \sigma)$. *In this case n is necessarily even.*

Given an involution of the first kind on a central simple algebra A, all the other involutions of the first kind on A can be obtained as follows:

Proposition 1.2.4. *Let A be a central simple algebra over a field F and let σ be an involution of the first kind on A.*
(1) For each unit $u \in A^\times$ such that $\sigma(u) = \pm u$, the map $\mathrm{Int}(u) \circ \sigma$ *is an involution of the first kind on A.*
(2) Conversely, for every involution σ' of the first kind on A, there exists some $u \in A^\times$, uniquely determined up to a factor in $F^\times$, such that

$$\sigma' = \mathrm{Int}(u) \circ \sigma \quad \text{and} \quad \sigma(u) = \pm u.$$

Then if $\sigma(u) = u$, we have

$$\begin{aligned} \mathrm{Sym}(A, \sigma') &= u\mathrm{Sym}(A, \sigma) = \mathrm{Sym}(A, \sigma)u^{-1}, \\ \mathrm{Skew}(A, \sigma') &= u\mathrm{Skew}(A, \sigma) = \mathrm{Skew}(A, \sigma)u^{-1}, \\ \mathrm{Alt}(A, \sigma') &= u\mathrm{Alt}(A, \sigma) = \mathrm{Alt}(A, \sigma)u^{-1}. \end{aligned}$$

If $\sigma(u) = -u$, we have

$$\mathrm{Sym}(A, \sigma') = u\mathrm{Skew}(A, \sigma) = \mathrm{Skew}(A, \sigma)u^{-1},$$
$$\mathrm{Skew}(A, \sigma') = u\mathrm{Sym}(A, \sigma) = \mathrm{Sym}(A, \sigma)u^{-1}.$$

(3) Suppose that $\sigma' = \mathrm{Int}(u) \circ \sigma$ where $u \in A^{\times}$ is such that $\sigma(u) = \pm u$. If $\mathrm{char}F \neq 2$, then σ and σ' are of the same type if and only if $\sigma(u) = u$. If $\mathrm{char}F = 2$, the involution σ' is symplectic if and only if $u \in \mathrm{Alt}(A, \sigma)$.

1.3 Quadratic Pairs

Let A be a central simple algebra of degree n over a field F. A *quadratic pair* on A is a couple (σ, f), where σ is an involution of the first kind on A and $f : \mathrm{Sym}(A, \sigma) \longrightarrow F$ is a linear map, subject to the following conditions:
(1) $\dim_F \mathrm{Sym}(A, \sigma) = n(n + 1)/2$ and $\mathrm{Trd}_A(\mathrm{Skew}(A, \sigma)) = \{0\}$.
(2) $f(x + \sigma(x)) = \mathrm{Trd}_A(x)$ for all $x \in A$.
If the characteristic of F is not 2, then a quadratic pair is an orthogonal involution with the map $f : \mathrm{Sym}(A, \sigma) \longrightarrow F$, defined by $f(x) = \frac{1}{2}\mathrm{Trd}_A(x)$. Quadratic pairs are interesting when the characteristic of F is 2. In this case a quadratic pair is a symplectic involution (which implies that n is even) with a map $f : \mathrm{Sym}(A, \sigma) \longrightarrow F$; condition (2) determines the value of f on the subspace $\mathrm{Symd}(A, \sigma)$ but not on $\mathrm{Sym}(A, \sigma)$. Therefore, there exist several quadratic pairs with the same symplectic involution. We recall a proposition from [14] for later use.

Lemma 1.3.1. *Let τ be an involution of A and let f be an F-linear form on $\mathrm{Sym}(A, \tau)$ such that $f\big(x + \tau(x)\big) = \mathrm{Trd}(x)$ for all $x \in A$. There exists an element $l \in A$ such that $f(s) = \mathrm{Trd}(ls)$ and $l + \tau(l) = 1$. The element l is uniquely determined up to additivity by an element of $\mathrm{Alt}(A, \tau)$. We take $l = 1/2$ if $\mathrm{char}F \neq 2$.*

Proof. Since the bilinear reduced trace form is nonsingular, every linear form $A \longrightarrow F$ is of the form $x \mapsto \mathrm{Trd}(ax)$ for some $a \in A$. Therefore, extending f arbitrarily to a linear form on A, we may find some $l \in A$ such that $f(s) = \mathrm{Trd}(ls)$ for all $s \in \mathrm{Sym}(A, \tau)$. If $l, l' \in A$ both satisfy this relation, then $\mathrm{Trd}((l - l')s) = 0$ for all $s \in \mathrm{Sym}(A, \tau)$, hence $l - l' \in \mathrm{Alt}(A, \tau)$. The second condition of the definition of a quadratic pair yields

$$\mathrm{Trd}(l(x + \tau(x))) = \mathrm{Trd}(x) \quad \text{for} \quad x \in A.$$

Since $\mathrm{Trd}(l\tau(x)) = \mathrm{Trd}(x\tau(l))$, it follows that

$$\mathrm{Trd}((l+\tau(l))x) = \mathrm{Trd}(x) \quad \text{for} \quad x \in A,$$

hence $l + \tau(l) = 1$ since the bilinear reduced trace form on A is nonsingular. □

Suppose that A has an orthogonal involution τ. In [14] a method to construct a quadratic pair is described: we take an element $a \in A$ such that $a + \tau(a)$ is invertible. Then a determines a quadratic pair (σ_a, f_a) as follows: let $g := a + \tau(a) \in A^\times$ and define $\sigma_a = \mathrm{Int}(g^{-1}) \circ \tau$ and $f_a(s) = \mathrm{Trd}_A(g^{-1}as)$ for $s \in \mathrm{Sym}(A, \sigma_a)$. The couple (σ_a, f_a) is a quadratic pair on A. Moreover, every quadratic pair is of this form:

Theorem 1.3.2. *Let τ be an orthogonal involution on A. Every quadratic pair on A is of the form (σ_a, f_a) for some $a \in A$ such that $a + \tau(a) \in A^\times$. If $a, b \in A$ are such that $a+\tau(a) \in A^\times$ and $b+\tau(b) \in A^\times$, then $(\sigma_a, f_a) = (\sigma_b, f_b)$ if and only if there exists $\lambda \in F^\times$ and $c \in \mathrm{Alt}(A, \tau)$ such that $a = \lambda b + c$.*

This theorem shows that quadratic pairs on A are in one-to-one correspondence with equivalence classes of elements $a + \mathrm{Alt}(A, \tau) \in A/\mathrm{Alt}(A, \tau)$ such that $a + \tau(a)$ is invertible, modulo multiplication by a factor in $F^\times$. In the particular case where $A = M_n(F)$ and $\tau = t$ is the transpose involution, the elements in $A/\mathrm{Alt}(A, \tau)$ may be regarded as quadratic forms of dimension n, by identifying $a + \mathrm{Alt}(A, \tau)$ with the quadratic form $q(X) = XaX^t$ where $X = (x_1, \dots, x_n)$. The matrix $a + a^t$ is invertible if and only if the corresponding quadratic form is nonsingular (of even dimension if $\mathrm{char}F = 2$). Therefore, quadratic pairs on $M_n(F)$ are in one to one correspondence with equivalence classes of nonsingular quadratic forms of dimension n modulo a factor in $F^\times$ (with n even if $\mathrm{char}F = 2$).

We give a construction similar to the one in 1.3.2; instead of an algebra with orthogonal involution, we consider an algebra with symplectic involution. Let A be a central simple algebra over F and ς a symplectic involution on A. Every element $a \in A$ such that $a - \varsigma(a) \in A^\times$ determines a quadratic pair (σ_a^s, f_a^s). More precisely, let $g := a - \varsigma(a)$ and define $\sigma_a^s := \mathrm{Int}\,(g^{-1}) \circ \varsigma$, $f_a^s(s) = \mathrm{Trd}_A(g^{-1}as)$ for $s \in \mathrm{Sym}(A, \sigma_a^s)$. It follows from 1.2.4 that σ_a^s is orthogonal if $\mathrm{char}F \neq 2$ and symplectic if $\mathrm{char}F = 2$. We compute $f_a^s(x+\sigma_a^s(x))$

for $x \in A$:

$$\begin{aligned} f_a^s(x+\sigma_a^s(x)) &= \mathrm{Trd}_A(g^{-1}a(x+\sigma_a^s(x))) = \\ &= \mathrm{Trd}_A(g^{-1}ax) + \mathrm{Trd}_A(g^{-1}ag^{-1}\varsigma(x)g) = \\ &= \mathrm{Trd}_A(g^{-1}ax) + \mathrm{Trd}_A(ag^{-1}\varsigma(x)) = \\ &= \mathrm{Trd}_A(g^{-1}ax) + \mathrm{Trd}_A(-xg^{-1}\varsigma(a)) = \\ &= \mathrm{Trd}_A(xg^{-1}(a-\varsigma(a))) = \mathrm{Trd}_A(x), \end{aligned}$$

hence (σ_a^s, f_a^s) is a quadratic pair.

Theorem 1.3.3. *Let ς be a symplectic involution on A. Every quadratic pair on A is of the form (σ_a^s, f_a^s) for some $a \in A$ such that $a - \varsigma(a) \in A^\times$. If $a, b \in A^\times$ are such that $a-\varsigma(a), b-\varsigma(b) \in A^\times$, then $(\sigma_a^s, f_a^s) = (\sigma_b^s, f_b^s)$ if and only if there exists $\lambda \in F^\times$ and $c \in \mathrm{Symd}(A,\varsigma)$ such that $a = \lambda b + c$.*

Proof. Let (σ, f) be a quadratic pair on A. By 1.2.4, there exists an element $g \in A^\times$ such that $\mathrm{Int}(g^{-1})\circ\varsigma = \sigma$ and $\varsigma(g) = -g$ if char $F \neq 2$, $g \in \mathrm{Alt}(A,\varsigma)$ if char $F = 2$. Moreover, 1.3.1 yields an element $l \in A$ such that $\sigma(l) + l = 1$ and $f(s) = \mathrm{Trd}_A(ls)$ for all $s \in \mathrm{Sym}(A,\sigma)$. Let $a := gl \in A$. We have $a - \varsigma(a) = gl - \varsigma(l)\varsigma(g) = gl - g\sigma(l)g^{-1}(-g) = gl + g\sigma(l) = g$ hence $\sigma_a^s = \sigma$. Moreover, for $s \in \mathrm{Sym}(A,\sigma)$, $f_a^s(s) = \mathrm{Trd}_A(g^{-1}as) = \mathrm{Trd}_A(ls) = f(s)$, hence $(\sigma, f) = (\sigma_a^s, f_a^s)$. Suppose now that $a, b \in A$ are such that $a-\varsigma(a)$, $b-\varsigma(b)$ are invertible and that $(\sigma_a^s, f_a^s) = (\sigma_b^s, f_b^s)$. Writing $g := a-\varsigma(a)$, $h := b-\varsigma(b)$, we have $\sigma_a^s = \mathrm{Int}(g^{-1})\circ\varsigma$ and $\sigma_b^s = \mathrm{Int}(h^{-1})\circ\varsigma$, hence the equality $\sigma_a^s = \sigma_b^s$ yields $g = \lambda h$ for some $\lambda \in F^\times$. Since $f_a^s = f_b^s$, we have $\mathrm{Trd}_A(g^{-1}as) = \mathrm{Trd}_A(h^{-1}bs)$ for $s \in \mathrm{Sym}(A,\sigma_a^s)$. Since $\mathrm{Sym}(A,\sigma_b^s) = \mathrm{Skew}(A,\varsigma)h$ and $g = \lambda h$, it follows that $\lambda^{-1}\mathrm{Trd}_A(az) = \mathrm{Trd}_A(bz)$ for $z \in \mathrm{Skew}(A,\varsigma)$, hence, by 1.2.2, $a - \lambda b \in \mathrm{Symd}(A,\varsigma)$. □

1.4 Generalized Quadratic Forms

Let D be a central division algebra over a field F with an involution $\sigma : x \mapsto \overline{x}$. Let V be a finite dimensional right vector space over D. A F-bilinear form

$$k : V \times V \to D,$$

is *sesquilinear* if $k(xa, yb) = \overline{a}k(x,y)b$ for all x, $y \in V$, a, $b \in D$. The additive group of such maps will be denoted by $\mathrm{Sesq}_\sigma(V,D)$. For any $k \in \mathrm{Sesq}_\sigma(V,D)$

we write

$$k^*(x,y)=\overline{k(y,x)}.$$

Let $\varepsilon\in F^\times$ be such that $\varepsilon\overline{\varepsilon}=1$. A sesquilinear form k such that $k=\varepsilon k^*$ is called *ε-hermitian* and the set of such forms on V will be denoted by $\operatorname{Herm}^\varepsilon_\sigma(V,D)$. Elements of

$$\operatorname{Alt}^\varepsilon_\sigma(V,D)=\{g=f-\varepsilon f^*\mid f\in\operatorname{Sesq}_\sigma(V,D)\},$$

are *ε-alternating forms.* We obviously have $\operatorname{Alt}^{-\varepsilon}_\sigma(V,D)\subset\operatorname{Herm}^\varepsilon_\sigma(V,D)$. We set

$$\mathrm{Q}^\varepsilon_\sigma(V,D)=\operatorname{Sesq}_\sigma(V,D)/\operatorname{Alt}^\varepsilon_\sigma(V,D),$$

and refer to elements of $\mathrm{Q}^\varepsilon_\sigma(V,D)$ as *(ε,σ)-quadratic forms.* We recall that (ε,σ)-quadratic forms were introduced by Tits [21], see also Wall [22], Bak [2] or Scharlau [19, Chapter 7].
For any algebra A with involution τ, let $\operatorname{Sym}^\varepsilon(A,\tau)=\{a\in A\mid a=\varepsilon\tau(a)\}$ and $\operatorname{Alt}^\varepsilon(A,\tau)=\{a\in A\mid a=c-\varepsilon\tau(c),\ c\in A\}$. To any class $\theta=[k]\in\mathrm{Q}^\varepsilon_\sigma(V,D)$, represented by $k\in\operatorname{Sesq}_\sigma(V,D)$, we associate a quadratic map

$$q_\theta: V\to D/\operatorname{Alt}^\varepsilon(D,\sigma),\quad q_\theta(x)=[k(x,x)],$$

where $[d]$ denotes the class of d in $D/\operatorname{Alt}^\varepsilon(D,\sigma)$. The ε-hermitian form

$$b_\theta(x,y)=k(x,y)+\varepsilon k^*(x,y)=k(x,y)+\varepsilon\overline{k(y,x)},$$

depends only on the class θ of k in $\mathrm{Q}^\varepsilon_\sigma(V,D)$. We say that b_θ is the *polarization* of q_θ.

Proposition 1.4.1. *The pair (q_θ,b_θ) satisfies the following formal properties:*

$$\begin{array}{lcl} q_\theta(x+y) & = & q_\theta(x)+q_\theta(y)+[b_\theta(x,y)],\\ q_\theta(xd) & = & \overline{d}q_\theta(x)d,\\ b_\theta(x,x) & = & q_\theta(x)+\varepsilon\overline{q_\theta(x)}, \end{array}\qquad(1.1)$$

for all $x,\ y\in V,\ d\in D$. Conversely, given any pair (q,b), $q:V\to D/\operatorname{Alt}^\varepsilon(D,\sigma)$, $b\in\operatorname{Herm}^\varepsilon_\sigma(V,D)$ satisfying (1.1)*, there exists a unique $\theta\in\mathrm{Q}^\varepsilon_\sigma(V,D)$ such that $q=q_\theta$, $b=b_\theta$.*

Proof. The formal properties are straightforward to verify. For the converse see [22, Theorem 1]. ☐

Proposition 1.4.2. *If the characteristic of F is different from 2, then a generalized quadratic form θ is uniquely determined by its polar form b_θ.*

Proof. See [22, p. 245]. □

Example 1.4.3. Let $D = F$, $\sigma = Id_F$ and $\varepsilon = 1$. Then sesquilinear forms are F-bilinear forms, $\operatorname{Alt}^\varepsilon(D,\sigma) = 0$ and a (σ,ε)-quadratic form is a (classical) quadratic form. We denote the set of bilinear forms on V by $\operatorname{Bil}(V,F)$. Accordingly, we speak of ε-symmetric bilinear forms instead of ε-hermitian forms.

Example 1.4.4. Let D be a division algebra with involution σ and let V be a finite dimensional (right) vector space over D. We use a basis of V to identify V with D^n and $\operatorname{End}_D(V)$ with the algebra $M_n(D)$ of $(n \times n)$-matrices with entries in D. For any $(n \times m)$-matrix $x = (x_{ij})$, let $x^* = \overline{x}^t$, where t is the transpose and $\overline{x} = (\overline{x}_{ij})$. In particular the map $a \mapsto a^*$ is an involution of $A = M_n(D)$. If we write elements of D^n as column vectors $x = (x_1, \dots, x_n)^t$ any sesquilinear form k over D^n can be expressed as $k(x,y) = x^*ay$, with $a \in M_n(D)$, and $k^*(x,y) = x^*a^*y$. We write $\operatorname{Alt}_n(D) = \{a = b - \varepsilon b^*\} \subset M_n(D)$, so that $\mathrm{Q}^\varepsilon_\sigma(V,D) = M_n(D)/\operatorname{Alt}_n(D)$.

For any left (right) D-space V we denote by ${}^\sigma V$ the space V viewed as right (left) D-space through the involution σ. If ${}^\sigma x$ is the element x viewed as an element of ${}^\sigma V$, we have ${}^\sigma x d ={}^\sigma \big(\sigma(d)x\big)$. Let V^* be the dual ${}^\sigma\operatorname{Hom}_D(V,D)$ as a right D-module, i.e., $({}^\sigma f d)(x) = {}^\sigma(\overline{d}f)(x)$, $x \in V$, $d \in D$. Any sesquilinear form $k \in \operatorname{Sesq}_\sigma(V,D)$ induces a D-module homomorphism $\widehat{k} : V \to V^*$, $x \mapsto k(x,-)$. Conversely any D-homomorphism $g : V \to V^*$ induces a sesquilinear form $k \in \operatorname{Sesq}_\sigma(V,D)$, $k(x,y) = g(x)(y)$ and the additive groups $\operatorname{Sesq}_\sigma(V,D)$ and $\operatorname{Hom}_D(V,V^*)$ can be identified through the map $k \mapsto \widehat{k}$. For any $f : V \to V'$, let $f^* : V'^* \to V^*$ be the transpose, viewed as a homomorphism of right vector spaces. We identify V with V^{**} through the map $v \mapsto v^{**}$, $v^{**}(f) = \overline{f(v)}$. Then, for any $f \in \operatorname{Hom}_D(V,V^*)$, f^* is again in $\operatorname{Hom}_D(V,V^*)$ and $\widehat{k}^* = \widehat{k^*}$. A (σ,ε)-quadratic form q_θ is called *nonsingular* if its polar form b_θ induces an isomorphism $\widehat{b_\theta}$. A pair (V,q_θ) with q_θ nonsingular is called a *(σ,ε)-quadratic space* . For any vector space W, the *hyperbolic space* $V = W \oplus W^*$ equipped with the quadratic form q_θ, $\theta = [k]$ with

$$k\big((p,q),(p',q')\big) = q(p'),$$

is nonsingular. There is an obvious notion of orthogonal sum $V \perp V'$ and a quadratic space decomposes whenever its polarization does. Most of the classical theory of quadratic spaces extends to (σ,ε)-quadratic spaces. For example Witt cancellation holds and any (σ,ε)-quadratic space decomposes uniquely (up to isomorphism) as the orthogonal sum of its anisotropic part with a hyperbolic space. Moreover, if we exclude the case $\sigma = 1$ and $\varepsilon = -1$, any (σ,ε)-quadratic space has an orthogonal basis. A *similitude* of (σ,ε)-quadratic spaces $t : (V, q) \xrightarrow{\sim} (V', q')$ is a D-linear isomorphism $V \xrightarrow{\sim} V'$ such that $q'(tx) = \mu(t)q(x)$ for some $\mu(t) \in F^\times$. The element $\mu(t)$ is called the *multiplier* of the similitude. Similitudes with multipliers equal to 1 are *isometries*. We briefly recall the construction of the Witt group. The orthogonal sum defines an addition on the set $\mathfrak{I}$ of isometry classes of (σ,ε)-quadratic spaces. With this addition the set $\mathfrak{I}$ is a *monoid*, i.e. the addition is associative, commutative and has a zero element. There is a canonical procedure to associate a group G (called the *Grothendieck group*) to a monoid H, the so-called Grothendieck construction. We refer to Scharlau (see [19]) for the description of this construction. The Grothendieck group of $\mathfrak{I}$ is denoted by $\mathrm{KU}^\varepsilon(\mathrm{D},\sigma)$. Let $\mathfrak{H}$ be the subgroup of $\mathrm{KU}^\varepsilon(\mathrm{D},\sigma)$ generated by the isometry classes of hyperbolic spaces. The *Witt group* of (σ,ε)-quadratic spaces is the quotient

$$W^\varepsilon(D,\sigma) := \mathrm{KU}^\varepsilon(\mathrm{D},\sigma)/\mathfrak{H}.$$

1.5 Generalized Quadratic Forms and Quadratic Pairs

Relations between quadratic pairs and generalized quadratic forms are discussed in [9], [10] and [15]. We recall how both notions are connected. We assume that σ_θ is an involution of the first kind, so that $\varepsilon = \pm 1$.

Theorem 1.5.1. *Let (V,θ), $\theta = [k]$ be a (σ,ε)-quadratic space over D and let $h = \widehat{k} + \varepsilon\widehat{k}^* : V \xrightarrow{\sim} V^*$. Let $A = \mathrm{End}_D(V)$. The F-linear form*

$$f_\theta : \mathrm{Sym}(A,\sigma_\theta) \to F, \quad f_\theta(s) = \mathrm{Trd}(h^{-1}\widehat{k}s),\ s \in \mathrm{Sym}(A,\sigma_\theta),$$

depends only on the class θ and satisfies $f_\theta\big(x + \sigma_\theta(x)\big) = \mathrm{Trd}(x)$. Thus to any $\theta \in \mathrm{Q}^\varepsilon_\sigma(V,D)$ is associated a quadratic pair $(\sigma_\theta, f_\theta)$ on A.
Conversely, let (τ, f) be a quadratic pair on $A = \mathrm{End}_D(V)$.
1) There exists a (σ,ε)-quadratic space θ on V such that $\tau = \sigma_\theta$ and $f = f_\theta$.
2) $(\sigma_\theta, f_\theta) = (\sigma_{\theta'}, f_{\theta'})$ if and only if $\theta' = \lambda\theta$ for $\lambda \in F$. In this case we call θ

and θ' similar*(similarity is an equivalence relation).*
3) If $\tau = \sigma_\theta$ *and* $f = f_\theta$ *with* $f_\theta(s) = \operatorname{Trd}(us)$*, the class of* u *in* $A/\operatorname{Alt}(A, \sigma_\theta)$ *is uniquely determined by* θ*.*

Proof. The first claim follows from 1.2.2 and the fact that if $k \in \operatorname{Alt}^{\varepsilon}_{\sigma}(V, D)$ then $h^{-1}\widehat{k} \in \operatorname{Alt}^{1}_{\sigma_\theta}(V, D)$. We prove the identity for f_θ:

$$\begin{aligned}
f_\theta\big(x + \sigma_\theta(x)\big) &= \operatorname{Trd}\big(h^{-1}\widehat{k}(x + \sigma_\theta(x))\big)\\
&= \operatorname{Trd}\big(h^{-1}\widehat{k}x\big) + \operatorname{Trd}\big(h^{-1}\widehat{k}h^{-1}x^*h\big)\\
&= \operatorname{Trd}\big(h^{-1}\widehat{k}x\big) + \operatorname{Trd}\big(\widehat{k}h^{-1}x^*\big)\\
&= \operatorname{Trd}\big(h^{-1}\widehat{k}x\big) + \operatorname{Trd}\big(x(h^{-1})^*\widehat{k}^*\big)\\
&= \operatorname{Trd}\big(h^{-1}\widehat{k}x\big) + \operatorname{Trd}\big(h^{-1}\varepsilon\widehat{k}^*x\big) = \operatorname{Trd}(x).
\end{aligned}$$

For the converse, we observe that the second and third assertions can be proved exactly as in 1.3.3. We prove 1) for completeness. Let $\tau(x) = h^{-1}x^*h$, $h = \varepsilon h^* : V \overset{\sim}{\to} V^*$. Let $f(s) = \operatorname{Trd}(us)$ with $u + \tau(u) = 1$ and let $k \in \operatorname{Sesq}_\sigma(V, D)$ be such that $\widehat{k} = hu : V \to V^*$. We set $\theta = [k]$. It is then straightforward to check that $h = k + \varepsilon k^*$. □

We say that an involution τ of A is a *q-involution* if τ is adjoint to the polar b_θ of a generalized quadratic form θ. We write $\tau = \sigma_\theta$. Two algebras with q-involutions are *isomorphic* if the isomorphism is induced by a similitude of the corresponding generalized quadratic forms.

Proposition 1.5.2. *Let* $\phi : \big(\operatorname{End}_D(V), \sigma_\theta\big) \overset{\sim}{\to} \big(\operatorname{End}_D(V'), \sigma_{\theta'}\big)$ *be an isomorphism of algebras with involution. Let* $f_\theta(s) = \operatorname{Trd}(us)$ *and* $f_{\theta'}(s') = \operatorname{Trd}(u's')$*. The following conditions are equivalent:*
1) ϕ *is an isomorphism of algebras with q-involutions.*
2) $f_{\theta'}\big(\phi(s)\big) = f_\theta(s)$ *for all* $s \in \operatorname{Sym}\big(\operatorname{End}_D(V), \sigma_\theta\big)$*.*
3) $[\phi(u)] = [u'] \in \operatorname{End}_D(V')/\operatorname{Alt}\big(\operatorname{End}_D(V'), \sigma_{\theta'}\big)$*.*

Proof. The implication 1) ⇒ 2) is clear. We check that 2) ⇒ 3). Let ϕ be induced by a similitude $t : (V, b_\theta) \overset{\sim}{\to} (V', b_{\theta'})$. Since $f_{\theta'}(\phi(s)) = f_\theta(s)$, we have $\operatorname{Trd}(t^{-1}u'ts) = \operatorname{Trd}(u'tst^{-1}) = \operatorname{Trd}(us)$ for all $s \in \operatorname{Sym}\big(\operatorname{End}_D(V), \sigma_\theta\big)$, hence $[\phi(u)] = [u']$. The implication 3) ⇒ 1) follows from the fact that u can be chosen as $h^{-1}\widehat{k}$, $h = \widehat{k} + \varepsilon\widehat{k}^*$. □

Chapter 2

Quadratic Quaternion Forms

2.1 Quaternions

We call a central simple algebra of degree 2 over a field F a *quaternion algebra* over F. If the characteristic is different from 2, every quaternion algebra D has a basis $(1, i, j, k)$ subject to the relations

$$i^2 \in F^\times, \quad j^2 \in F^\times, \quad ij = k = -ji.$$

Such a basis is called a *quaternion basis*; if $i^2 = a$ and $j^2 = b$, the quaternion algebra D is denoted $D = (a, b)_F$. Conversely, for any $a, b \in F^\times$ the 4-dimensional F-algebra D with basis $(1, i, j, k)$ where the multiplication is defined through the relations $i^2 = a$, $j^2 = b$, $ij = k = -ji$ is central simple and is therefore a quaternion algebra $(a, b)_F$.
If $x = x_1 \cdot 1 + x_2 \cdot i + x_3 \cdot j + x_4 \cdot k \in D$, then $\mathrm{Trd}_D(x) = 2x_1$ and $\mathrm{Nrd}_D(x) = x_1^2 - ax_2^2 - bx_3^2 + abx_4^2$.
If $\mathrm{char}F = 2$, every quaternion algebra D has a basis $(1, u, v, w)$ subject to the relations

$$u^2 + u \in F, \quad v^2 \in F^\times, \quad uv = w = vu + v.$$

Such a basis is called a *quaternion basis in characteristic* 2. If $u^2 + u = a$ and $v^2 = b$, the quaternion algebra D is denoted $D = [a, b)_F$. Conversely, for all $a \in F$, $b \in F^\times$, the relations $u^2 + u = a$, $v^2 = b$ and $uv = w = vu + v$ give the span of $1, u, v, uv$ the structure of a quaternion algebra. If $x = x_1 \cdot 1 + x_2 \cdot u + x_3 \cdot v + x_4 \cdot w \in D$, then $\mathrm{Trd}_D(x) = x_2$ and $\mathrm{Nrd}_D(x) = x_1^2 + x_1x_2 + x_2^2a + x_3^2b + x_3x_4b + x_4^2ab$.

Let $\gamma : D \longrightarrow D$ be the F-linear map defined by $\gamma(x) := \bar{x} := \mathrm{Trd}_D(x) - x$ for $x \in D$. Explicitly, for $x_1, x_2, x_3, x_4 \in F$,

$$\gamma(x_1 + x_2 i + x_3 j + x_4 k) = x_1 - x_2 i - x_3 j - x_4 k$$

if char$F \neq 2$ and

$$\gamma(x_1 + x_2 u + x_3 v + x_4 w) = x_1 + x_2(u + 1) + x_3 v + x_4 w$$

if char$F = 2$.

Example 2.1.1. For the *split quaternion algebra* $D = M_2(F)$ (in arbitrary characteristic),

$$\gamma \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}.$$

Direct computations show that γ is an involution, called the *quaternion conjugation* or the *canonical involution.* If char$F \neq 2$, then $\mathrm{Sym}(D, \gamma) = F$ and $\mathrm{Skew}(D, \gamma)$ has dimension 3. If char$F = 2$, then $\mathrm{Sym}(D, \gamma)$ is spanned by $1, v, w$ which have reduced trace equal to zero. Therefore, the involution γ is symplectic in every characteristic.

Proposition 2.1.2. *The canonical involution γ on a quaternion algebra D is the unique symplectic involution on D. Every orthogonal involution σ on D is of the form*

$$\sigma = \mathrm{Int}(u) \circ \gamma,$$

where u is an invertible quaternion in $\mathrm{Skew}(D, \gamma) \backslash F$ *which is uniquely determined by σ up to a factor in $F^\times$.*

Proof. It follows from 1.2.4 that every involution of the first kind σ on D has the form $\sigma = \mathrm{Int}(u) \circ \gamma$, where u is a unit such that $\gamma(u) = \pm u$. Suppose that σ is symplectic. If char$F = 2$, Proposition 1.2.4 shows that $u \in \mathrm{Alt}(D, \gamma) = F$, hence $\sigma = \gamma$. Similarly, if char$F \neq 2$, Proposition 1.2.4 shows that $\gamma(u) = u$, hence $u \in F^\times$ and $\sigma = \gamma$. □

We shall use the following presentation of a quaternion algebra D over a field F, which is valid for fields of any characteristic. Let K be a maximal subfield of D which is a quadratic Galois extension of F and let $\sigma : x \mapsto \overline{x}$ be the nontrivial automorphism of K. Let $j \in K \setminus F$ be an element of trace 1, so that $K = F(j)$ with $j^2 = j + \lambda$, $\lambda \in F$. Let $\ell \in D$ be such that

$\ell x\ell^{-1} = \overline{x}$ for $x \in K$, $\ell^2 = \mu \in F^\times$. The elements $\{1, j, \ell, \ell j\}$ form a basis of D and $D = K \oplus \ell K$ is also denoted $[K, \mu)$. The canonical involution $\gamma : D \to D$, $\gamma(d) = \mathrm{Trd}_D(d) - d = \overline{d}$ extends the automorphism σ of K. The element $\mathrm{N}(d) = d\gamma(d) = \gamma(d)d$ is the reduced norm of d. We observe that the quaternion algebras $[K, \mu)$ and $[K, N(u)\mu)$ are isomorphic for all $u \in K^\times$.

2.2 Seip-Hornix Forms

Let $D = [K, \mu)$ be a quaternion division algebra. Let $a, b \in D$. We define an equivalence relation on D by setting $a \equiv b$ if, and only if $a - b \in F$. Let V be an n-dimensional right vector space over D.
A *Seip-Hornix form (S-H-form) L on V* (see [20]) is a map

$$L : V \longrightarrow D$$

which satisfies the following conditions:
1. Let $\Phi : V \times V \longrightarrow D$ be defined by $\Phi(v, w) := L(v + w) - L(v) - L(w)$ for every $v, w \in V$. Then $\Phi(v_1 + v_2, w) \equiv \Phi(v_1, w) + \Phi(v_2, w)$ for every $v_1, v_2, w \in V$.
2. $L(v\lambda) \equiv \bar{\lambda}L(v)\lambda$ for every $v \in V$, $\lambda \in D$.
In this section we prove that S-H-forms correspond to a generalized quadratic form over quaternions. We may view the n-dimensional D-vector space V as a $2n$-dimensional vector space V^0 over K (by restriction of scalars). Let T be the map $T : V^0 \longrightarrow V^0$ defined by $T(v) := v\ell$. The map T is a K-*semilinear automorphism* of V^0 (which means that T is additive, invertible and $T(v\lambda) = T(v)\overline{\lambda}$ for all $v \in V^0$, $\lambda \in K$) such that $T^2 = \mu$.
Following [20], we write $L(v) = P(v) + \ell R(v)$ with $P, R : V^0 \longrightarrow K$. In the same way $\Phi(v, w) = p(v, w) + \ell r(v, w)$ with $p, r : V^0 \times V^0 \longrightarrow K$. At this point we recall an useful theorem (see [20, 1.3] and compare with [7]):

Theorem 2.2.1. *Let $L : V \longrightarrow D$ be a S-H-form, $D = K \oplus \ell K$, $\rho \in K$ and $\rho + \bar{\rho} = 1$. Then we have $L(v) = q(v) - \rho r(T(v), v) + \ell R(v)$ for $v \in V$, where R is a quadratic form over K, r is the polarization of R, and q is a map $V \longrightarrow F$.*

Remark 2.2.2. It is equivalent to consider a vector space V of dimension n over D or a vector space V^0 of dimension $2n$ over K together with a semilinear automorphism T such that $T^2 = \mu \in F^\times$: we may consider V

as a $2n$-dimensional vector space V^0 over K (by restriction of scalars) and define $T(v) := v\ell$; conversely, given a vector space U over K, together with a semilinear automorphism T such that $T^2 = \mu \in F^\times$, we define the structure of a right D-module on U, $D = [K, \mu)$, by putting $x\ell := T(x)$.

Let q be a quadratic form on a space U over K and let $\lambda \in K^\times$. A semilinear automorphism t of U such that $q(tx) = \lambda\overline{q(x)}$ for all $x \in U$ is a *semilinear similitude* of (U, q), with *multiplier* λ. In particular $T(x) = x\ell$ is a semilinear similitude of R on V^0, such that $T^2 = \mu$ and with multiplier $-\mu$. The following observations (compare Seip-Hornix [20, p. 328]) will be used later:

Proposition 2.2.3. *Let R be a quadratic form over the K-vector space U and let T be a semilinear similitude of U with multiplier $\lambda \in K^\times$ and such that $T^2 = \mu$. Then:*
1) $\mu \in F$,
2) For any $\xi \in K$ and $x \in U$, let $\rho_\xi(x) = x\xi$. There exists $\nu \in K^\times$ such that $T' = \rho_\nu \circ T$ satisfies ${T'}^2 = \mu'$ and $R(T'x) = -\mu'\overline{R(x)}$ for some $\mu' \in F^\times$.

Proof. The first claim follows from $T^3 = T^2 \circ T = T \circ T^2$. For the second we may assume that $\lambda \neq \mu$ (if $\lambda = \mu$ replace T by $T \circ \rho_k$ for an appropriate k). For $\nu = (1 - \mu\lambda^{-1})$ we have $\mu' = 2\mu - \lambda - \overline{\lambda}$. □

Theorem 2.2.4. *There is exactly one mapping $h : V \times V \longrightarrow D$, which is sesquilinear over D (with respect to $^-$), antihermitian if* char$F \neq 2$, *hermitian if* char$F = 2$, *and for which*

$$h(v, v) = L(v) - \overline{L(v)} \quad \forall v \in V.$$

Proof. Uniqueness: let h_1, h_2 be two such mappings. Then $\Delta := h_1 - h_2$ is a mapping with the same properties, with $\Delta(v, v) = 0$. This implies that $\Delta : V \times V \longrightarrow \mathrm{Sym}(D, ^-)$. If $\Delta(v, w) \neq 0$ for some v, $w \in V$, then clearly $\Delta(V, V) = D$. It follows from $\mathrm{Sym}(D, ^-) \neq D$, that $\Delta : V \times V \longrightarrow \{0\}$.
Existence: The map $h(v, w) = -r(T(v), w) + \ell r(v, w)$, with $D = K \oplus \ell K$ and r, T as before, has the required properties. □

We call h the *associated sesquilinear form.* We introduce some conventions. The subspace of V generated by the vectors $e_1, \ldots, e_k$ (k is an integer) is denoted with $< e_1, \ldots, e_k >_D$. We call Φ *nondegenerate* if $\Phi(v, w) \in F$ for every $w \in V$ implies $v = 0$. In that case L is called *nondegenerate.*

Remark 2.2.5. The form L is nondegenerate if, and only if r is nondegenerate if, and only if h is nondegenerate.

Lemma 2.2.6. *Let $L : V \longrightarrow D$ be a S-H-form. Suppose that Φ, h are defined as before. Then $h(v,w) \equiv \Phi(v,w)$.*

Proof. $h(v,w) - \Phi(v,w) = h(v+w, v+w) - h(v,v) - h(w,w) - h(w,v) - L(v+w) + L(v) + L(w) = L(v+w) - \overline{L(v+w)} - L(v) + \overline{L(v)} - L(w) + \overline{L(w)} + \overline{h(v,w)} - L(v+w) + L(v) + L(w) = \overline{h(v,w) - \Phi(v,w)}$ and $h(v,w) - \Phi(v,w) = -r(T(v),w) - p(v,w) \in K$. ☐

Theorem 2.2.7. *Let $L : V \longrightarrow D$ be a S-H-form. Then there is a sesquilinear form $k : V \times V \longrightarrow D$ such that $L(v) \equiv k(v,v)$.*

Proof. Let $L : V \longrightarrow D$ be a S-H-form, $h : V \times V \longrightarrow D$ its associated sesquilinear form, and $e_1, \dots, e_n$ an orthogonal (with respect to h) basis of V. We claim that for $v = e_1\lambda_1 + \cdots + e_n\lambda_n$ ($\lambda_i \in D$), $L(v) \equiv \sum_{i=1}^{n} \overline{\lambda_i} L(e_i)\lambda_i$. In other words, $L(v) \equiv k(v,v)$, with k the sesquilinear form with matrix

$$\begin{pmatrix} L(e_1) & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & L(e_{n-1}) & 0 \\ 0 & \dots & 0 & L(e_n) \end{pmatrix}$$

with respect to the basis $e_1, \dots, e_n$. We prove this assertion by induction. If $n = 1$ the claim is trivial. If $n > 1$ then we write $v = w + e_n\lambda_n$, with $w \in < e_1, \dots, e_{n-1} >_D$. Then $L(v) = L(w + e_n\lambda_n) \equiv L(w + e_n\lambda_n) - \overline{\Phi(w, e_n\lambda_n)}$, because $0 = h(w, e_n) \equiv \overline{\Phi(w, e_n\lambda_n)}$.
On the other hand, $\overline{\Phi(w, e_n\lambda_n)} = \overline{L(w + e_n\lambda_n)} - \overline{L(w)} - \overline{L(e_n\lambda_n)}$ and
$0 = L(w) - L(w) + L(e_n\lambda_n) - L(e_n\lambda_n)$.
Hence $L(v) \equiv L(w + e_n\lambda_n) - \overline{L(w + e_n\lambda_n)} - L(w) + \overline{L(w)} - L(e_n\lambda_n) + \overline{L(e_n\lambda_n)} + L(w) + L(e_n\lambda_n)$. We use the fact that $h(z,z) = L(z) - \overline{L(z)}$ and we obtain $L(v) \equiv L(w) + L(e_n\lambda_n) + h(w + e_n\lambda_n, w + e_n\lambda_n) - h(w,w) - h(e_n\lambda_n, e_n\lambda_n)$. But w and e_n are orthogonal, hence $L(v) \equiv L(w) + \overline{\lambda_n} L(e_n)\lambda_n$. The claim follows by induction. ☐

We say that two S-H-forms L_1 and L_2 defined on the spaces V_1 and V_2 over D are *equivalent* if there exists an isomorphism $t : V_1 \longrightarrow V_2$ such that $L_1(v_1) + F = L_2(t(v_1)) + F$ for all $v_1 \in V_1$. The equivalence class of L is denoted by $[L]$. S-H-forms are related to generalized quadratic forms as follows. Let V be an n-dimensional right D-vector space. We call a $(-1,\gamma)-$quadratic form $\theta = [k]$ a *quadratic quaternion form.* Since $\mathrm{Alt}_{\gamma}^{-1}(D) = F$, the map

q_θ associated to θ (see 1.1) has values in D/F and $q_\theta(x) = [k(x,x)]$. The polarization b_θ of q_θ,

$$b_\theta(x,y) = k(x,y) - k^*(x,y) = k(x,y) - \overline{k(y,x)},$$

is hermitian if char $= 2$, skew-hermitian if char $\neq 2$.

Theorem 2.2.8. *1) Let $L : V \longrightarrow D$ be a S-H-form. There exists a unique quadratic quaternion form $\theta = [k] \in \mathrm{Q}_\gamma^{-1}(V,D)$ such that $[L] = q_\theta$.*
2) For every quadratic quaternion form $\theta \in \mathrm{Q}_\gamma^{-1}(V,D)$ there exists a S-H-form $L : V \longrightarrow D$ with $[L] = q_\theta$.

Proof. The first assertion is a corollary of 2.2.7. The second statement follows choosing $L(x) := k(x,x)$. □

Remark 2.2.9. We have the relations $q_\theta(x) = L(x) + F$ and $b_\theta = h$.

Let D be a quaternion algebra and $A \simeq \mathrm{End}_D(V)$. We call a quadratic pair (σ, f) on A *quaternionic quadratic pair.*

Proposition 2.2.10. *Let $A = \mathrm{End}_D(V)$ and let (σ, f) be a quadratic pair on A. Then there exists a quadratic quaternion form $\theta \in \mathrm{Q}_\gamma^{-1}(V,D)$ such that $(\sigma, f) = (\sigma_\theta, f_\theta)$.*

Proof. The assertion follows directly from 1.5.1. The form θ is determined up to the multiplication by a scalar $\lambda \in F^\times$. In fact σ_θ determines the polar b_θ up to λ and f_θ determines u. We have $\theta = [\widehat{b_\theta u}]$. □

Example 2.2.11. We also consider quadratic pairs on $A = M_n(D)$, the F-algebra of $n \times n$ matrices with entries in D. In A there is a canonical symplectic involution, namely $x \mapsto \varsigma(x) := x^* := \overline{x}^t$. Let (σ, f) be a quadratic pair on $A = M_n(D)$. There exists a quadratic quaternion form $\theta \in \mathrm{Q}_\gamma^{-1}(D^n, D)$ such that $(\sigma, f) = (\sigma_\theta, f_\theta)$. We use 1.3.3: every quadratic pair on $M_n(D)$ is of the form (σ_a^s, f_a^s), where $a \in M_n(D)$ and $a - a^*$ is invertible. Hence $(\sigma, f) = (\sigma_\theta, f_\theta)$, where $\theta = [k]$, k is the (nondegenerate) sesquilinear form $k : D^n \times D^n \longrightarrow D$ defined by $k(x,y) = x^*ay$.

Let $\theta = [k] \in \mathrm{Q}_\gamma^{-1}(V,D)$ be a quadratic quaternion form. We know (see 2.2.1 and 2.2.8) that

$$q_\theta(v) = -\rho r(T(v), v) + lR(v) + F,$$

for $v \in V$, where R is a quadratic form over K, r is the polarization of R, $\rho \in K$ and $\rho+\bar{\rho}=1$. We look at relations between θ and R. The sesquilinear form $k : V \times V \to D$ can be decomposed as

$$k(x,y) = f_1(x,y) + \ell f_2(x,y)$$

with $f_1 : V \times V \to K$ and $f_2 : V \times V \to K$. The following properties of f_1 and f_2 are straightforward.

Lemma 2.2.12. *1)* $f_1 \in \mathrm{Sesq}_\gamma(V,K)$, $f_2 \in \mathrm{Sesq}_1(V,K) = \mathrm{Bil}(V,K)$.
2) $k^* = f_1^* - \ell f_2^t$, *where* $f_1^*(x,y) = \overline{f_1(y,x)}$ *and* $f_2^t(x,y) = f_2(y,x)$.

The sesquilinearity of k implies the following identities:

$$\begin{array}{rclrcl} f_2(x\ell,y) &=& -f_1(x,y), & f_2(x,y\ell) &=& \overline{f_1(x,y)} \\ f_1(x\ell,y) &=& -\mu f_2(x,y), & f_1(x,y\ell) &=& \mu\overline{f_2(x,y)} \\ f_1(x\ell,y\ell) &=& -\mu\overline{f_1(x,y)}, & f_2(x\ell,y\ell) &=& -\mu\overline{f_2(x,y)} \end{array} \tag{2.1}$$

We recall that it is equivalent to have a D-vector space V or a pair (V^0,T) (see 2.2.2), where V^0 denotes the K-vector space obtained from V by restricting scalars to K and T the map $x \mapsto x\ell$.

Lemma 2.2.13. *Let* V *be a vector space over* D*. 1) Let* $f_1 : V^0 \times V^0 \to K$ *be a sesquilinear form over* K*. The form*

$$f(x,y) = f_1(x,y) - \ell\mu^{-1} f_1(Tx,y)$$

is sesquilinear over D *if and only if* $f_1(Tx,Ty) = -\mu\overline{f_1(x,y)}$.
2) Let $f_2 : V^0 \times V^0 \to K$ *be a bilinear form over* K*. The form*

$$f(x,y) = -f_2(Tx,y) + \ell f_2(x,y)$$

is sesquilinear over D *if and only if* $f_2(Tx,Ty) = -\mu\overline{f_2(x,y)}$.

Proof. The two claims follow from the identities (2.1). □

Let $\theta = [k]$ be a quadratic quaternion form. The class $[f_1] \in \mathrm{Q}_\gamma^{-1}(V^0,K)$ is a $(\gamma,-1)$-quadratic space and $[f_2]$ is an $(Id,1)$-quadratic space on V^0 over K, i.e. a (classical) quadratic form R with $R(x) = f_2(x,x)$. Let $K = F(j)$ with $j^2 = j + \lambda$. Let $r(x,y) = f_2(x,y) + f_2(y,x)$ be the polarization of R.

Proposition 2.2.14. *1)* $q_{[f_1]}(x) = -jr(x, Tx) + F$.
2) $q_\theta(x) = -jr(x, Tx) + \ell R(x) + F$.
3) The map T *is a semilinear similitude of* $\big(V^0, R\big)$ *with multiplier* $-\mu$.

Proof. It follows from the relations (2.1) that

$$\overline{f_1(x,x)} - f_1(x,x) = f_2(x, Tx) + f_2(Tx, x) = r(x, Tx) \qquad (2.2)$$

and obviously this relation determines $f_1(x,x)$ up to a function with values in $\mathrm{Sym}^1(K,\gamma)$. Since $\mathrm{Sym}^1(K,\gamma) = \mathrm{Alt}^{-1}(K,\gamma)$, the class $[f_1]$ is determined by the equation (2.2). Since $\overline{r(x,Tx)} = -r(x,Tx)$ by the relations (2.1), we have $\overline{-jr(x,Tx)} - \big(-jr(x,Tx)\big) = r(x,Tx)$ and 1) follows. The second claim follows from 1) and 3) is again a consequence of the identities (2.1). ☐

Corollary 2.2.15. *Any pair* $\big(R, T\big)$ *with* $R \in \mathrm{Q}_1^{-1}(U, K)$ *(i.e.* R *is a (classical) quadratic form) and* T *a semilinear similitude with multiplier* $-\mu \in F^\times$ *and such that* $T^2 = \mu$*, determines the structure of a quadratic quaternion form on* U *over* $D = [K, \mu)$.

Remark 2.2.16. We observe that it is equivalent to have a quadratic quaternion space in $\mathrm{Q}_\gamma^{-1}(V, D)$ or a triple $\big(V^0, R, T\big)$.

We conclude this section describing a notion which shall be used later. An invertible D-linear map $t : V \longrightarrow V$ is called an *isometry* if $q_\theta(t(x)) = q_\theta(x)$ for all $x \in V$. The group of isometries is denoted by $\mathrm{O}(D, \theta)$.

Proposition 2.2.17. *A map* $t : V \longrightarrow V$ *is an element of* $\mathrm{O}(D,\theta)$ *if, and only if* $t \in O^+(V^0, R)$ *and* $tT = Tt$.

Proof. The proposition is a reformulation of [20, 4.1]. ☐

2.3 An Exact Sequence of Lewis

In this section we apply the formalism of quadratic quaternion forms to obtain a characteristic free version of an exact sequence due to Lewis for fields of characteristic not 2 (see [16]).

Let $\theta = [k] \in \mathrm{Q}_\gamma^{-1}(V, D)$ be a quadratic quaternion form, $D = [K, \mu)$. We decompose the sesquilinear form $k : V \times V \to D$ as before:

$$k(x, y) = f_1(x, y) + \ell f_2(x, y)$$

with $f_1 : V \times V \to K$ and $f_2 : V \times V \to K$.

Proposition 2.3.1. *The assignments $k \mapsto f_1$ and $k \mapsto f_2$ induce homomorphisms of groups $\pi_1 : W^{\varepsilon}(D, -) \to W^{\varepsilon}(K, -)$ and $\pi_2 : W^{-\varepsilon}(D, -) \to W^{\varepsilon}(K, Id)$.*

Proof. The assignments are compatible with orthogonal sums and with Witt equivalence. ☐

Let $i \in K^{\times}$ be such that $\sigma(i) = -i$ (take $i = 1$ if $\mathrm{char} F = 2$). The map $k \mapsto ik$ induces an isomorphism $s : W^{\varepsilon}(K, -) \xrightarrow{\sim} W^{-\varepsilon}(K, -)$ (called *scaling*). For any space U over K, let $U_D = U \otimes_K D$. We identify U_D with $U \oplus U\ell$ through the map $u \otimes (x + \ell y) \mapsto (ux, u\overline{y}l)$ and get a natural D-module structure on $U_D = U \oplus U\ell$. Any K-sesquilinear form k on U extends to a D-sesquilinear form k_D on U_D through the formula

$$k_D(x \otimes a, y \otimes b) = \overline{a} k(x, y) b$$

for x, $y \in U$ and a, $b \in D$.

Lemma 2.3.2. *The assignment $k \mapsto (ik)_D$ induces a homomorphism*

$$\beta : W^{\varepsilon}(K, -) \to W^{-\varepsilon}(D, -)$$

Proof. Let $\widetilde{k} = (ik)_D$. We have $(\widetilde{k})^* = -\widetilde{k}^*$. ☐

Theorem 2.3.3 (Lewis). *With the notations above, the sequence*

$$W^{\varepsilon}(D, -) \xrightarrow{\pi_1} W^{\varepsilon}(K, -) \xrightarrow{\beta} W^{-\varepsilon}(D, -) \xrightarrow{\pi_2} W^{\varepsilon}(K, Id)$$

is exact.

Proof. This is essentially the proof given in Appendix 2 of [3] with some changes due to the use of generalized quadratic forms, instead of hermitian forms. We first check that the sequence is a complex. Let $[k] \in Q^{\varepsilon}_{\sigma}(V, D)$ and let $V^0 = U$. We write elements of $U_D = U \oplus U\ell$ as pairs $(x, y\ell)$ and decompose $k_D = P + \ell Q$. By definition we have $\beta\pi_1([k]) = [\beta(P)]$ and

$$\begin{aligned} \beta(P)\big((x_1, y_1), (x_2, y_2)\big) \;=\; & i\big(P(x_1, x_2) + P(x_1, y_2)\ell + \ell P(y_1, x_2) \\ & + \ell P(y_1, y_2)\ell\big). \end{aligned}$$

Let $(x\ell, x\ell) \in U \oplus U\ell$. We get $\beta(P)\big((x\ell, x\ell), (x\ell, x\ell)\big) = 0$ hence $W = \{(x\ell, x\ell)\} \subset U \oplus U\ell$ is totally isotropic. It is easy to see that $W \subset W^{\perp}$, so that $[\beta(P)]$ is hyperbolic and $\beta \circ \pi_1 = 0$. Let $[g] \in \mathrm{Q}_\sigma^\varepsilon(U, K)$. The subspace $W = \{(x, 0) \in U \oplus U\ell\}$ is totally isotropic for $\pi_2\beta([g])$ and $W \subset W^\perp$. Hence $\pi_2\beta([g]) = 0$. We now prove exactness at $W^\varepsilon(K, -)$. Since the claim is known if $\mathrm{char} F \neq 2$, we may assume that $\mathrm{char} F = 2$ and $\varepsilon = 1$. Let $[g] \in \mathrm{Q}_\sigma^\varepsilon(U, K)$ be anisotropic such that $\beta([g]) = 0 \in W^{-\varepsilon}(D, -)$. In particular $\beta([g]) \in \mathrm{Q}_\sigma^{-\varepsilon}(U_D, D)$ is isotropic. Hence there exist elements x_1, $x_2 \in U$ such that $[\widetilde{g}]\big((x_1, x_2\ell), (x_1, x_2\ell)\big) = 0$. This implies (in $\mathrm{char} F = 2$) that

$$g(x_1, x_1) + \mu\overline{g(x_2, x_2)} \in F, \quad g(x_1, x_2)\ell + \ell g(x_2, x_1) = 0. \tag{2.3}$$

Let V_1 be the K-subspace of V generated by the vectors x_1 and x_2. Since $[g]$ is anisotropic, $[g] = [g_1] \perp [g_2]$ with $g_1 = g|_{V_1}$. We make V_1 into a D-space by putting

$$(x_1a_1 + x_2a_2)\ell = \mu x_2\overline{a}_1 + x_1\overline{a}_2.$$

To see that the action is well-defined, it suffices to show that $\dim_K V_1 = 2$. The elements x_1 and x_2 cannot be zero since $[g]$ is anisotropic, so assume $x_2 = x_1c$, $c \in K^\times$. Then (2.3) implies $g(x_1, x_1) + \mu c\overline{c}g(x_1, x_1) \in F$, which contradicts the fact that g is anisotropic. Let $g_1(x_1, x_1) + \mu\overline{g_1(x_2, x_2)} = z \in F$. Let $f \in \mathrm{Sesq}_\sigma(V_1, K)$. Replacing g_1 by $g_1 + f + f^*$ defines the same class in $\mathrm{Q}_\sigma^\varepsilon(V_1, K)$ (recall that $\mathrm{char} F = 2$). Choosing f as

$$f(x_1, x_1) = jz,\ f(x_2, x_2) = 0,\ f(x_1, x_2) = f(x_2, x_1) = 0,$$

we may assume that

$$g_1(x_1, x_1) + \mu\overline{g_1(x_2, x_2)} = 0, \quad g_1(x_1, x_2)\ell + \ell g_1(x_2, x_1) = 0. \tag{2.4}$$

By (2.2.13) we may extend g_1 to a sesquilinear form

$$g'(x, y) = g_1(x, y) + \ell\mu^{-1}g_1(x\ell, y)$$

over D if g_1 satisfies

$$g_1(x\ell, y\ell) = -\mu\overline{g_1(x, y)}.$$

This can easily be checked using (2.4) (and the definition of $x\ell$). Then g_1 is in the image of π_1. Exactness at $W^\varepsilon(K, -)$ now follows by induction on the dimension of U. We finally check exactness at $W^{-\varepsilon}(D, -)$. Let $[k]$ be anisotropic such that $\pi_2([k]) = 0$ in $W^{-\varepsilon}(K, Id)$. In particular $\pi_2([k])$ is isotropic; let $x \neq 0$ be such that $\pi_2k(x, x) = 0$ and let W be the D-subspace of V generated by x. Since $[k]$ is anisotropic, $[k'] = [k|_W]$ is nonsingular and

$[k] = [k'] \perp [k'']$. The condition $\pi_2 k(x,x) = 0$ implies $k(x,x) \in K$. Let W_1 be the K-subspace of W generated by x. Define $g : W_1 \times W_1 \to K$ by $g(xa, xb) = k(xa, xb)i^{-1}$ for a, $b \in K$. Then clearly $[g]$ defines an element of $W^{\varepsilon}(K, -)$ and $\beta(g) = k'$. Once again exactness follows by induction on the dimension of V. $\square$

Chapter 3

The Clifford Algebra

3.1 The Clifford Algebra

Let σ be an involution of the first kind on D and let θ be a nonsingular (σ, ε)-quadratic form on V. Let σ_θ be the corresponding q-involution on $A = \operatorname{End}_D(V)$. We assume in this section that over a splitting $A \otimes_F \tilde{F} \xrightarrow{\sim} \operatorname{End}_{\tilde{F}}(M)$ of A, $\theta_{\tilde{F}} = \theta \otimes 1_{\tilde{F}}$ is a $(Id, 1)$-quadratic form $\tilde{q}$ over $\tilde{F}$, i.e. $\theta_{\tilde{F}}$ is a (classical) quadratic form. In other words this means that σ_θ is orthogonal if $\operatorname{char} F \neq 2$ and symplectic if $\operatorname{char} F = 2$. We recall the definition of the Clifford algebra $\operatorname{Cl}(V, \theta)$, following Tits (see [21, 4.1]). Given (V, θ) as above, let $\theta = [k]$, $k \in \operatorname{Sesq}_\sigma(V, D)$, $b_\theta = k + \varepsilon k^*$ and $h = \widehat{b_\theta} \in \operatorname{Hom}_D(V, V^*)$. Let

$$A = \operatorname{End}_D(V), \quad B = \operatorname{Sesq}_\sigma(V, D) \quad \text{and} \quad B' = V \otimes_D {}^{\sigma}V.$$

We identify the vector spaces A with $V \otimes_D {}^{\sigma}V^*$ through the canonical isomorphism $(x \otimes {}^{\sigma}f)(v) = xf(v)$ and B with $V^* \otimes_D {}^{\sigma}V^*$ through $(f \otimes {}^{\sigma}g)(x, y) = \overline{g(x)}f(y)$. The isomorphism h can be used to define further isomorphisms:

$$\varphi_\theta : B' = V \otimes_D {}^{\sigma}V \xrightarrow{\sim} A = \operatorname{End}_D(V), \ \varphi_\theta : x \otimes y \mapsto x \otimes h(y),$$

and

$$\psi_\theta : A \xrightarrow{\sim} B, \ \psi_\theta : x \otimes {}^{\sigma}f \mapsto h(x) \otimes {}^{\sigma}f.$$

We use φ_θ and ψ_θ to define maps $B' \times B \to A$, $(b', b) \mapsto b'b$ and $A \times B' \to B'$, $(a, b') \mapsto ab'$:

$$(x \otimes {}^{\sigma}y)(h(v) \otimes {}^{\sigma}g) = x b_\theta(y, v) \otimes {}^{\sigma}g \text{ and } (x \otimes {}^{\sigma}f)(v \otimes {}^{\sigma}w) = xf(v) \otimes {}^{\sigma}w.$$

Furthermore, let $\tau_\theta = \varphi_\theta^{-1}\sigma_\theta\varphi_\theta : B' \to B'$ be the transport of the involution σ_θ on A. We have $\tau_\theta(x \otimes {}^\sigma y) = \varepsilon y \otimes {}^\sigma x$. Let $S_1 = \{s_1 \in B' \mid \tau_\theta(s_1) = s_1\}$. We have $S_1 = \big(\mathrm{Alt}^\varepsilon(V, D)\big)^\perp$ for the pairing $B' \times B \to F$, $(b', b) \mapsto \mathrm{Trd}_A(b'b)$. Let Sand be the bilinear map $B' \otimes B' \times B \to B'$ defined by $\mathrm{Sand}(b_1' \otimes b_2', b) = b_2' b b_1'$. The *Clifford algebra* $\mathrm{Cl}(V, \theta)$ of the quadratic space (V, θ) is the quotient of the tensor algebra of the F-module B' by the ideal I generated by the sets

$$\begin{aligned} I_1 &= \{s_1 - \mathrm{Trd}_A(s_1 k)1,\ s_1 \in S_1\}, \\ I_2 &= \{c_1 - \mathrm{Sand}(c_1, k) \mid \mathrm{Sand}\big(c_1, \mathrm{Alt}^\varepsilon(V, D)\big) = 0\}. \end{aligned}$$

The Clifford algebra $\mathrm{Cl}(V, \theta)$ has a canonical involution τ induced by the map τ_θ. We have $\mathrm{Cl}(V, \theta) \otimes_F \widetilde{F} = \mathrm{Cl}(V \otimes_F \widetilde{F}, \theta \otimes 1_{\widetilde{F}})$ for any field extension $\widetilde{F}$ of F and $\mathrm{Cl}(V, q)$ is the even Clifford algebra $C_0(V, q)$ of (V, q) if $D = F$ ([21, Théorème 2]).

In [14, §8] the Clifford algebra $C(A, \sigma_\theta, f_\theta)$ of the triple $(A, \sigma_\theta, f_\theta)$ is defined as the quotient of the tensor algebra $T(A)$ of the F-space A by the ideal generated by the sets

$$\begin{aligned} J_1 &= \{s - \mathrm{Trd}_A(us),\ s \in \mathrm{Sym}(A, \sigma_\theta)\}, \\ J_2 &= \{c - \mathrm{Sand}'(c, u),\ c \in A \otimes A \text{ with } \mathrm{Sand}'\big(c, \mathrm{Alt}(A, \sigma_\theta)\big) = 0\}, \end{aligned}$$

where $u = \widehat{b_\theta}^{-1}\widehat{k}$ and $\mathrm{Sand}' : (A \otimes A, A) \to A$ is defined as $\mathrm{Sand}'(a \otimes b, x) = axb$. The two definitions give in fact isomorphic algebras:

Proposition 3.1.1. *The isomorphism* $\varphi_\theta : V \otimes_D {}^\sigma V \xrightarrow{\sim} \mathrm{End}_D(V)$ *induces an isomorphism* $\mathrm{Cl}(V, \theta) \xrightarrow{\sim} C(A, \sigma_\theta, f_\theta)$.

Proof. We shall check that φ_θ maps I_i to J_i ($i = 1, 2$). By definition of τ_θ and S_1, $s = \varphi_\theta(s_1)$ is a symmetric element of A. On the other hand we have by definition of the pairing $B' \times B \to A$,

$$\begin{aligned} \mathrm{Trd}_A(s_1 k) &= \mathrm{Trd}_A\big(\varphi_\theta(s_1)\psi_\theta^{-1}(k)\big) \\ &= \mathrm{Trd}_A\big(sh^{-1}\widehat{k}\big) = \mathrm{Trd}_A(su) = \mathrm{Trd}_A(us), \end{aligned}$$

hence φ_θ maps I_1 to J_1. Let $c_1 = (x \otimes y) \otimes (v \otimes w)$ be a pure tensor in $B' \otimes B'$. Then (see [21, 4.1]) $\mathrm{Sand}(c_1, k) = xk(v, y) \otimes w := r$. On the other hand

$$\mathrm{Sand}'\big((x \otimes h(y)) \otimes (v \otimes h(w)), u\big) = xk(v, y) \otimes h(w) = \varphi_\theta(r),$$

hence φ_θ maps I_2 to J_2. $\square$

In particular we have $C\big(\mathrm{End}_F(V), \sigma_q, f_q\big) = C_0(V, q)$ for a quadratic space (V, q) over F. It is convenient to use both definitions of the Clifford algebra of a generalized quadratic space.

Corollary 3.1.2. *Let θ and θ' be generalized quadratic forms with $\theta' = \lambda\theta$ for $\lambda \in F^\times$. Then*

$$\mathrm{Cl}(V, \theta) \xrightarrow{\sim} \mathrm{Cl}(V, \theta').$$

Proof. Since $\theta' = \lambda\theta$, it follows from 1.5.1 that $(\sigma_\theta, f_\theta) = (\sigma_{\theta'}, f_{\theta'})$. Hence

$$\mathrm{Cl}(V, \theta) \xrightarrow{\sim} C(\mathrm{End}_D(V), \sigma_\theta, f_\theta) \xrightarrow{\sim} C(\mathrm{End}_D(V), \sigma_{\theta'}, f_{\theta'}) \xrightarrow{\sim} \mathrm{Cl}(V, \theta').$$

□

3.2 The Clifford Algebra of a Quadratic Quaternion Form

Let $D = [K, \mu) = K \oplus \ell K$ be a quaternion algebra with conjugation σ. Let V be a D-module and let V^0 be V as a right vector space over K (through restriction of scalars). Let $T : V^0 \to V^0$, $Tx = x\ell$. We have $\mathrm{End}_D(V) \subset \mathrm{End}_K(V^0)$ and

$$\mathrm{End}_D(V) = \{f \in \mathrm{End}_K(V^0) \mid fT = Tf\}.$$

Let θ be a quaternion quadratic form and let $q_\theta(x) = -\rho r(T(x), x) + \ell R(x) + F$, as in 2.2.1. Then $R : V^0 \longrightarrow K$ is a nondegenerate quadratic form.

Proposition 3.2.1. *We have $\sigma_R|_{\mathrm{End}_D(V)} = \sigma_\theta$ and $f_\theta = f_R|_{\mathrm{End}_D(V)}$.*

Proof. We have an embedding $D \hookrightarrow M_2(K)$, $a + \ell b \mapsto \begin{pmatrix} a & \mu\overline{b} \\ b & \overline{a} \end{pmatrix}$ and conjugation given by $x \mapsto x^* = c^{-1}x^t c$, $c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The choice of a basis of V over D identifies V with D^n, V^0 with K^{2n}, $\mathrm{End}_D(V)$ with $M_n(D)$ and $\mathrm{End}_K(V^0)$ with $M_{2n}(K)$, where $n = \dim_D V$. We further identify V and V^* through the choice of the dual basis. We embed any element $x = x_1 + \ell x_2 \in M_{k,l}(D)$, $x_i \in M_{k,l}(K)$ in $M_{2k,2l}(K)$ through the map $\iota : x \mapsto \xi = \begin{pmatrix} x_1 & \mu\overline{x_2} \\ x_2 & \overline{x_1} \end{pmatrix}$. In particular D^n is identified with a subspace of the space of $(2n \times 2)$-matrices over K. Then $D \subset M_2(K)$ operates on the right through (2×2)-matrices and $M_n(D) \subset M_{2n}(K)$ operates on the left through $(2n \times 2n)$-matrices. With the notations of Example (1.4.4) we have $\iota(x^*) = \mathrm{Int}(c^{-1})(x^t)$. Any D-sesquilinear form k on D^n can be written as $k(x,y) = x^*ay$, where $a \in M_n(D)$, as in (1.4.4). Let $a = a_1 + \ell a_2$, $a_i \in M_n(K)$ and let

$$\alpha = \iota(a) = \begin{pmatrix} a_1 & \mu\overline{a_2} \\ a_2 & \overline{a_1} \end{pmatrix}.$$

Let $\eta = \iota(y)$, $y = y_1 + \ell y_2$. We have

$$k(x,y) = x^*ay = \xi^*\alpha\eta = \begin{pmatrix} x_1 & \mu\overline{x_2} \\ x_2 & \overline{x_1} \end{pmatrix}^* \begin{pmatrix} a_1 & \mu\overline{a_2} \\ a_2 & \overline{a_1} \end{pmatrix} \begin{pmatrix} y_1 & \mu\overline{y_2} \\ y_2 & \overline{y_1} \end{pmatrix}.$$

On the other side it follows from $q_\theta(x) = -\rho r(T(x),x) + \ell R(x) + F$ that $R(x) = \xi^t\rho\xi$ with

$$\rho = \begin{pmatrix} a_2 & \overline{a_1} \\ -a_1 & -\mu\overline{a_2} \end{pmatrix}.$$

Assume that $\theta = [k]$, so that σ_θ corresponds to the involution $\mathrm{Int}(\gamma^{-1}) \circ *$, where $\gamma = \alpha - \alpha^*$. Similarly σ_R corresponds to the involution $\mathrm{Int}(\widetilde{\rho}^{-1}) \circ t$, where $\widetilde{\rho} = \rho + \rho^t$. We obviously have $\rho = c\alpha$ with $c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so that $\rho^t = \alpha^tc^t = -\alpha^tc = -c\alpha^*$ and $\rho + \rho^t = c(\alpha - \alpha^*)$ or $c\gamma = \widetilde{\rho}$. Now $^* = \mathrm{Int}(c^{-1}) \circ t$ implies $\sigma_R|_{M_n(D)} = \sigma_\theta$. We finally check that $f_\theta = f_R|_{\mathrm{Sym}(M_n(D),\sigma_\theta)}$. We have $f_\theta(s) = \mathrm{Trd}_{M_n(D)}(\gamma^{-1}\alpha s)$ and $f_R(s) = \mathrm{Trd}_{M_{2n}(K)}(\widetilde{\rho}^{-1}\rho s)$, hence the claim, since $\rho = c\alpha$ and $\widetilde{\rho} = c\gamma$ implies $\gamma^{-1}\alpha = \widetilde{\rho}^{-1}\rho$. □

Corollary 3.2.2. *The embedding* $\mathrm{End}_D(V) \hookrightarrow \mathrm{End}_K(V^0)$ *induces*
1) an isomorphism $\big(\mathrm{End}_D(V), \sigma_\theta, f_\theta\big) \otimes K \xrightarrow{\sim} \big(\mathrm{End}_K(V^0), \sigma_R, f_R\big)$,
2) an isomorphism $C\big(\mathrm{End}_D(V), \sigma_\theta, f_\theta\big) \otimes K \xrightarrow{\sim} C_0(V^0, R)$.

Corollary 3.2.3. *Let* $\theta \in \mathrm{Q}_\gamma^{-1}(V,D)$ *be a quadratic quaternion form and assume that* $\dim_D(V) = n$. *Then* $\dim_F \mathrm{Cl}(V,\theta) = 2^{2n-1}$.

The semilinear automorphism $T : V^0 \xrightarrow{\sim} V^0$, $T(x) = x\ell$, is a semilinear similitude with multiplier $-\mu$ of the quadratic form R, such that $T^2 = \mu$.

Lemma 3.2.4. *Any semilinear similitude* $t : V^0 \longrightarrow V^0$ *with multiplier* $\lambda \in F^\times$ *induces a semilinear automorphism* $C_0(t)$ *of the even Clifford algebra* $C_0(V^0, R)$ *such that*

$$C_0(t)(xy) = \lambda^{-1}t(x)t(y) \text{ for } x,\ y \in V^0.$$

In particular the map T induces a semilinear automorphism $C_0(T)$ of the even Clifford algebra $C_0(V^0, R)$ such that

$$C_0(T)(xy) = (-\mu)^{-1}T(x)T(y) \text{ for } x,\, y \in V^0,$$

and $C_0(T)^2 = Id$.

Proof. This follows (for example) as in [14, (13.1)]. ☐

We prove a proposition in the spirit of [11].

Proposition 3.2.5.

$$C\big(\mathrm{End}_D(V), \sigma_\theta, f_\theta\big) = \{c \in C_0(V^0, R) \mid C_0(T)(c) = c\}.$$

Proof. The claim follows from the defining relations of $C\big(\mathrm{End}_D(V), \sigma_\theta, f_\theta\big)$, given by J_1, J_2, which are subspaces of the tensor algebra $T(\mathrm{End}_D(V))$ and the fact that

$$\mathrm{End}_D(V) = \{f \in \mathrm{End}_K(V^0) \mid T^{-1}fT = f\}.$$

☐

We call $C\big(\mathrm{End}_D(V), \sigma_\theta, f_\theta\big)$ or equivalently $\mathrm{Cl}(V, \theta)$ the *(generalized even) Clifford algebra of the quadratic quaternion space* (V, θ).

3.3 The Discriminant

We define the discriminant of a quadratic pair (as in [14, (7), Chap. II]) and the discriminant of a generalized quadratic form. Then we compare the two notions. The cases char $\neq 2$ and char $= 2$ must be treated separately. We begin with the discriminant of a quadratic pair.

First, assume that the characteristic is different from 2. Let σ be an orthogonal involution on a central simple algebra A of even degree $n = 2m$ over F. The *determinant* of σ is the square class of the reduced norm of any alternating unit:

$$\det(\sigma) := \mathrm{Nrd}_A(a) \cdot F^{\times 2} \in F^\times / F^{\times 2} \text{ for } a \in \mathrm{Alt}(A, \sigma) \cap A^\times,$$

and the *discriminant* of σ is the signed determinant:

$$\operatorname{disc}(\sigma) := (-1)^m \det(\sigma) \in F^\times / F^{\times 2}.$$

Now assume that char$F = 2$. Let (σ, f) be a quadratic pair on a central simple algebra A of even degree $n = 2m$ over F. By 1.3.1, there exists an element $l \in A$ such that $f(ls) = \operatorname{Trd}_A(ls)$ for all $s \in \operatorname{Sym}(A, \sigma)$. Let $\wp(x) := x^2 + x$ for $x \in F$. The *determinant* of (σ, f) is

$$\det(\sigma, f) := \operatorname{Srd}_A(l) + \wp(F) \in F/\wp(F),$$

and the *discriminant* of (σ, f) is

$$\operatorname{disc}(\sigma, f) := \det(\sigma, f) + \frac{m(m-1)}{2} \in F/\wp(F).$$

The discriminant describes the center of the Clifford algebra (see [14]).

Proposition 3.3.1. *Let (σ, f) be a quadratic pair on a central simple algebra A of even degree $n = 2m$ over F. Let Z be the center of $C\big(A, \sigma, f\big)$.*
1) If char$F \neq 2$, *$Z \simeq F[X]/(X^2 - \delta)$ where $\delta \in F^\times$ is a representative of the discriminant* disc(σ).
2) If char$F = 2$, *$Z \simeq F[X]/(X^2 + X + \delta)$ where $\delta \in F$ is a representative of the discriminant* disc(σ, f).

Let $\theta = [k]$ be a nonsingular (σ, ε)-quadratic form on the D-vector space V. First, assume that the characteristic is different from 2 and that σ_θ is an orthogonal involution on the central simple algebra $A = \operatorname{End}_D(V)$ of even degree $n = 2m$. The *discriminant* of θ is

$$\operatorname{disc}(\theta) := (-1)^m \operatorname{Nrd}_A(\widehat{a}^{-1}\widehat{k}) \in F^\times / F^{\times 2},$$

where a is an invertible element in $\operatorname{Alt}^\varepsilon_\sigma(V, D)$.
Now suppose that the characteristic is 2 and that $(\sigma_\theta, f_\theta)$ is a quadratic pair. The *Arf invariant* of $\theta = [k]$ is

$$\operatorname{Arf}(\theta) := \operatorname{Srd}_A(\widehat{b_\theta}^{-1}\widehat{k}) + \frac{m(m-1)}{2} \in F/\wp(F).$$

In [21] the center of $\operatorname{Cl}(V, \theta)$ is described with the discriminant and the Arf invariant.

Proposition 3.3.2. *Let $\theta = [k]$ be a nonsingular (σ,ε)-quadratic form on the D-vector space V. Let $(\sigma_\theta, f_\theta)$ be a quadratic pair on the central simple algebra $A = \mathrm{End}_D(V)$ of even degree $n = 2m$ over F. Let Z be the center of* $\mathrm{Cl}(V,\theta)$.
1) If $\mathrm{char} F \neq 2$, *$Z \simeq F[X]/(X^2 - \delta)$ where $\delta \in F^\times$ is a representative of the discriminant* $\mathrm{disc}(\theta)$.
2) If $\mathrm{char} F = 2$, *$Z \simeq F[X]/(X^2 + X + \delta)$ where $\delta \in F$ is a representative of the Arf invariant* $\mathrm{Arf}(\theta)$.

Corollary 3.3.3. *Let θ be a nonsingular (σ,ε)-quadratic form on V and let $(\sigma_\theta, f_\theta)$ be the corresponding quadratic pair on* $\mathrm{End}_D(V)$. *Then* $\mathrm{disc}(\theta) = \mathrm{disc}(\sigma_\theta)$ *if* $\mathrm{char} F \neq 2$ *and* $\mathrm{Arf}(\theta) = \mathrm{disc}(\sigma_\theta, f_\theta)$ *if* $\mathrm{char} F = 2$.

Proof. From 3.1.1 we have $\mathrm{Cl}(V,\theta) \xrightarrow{\sim} C(A, \sigma_\theta, f_\theta)$. In particular, the centers are isomorphic. The assertion follows from the description of the centers given above. □

Finally we assume that θ is a quadratic quaternion form on the vector space V over $D = [K, \mu)$. In this case we may compute the discriminant as in the case of classical quadratic forms, with orthogonal bases (see [20, p.346]): let $e_1, \dots, e_n$ be an orthogonal (with respect to b_θ) basis of V. We use the notation $N(x) := x\overline{x}$ for $x \in D$. If $\mathrm{char} F \neq 2$, we have

$$\mathrm{disc}(\theta) = (-1)^n \prod_{i=1}^{n} N\big(h(e_i, e_i)\big) \cdot F^{\times 2} \in F^\times / F^{\times 2};$$

if $\mathrm{char} F = 2$, we have

$$\mathrm{Arf}(\theta) = \sum_{i=1}^{n} N\big(h(e_i, e_i)^{-1} q_\theta(e_i)\big) + \wp(F) \in F/\wp(F).$$

We compare the discriminant (resp. the Arf invariant) of θ with the one of R.

Corollary 3.3.4. *If* $\mathrm{char} F \neq 2$, *then*

$$\mathrm{disc}(\theta) K^{\times 2} = \mathrm{disc}(R).$$

If $\mathrm{char} F = 2$, *then*

$$\mathrm{Arf}(\theta) + \wp(K) = \mathrm{Arf}(R).$$

Proof. The assertion follows from $C\big(\mathrm{End}_D(V), \sigma_\theta, f_\theta\big) \otimes K \xrightarrow{\sim} C_0(V^0, R)$, because the center Z of $C\big(\mathrm{End}_D(V), \sigma_\theta, f_\theta\big)$ tensored with K is isomorphic to the center of $C_0(V^0, R)$. □

Chapter 4

Triality

4.1 Triality for Semilinear Similitudes

Let t be a semilinear similitude of a quadratic space (U, q) of even dimension over K. Assume that $\operatorname{disc}(q)$ is trivial, so that the even Clifford algebra $C_0(U, q)$ of (U, q) decomposes as product of two K-algebras $C^+(U, q)$ and $C^-(U, q)$. We say that t is *proper* if $C_0(t)\big(C^{\pm}(U, q)\big) \subset C^{\pm}(U, q)$ and we say that t is *improper* if $C_0(t)\big(C^{\pm}(U, q)\big) \subset C^{\mp}(U, q)$. For any semilinear similitude t, let $d(t) = 1$ is t if proper and $d(t) = -1$ if t is improper.

Lemma 4.1.1. *Let t_i be a semilinear similitude of (U_i, q_i), with multiplier μ_i $i = 1, 2$. Suppose that $\mu_1 = \mu_2$. Then we have $d(t_1 \perp t_2) = d(t_1)d(t_2)$.*

Proof. We assume that $\operatorname{disc}(q_i)$, $i = 1, 2$, is trivial. Let e_i be an idempotent generating the center Z_i of $C_0(q_i)$. We have $C_0(t_i)(e_i) = e_i$ if t_i is proper and $C_0(t_i)(e_i) = 1 - e_i$ if t_i is improper. The idempotent $e = e_1 + e_2 - 2e_1e_2 \in C_0(q_1 \perp q_2)$ generates the center of $C_0(q_1 \perp q_2)$ (see for example [13, (2.3), Chap. IV]) and the claim follows by case checking. ☐

Let $\mathfrak{C}$ be a Cayley algebra over F with conjugation $\pi : x \mapsto \overline{x}$ and norm $\mathfrak{n}\colon x \mapsto x\overline{x}$. The new multiplication $x \star y = \overline{x}\,\overline{y}$ satisfies

$$x \star (y \star x) = (x \star y) \star x = \mathfrak{n}(x)y \tag{4.1}$$

for x, $y \in \mathfrak{C}$. Further, the polar form $b_{\mathfrak{n}}$ is *associative* with respect to $\star$, in the sense that

$$b_{\mathfrak{n}}(x \star y, z) = b_{\mathfrak{n}}(x, y \star z).$$

Proposition 4.1.2. *For* x, $y \in \mathfrak{C}$, *let* $r_x(y) = y \star x$ *and* $\ell_x(y) = x \star y$. *The map* $\mathfrak{C} \to \operatorname{End}_F(\mathfrak{C} \oplus \mathfrak{C})$ *given by*

$$x \mapsto \begin{pmatrix} 0 & \ell_x \\ r_x & 0 \end{pmatrix}$$

induces isomorphisms $\alpha\colon \big(C(\mathfrak{C}, \mathfrak{n}), \tau\big) \xrightarrow{\sim} \big(\operatorname{End}_F(\mathfrak{C} \oplus \mathfrak{C}), \sigma_{\mathfrak{n}\perp\mathfrak{n}}\big)$ *and*

$$\alpha_0\colon \big(C_0(\mathfrak{C}, \mathfrak{n}), \tau_0\big) \xrightarrow{\sim} \big(\operatorname{End}_F(\mathfrak{C}), \sigma_{\mathfrak{n}}\big) \times \big(\operatorname{End}_F(\mathfrak{C}), \sigma_{\mathfrak{n}}\big), \tag{4.2}$$

of algebras with involution.

Proof. See [14]. ☐

Assume from now on that $\mathfrak{C}$ is defined over a field K which is quadratic Galois over F. Any proper semilinear similitude t of $\mathfrak{n}$ induces a semilinear automorphism $C_0(t)$ of the even Clifford algebra $\big(C_0(\mathfrak{C}, \mathfrak{n}), \tau_0\big)$, which does not permute the two components of the center of $C_0(\mathfrak{C}, \mathfrak{n})$. Thus $\alpha_0 \circ C_0(t) \circ \alpha_0^{-1}$ is a pair of semilinear automorphisms of $\big(\operatorname{End}_K(\mathfrak{C}), \sigma_{\mathfrak{n}}\big)$. It follows as in (1.5.2) that, for any quadratic space (V, q), semilinear automorphisms of $(\operatorname{End}_K(V), \sigma_q, f_q)$ are of the form $f \mapsto gfg^{-1}$, where g is a semilinear similitude of q.

Proposition 4.1.3. *For any proper semilinear similitude* t_1 *of* $\mathfrak{n}$ *with multiplier* μ_1, *there exist proper semilinear similitudes* t_2, t_3 *with multipliers* μ_2, μ_3 *respectively, such that*

$$\alpha_0 \circ C_0(t_1) \circ \alpha_0^{-1} = \big(\operatorname{Int}(t_2), \operatorname{Int}(t_3)\big)$$

and

$$\begin{aligned} \mu_3^{-1} t_3(x \star y) &= t_1(x) \star t_2(y), \\ \mu_1^{-1} t_1(x \star y) &= t_2(x) \star t_3(y) \\ \mu_2^{-1} t_2(x \star y) &= t_3(x) \star t_1(y). \end{aligned} \tag{4.3}$$

Let t_1 *be an improper semilinear similitude with multiplier* μ_1. *There exist improper semilinear similitudes* t_2, t_3 *with multipliers* μ_2, μ_3 *respectively, such that*

$$\begin{aligned} \mu_3^{-1} t_3(x \star y) &= t_1(y) \star t_2(x), \\ \mu_1^{-1} t_1(x \star y) &= t_2(y) \star t_3(x) \\ \mu_2^{-1} t_2(x \star y) &= t_3(y) \star t_1(x). \end{aligned}$$

The pair (t_2, t_3) *is determined by* t_1 *up to a factor* (λ, λ^{-1}), $\lambda \in K^\times$, *and we have* $\mu_1\mu_2\mu_3 = 1$. *Furthermore, any of the formulas in* (4.3) *implies the two others.*

Proof. The proof given in [14, (35.4)] for similitudes can also be used for semilinear similitudes. □

Remark 4.1.4. The class of two of the t_i, $i = 1, 2, 3$, modulo $K^\times$ is uniquely determined by the class of the third t_i.

Corollary 4.1.5. *Let T_1 be a proper semilinear similitude of $(\mathfrak{C}, \mathfrak{n})$ such that $T_1^2 = \mu_1$, $\mu_1 \in K^\times$ and with multiplier $-\mu_1$. There exist elements $a_i \in K^\times$, $i = 1, 2, 3$, and proper semilinear similitudes T_i of $(\mathfrak{C}, \mathfrak{n})$, with $T_i^2 = \mu_i$, $\mu_i \in K^\times$ and with multiplier $-\mu_i$, $i = 2, 3$, such that $a_i\overline{a_i}\mu_i = \mu_{i+1}\mu_{i+2}$ and*

$$\begin{aligned} a_3T_3(x \star y) &= T_1(x) \star T_2(y), \\ a_1T_1(x \star y) &= T_2(x) \star T_3(y), \\ a_2T_2(x \star y) &= T_3(x) \star T_1(y). \end{aligned}$$

The class of any T_i modulo $K^\times$ determines the two other classes and the μ_i's are determined up to norms from $K^\times$. Furthermore any of the three formulas determines the two others.

Proof. Counting indices modulo 3, we have relations

$$T_i(x) \star T_{i+1}(y) = b_{i+2}T_{i+2}(x \star y), \quad b_i \in K^\times,$$

in view of (4.1.3). Since T_1^2 is central, it follows that T_i^2 $(i = 2, 3)$ are central. If we replace all T_j by $T_j \circ \rho_{\nu_j}$, $\nu_j \in K^\times$, we get new constants a_i. The claim then follows from (2.2.3). □

Remark 4.1.6. In the situation of the corollary above, if

$$a_iT_i(x \star y) = T_{i+1}(x) \star T_{i+2}(y),$$

then $\overline{a_i} = -a_i$: taking the norm, we obtain $a_i^2\mu_i = -\mu_{i+1}\mu_{i+2}$ and together with $a_i\overline{a_i}\mu_i = \mu_{i+1}\mu_{i+2}$, we get $\overline{a_i} = -a_i$.

4.2 Triality for Quadratic Quaternion Forms

Let $D_1 = K \oplus \ell_1 K = [K, \mu_1)$ be a quaternion algebra over F and let $\theta_1 \in \mathrm{Q}_\gamma^{-1}(V_1, D_1)$ be a nonsingular quaternion quadratic form of dimension 4 over D_1. Let $q_{\theta_1}(x) = -\rho r_1(T_1(x), x) + \ell R_1(x) + F$ as in 2.2.1, so that R_1 is a 8-dimensional (classical) quadratic form on V_1^0 over K. The map $T_1 : V_1^0 \to V_1^0$,

$T_1(x) = x\ell_1$, is a (proper, by 4.1.1) semilinear similitude of $\big(V_1^0, R_1\big)$ with multiplier $-\mu_1$ and such that $T_1^2 = \mu_1$. We assume from now on that the quadratic form R_1 is the norm form $\mathfrak{n}$ of a Cayley algebra $\mathfrak{C}$ over K. In view of (4.1.5) T_1 induces two semilinear similitudes T_2, resp. T_3, with multipliers μ_2, resp. μ_3, which in turn define a quaternion quadratic space (V_2, θ_2) of dimension 4 over $D_2 = [K, \mu_2)$, resp. a quaternion quadratic space (V_3, θ_3) of dimension 4 over $D_3 = [K, \mu_3)$: we set $V_i^0 = V_1^0$, we define a D_i-vector space structure on V_i^0 by $x\ell_i = T_i(x)$ and we denote the new D_i-vector space by V_i, $i = 2, 3$; the quadratic quaternion forms are defined by

$$q_{\theta_i}(x) = -\rho\, r_1(T_i(x), x) + \ell_i R_1(x) + F,\ \ i = 2, 3.$$

Theorem 4.2.1. *1)* $[D_1][D_2][D_3] = 1 \in \operatorname{Br}(F)$.
2) The restriction of $\alpha_0 : C_0\big(\mathfrak{C}, \mathfrak{n}\big) \xrightarrow{\sim} \operatorname{End}_K(\mathfrak{C}) \times \operatorname{End}_K(\mathfrak{C})$ *to* $Cl(V_i, \theta_i)$ *induces isomorphisms*

$$\alpha_i : \big(Cl(V_i, \theta_i), \tau\big) \xrightarrow{\sim} \big(\operatorname{End}_{D_{i+1}}(V_{i+1}), \sigma_{\theta_{i+1}}\big) \times \big(\operatorname{End}_{D_{i+2}}(V_{i+2}), \sigma_{\theta_{i+2}}\big).$$

Proof. The first claim follows from the fact that $\mu_1\mu_2 = \mu_3 \operatorname{Nrd}_{D_3}(a_3)$. In fact it is a particular case of [14, 42.7]. We compute $\big(Cl(V_1, \theta_1), \tau\big)$. We use the following fact (see (3.2.5)):

$$Cl(V_1, \theta_1) \simeq C\big(\operatorname{End}_{D_1}(V_1), \sigma_{\theta_1}, f_{\theta_1}\big) = \{c \in C_0(V_1^0, R_1) \mid C_0(T_1)(c) = c\}.$$

The image of the algebra with involution $(C_0(V^0, R_1), \tau_0) = (C_0(\mathfrak{C}, \mathfrak{n}), \tau_0)$ under α_0 is $\big(\operatorname{End}_K(\mathfrak{C}), \sigma_{\mathfrak{n}}\big) \times \big(\operatorname{End}_K(\mathfrak{C}), \sigma_{\mathfrak{n}}\big)$ (see 4.1.2). Moreover (see 4.1.3), we have $\alpha_0 \circ C_0(T_1) \circ \alpha_0^{-1} = \big(\operatorname{Int}(T_2), \operatorname{Int}(T_3)\big)$. Since for $i = 1, 2, 3$ we have $\operatorname{End}_{D_i}(V_i) = \{f \in \operatorname{End}_K(\mathfrak{C}) \mid T_i^{-1} f T_i = f\}$ and $\sigma_{\mathfrak{n}}|_{\operatorname{End}_{D_i}(V_i)} = \sigma_{\theta_i}$ (see 3.2.1), the proof of 2) is completed. □

We call a triple of quadratic quaternion forms $(\theta_1, \theta_2, \theta_3)$ over a triple of vector spaces (V_1, V_2, V_3) over a triple of quaternion algebras (D_1, D_2, D_3) with an isomorphism

$$\big(C(V_1, \theta_1), \tau\big) \xrightarrow{\sim} \big(\operatorname{End}_{D_2}(V_2), \sigma_{\theta_2}\big) \times \big(\operatorname{End}_{D_3}(V_3), \sigma_{\theta_3}\big) \tag{4.4}$$

a *quaternionic trialitarian triple.* In the preceding theorem the quadratic quaternion forms are defined by

$$q_{\theta_i}(x) = -\rho\, r(T_i(x), x) + \ell_i R(x) + F,$$

where R is the norm form $\mathfrak{n}$ of a Cayley algebra $\mathfrak{C}$ over K and $\rho \in K^\times$ is such that $\rho + \bar\rho = 1$. This fact is not peculiar, as the next theorem shows.

Theorem 4.2.2. *Let $(\theta_1, \theta_2, \theta_3)$ be a quaternionic trialitarian triple. Then there exists a quadratic field extension K of F, $K \subset D_i$ ($i = 1, 2, 3$) such that the following holds. Let V_i^0 be the vector space V_i viewed as K-vector space, $i = 1, 2, 3$. Then we may identify V_1^0, V_2^0 and V_3^0 with the K-vector space structure of a Cayley algebra $\mathfrak{C}$ over K with norm form $\mathfrak{n}$, such that in the decomposition of 2.2.1,*

$$q_{\theta_i}(x) = -\rho\ r(T_i(x), x) + \ell_i R(x) + F,\ \ i = 1, 2, 3,$$

we have $R = \mathfrak{n}$ (r is the polarization of R and $\rho \in K^\times$ with $\rho + \bar{\rho} = 1$).

Proof. The fact that

$$\bigl(C(V_i, \theta_i), \tau\bigr) \xrightarrow{\sim} \bigl(\mathrm{End}_{D_{i+1}}(V_{i+1}), \sigma_{\theta_{i+1}}\bigr) \times \bigl(\mathrm{End}_{D_{i+2}}(V_{i+2}), \sigma_{\theta_{i+2}}\bigr)$$

implies, by [14, (42.7)], that $[D_1][D_2][D_3] = 1 \in \mathrm{Br}(F)$. Hence, by [14, (16.30)], we may assume that there is a quadratic extension K of F contained in D_1, D_2 and D_3. We have (see 2.2.1) that

$$q_{\theta_i} = -\rho\ r_i(T_i(x), x) + \ell_i R_i(x) + F.$$

From 3.2.2, tensoring 4.4 with K, it follows that R_i has trivial discriminant and trivial Clifford invariant. Hence, by [14, (35.2)], R_i is similar to the norm form $\mathfrak{n}_i$ of a Cayley algebra $\mathfrak{C}_i$ over K. Replacing ℓ_i by a multiple (which does not change the isometry class of D_i), we may assume that $R_i = \mathfrak{n}_i$. By triality (see [14, (42.7)]), we may take $\mathfrak{n}_1 = \mathfrak{n}_2 = \mathfrak{n}_3$. □

The fact that $[D_1][D_2][D_3] = 1 \in \mathrm{Br}(F)$ is not only necessary, but also sufficient.

Theorem 4.2.3. *Let D_i, $i = 1, 2, 3$, be quaternion algebras over F such that $[D_1][D_2][D_3] = 1 \in \mathrm{Br}(F)$. Then there exist a quaternionic trialitarian triple $(\theta_1, \theta_2, \theta_3)$ corresponding to the three given quaternion algebras. As matrix of θ_i we can choose*

$$-u\rho \mathrm{diag}(1, 1, 1, \mu_{i+1}^{-1}),$$

for $u \in K^\times$ such that $\overline{u} = -u$.

Proof. We give a constructive proof. We may assume that the D_i contain a common separable quadratic field K and that $D_i = [K, \mu_i)$, $\mu_i \in F^\times$ with $\mu_1\mu_2\mu_3 \in F^{\times 2}$. We may also assume that $\mu_1\mu_2\mu_3 = 1$. Let $\mathfrak{C}_s$ be the split

Cayley algebra over F, with basis $(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8)$ and multiplication table

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
u_1	0	u_7	$-u_6$	u_1	$-u_8$	0	0	0
u_2	$-u_7$	0	u_5	u_2	0	$-u_8$	0	0
u_3	u_6	$-u_5$	0	u_3	0	0	$-u_8$	0
u_4	0	0	0	u_4	u_5	u_6	u_7	0
u_5	$-u_4$	0	0	0	0	u_3	$-u_2$	u_5
u_6	0	$-u_4$	0	0	$-u_3$	0	u_1	u_6
u_7	0	0	$-u_4$	0	u_2	$-u_1$	0	u_7
u_8	u_1	u_2	u_3	0	0	0	0	u_8

We have that $1 = u_4 + u_8$ and the conjugation map π is given by $\pi(u_i) = -u_i$ for $i \neq 4, 8$ and $\pi(u_4) = u_8$. The sets $\{u_1, \dots, u_4\}$ and $\{u_5, \dots, u_8\}$ span complementary totally isotropic subspaces of $\mathfrak{C}_s$ and $b_{\mathfrak{n}}(u_i, u_j) = \delta_{i+4,j}$ (indices modulo 8). We define the diagonal matrices

$$A_i = \operatorname{diag}(\mu_i, \mu_i, \mu_i, \mu_{i+2}^{-1}), \quad B_i = \operatorname{diag}(1, 1, 1, \mu_{i+1}^{-1})$$

in $M_4(F)$ and let $S_i = \begin{pmatrix} 0 & A_i \\ B_i & 0 \end{pmatrix} \in M_8(F)$, $i = 1, 2, 3$. We also write S_i for the F-automorphism of $\mathfrak{C}_s$ induced by S_i with respect to the basis $(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8)$. Then (see [14, (43.12)]) S_i is a similitude of $\mathfrak{C}_s$ over F with multiplier μ_i such that 1) $\mu_3^{-1} S_3(x \star y) = S_1(x) \star S_2(y)$ and 2) $S_i^2 = \mu_i$. The verification of the two properties above is a straightforward calculation. Let $\mathfrak{C} = K \otimes \mathfrak{C}_s$. We denote the norm form on $\mathfrak{C}$ with R. The quadratic form R is hyperbolic: this is clear by construction. Let $u \in K^\times$ be such that $\overline{u} = -u$. The semilinear similitudes $T_i(k \otimes x) = u\overline{k} \otimes S_i(x)$, $i = 1$, 2, 3, satisfy

$$a_3 T_3(x \star y) = T_1(x) \star T_2(y)$$

with $a_3 = u\mu_3^{-1}$ (we use the same notation $\star$ in $\mathfrak{C}_s$ and in $\mathfrak{C}$). We have that $T_i^2(x) = -u^2 \mu_i x$, hence, since $-u^2 = u\overline{u}$ is a norm, we can define an action of D_i on $\mathfrak{C}$. We denote $\mathfrak{C}$ viewed as D_i-vector space by V_i. The vectors $1 \otimes u_1, \dots, 1 \otimes u_4$ form a basis $\mathfrak{B}$ of V_i. We have that $q_{\theta_i}(x) = -\rho\ r(T_i(x), x) + \ell_i R(x) + F$ and $b_{\theta_i}(x, y) = -r(T_i(x), y) + \ell_i r(x, y)$. We consider the quadratic quaternion forms θ_i defined by $(\mathfrak{C}, R, T_i)$. The basis $\mathfrak{B}$ is orthogonal with respect to b_{θ_i}. Actually, the matrix of θ_i with respect to $\mathfrak{B}$ is

$$L_i = -u\rho\mathrm{diag}(1, 1, 1, \mu_{i+1}^{-1}).$$

The triple $(\theta_1, \theta_2, \theta_3)$ is a quaternionic trialitarian triple, as desired. Moreover, the exact sequence of Lewis confirms us that L_i comes from a quadratic form, because R is hyperbolic. ☐

We summarize the assertions of the theorems above in a corollary.

Corollary 4.2.4. *Let D_1, D_2 and D_3 be quaternion algebras over F. The following statements are equivalent:*
1) *There is a quaternionic trialitarian triple $(\theta_1, \theta_2, \theta_3)$ with $\theta_i \in \mathrm{Q}_\gamma^{-1}(V_i, D_i)$, for some D_i-vector spaces V_i $(i = 1, 2, 3)$.*
2) $[D_1][D_2][D_3] = 1 \in \mathrm{Br}(F)$.
3) *There is a quadratic extension K of F which splits D_1, D_2 and D_3.*
4) *There is a quadratic extension K of F in D_1 and a quadratic quaternion form θ_1 given by (R, T_1), such that R is the norm form $\mathfrak{n}$ of a Cayley algebra $\mathfrak{C}$ over K and the induced semilinear similitudes T_2 and T_3 define two quaternion algebras isomorphic to D_2 and D_3, respectively.*

Proof. $1 \Rightarrow 2$: this is the first assertion of 4.2.1.
$2 \Rightarrow 3$: this is a classical result due to Albert, see for example [14, 16.30].
$3 \Rightarrow 4$: this follows from the proof of 4.2.3.
$4 \Rightarrow 1$: this is the second part of 4.2.1.

☐

Let $\alpha_1, \dots, \alpha_n \in K$. If $\mathrm{char}K \neq 2$ we denote by $< \alpha_1, \dots, \alpha_n >$ the diagonal quadratic form

$$< \alpha_1, \dots, \alpha_n > := \alpha_1 x_1^2 + \dots + \alpha_n x_n^2.$$

We also define the *n-fold Pfister quadratic form* $\ll \alpha_1, \dots, \alpha_n \gg$ by

$$\ll \alpha_1, \dots, \alpha_n \gg := < 1, -\alpha_1 > \otimes \dots \otimes < 1, -\alpha_n > .$$

If $\mathrm{char}K = 2$, the quadratic forms $[\alpha_1, \alpha_2]$ and $[\alpha_1]$ are defined by

$$[\alpha_1, \alpha_2] := \alpha_1 x_1^2 + x_1 x_2 + \alpha_2 x_2^2 \quad \text{and} \quad [\alpha_1] := \alpha_1 x_1^2,$$

and the *n-fold Pfister quadratic form* $\ll \alpha_1, \dots, \alpha_n]]$ by

$$\ll \alpha_1, \dots, \alpha_n]] := \ll \alpha_1, \dots, \alpha_{n-1} \gg \otimes [1, \alpha_n].$$

We observe that the norm $\mathfrak{n}$ of a Cayley algebra $\mathfrak{C}$ over K is a 3-fold Pfister form (in any characteristic). If $\theta_1 \in \mathrm{Q}_{\gamma}^{-1}(V_1, D_1)$ is a hyperbolic quadratic quaternion form on the 4-dimensional vector space V_1 and K is a splitting field for D_1, then in the decomposition

$$q_{\theta_1}(x) = -\rho\ r(T_1(x), x) + \ell_1 R(x) + F,$$

the quadratic form R is always a 3-fold Pfister (hyperbolic) form with values in K. However, in general, the following situation can occur: we may have a quadratic quaternion form $\theta_1 \in \mathrm{Q}_{\sigma}^{\varepsilon}(V_1, D_1)$ with decompositions

$$q_{\theta_1}(x) = -\rho\ r(T_1(x), x) + \ell_1 R(x) + F \text{ with respect to } K, \tag{4.5}$$

$$q_{\theta_1}(x) = -\rho'\ r'(T_1'(x), x) + \ell_1' R'(x) + F \text{ with respect to } K', \tag{4.6}$$

where K and K' are splitting fields of D_1 and it is possible that R is a Pfister form but R' needs not to be a Pfister form. For example, one can take a quaternionic trialitarian triple $(\theta_1, \theta_2, \theta_3)$ over a triple of vector spaces defined over a triple of quaternion division algebras (D_1, D_2, D_3): then there is a common splitting field K for D_1, D_2 and D_3. In this situation, in the equality 4.5, the quadratic form R is a 3-fold Pfister form. On the other hand, assume that K' is a splitting field for D_1 but it is not a splitting field for, say, D_2. Then, tensoring the equation 4.4 with K' we get, on the right hand side, something which is not split, i.e. R' is not a 3-fold Pfister form.

Example 4.2.5 *(*char$F \neq 2$*).* Let $D = [K, \mu)$ be a quaternion algebra over the filed F. Let $q =< \lambda_1, \ldots, \lambda_n >$ be a diagonal quadratic form on F^n. Let $k : D^n \times D^n \longrightarrow D$ be the sesquilinear form given by the diagonal form ℓq. Then the quadratic form R on K^{2n} corresponding to $\theta = [k]$ is given by the diagonal form $< 1, -\mu > \otimes q$. In particular we get the 3-fold Pfister form $<< a, b, \mu >>$ choosing for q the norm form of a quaternion algebra $(a, b)_F$.

Example 4.2.6 *(*char$F = 2$*).* Let $D = [K, \mu)$ be a quaternion algebra over F and let $j \in K \setminus F$ be an element of trace 1, so that $K = F(j)$ with $j^2 = j + \lambda$, $\lambda \in F$. Let $b =< \lambda_1, \ldots, \lambda_n >$ be a bilinear diagonal form on F^n, i.e., $b(x, y) = \sum \lambda_i x_i y_i$. Let $k = (j + \ell)b$ on D^n, $\theta = [k]$. Let $q : F^n \longrightarrow F$ be the quadratic form defined by $q(x) = b(x, x)$. Then the quadratic form R over $K = F(j)$ corresponding to $\theta = [k]$ is given by $R = q \otimes [1, \mu]$. In particular, for $q =< 1, a, c, ac >$, we get the 3-fold Pfister form $<< a, c, \mu]]$.

Remark 4.2.7. In the proof of 4.2.3 and in the examples above the associated quadratic form is always a hyperbolic 3-fold Pfister form: this follows from the exact sequence of Lewis, in fact θ is constructed from a quadratic form.

We shall show that if two quadratic quaternion forms θ_1 and θ_1' of dimension 4 over a quaternion algebra D_1 have isomorphic Clifford algebras with involution, then θ_1 and θ_1' are similar.

Theorem 4.2.8. *Let* θ_1 *and* θ_1' *be quadratic quaternion forms of dimension* 4 *over* D_1*. If there are quaternionic trialitarian triples* $(\theta_1, \theta_2, \theta_3)$ *and* $(\theta_1', \theta_2, \theta_3)$*, then* $\theta_1 = \xi\theta_1'$ *for some* $\xi \in F^\times$*.*

Proof. We have $a_1T_1(x \star y) = T_2(x) \star T_3(y)$ and $a_1'T_1'(x \star y) = T_2(x) \star T_3(y)$, for all $x, y \in \mathfrak{C}$. Hence $T_1 = \xi T_1'$, with $\xi = \frac{a_1'}{a_1}$. The theorem follows from the fact that $\xi = \overline{\xi}$. ☐

Suppose that $\mathrm{char} F \neq 2$. Then we may view the preceding theorem as a particular case of the following, more general, assertion concerning algebras with involution.

Theorem 4.2.9. *Two central simple algebras with involution* (A, σ) *and* (A', σ') *of degree* 8 *over a field* F *with trivial discriminant are isomorphic if and only if their Clifford algebras are isomorphic as algebras with involution.*

Proof. If both algebras have as Clifford the algebra with involution $(B, \sigma_B) \times (C, \sigma_C)$, then, by triality (see [14, 42.3]), the Clifford algebra of (B, σ_B) is $(C, \sigma_C) \times (A, \sigma) \simeq (C, \sigma_C) \times (A', \sigma')$. Since this decomposition is unique, it follows directly that $(A, \sigma) \simeq (A', \sigma')$ or $(A, \sigma) \simeq (C, \sigma_C) \simeq (A', \sigma')$, as desired. ☐

4.3 Geometric Triality

In this section we prove some geometric results concerning a quaternionic trialitarian triple $(\theta_1, \theta_2, \theta_3)$. If the triple of quaternion algebras (D_1, D_2, D_3) is $(M_2(F), M_2(F), M_2(F))$, then we are in the situation considered in [14, (42.7)]. We assume here that D_1 is a division algebra. We decompose θ_1 as in 2.2.1 ($\rho \in K^\times$ with $\rho + \bar{\rho} = 1$):

$$q_{\theta_1}(x) = -\rho r(T_1(x), x) + \ell_1 R(x) + F,$$

and we assume that R is a 3-fold Pfister form. We look at the isotropic subspaces of our quadratic quaternion forms. We recall that a vector $v \in V_1$ is *isotropic* if $q_{\theta_1}(v) = 0$, a subspace U_1 of V_1 is *isotropic* if $q_{\theta_1}(U_1) = 0$ and θ_1 is *isotropic* if there exists an isotropic vector $v \in V_1$.

Lemma 4.3.1. *If θ_1 is isotropic, then R is hyperbolic.*

Proof. Clearly, if θ_1 is isotropic, then R is also isotropic. Since R is the norm form $\mathfrak{n}$ of a Cayley algebra $\mathfrak{C}$ over K, it follows that R is also hyperbolic (see [14, 33.23]). ☐

The totally isotropic subspaces of $V^0 = \mathfrak{C}$, the Cayley algebra over K with norm R, are described in [4]. We summarize some results which will be used later. We also use the terminology of [4], calling *point*, *line* and *space* the $1, 2, 4$-dimensional totally isotropic subspaces of $\mathfrak{C}$.

Proposition 4.3.2. *1. Every space is of the form $a \star \mathfrak{C}$ or $\mathfrak{C} \star a$, for some isotropic element $a \in \mathfrak{C}$, determined up to a nonzero scalar.*
2. Two spaces $a \star \mathfrak{C}$, $b \star \mathfrak{C}$ (resp. $\mathfrak{C} \star a, \mathfrak{C} \star b$) are equal if, and only if a, b are linearly dependent.
3. If $a, b \in \mathfrak{C}$ generate a line, then $a \star \mathfrak{C} \cap b \star \mathfrak{C}$ and $\mathfrak{C} \star a \cap \mathfrak{C} \star b$ are lines.

In this context we have the following geometric triality.

Proposition 4.3.3. *1. The map ϱ, defined by*

$$\varrho(aK) = \mathfrak{C} \star a, \;\; \varrho(\mathfrak{C} \star a) = a \star \mathfrak{C}, \;\; \varrho(a \star \mathfrak{C}) = aK,$$

on the set formed by all points and spaces, has the property $\varrho^3 = 1$.
2. The map λ, defined on the set of lines Λ, sending the line L generated by a, b to

$$\lambda(L) = \mathfrak{C} \star a \cap \mathfrak{C} \star b,$$

is a permutation of Λ and has the property $\lambda^3 = 1$.

Lemma 4.3.4. *Let $D = [K, \mu)$ be a quaternion algebra and suppose that V is a D-vector space. Let $x \in V$ be a nonzero vector. If D is a division algebra, then x and $T(x)$ are linearly independent vectors of V^0.*

Proof. If there exist scalars $\lambda_1, \lambda_2 \in K$ (λ_1, λ_2 not both zero), with $x\lambda_1 + T(x)\lambda_2 = 0$, then $\mu = \kappa\overline{\kappa}$, with $\kappa \in K$, i.e. D is split. ☐

Now we obtain two results about isotropic subspaces. First we consider the case with θ_1 hyperbolic. This case was already studied before (see [1], [6], [21]).

Proposition 4.3.5. *Let θ_1 be a hyperbolic quadratic quaternion form defined on a vector space V_1 over the division algebra D_1. The form θ_1 is part of a quaternionic trialitarian triple $(\theta_1, \theta_2, \theta_3)$: one of the forms (say θ_2) is hyperbolic and the corresponding quaternion algebra (say D_2) is isomorphic to D_1; the quaternion algebra of the other form (say D_3) is split.*

Proof. Let U_1 be an isotropic subspace of V_1 of dimension 2 over D_1. Then U_1 is a space in $\mathfrak{C}$, with $T_1(U_1) = U_1$. We suppose that $U_1 = \mathfrak{C} \star a$ (the case $U_1 = a \star \mathfrak{C}$ is similar). Then

$$U_1 = T_1(U_1) = T_1(\mathfrak{C} \star a) = \mathfrak{C} \star T_3(a).$$

Hence $T_3(a) = a\xi$, $\xi \in K$, i.e. D_3 is split. On the other hand the space $U_2 = a \star \mathfrak{C}$ is stable under T_2, i.e. U_2 is a subspace of V_2 of dimension 2. Since for any $c \in \mathfrak{C}$ we have

$$q_{\theta_2}(a \star c) \equiv -\rho r(T_2(a \star c), a \star c) + \ell_2 R(a \star c) = -\rho a_2^{-1} r(a\xi \star T_1(c), a \star c) = 0,$$

it follows that θ_2 is hyperbolic.
The isomorphism $D_1 \simeq D_2$ follows from the fact that D_3 is split and from the equation $[D_1][D_2][D_3] = 1 \in \mathrm{Br}(F)$. □

The map λ is a permutation of Λ with $\lambda^3 = 1$. We show that in fact λ sends *points* (i.e 1-dimensional isotropic D_i-subspaces) of θ_i to points of θ_{i+1} ($i = 1, 2, 3$). We observe that a 1-dimensional D_i-subspace U is a 2-dimensional K-subspace stable under T_i.

Proposition 4.3.6. *Let $(\theta_1, \theta_2, \theta_3)$ be a quaternionic trialitarian triple over a triple of division algebras (D_1, D_2, D_3). Then λ maps points of θ_i to points of θ_{i+1} ($i = 1, 2, 3$) and has the property $\lambda^3 = 1$.*

Proof. We assume that U_1 is a point of θ_1 generated (over K) by $x, T_1(x)$. The 2-dimensional totally isotropic subspace $U_2 = \mathfrak{C} \star x \cap \mathfrak{C} \star T_1(x)$ is a point for θ_2: $T_2(U_2) \subseteq T_2(\mathfrak{C} \star x) \cap T_2(\mathfrak{C} \star T_1(x)) = \mathfrak{C} \star T_1(x) \cap \mathfrak{C} \star x = U_2$, hence U_2 is a 1-dimensional space over D_2. Moreover, any $u \in U_2$ is of the form $u = c \star x = c' \star T_1(x)$ for certain $c, c' \in \mathfrak{C}$, hence we have:

$$\begin{aligned} q_{\theta_2}(u) &\equiv -\rho r(T_2(u), u) + \ell_2 R(u) \\ &= -\rho r(T_2(u), u) \\ &= -\rho r(T_2(c \star x), c' \star T_1(x)) \\ &= -\rho a_2^{-1} r(T_3(c) \star T_1(x), c' \star T_1(x)) \\ &= -\rho a_2^{-1} r(T_3(c), T_1(x) \star c' \star T_1(x)) = 0. \end{aligned}$$

In a similar way we can construct a point for θ_3. □

Corollary 4.3.7. *Let* $(\theta_1, \theta_2, \theta_3)$ *be a quaternionic trialitarian triple and suppose that* (D_1, D_2, D_3) *is a triple of division algebras. If* θ_i *is isotropic, then also* θ_{i+1} *and* θ_{i+2} *are isotropic, and all the forms have Witt index* 1.

Proof. The Witt index can not be equal 2, otherwise one quaternion algebra would be split (see 4.3.5). □

Corollary 4.3.8. *Let* $(\theta_1, \theta_2, \theta_3)$ *be a quaternionic trialitarian triple and suppose that* (D_1, D_2, D_3) *is a triple of division algebras. If* θ_i *is anisotropic, then also* θ_{i+1} *and* θ_{i+2} *are anisotropic.*

4.4 The Spinor Group

There is a nice description of the spinor group of the norm of a Cayley algebra over a field of characteristic not 2 (see [14]). Here we describe the spinor group of a quadratic quaternion form. First of all we treat the case of the norm of a Cayley algebra, in a characteristic free way. We recall that the *spinor group* of $(\mathfrak{C}, \mathfrak{n})$ is defined by

$$\mathrm{Spin}(\mathfrak{C}, \mathfrak{n}) := \{c \in C_0(\mathfrak{n}) \mid c \cdot \tau(c) = 1 \quad \text{and} \quad c \cdot x \cdot c^{-1} \in \mathfrak{C} \quad \forall x \in \mathfrak{C}\},$$

where τ is the involution induced by the identity on $\mathfrak{C}$ and the symbol $\cdot$ denotes the multiplication in the Clifford algebra. Let $r_x(y) := y \star x$ and $l_x(y) := x \star y$. Then the map $\mathfrak{C} \longrightarrow \mathrm{End}_F(\mathfrak{C} \oplus \mathfrak{C})$ given by

$$x \mapsto \begin{pmatrix} 0 & l_x \\ r_x & 0 \end{pmatrix}$$

induces isomorphisms of algebras with involution:

$$\alpha : (C(\mathfrak{n}), \tau) \longrightarrow (\mathrm{End}_F(\mathfrak{C} \oplus \mathfrak{C}), \sigma_{\mathfrak{n} \perp \mathfrak{n}}),$$

and

$$\alpha_0 : (C_0(\mathfrak{n}), \tau) \longrightarrow (\mathrm{End}_F(\mathfrak{C}), \sigma_{\mathfrak{n}}) \times (\mathrm{End}_F(\mathfrak{C}), \sigma_{\mathfrak{n}}).$$

A triple $(t_1, t_2, t_3) \in \mathrm{O}^+(\mathfrak{n})^3$ will be called *related* if $t_1(x \star y) = t_2(x) \star t_3(y)$ for all $x, y \in \mathfrak{C}$. For $c \in \mathrm{Spin}(\mathfrak{C}, \mathfrak{n})$, we let $\alpha_S(c) = \begin{pmatrix} t_3 & 0 \\ 0 & t_2 \end{pmatrix}$.
Let $\chi_c(x) = c \cdot x \cdot c^{-1}$ be the *vector representation.* We define the group $\mathrm{RT}(\mathfrak{C}, \mathfrak{n}) := \{(t_1, t_2, t_3) \mid t_1, t_2, t_3 \in \mathrm{O}^+(\mathfrak{n}) : t_1(x \star y) = t_2(x) \star t_3(y)\}$.

Proposition 4.4.1. *The map*

$$\Psi : \mathrm{Spin}(\mathfrak{C}, \mathfrak{n}) \longrightarrow \mathrm{RT}(\mathfrak{C}, \mathfrak{n})$$

defined by $\Psi(c) := (\chi_c, t_2, t_3)$ *is an isomorphism of groups.*

Proof. For any $x \in \mathfrak{C}$, $c \cdot x = \chi_c(x) \cdot c$. Applying α, we get

$$t_3(x \star y) = \chi_c(x) \star t_2(y),$$

$$t_2(x \star y) = t_3(x) \star \chi_c(y),$$

and, by replacing x with $y \star x$ in the first equation, we get

$$\mathfrak{n}(y) t_3(x) = \chi_c(y \star x) \star t_2(y).$$

Multiplying with $t_2(y)$ on the left and viewing y as a generic element gives

$$t_2(y) \star t_3(x) = \chi_c(y \star x).$$

We know that $c \cdot \tau(c) = 1$. Applying α, we have

$$\begin{pmatrix} t_3 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} \sigma_{\mathfrak{n}}(t_3) & 0 \\ 0 & \sigma_{\mathfrak{n}}(t_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $t_i \sigma_{\mathfrak{n}}(t_i) = 1$ for $i = 2, 3$. If the characteristic of F is not 2 we can already conclude that t_2 and t_3 are orthogonal maps: $b_{\mathfrak{n}}(t_i(x), t_i(y)) = b_{\mathfrak{n}}(x, \sigma_{\mathfrak{n}}(t_i) t_i(y)) = b_{\mathfrak{n}}(x, y)$ for $i = 2, 3$. Now we suppose that the characteristic of F is 2. We prove that t_2 is a similitude.

$$\begin{aligned} \mathfrak{n}(t_2(x)) = \mathfrak{n}(t_2(1x)) = \mathfrak{n}(t_2(1 \star \overline{x})) &= \mathfrak{n}(t_3(1) \star \chi_c(\overline{x})) \\ &= \mathfrak{n}(t_3(1)) \mathfrak{n}(\chi_c(\overline{x})) = \lambda \mathfrak{n}(x), \end{aligned}$$

where $\lambda := \mathfrak{n}(t_3(1))$. Hence t_2 is a similitude with multiplier λ. Since $\lambda b_{\mathfrak{n}}(x, y) = b_{\mathfrak{n}}(t_2(x), t_2(y)) = b_{\mathfrak{n}}(x, y)$, it follows that $\lambda = 1$. In the same way, $t_3 \in \mathrm{O}^+(\mathfrak{n})$.

The map $\Psi : \mathrm{Spin}(\mathfrak{C}, \mathfrak{n}) \longrightarrow \mathrm{RT}(\mathfrak{C}, \mathfrak{n})$ defines an injective group homomorphism. The map Ψ is surjective, because given $(t_1, t_2, t_3) \in \mathrm{RT}(\mathfrak{C}, \mathfrak{n})$, we have $\Psi(c) = (t_1, t_2, t_3)$ for $\alpha(c) = \begin{pmatrix} t_3 & 0 \\ 0 & t_2 \end{pmatrix}$. ☐

Remark 4.4.2. Let $\mathrm{O}'(\mathfrak{n})$ be the *reduced orthogonal group* of $\mathfrak{n}$ (see [5, p. 49]). We observe that if (t_1, t_2, t_3) is a related triple in $\mathrm{O}^+(\mathfrak{n})^3$, then $t_1, t_2, t_3 \in \mathrm{O}'(\mathfrak{n})$: since the map Ψ is surjective, there exists an element $c \in \mathrm{Spin}(\mathfrak{C}, \mathfrak{n})$ with $t_1 = \chi_c \in \mathrm{O}'(\mathfrak{n})$; but by triality (t_3, t_1, t_2) and (t_2, t_3, t_1) are related triples too, hence $t_2, t_3 \in \mathrm{O}'(\mathfrak{n})$.

We shall consider the following case: let θ_1 be a quadratic quaternion form given by $\big(\mathfrak{C}, \mathfrak{n}, T_1\big)$, where $\mathfrak{n}$ is the norm of the Cayley algebra $\mathfrak{C}$. The *spinor group* of θ_1 is defined by

$$\mathrm{Spin}(\theta_1) := \mathrm{Spin}(\mathfrak{C}, \mathfrak{n}) \cap Cl(V_1, \theta_1).$$

Theorem 4.4.3. *The restriction of the map* Ψ *to* $\mathrm{Spin}(\theta_1)$ *induces an isomorphism*

$$\mathrm{Spin}(\theta_1) \simeq \{\text{related triples } (t_1, t_2, t_3) \in \ \mathrm{O}'(\mathfrak{n})^3 \text{ such that } t_i \in \mathrm{O}(D_i, \theta_i)\}.$$

Proof. Let $c \in \mathrm{Spin}(\theta_1)$. Then χ_c is an element of $\mathrm{O}'(\mathfrak{n})$. Moreover, since $c \in Cl(V_1, \theta_1)$, we have $\chi_c T_1 = T_1 \chi_c$. Let $t_1 := \chi_c$. By triality (see 4.2.1), we get two maps t_2 and t_3 in $O^+(\mathfrak{C}, \mathfrak{n})$ (uniquely determined up two multiplication with ± 1) such that $t_2 T_2 = T_2 t_2$ and $t_3 T_3 = T_3 t_3$, i.e. $t_2 \in \mathrm{O}(D_2, \theta_2)$ and $t_3 \in \mathrm{O}(D_3, \theta_3)$. □

Corollary 4.4.4. *Let* $(\theta_1, \theta_2, \theta_3)$ *be a quaternionic trialitarian triple. Triality induces isomorphisms*

$$\mathrm{Spin}(\theta_1) \cong \mathrm{Spin}(\theta_2) \cong \mathrm{Spin}(\theta_3).$$

4.5 The Chevalley Algebra

In [5], Chevalley defined a 24-dimensional commutative (nonassociative and without a unit) algebra C. He used the algebra C to construct the algebra $\mathfrak{C}$ of octonions. Reversing the construction as in [18], we define *the Chevalley algebra* C as the algebra of triples (a, b, c) of octonions with multiplication given by

$$(a, b, c) \cdot (x, y, z) := (b \star z + y \star c, c \star x + z \star a, a \star y + x \star b).$$

In this context triality consists in the existence of an algebra automorphism of C of period 3, namely $J : C \longrightarrow C$, defined by $J((a, b, c)) := (b, c, a)$.

We describe how triality for quadratic quaternion forms is reflected in this situation. Let $D_1 = K \oplus \ell_1 K = [K, \mu_1)$ be a quaternion algebra over F and let $\theta_1 \in \mathrm{Q}_\gamma^{-1}(V_1, D_1)$ be a quaternion quadratic space of dimension 4 over D_1. Let $q_{\theta_1}(x) \equiv -\rho r_1(T_1(x), x) + \ell R_1(x)$ as in 2.2.1, so that R_1 is a 3-fold Pfister form, i.e., the norm form $\mathfrak{n}$ of a Cayley algebra $\mathfrak{C}$ over K. In view of (4.1.5) T_1 induces two semilinear similitudes T_2, T_3 and there are three constants a_1, a_2 and a_3 such that

$$\begin{aligned} a_3T_3(x \star y) &= T_1(x) \star T_2(y), \\ a_1T_1(x \star y) &= T_2(x) \star T_3(y), \\ a_2T_2(x \star y) &= T_3(x) \star T_1(y). \end{aligned}$$

We call $\mathfrak{t} := a_1a_2a_3$ the *triality constant.* Let C be the Chevalley algebra over K constructed from $\mathfrak{C}$.

Proposition 4.5.1. *The maps $\varphi_i : C \longrightarrow C$ defined by $\varphi_i = (T_i, T_{i+1}, T_{i+2})$ (indices modulo 3, $i = 1, 2, 3$) satisfy $J \circ \varphi_i = \varphi_{i+1}$ and*

$$\mathfrak{t}\varphi_i(c_1 \cdot c_2) = \varphi_i(c_1) \cdot \varphi_i(c_2) \quad \forall\ c_1, c_2 \in C.$$

Proof. We only prove the second assertion for $\varphi_1 = (T_1, T_2, T_3)$.
$\mathfrak{t}\varphi_1((a, b, c) \cdot (x, y, z)) = (a_1T_1(b \star z + y \star c), a_2T_2(c \star x + z \star a), a_3T_3(b \star y + x \star b))$
$= (\ T_2(a) \star T_3(z) + T_2(y) \star T_3(c)\ ,\ T_3(c) \star T_1(x) + T_3(z) \star T_1(a)\ ,\ T_1(c) \star T_2(x) + T_1(z) \star T_2(a)\ ) = \varphi_1((a, b, c)) \cdot \varphi_1((x, y, z))$,
for all (a, b, c), $(x, y, z) \in C$. □

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Curriculum Vitae

I was born in Locarno, Switzerland, on May 6th, 1972. I attended primary school in Muralto, secondary school in Minusio and high school in Locarno, obtaining a mathematics/science diploma in 1991. In March 1997, I obtained the degree of Holder of the Diploma in Mathematics of the Swiss Federal Institute of Technology in Zurich (ETHZ). Since 1997, I have been working as a teaching assistant in the Group for Algebra and Topology at the ETHZ. In this group I began my doctoral studies under the supervision of Professor Max-Albert Knus. This work was completed in Winter 2001.