NOTES ON TRIALITY

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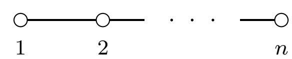
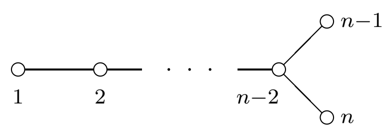
1. Introduction

Let $\overline{F}$ be an algebraically closed field of characteristic not 2 and 3. Simple Lie algebras $\mathfrak{L}$ or simple, connected, simply connected algebraic groups $\widetilde{G}$, or simple, connected, adjoint algebraic groups $\overline{G}$ over $\overline{F}$ are classified by their Dynkin diagrams $A_n, B_n, \ldots, E_8$. For example the diagram



of type A_n, $n \geq 1$, corresponds to $\mathfrak{L} = M_{n+1}(\overline{F})_0$, the Lie algebra of $(n+1 \times n+1)$-matrices of trace zero, or $\widetilde{G} = \mathrm{SL}_{n+1}$ or $\overline{G} = \mathrm{PGL}_{n+1}$; the diagram



of type D_l, $l \geq 3$, corresponds to $\mathfrak{L} = \mathrm{Skew}_{2l}(\overline{F})_0$, the Lie algebra of skew-symmetric matrices in $M_{2l}(\overline{F})$, or $\widetilde{G} = \mathrm{Spin}(2l)$ or $\overline{G} = \mathrm{PGO}_{2l}^{+}$.

Inner automorphisms of $\mathfrak{L}$, $\widetilde{G}$, or $\overline{G}$ i.e., those induced by conjugation with elements of $\overline{G}$, form a normal subgroup of the full automorphism group and the quotient group is isomorphic to the group of automorphisms of the Dynkin diagram. In most of the cases the group of automorphisms of the Dynkin diagram is either trivial or consists of two elements. For example the class of the automorphism $x \mapsto -x^t$ of $\mathfrak{L}$ is the nontrivial class in case A_n if $n \geq 2$. For D_n conjugation with $\mathrm{diag}(1, -1, \ldots, -1)$ (which is an element of PGO_{2l}, but not of PGO_{2l}^{+}) gives the non-inner class. The case of type D_4 is exceptional, in the sense that the group of automorphisms of the Dynkin diagram is S_3, the group of permutations of 3 objects. This phenomenon is known as triality. Our aim in these notes is to describe some avatars of triality, starting with the simple case of (8×8)-skew-symmetric matrices and ending with various twisted forms (in the sense of Galois cohomology), associated with PGO_8^{+}. In this case a certain "trialitarian" associative algebra places a central rôle. As shown by A. Weil, classical connected semisimple adjoint algebraic groups can be realized as automorphism groups of algebras with involution. The case D_4 was not considered by Weil. Trialitarian algebras can be used to fulfill Weil's program for D_4.

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Techniques from differents fields of algebra will be used. It will be impossible to prove everything, however we will try to "test" at least parts of the used techniques in concrete situations. In these notes we somehow work on a huge exercice around a very exceptional situation.

Triality also occurs in projective geometry (like duality). These geometric aspects are not touched here. A lovely introduction to triality (algebraic as well as geometric) is given in the paper "Octaves and triality", Nieuw. Arch. Wisk. (3) 8 (1960), 158-169, by van Blij and Springer.

The presentation given here owes much to the "Book of Involutions" published recently by the AMS as volume 44 of the Series Colloquium Publications. Historical remarks can be found in the quoted paper of van Blij and Springer or in the "Book of Involutions". Results in the generic situation come from work in preparation with Parimala and Sridharan. Special thanks are due to Frank DeMeyer who was willing to act as a guinea-pig for parts of these notes.

2. Matrices

Let $M_n(F)$ be the F-algebra of $(n \times n)$-matrices with entries in F. For any matrix $A = (a_{ij})$ we note a^t the *transpose*, $(a^t)_{ij} = a_{ji}$. The map $a \mapsto a^t$ is an *involution* of $M_n(F)$, i.e., an F-anti-automorphism of order 2. The set of skew-symmetric matrices

$$\mathrm{Skew}_n(F) = \{a \in M_n(F) \mid a^t = -a\}$$

is a Lie algebra for the Lie bracket $[x, y] = xy - yx$ induced by the multiplication of $M_n(F)$, since

$$(xy - yx)^t = y^t x^t - x^t y^t = yx - xy$$

The vector space $\mathrm{Skew}_n(F)$ over F has dimension $\frac{n(n-1)}{2}$. A basis is given by the skew-symmetric matrices $\mathcal{E}_{ij} = e_{ij} - e_{ji}$, $i < j$, where the e_{ij} are the standard matrix units in $M_n(F)$. Since $\mathcal{E}_{ij}\mathcal{E}_{jk} = e_{ik}$ for $i \neq j$, $j \neq k$ and $i \neq k$, the set $\{\mathcal{E}_{ij}\}$ generates $M_n(F)$ as an algebra if $n \geq 3$. The Lie algebras $\mathrm{Skew}_n(F)$ are simple[1] for $n \geq 3$, $(n \neq 4)$[2], and are, in the classification of simple Lie algebras, of type B_l if $n = 2l + 1$, resp. of type D_l if $n = 2l$. From now on we assume that $n = 2l$ is even.

To any $a \in \mathrm{GL}_n(F)$, the group of invertible matrices in $M_n(F)$, we associate the *inner automorphism* $\mathrm{Int}(a)(x) = axa^{-1}$ of $M_n(F)$ and $\mathrm{Int}(a) = 1$ if and only if a is a nonzero element of the center F. Since any automorphism of $M_n(F)$ is inner we may identify $\mathrm{Aut}_F\big(M_n(F)\big)$ with

$$\mathrm{PGL}_n(F) = \mathrm{GL}_n(F)/F^\times$$

[1] simple means simple over an algebraic closure $\overline{F}$ of F

[2] $\mathrm{Skew}_4(\overline{F}) \simeq M_3(\overline{F})_0 \times M_3(\overline{F})_0$, where $M_n(F)_0$ is the Lie algebra of $(n \times n)$-matrices with zero trace

We write $[a] \in \mathrm{PGL}_n(F)$ for the class of $a \in \mathrm{GL}_n(F)$. The *group of similitudes* is

$$\mathrm{GO}_n(F) = \{a \in \mathrm{GL}_n(F) \mid aa^t \in F^\times\}$$

and $m(a) = aa^t \in F^\times$ is the *multiplier* of the similitude a. If A is an algebra with involution σ, an automorphism of the pair (A, σ) is an automorphim α of A such that $\alpha\big(\sigma(a)\big) = \sigma\big(\alpha(a)\big)$ for all $a \in A$. For $a \in \mathrm{GO}_n(F)$ we have

$$(axa^{-1})^t = (a^{-1})^t x^t a^t = ax^t a^{-1}$$

hence $\mathrm{PGO}_n(F) = \mathrm{GO}_n(F)/F^\times$ acts as automorphisms of $\big(M_n(F), t\big)$. In fact

$$\mathrm{Aut}_F\big(M_n(F), t\big) = \mathrm{PGO}_n(F)$$

since, writing any automorphism of $\big(M_n(F), t\big)$ as $\mathrm{Int}(a)$, the condition

$$\mathrm{Int}(a)(x^t) = \big(\mathrm{Int}(a)(x)\big)^t$$

for all $x \in M_n(F)$ implies that $aa^t \in F^\times$. Elements of $\mathrm{Aut}_F\big(M_n(F), t\big)$ are also automorphisms of $\mathrm{Skew}_n(F)$ and, in fact,

$$\mathrm{Aut}_F\big(\mathrm{Skew}_n(F)\big) = \mathrm{PGO}_n(F)$$

if $n = 2l \geq 6$, but $n \neq 8$. To discuss the case $n = 8$ we need the notion of a proper similitude. For any similitude a we have $\det(aa^t) = \det(a)^2 = m(a)^{2l}$, so that $\det(a) = \pm\, m(a)^l$. Similitudes a with $\det(a) = m(a)^l$ form the subgroup $\mathrm{GO}_n^+(F)$ of $\mathrm{GO}_n(F)$ of *proper similitudes*. Similarly we get the subgroup $\mathrm{PGO}_n^+(F)$ of $\mathrm{PGO}_n(F)$ of classes of proper similitudes. Let s be a similitude which is not proper, for example $s = \mathrm{diag}(1, -1, \dots, -1)$, then $\mathrm{PGO}_n(F)$ is the disjoint union

$$\mathrm{PGO}_n(F) = \mathrm{PGO}_n^+(F) \cup [s]\,\mathrm{PGO}_n^+(F)$$

and $\mathbb{Z}/2\mathbb{Z} = \{1, [s]\}$ acts on $\mathrm{PGO}_n^+(F)$ through conjugation with $[s]$. Thus $\mathrm{PGO}_n(F)$ is the *semidirect product* $\mathrm{PGO}_n(F) = \mathrm{PGO}_n^+(F) \rtimes \mathbb{Z}/2\mathbb{Z}$. In particular,

$$\mathrm{Aut}_F\big(\mathrm{Skew}_n(F)\big) = \mathrm{PGO}_n^+(F) \rtimes \mathbb{Z}/2\mathbb{Z}$$

for $n = 2l$ even and $l \geq 5$. We recall that, if a group A acts as automorphisms on a group B, then the semidirect product $B \rtimes A$ consists of pairs (a, β), $a \in A$, $\beta \in B$ with the multiplication $(a, \beta)(a', \beta') = (a\beta(a'), \beta\beta')$. For example the group S_3 of permutations of 3 objects is the semidirect product of the alternating subgroup A_3 with S_2, $S_3 = A_3 \rtimes S_2$.

From now on we call $\mathrm{PGO}_n^+(F)$ the group of *inner automorphisms* of $\mathrm{Skew}_n(F)$. Automorphisms which are not inner are *outer automorphisms*. Thus conjugation with s as above is **not** an inner automorphism. In the case D_4, $(l = 4)$ we shall see that

$$\mathrm{Aut}_F\big(\mathrm{Skew}_8(F)\big) = \mathrm{PGO}_8^+(F) \rtimes S_3$$

Thus the Lie algebra $\mathrm{Skew}_8(F)$ has a group of outer automorphisms isomorphic to S_3, the group of permutation of 3 objects. Observe that if ϕ is an automorphism of order 3 of $\mathrm{Skew}_8(F)$ which is not inner, then it cannot be extended to an automorphism of $\big(M_8(F), t\big)$, since $\mathrm{Aut}_F\big(M_8(F), t\big) = \mathrm{PGO}_8^+(F) \rtimes S_2$, even if the set of skew-symmetric matrices generates $M_8(F)$ as an algebra. The action of S_3 on the Lie algebra $\mathrm{Skew}_8(F)$ and on the group $\mathrm{PGO}_n^+(F)$ is the first aspect of "triality" which we shall describe. A fundamental tool is the Clifford algebra, which we discuss soon. For this we need first to characterize different types of involutions.

3. The Type of an Involution

For any algebra A with involution σ, σ induces an automorphism of order ≤ 2 of the center of A. We shall only consider involutions which restrict to the identity on the center, so-called *involutions of the first kind*.

Typical examples can be given on the endomorphism algebra of a finite dimensional vector space. Let V be a finite dimensional vector space of even dimension n and let $b\colon V \times V \to F$ be a bilinear form, symmetric or skew-symmetric. If b is nonsingular we define the involution σ_b *adjoint to* b on $\mathrm{End}_F(V)$ through the identity:

$$b\big(\sigma_b(f)(x), y\big) = b\big(x, f(y)\big)$$

for x, $y \in V$ and $f \in \mathrm{End}_F(V)$.

If we identify $\mathrm{End}_F(V)$ with $M_n(F)$ through the choice of a basis $(e_1, \dots, e_n)$, and let $b = (b_{ij})$, with $b_{ij} = b(e_i, e_j)$, be the matrix of b, then $\sigma_b(f) = b^{-1} f^t b$. In particular $\sigma_b(f) = f^t$ if b is the diagonal form $\mathrm{diag}(1, 1, \dots, 1)$.

Observe that if σ is any F-linear involution of $M_n(F)$, $\sigma \circ t$ is an automorphism, hence of the form $\mathrm{Int}(u)$ and $\sigma(f) = u f^t u^{-1}$. The condition $\sigma^2 = 1$ implies $u^t = u$ or $u = -u^t$.

We say that $\sigma = \mathrm{Int}(u) \circ t$ is *orthogonal* if $u^t = u$ and is *symplectic* if $u^t = -u$. For any algebra A with F-linear involution σ, let

$$\mathrm{Skew}(A, \sigma) = \{x \in A \mid \sigma(x) = -x\} \text{ and } \mathrm{Sym}(A, \sigma) = \{x \in A \mid \sigma(x) = x\}$$

Since

$$\mathrm{Skew}\big(M_n(F), \mathrm{Int}(u) \circ t\big) = \begin{cases} u \cdot \mathrm{Skew}\big(M_n(F), t\big) & \text{if } u = u^t \\ u \cdot \mathrm{Sym}\big(M_n(F), t\big) & \text{if } u = -u^t \end{cases}$$

we have

$$\dim_F \mathrm{Skew}\big(M_n(F), \sigma\big) = \begin{cases} \frac{n(n-1)}{2} & \text{if } \sigma \text{ is orthogonal} \\ \frac{n(n+1)}{2} & \text{if } \sigma \text{ is symplectic} \end{cases} \tag{1}$$

Thus the notion of an orthogonal (resp. symplectic) involution is stable under scalar extension and we define the type of an involution σ on a central simple algebra A as the type of $\sigma \otimes 1_E$ over some splitting field E of the algebra.

4. The Clifford Algebra and the Lie Algebra of a Quadratic Space

Let $q : V \to F$ be a quadratic form on V, with associated polar form

$$b_q(x, y) = q(x + y) - q(x) - q(y).$$

We call the pair (V, q) a *quadratic space* if b_q is nonsingular. We write σ_q for the involution associated with b_q. As in Section 1, the space

$$\begin{aligned} \operatorname{Skew}\bigl(\operatorname{End}_F(V), \sigma_q\bigr) &= \{f \in \operatorname{End}_F(V) \mid \sigma_q(f) = -f\} \\ &= \{f \in \operatorname{End}_F(V) \mid b_q\bigl(x, f(y)\bigr) + b_q\bigl(f(x), y\bigr) = 0\} \end{aligned}$$

is a Lie subalgebra (of dimension $\frac{n(n-1)}{2}$) of $\operatorname{End}_F(V)$ for the Lie bracket $[f, g] = f \circ g - g \circ f$ of $\operatorname{End}_F(V)$. We write $\mathfrak{o}(V, q) = \operatorname{Skew}\bigl(\operatorname{End}_F(V), \sigma_q\bigr)$ and call $\mathfrak{o}(V, q)$ the Lie algebra of (V, q). If $q =< 1, 1, \ldots, 1 >$, i.e.,

$$q(\sum_i x_i e_i) = \sum_i x_i^2$$

with respect to a basis $(e_1, \ldots, e_n)$, then $\mathfrak{o}(V, q) = \operatorname{Skew}_n(F)$.

Let $C(V, q)$ be the *Clifford algebra* of the quadratic space (V, q). We recall that $C(V, q) = TV/I$ where TV is the tensor algebra of V and I is the ideal of TV generated by the elements $x \otimes x - q(x) \cdot 1$, $x \in V$. The canonical map $V \to C(V, q)$ is injective and the image of V in $C(V, q)$ (which we identify with V) generates $C(V, q)$ as an algebra. The *even Clifford algebra* $C_0(V, q)$ is the subalgebra of $C(V, q)$ generated by even products of elements of V.

Example 2. Assume that $\dim V$ is even and that $q =< 1, 1, \ldots, 1 >$ with respect to some basis $(e_1, \ldots, e_n)$ of V. Then $(1,\ e_1, \ldots, e_n,\ e_i e_j,\ i < j,\ e_{i_1} e_{i_2} \cdots e_{i_j},\ i_1 < i_2 < \ldots i_j, 1 \leq j \leq n,\ \ldots, e_1 e_2 \cdots e_n)$ is a basis of $C(V, q)$ and the relations

$$e_i^2 = 1,\ 1 \leq i \leq n,\ e_i e_j + e_j e_i = 0,\ i \neq j$$

hold in $C(V, q)$. The algebra $C(V, q)$ is central over F. The set $\{e_i e_j, i < j\}$ generates the even algebra $C_0(V, q)$ and the element $z = e_1 e_2 \cdots e_n$, which satisfies $z^2 = 1$, generates the center of $C_0(V, q)$. Note that the assumption n even only matters for the claims about the center. We write $C(V, q) = C(n)$ and $C_0(V, q) = C_(n)$.

The properties of the Clifford algebra which we shall need are summarized in the following:

Proposition 3. *Let (V,q) be a nonsingular quadratic space of even dimension $n=2l$.*
1) The F-algebra $C(V,q)$ is central simple of dimension 2^n and has a unique involution τ which is the identity on V.
2) The center Z of the even Clifford algebra is a separable quadratic extension of F, of the form $Z=F(\sqrt{\delta})$, $\delta=(-1)^l\det(b_q)$ the signed discriminant of q. If Z is a field, $C_0(V,q)$ is central simple over Z of dimension $2^{2(l-1)}$; if $Z\simeq F\times F$, $C_0(V,q)$ is the direct product of two central simple algebras over F of dimension $2^{2(l-1)}$. The involution τ restricts to an involution τ_0 of $C_0(V,q)$ which is the identity on Z if l is congruent to 0 modulo 2; as a Z-linear involution, τ is of orthogonal type if l is congruent to 0 modulo 4 and of symplectic type if l is congruent to 2 modulo 4.

Reference. For a proof see for example the book of Scharlau on Quadratic and Hermitian Forms. □

The Lie algebra $\mathfrak{o}(V,q)$ can be identified with a Lie subalgebra of $\operatorname{Skew}(C_0(V,q),\tau_0)$, as we now show:

Lemma 4. *For x, y, $z\in V$ we have in $C(V,q)$:*

$$[[x,y],z]=2\big(xb_q(y,z)-yb_q(x,z)\big)\in V.$$

Proof. This is a direct computation based on the fact that for v, $w\in V$, $b_q(v,w)=vw+wv$ in $C(V,q)$: one finds

$$\begin{aligned}[[x,y],z]&=(xyz+xzy+yzx+zyx)\\&\quad-(yxz+yzx+xzy+zxy)\\&=2\big(xb_q(y,z)-yb_q(x,z)\big)\in V.\end{aligned}$$

for x, y, $z\in V$. □

Let $[V,V]\subset C(V,q)$ be the subspace spanned by the brackets $[x,y]=xy-yx$ for x, $y\in V$. In view of Lemma 4 we may define a linear map

$$\operatorname{ad}\colon[V,V]\to\operatorname{End}_F(V)$$

by: $\operatorname{ad}_\xi(z)=[\xi,z]$ for $\xi\in[V,V]$ and $z\in V$. Lemma 4 yields:

$$\operatorname{ad}_{[x,y]}=2\big(x\otimes\hat{b}_q(y)-y\otimes\hat{b}_q(x)\big)\quad\text{for } x,\ y\in V,\tag{5}$$

$\hat{b}_q\colon V\xrightarrow{\sim}V^*$ denoting the isomorphism $x\mapsto b_q(x,-)$.

Lemma 6. *The subspace $[V,V]$ is a Lie subalgebra of* $\operatorname{Skew}\big(C_0(V,q),\tau\big)$, *and* ad *induces an isomorphism of Lie algebras:*

$$\operatorname{ad}\colon[V,V]\xrightarrow{\sim}\mathfrak{o}(V,q).$$

Proof. Jacobi's identity yields for x, y, u, $v \in V$:

$$[[u, v], [x, y]] = [[[x, y], v], u] - [[[x, y], u], v].$$

Since Lemma 4 shows that $[[x, y], z] \in V$ for all x, y, $z \in V$, it follows that

$$[[u, v], [x, y]] \in [V, V].$$

Therefore, $[V, V]$ is a Lie subalgebra of $\operatorname{Skew}\bigl(C_0(V, q), \tau_0\bigr)$. Jacobi's identity also yields:

$$\mathrm{ad}_{[\xi,\zeta]} = [\mathrm{ad}_\xi, \mathrm{ad}_\zeta] \text{ for } \xi,\, \zeta \in [V, V],$$

hence ad is a Lie algebra homomorphism. From (5) it follows for x, y, u, $v \in V$ that:

$$\begin{aligned} b_q\bigl(\mathrm{ad}_{[x,y]}(u), v\bigr) &= 2\bigl(b_q(x, v)b_q(y, u) - b_q(y, v)b_q(x, u)\bigr) \\ &= -b_q\bigl(u, \mathrm{ad}_{[x,y]}(v)\bigr), \end{aligned}$$

hence $\mathrm{ad}_{[x,y]} \in \mathfrak{o}(V, q)$. Therefore, we may consider ad as a map:

$$\mathrm{ad}\colon [V, V] \to \mathfrak{o}(V, q).$$

It only remains to prove that this map is bijective. By going to an algebraic closure of F wqe may assume that $q =< 1, 1, \dots, 1 >$ with respect to some basis $(e_1, \dots, e_n)$, in which case the claim follows from the computations in the next Example. □

Example 7. Assume that $q =< 1, 1, \dots, 1 >$ with respect to some basis $(e_1, \dots, e_n)$ of V. Then $(e_ie_j,\ i < j)$, is a basis of $[V, V]$, since $[e_i, e_j] = 2e_ie_j$, and ad^{-1} identifies the skew-symmetric matrices $\mathcal{E}_{ij}$ with the elements $\frac{1}{2}e_ie_j$ of $C_0(V, q)$, since

$$\mathrm{ad}_{[e_i,e_j]}(e_k) = 2\,\mathrm{ad}_{e_ie_j}(e_k) = 4(e_i\delta_{jk} - e_j\delta_{ik}) = 4\mathcal{E}_{ij}e_k$$

(in the last formula we view e_k as a column vector with entry 1 in k-th position and zero entries elsewhere). Thus, through ad^{-1}, any skew-symmetric matrix $\sum_{i<j} u_{ij}\mathcal{E}_{ij}$ is mapped to $\frac{1}{2}\sum_{i<j} u_{ij}e_ie_j$.

We have more in dimension 8:

Lemma 8. *Let Z be the center of the even Clifford algebra $C_0(q)$. If V has dimension 8, the embedding $[V, V] \subset \operatorname{Skew}\bigl(C_0(q), \tau\bigr)$ induces a canonical isomorphism of Lie algebras over Z $[V, V] \otimes Z \xrightarrow{\sim} \operatorname{Skew}\bigl(C_0(q), \tau\bigr)$. Thus the adjoint representation induces an isomorphism $\mathrm{ad}\colon \operatorname{Skew}\bigl(C_0(q), \tau\bigr) \xrightarrow{\sim} \mathfrak{o}(q) \otimes Z$.*

Proof. Since $\dim_F V = 8$, τ_0 is of orthogonal type as a Z-linear involution (see Proposition 3) and $\dim_Z \operatorname{Skew}\bigl(C_0(V, q), \tau_0\bigr) = 28$. Fixing an orthogonal basis of V, it is easy to check that $[V, V]$ and Z are linearly disjoint over F in $C_0(q)$, so that the canonical map $[V, V] \otimes Z \to \operatorname{Skew}\bigl(C_0(q), \tau\bigr)$ is injective. It is surjective by dimension count. □

Similitudes of the quadratic space (V,q) are linear automorphisms $f \in \mathrm{Aut}_F(V)$ with $q\big(f(x)\big) = m(f)q(x)$, where $m(f) \in F^\times$ is the *multiplier* of the similitude. They form a group $\mathrm{GO}(V,q)$ which can be identified with $\mathrm{GO}_n(F)$ if $q = < 1, 1, \dots, 1 >$. A similitude f is *proper* if $\det(f) = m(f)^{n/2}$ (recall that we assume $\dim_F V = 2l$ even). Proper similitudes form a normal subgroup $\mathrm{GO}^+(V,q)$ of $\mathrm{GO}(V,q)$ of index 2. Similitudes are *isometries* if they have multiplier equal to 1.

It readily follows from the definition of Clifford algebras that isometries of (V,q) induce automorphisms of $C(V,q)$. For similitudes we have:

Proposition 9. *Any similitude $f \in \mathrm{GO}(V,q)$ induces an automorphism $C(f)$ of $C_0(V,q)$ such that*

$$C(f)(xy) = m(f)^{-1} f(x) f(y)$$

for x, $y \in V$. The automorphism $C(f)$ restricts to the identity of the center Z of $C_0(V,q)$ if and only if f is proper. Further we have

$$\mathrm{ad} \circ C(f) = \mathrm{Int}(f) \circ ad$$

on $[V,V]$.

Proof. The algebra $C_0(V,q)$ can be identified with $T(V \otimes V)/(I,J)$ where I is the ideal generated by the set $\{x \otimes x - q(x) \cdot 1,\ x \in V\}$ and J is the ideal generated by the set $\{y \otimes (x \otimes x - q(x) \cdot 1) \otimes z,\ x, y, z \in V\}$. The map

$$\widetilde{f} = m(f)^{-1} f \otimes f \colon V \otimes V \to V \otimes V$$

extends to an automorphism $T(\widetilde{f})$ of $T(V \otimes V)$ which maps I to itself since

$$m(f)^{-1} f(x) \otimes f(x) - q(x) \cdot 1 = m(f)^{-1}\big(f(x) \otimes f(x) - q\big(f(x)\big) \cdot 1\big)$$

Similarly $T(\widetilde{f})$ maps J to itself. The map induced by $T(\widetilde{f})$ on $C_0(V,q)$ is the desired map $C(f)$. For the claim on the center we may assume (by going to an algebraic closure) that $q = \mathrm{diag}(1, 1, \dots, 1)$ with respect to a base $(e_1, e_2, \dots, e_n)$. Then

$$C(f)(e_1 e_2 \cdots e_n) = m(f)^{-l} \det(f) e_1 e_2 \cdots e_n$$

hence the claim about the center. For the last claim we have, using the identity in Lemma 4,

$$\big(\mathrm{ad} \circ C(f)\big)\big(\big([x,y]\big)(z) = 2m(f)^{-1}\Big(\big(f(x) b_q(f(y), z\big) - f(y) b_q\big(f(x), z\big)\Big)$$

and

$$\begin{aligned}\big(\mathrm{Int}(f) \circ ad\big)\big([x,y]\big)(z) &= 2f\big(x b_q(y, f^{-1}(z)) - y b_q(x, f^{-1}(z))\big) \\ &= 2\big(f(x) b_q(y, f^{-1}(z)) - f(y) b_q(x, f^{-1}(z))\big)\end{aligned}$$

so that the claim follows from

$$m(f)^{-1} b_q\big((f(y), z\big) = b_q(y, f^{-1}(z))\big).$$

□

Example 10. If $q = < 1, 1, \ldots, 1 >$ and $(e_1, e_2, \ldots, e_n)$ is an orthogonal basis, and if we identify $\mathrm{Skew}_n(F)$ with $[V, V]$ through ad^{-1} (see Example 7), then for $b \in \mathrm{GO}_n(F)$ and $\mathcal{U}$ skew-symmetric, $C(b)(\mathcal{U}) = b\mathcal{U}b^{-1}$.

For any $\lambda \in F^\times$, $\lambda \cdot 1_V$ is a similitude with multiplier $m(\lambda \cdot 1_V) = \lambda^2$, so that $C(\lambda \cdot 1_V)$ acts trivially on $C_0(V, q)$ and we have an induced action of $\mathrm{PGO}(V, q) = \mathrm{GO}(V, q)/F^\times$, resp. of $\mathrm{PGO}^+(V, q) = \mathrm{GO}^+(V, q)/F^\times$. Observe that the homomorphism

$$C\colon\ \mathrm{PGO}(V, q) \to \mathrm{Aut}_F\big(C_0(V, q), \tau_0\big)$$

is injective if $\dim V \geq 3$, in view of Proposition 9 and the fact that $\mathfrak{o}(V, q)$ generates $\mathrm{End}_F(V)$ as an algebra. It is a nontrivial result of Wonenburger that C is an isomorphim up to dimension 6.

5. The Octonions

In this section we restrict to a quadratic space V of dimension 8 and quadratic form $q = < 1, 1, \ldots, 1 >$. We use the notations $\mathfrak{o}(8)$ for $\mathrm{Skew}_8(F)$ and $C(8)$, (resp. $C_0(8)$) for $C(V, q)$, (resp. $C_0(V, q)$). We take as a model the octonion algebra $\mathbb{O}$ with norm $n = < 1, 1, \ldots, 1 >$ and start with an explicit description of $\mathbb{O}$.

Let $\mathbb{H}$ be the *quaternion algebra* with standard basis $e_1 = 1$, $e_2 = i$, $e_3 = j$, $e_4 = k = ij$ and relations $i^2 = -1$, $j^2 = -1$, $ij + ji = 0$. We denote $a \mapsto \overline{a}$ the conjugation on $\mathbb{H}$. The norm form $x \mapsto x\overline{x}$ of $\mathbb{H}$ is isometric to the diagonal form $< 1, 1, 1, 1 >$ with respect to the basis $(e_1, \ldots, e_4)$. Let $\mathbb{O}$ be the *octonion algebra* $\mathbb{H} \oplus v\mathbb{H}$ with multiplication rule $v^2 = -1$ and

$$(a + vb) \cdot (c + vd) = ac - d\overline{b} + v(\overline{a}d + cb)$$

The algebra $\mathbb{O}$ is not associative anymore. It only satisfies the weaker *alternative* rule

$$(xx)y = x(xy) \text{ and } x(yy) = (xy)y$$

for all $x, y \in \mathbb{O}$. An element a such that $(xy)a = x(ya)$ holds for all $x, y \in \mathbb{O}$ lies necessarily in F. The conjugation of the quaternion algebra extends to a conjugation

$$\pi\colon x = a + vb \mapsto \overline{x} = \overline{a} - vb$$

of $\mathbb{O}$, satisfying $\pi(xy) = \pi(x)\pi(y)$. The norm $n(x) = x\overline{x} = \overline{x}x$ is a multiplicative quadratic form (i.e., $n(xy) = n(x)n(y)$) on $\mathbb{O}$. We complete the given basis of $\mathbb{H}$ to a basis of $\mathbb{O}$ by putting $e_5 = v$, $e_6 = vi$, $e_7 = vj$ and $e_8 = vk$. With respect to this basis the multiplication table of $\mathbb{O}$ is

(11)

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	e_2	$-e_1$	e_4	$-e_3$	$-e_6$	e_5	$-e_8$	e_7
e_3	e_3	$-e_4$	$-e_1$	e_2	$-e_7$	e_8	e_5	$-e_6$
e_4	e_4	e_3	$-e_2$	$-e_1$	$-e_8$	$-e_7$	e_6	e_5
e_5	e_5	e_6	e_7	e_8	$-e_1$	$-e_2$	$-e_3$	$-e_4$
e_6	e_6	$-e_5$	$-e_8$	e_7	e_2	$-e_1$	$-e_4$	e_3
e_7	e_7	e_8	$-e_5$	$-e_6$	e_3	e_4	$-e_1$	$-e_2$
e_8	e_8	$-e_7$	e_6	$-e_5$	e_4	$-e_3$	e_2	$-e_1$

and the *norm* n of $\mathbb{O}$ with respect to the same basis is the diagonal form $< 1, 1, \dots, 1 >$.

6. Local triality

We now describe triality for the Lie algebra $\mathfrak{o}(8)$ (local triality) following the Book of Involution. Let $\mathbb{O}$ be the octonion algebra with norm $n =< 1, 1, \dots, 1 >$. The multiplication

$$x \star y = \overline{x} \cdot \overline{y},$$

where $(x, y) \mapsto x \cdot y$, x, $y \in \mathbb{O}$, is multiplication in $\mathbb{O}$, satisfies

$$x \star (y \star x) = (x \star y) \star x = n(x)y \tag{12}$$

for x, $y \in \mathbb{O}$. Further b_n is *associative*, in the sense that

$$b_n(x \star y, x) = b_n(x, y \star z).$$

Proposition 13. *Let $r_x(y) = y \star x$ and $\ell_x(y) = x \star y$. The map $\mathbb{O} \to \operatorname{End}_F(\mathbb{O} \oplus \mathbb{O})$ given by*

$$x \mapsto \begin{pmatrix} 0 & \ell_x \\ r_x & 0 \end{pmatrix}$$

induces isomorphisms

$$\alpha \colon \big(C(8), \tau\big) \xrightarrow{\sim} \big(\operatorname{End}_F(\mathbb{O} \oplus \mathbb{O}), \sigma_{n \perp n}\big)$$

and

$$\alpha_0 \colon \big(C_0(8), \tau_0\big) \xrightarrow{\sim} \big(\operatorname{End}_F(\mathbb{O}), \sigma_n\big) \times \big(\operatorname{End}_F(\mathbb{O}), \sigma_n\big), \tag{14}$$

of algebras with involution.

Proof. We have $r_x \circ \ell_x(y) = \ell_x \circ r_x(y) = n(x) \cdot y$ by (12). Thus the existence of the map α follows from the universal property of the Clifford algebra. The fact that α is compatible with involutions is equivalent to

$$b_n\big(x \star (z \star y), u\big) = b_n\big(z, y \star (u \star x)\big)$$

for all x, y, z, u in S. This formula follows from the associativity of b_n, since

$$b_n\big(x \star (z \star y), u\big) = b_n(u \star x, z \star y) = b_n\big(z, y \star (u \star x)\big).$$

The map α is an isomorphism by dimension count, since $C(8)$ is central simple. □

From now on we use the basis of $\mathbb{O}$ given above to identify $\mathbb{O}$ with F^8, n with $< 1, \ldots, 1 >$, σ_n with transpose and $\mathrm{End}_F(\mathbb{O})$ with $M_8(F)$.

Through ad^{-1} we have identified $\mathfrak{o}(8)$ with $[\mathbb{O}, \mathbb{O}]$ inside of $C_0(8)$. Thus we get an (injective) homomorphism

$$\alpha_0|_{[\mathbb{O},\mathbb{O}]} \circ \mathrm{ad}^{-1} : \mathfrak{o}(8) \to \mathfrak{o}(8) \times \mathfrak{o}(8).$$

(The fact that the image lies in $\mathfrak{o}(8) \times \mathfrak{o}(8)$ follows from the fact that α_0 is an isomorphism of algebras with involution.) For any $\lambda \in \mathfrak{o}(8)$ let

$$\alpha_0|_{[\mathbb{O},\mathbb{O}]} \circ \mathrm{ad}^{-1}(\lambda) = (\lambda^+, \lambda^-).$$

Proposition 15 (Local triality)**.** *For any $\lambda \in \mathfrak{o}(8)$, there exist elements λ^+, $\lambda^- \in \mathfrak{o}(8)$ such that*

$$\lambda^+(x \star y) = \lambda(x) \star y + x \star \lambda^-(y), \tag{1}$$

$$\lambda^-(x \star y) = \lambda^+(x) \star y + x \star \lambda(y), \tag{2}$$

$$\lambda(x \star y) = \lambda^-(x) \star y + x \star \lambda^+(y) \tag{3}$$

for all x, $y \in \mathfrak{o}(n)$. Furthermore the pair (λ^+, λ^-) is uniquely determined by the first relation.

Proof. Let $\xi = \mathrm{ad}^{-1}(\lambda)$. Since α_0 is an isomorphism of algebras we have $\alpha_0 \circ \mathrm{ad}|_\xi = \mathrm{ad}|_{\alpha_0(\xi)} \circ \alpha_0$, hence

$$(\alpha_0 \circ \mathrm{ad}^{-1})(\lambda)(x) = \begin{pmatrix} 0 & \ell_{\lambda x} \\ r_{\lambda x} & 0 \end{pmatrix} = \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix} \begin{pmatrix} 0 & \ell_x \\ r_x & 0 \end{pmatrix} - \begin{pmatrix} 0 & \ell_x \\ r_x & 0 \end{pmatrix} \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix}$$

or

$$\lambda^+(x \star y) - x \star \lambda^-(y) = \lambda(x) \star y$$
$$\lambda^-(y \star x) - \lambda^+(y) \star x = y \star \lambda(x).$$

This gives the formulas (1) and (2). From (1) we obtain

$$b_n\big(\lambda^+(x \star y), z\big) = b_n\big(\lambda(x) \star y, z\big) + b_n\big(x \star \lambda^-(y), z\big).$$

Since $b_n(x \star y, z) = b_n(x, y \star z)$ and since λ^-, λ and λ^+ are in $\mathfrak{o}(8)$, this implies

$$-b_n\bigl(x, y \star \lambda^+(z)\bigr) = -b_n\bigl(x, \lambda(y \star z)\bigr) + b_n\bigl(x, \lambda^-(y) \star z\bigr)$$

for all x, y, and z in $\mathfrak{o}(8)$, hence (3). We finally check that, given λ, the pair λ^+, λ^- are uniquely determined by (1). It suffices to check that the only pair of linear maps λ_1, $\lambda_2 \in \mathrm{End}_F(\mathbb{O})$ satisfying

$$\lambda_1(x \star y) = x \star \lambda_2(y)$$

for all $x, y \in \mathbb{O}$ is the pair $(0,0)$. Going back to the multiplication of $\mathbb{O}$, we have $\lambda_1(\overline{xy}) = \overline{\overline{x}\lambda_2(y)}$. Then $x = 1$ implies $\lambda_1(\overline{y}) = \overline{\lambda_2(y)}$, so that $\lambda_1(\overline{xy}) = \overline{x}\lambda_1(\overline{y})$ and $\lambda_1(x) = xa$ for $a = \lambda_1(1)$. This finally implies $(xy)a = x(ya)$ for all x, $y \in \mathbb{O}$ and a lies in F. However $\lambda_1(x) = ax$ for $a \in F$ only lies in $\mathfrak{o}(8)$ if $a = 0$. □

Let d_ρ, resp. d_{ρ^2} be the endomorphisms of $\mathfrak{o}(8)$ defined by $\lambda^+ = d_\rho(\lambda)$ and $\lambda^- = d_{\rho^2}(\lambda)$ for $\lambda \in \mathfrak{o}(8)$, so that $\alpha_0 \circ \mathrm{ad}^{-1} = (d_\rho, d_{\rho^2})$.

Corollary 16. *The endomorphisms d_ρ and d_{ρ^2} are automorphisms of $\mathfrak{o}(8)$ (as a Lie algebra) and satisfy*

$$(d_\rho)^2 = d_{\rho^2} \text{ and } (d_\rho)^3 = 1.$$

Proof. The claims follow from uniqueness in Proposition 15. □

The conjugation π of $\mathbb{O}$ induces an automorphism d_π of $\mathfrak{o}(8)$, $d_\pi \colon f \mapsto \pi f \pi$, and an automorphism $C(\pi)$ of $C(8)$ which is of the form $\mathrm{Int}(e)$, with e the image of $1_{\mathbb{O}}$ in $C(8)$.

Remark 17. More generally, let $v \in V$ be such that $q(v) \neq 0$ and let π_v be the reflection in V with respect to v, i.e.,

$$\pi_v(x) = \frac{b_q(x,v)}{q(v)}v - x.$$

Then $C(\pi_v) = \mathrm{Int}(v)$ in $C(V,q)$.

Proposition 18. *The relations*

$$(d_\pi)^2 = 1 \text{ and } d_\pi \circ d_\rho = d_{\rho^2} \circ d_\pi$$

hold in $\mathrm{Aut}_F\bigl(\mathfrak{o}(8)\bigr)$ and $\{d_\pi, d_\rho\}$ generate a subgroup isomorphic to S_3.

Proof. The first relation is obvious. We check the second one. Since $C(\pi) = \mathrm{Int}(e)$ for $e = 1_{\mathbb{O}}$, we have

$$\alpha \circ C(\pi) \circ \alpha^{-1} = \mathrm{Int}\bigl(\alpha(e)\bigr)$$

Plugging in the definition of α we get for $\left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right) \in M_{16}(F)$,

$$\bigl(\alpha \circ C(\pi) \circ \alpha^{-1}\bigr)\left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right) = \left(\begin{smallmatrix} \pi g \pi & 0 \\ 0 & \pi f \pi \end{smallmatrix}\right).$$

On the other hand we know that $\mathrm{ad}\circ C(\pi)\circ\mathrm{ad}^{-1}=\mathrm{Int}(\pi)$ on $\mathfrak{o}(8)$ by Proposition 9. Thus, since $(\alpha_0\circ ad^{-1})(\lambda)=\big(d_\rho(\lambda),d_{\rho^2}(\lambda)\big)$,

$$\begin{aligned}
\big(\mathrm{ad}\circ C(\pi)\circ\mathrm{ad}^{-1}\big)\big(d_\rho(\lambda),d_{\rho^2}(\lambda)\big) &= \big(\pi d_{\rho^2}(\lambda)\pi,\pi d_\rho(\lambda)\pi\big)\\
&= \big(\alpha\circ C(\pi)\circ\mathrm{ad}^{-1}\big)(\lambda)\\
&= \big(\alpha\circ\mathrm{ad}^{-1}\circ\mathrm{ad}\circ C(\pi)\circ\mathrm{ad}^{-1}\big)(\lambda)\\
&= \big(\alpha\circ\mathrm{ad}^{-1}\big)(\pi\lambda\pi)\\
&= \big(d_\rho(\pi\lambda\pi),d_{\rho^2}(\pi\lambda\pi)\big)
\end{aligned}$$

hence the second relation. Thus we get get a homomorphism $S_3\to\mathrm{Aut}_F\big(\mathfrak{o}(8)\big)$. The fact that it is injective follows from the explicit formulas given in the next section. $\square$

7. Triality for Generic Matrices

Explicit formulas can be given for d_ρ, d_ρ^2 and d_π, using generic matrices. Computing the induced action on the Dynkin diagram, we shall see that the action is not inner. Let x_{ij}, i, $j=1,\dots,8$, be indeterminates and let $F(x_{ij})$ be the quotient field of the polynomial ring $F[x_{ij}]$ in the inderminates x_{ij}. The (8×8)-matrix $X=\sum_{i,j}x_{ij}E_{ij}\in M_8\big(F(x_{ij})\big)$ is the *generic* (8×8)-*matrix* and the matrix $\mathcal{X}=\sum_{i<j}x_{ij}\mathcal{E}_{ij}$ is the *generic skew-symmetric matrix* and lies in $\mathfrak{o}(8)\otimes F(x_{ij})$. We compute the image of $\mathcal{X}$ under the automorphisms d_ρ and d_π of $\mathfrak{o}(8)\otimes F(x_{ij})$. The element $\mathcal{E}_{ij}$ corresponds to the product $\frac{1}{2}e_ie_j$ in the Clifford algebra $C_0(8)$, through the identification of $\mathfrak{o}(8)$ with $[\mathbb{O},\mathbb{O}]\subset C_0(8)$ given in Example 7. Thus the image of $\mathcal{E}_{ij}$ under d_ρ is the matrix of the automorphism $u\mapsto e_i\star(u\star e_j)=\overline{e_i}\cdot(e_j\cdot u)$ of the space $\mathbb{O}$. Straightforward explicit calculations using the multiplication table (11) show that $\mathcal{X}=\sum_{i<j}x_{ij}\mathcal{E}_{ij}$ has as images under $\alpha_0\circ\mathrm{ad}^{-1}$ the skew-symmetric matrices

$$d_\rho(\mathcal{X})=\frac{1}{2}\left(\begin{array}{c|c|c|c|c|c|c|c}
 & \begin{smallmatrix}-x_{12}+x_{34}\\-x_{56}-x_{78}\end{smallmatrix} & \begin{smallmatrix}-x_{13}-x_{24}\\-x_{57}+x_{68}\end{smallmatrix} & \begin{smallmatrix}-x_{14}+x_{23}\\-x_{58}-x_{67}\end{smallmatrix} & \begin{smallmatrix}-x_{15}+x_{26}\\+x_{37}+x_{48}\end{smallmatrix} & \begin{smallmatrix}-x_{16}-x_{25}\\-x_{38}+x_{47}\end{smallmatrix} & \begin{smallmatrix}-x_{17}+x_{28}\\-x_{35}-x_{46}\end{smallmatrix} & \begin{smallmatrix}-x_{18}-x_{27}\\x_{36}-x_{45}\end{smallmatrix}\\ \hline
 & & \begin{smallmatrix}-x_{14}+x_{23}\\+x_{58}+x_{67}\end{smallmatrix} & \begin{smallmatrix}x_{13}+x_{24}\\-x_{57}+x_{68}\end{smallmatrix} & \begin{smallmatrix}x_{16}+x_{25}\\-x_{38}+x_{47}\end{smallmatrix} & \begin{smallmatrix}-x_{15}+x_{26}\\-x_{37}-x_{48}\end{smallmatrix} & \begin{smallmatrix}x_{18}+x_{27}\\+x_{36}-x_{45}\end{smallmatrix} & \begin{smallmatrix}-x_{17}+x_{28}\\+x_{35}+x_{46}\end{smallmatrix}\\ \hline
 & & & \begin{smallmatrix}-x_{12}+x_{34}\\+x_{56}+x_{78}\end{smallmatrix} & \begin{smallmatrix}x_{17}+x_{28}\\+x_{35}-x_{46}\end{smallmatrix} & \begin{smallmatrix}-x_{18}+x_{27}\\+x_{36}+x_{45}\end{smallmatrix} & \begin{smallmatrix}-x_{15}-x_{26}\\+x_{37}-x_{48}\end{smallmatrix} & \begin{smallmatrix}+x_{16}-x_{25}\\+x_{38}+x_{47}\end{smallmatrix}\\ \hline
 & & & & \begin{smallmatrix}x_{18}-x_{27}\\+x_{36}+x_{45}\end{smallmatrix} & \begin{smallmatrix}x_{17}+x_{28}\\-x_{35}+x_{46}\end{smallmatrix} & \begin{smallmatrix}-x_{16}+x_{25}\\+x_{38}+x_{47}\end{smallmatrix} & \begin{smallmatrix}-x_{15}-x_{26}\\-x_{37}+x_{48}\end{smallmatrix}\\ \hline
 & & & & & \begin{smallmatrix}x_{12}+x_{34}\\+x_{56}-x_{78}\end{smallmatrix} & \begin{smallmatrix}x_{13}-x_{24}\\+x_{57}+x_{68}\end{smallmatrix} & \begin{smallmatrix}x_{14}+x_{23}\\+x_{58}-x_{67}\end{smallmatrix}\\ \hline
 & & & & & & \begin{smallmatrix}x_{14}+x_{23}\\-x_{58}+x_{67}\end{smallmatrix} & \begin{smallmatrix}-x_{13}+x_{24}\\+x_{57}+x_{68}\end{smallmatrix}\\ \hline
 & & & & & & & \begin{smallmatrix}x_{12}+x_{34}\\-x_{56}+x_{78}\end{smallmatrix}\\ \hline
 & & & & & & &
\end{array}\right)$$

and

$$d_{\rho^2}(\mathcal{X}) = \frac{1}{2}\left(\begin{array}{c|c|c|c|c|c|c|c}
 & \substack{-x_{12}-x_{34}\\+x_{56}+x_{78}} & \substack{-x_{13}+x_{24}\\+x_{57}-x_{68}} & \substack{-x_{14}-x_{23}\\+x_{58}+x_{67}} & \substack{-x_{15}-x_{26}\\-x_{37}-x_{48}} & \substack{-x_{16}+x_{25}\\+x_{38}-x_{47}} & \substack{-x_{17}-x_{28}\\+x_{35}+x_{46}} & \substack{-x_{18}+x_{27}\\-x_{36}+x_{45}} \\ \hline
 & & \substack{x_{14}+x_{23}\\+x_{58}+x_{67}} & \substack{-x_{13}+x_{24}\\-x_{57}+x_{68}} & \substack{-x_{16}+x_{25}\\-x_{38}+x_{47}} & \substack{x_{15}+x_{26}\\-x_{37}-x_{48}} & \substack{-x_{18}+x_{27}\\+x_{36}-x_{45}} & \substack{x_{17}+x_{28}\\+x_{35}+x_{46}} \\ \hline
 & & & \substack{x_{12}+x_{34}\\+x_{56}+x_{78}} & \substack{-x_{17}+x_{28}\\+x_{35}-x_{46}} & \substack{x_{18}+x_{27}\\+x_{36}+x_{45}} & \substack{x_{15}-x_{26}\\+x_{37}-x_{48}} & \substack{-x_{16}-x_{25}\\+x_{38}+x_{47}} \\ \hline
 & & & & \substack{-x_{18}-x_{27}\\+x_{36}+x_{45}} & \substack{-x_{17}+x_{28}\\-x_{35}+x_{46}} & \substack{x_{16}+x_{25}\\+x_{38}+x_{47}} & \substack{x_{15}-x_{26}\\-x_{37}+x_{48}} \\ \hline
 & & & & & \substack{-x_{12}+x_{34}\\+x_{56}-x_{78}} & \substack{-x_{13}-x_{24}\\+x_{57}+x_{68}} & \substack{-x_{14}+x_{23}\\+x_{58}-x_{67}} \\ \hline
 & & & & & & \substack{-x_{14}+x_{23}\\-x_{58}+x_{67}} & \substack{x_{13}+x_{24}\\+x_{57}+x_{68}} \\ \hline
 & & & & & & & \substack{-x_{12}+x_{34}\\-x_{56}+x_{78}} \\ \hline
 & & & & & & &
\end{array}\right)$$

Since the conjugation map of $\mathbb{O}$ is given by the diagonal matrix $P = \operatorname{diag}(1, -1, \ldots, -1)$ we have

$$d_\pi(\mathcal{X}) = P\mathcal{X}P = \sum_{1<j} -x_{1j}\mathcal{E}_{1j} + \sum_{1<i<j} x_{ij}\mathcal{E}_{ij}.$$

For any skew-symmetric matrix $\mathcal{U}$ we get $d_\alpha(\mathcal{U})$ for $\alpha = \rho$, ρ^2 and π by specializing $\mathcal{X}$ to $\mathcal{U}$. This shows that S_3 acts faithfully on $\mathfrak{o}(8)$.

Remark 19. The elements f of $\mathfrak{o}(8)$ fixed under the action of S_3 are such that

$$f(x \star y) = f(x) \star y + x \star f(y)$$

for all x, $y \in \mathbb{O}$. Such f are *derivations* of $\mathbb{O}$ and they define a Lie algebra of type G_2.

8. Triality and the Dynkin Diagram

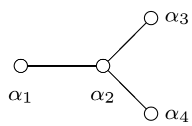
In this section we apply classical results about semisimple Lie algebras for algebras of type D_4. References are the books on Lie algebras of Jacobson, Bourbaki and Humphreys. Let $\mathfrak{L}$ be a simple Lie algebra of type D_l, $l \geq 3$, over an **algebraically closed field** F, for example $\operatorname{Skew}_n(F)$. A Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{L}$ is a commutative subalgebra such that $[x, \mathfrak{H}] \subset \mathfrak{H}$ implies $x \in \mathfrak{H}$. Two Cartan subalgebras are conjugate in $\mathfrak{L}$. For each such $\mathfrak{H}$ there is a direct sum decomposition (as vector space)

$$\mathfrak{L} = \mathfrak{H} \oplus \left(\oplus_\alpha \mathfrak{L}_\alpha\right) \tag{20}$$

where the $\mathfrak{L}_\alpha$ are eigenspaces for $\operatorname{ad}|_H$, i.e.,

$$[h, x_\alpha] = \alpha(h)x_\alpha,\ x_\alpha \in \mathfrak{L}_\alpha,$$

corresponding to nonzero linear forms, $h \mapsto \alpha(h)$ on $\mathfrak{H}$, (called the *roots*). The algebra $\mathfrak{H}$ is l-dimensional and all the $\mathfrak{L}_\alpha$ are one-dimensional. There are $2l(l-1)$ roots, which can be described as follows: The restriction of the *Killing form* $(x, y) \mapsto k(x, y) = \operatorname{Tr}(\operatorname{ad} x \circ \operatorname{ad} y)$ of $\mathfrak{L}$ to $\mathfrak{H}$ is nonsingular. Fix an orthonormal basis $(h_1, \ldots, h_l)$ of $\mathfrak{H}$ and let $(e_1, \ldots, e_l)$ be the dual basis in $\mathfrak{H}^* = \mathrm{Hom}_F(\mathfrak{H}, F)$ of $\mathfrak{H}$. Then the set of roots is $\{\pm e_i \pm e_j,\ i < j\}$. Among these roots, there are l *simple* roots $\{\alpha_i = e_i - e_{i+1},\ i = 1, \dots l-1,\ \alpha_l = e_{l-1} + e_l\}$. Simple roots can be characterized as follows: a root $\alpha = \sum \lambda_i e_i$ is *positive* is the first nonzero λ_i is positive (assume that they are in $\mathbb{Q}$) and α is simple if it is positive and not the sum $\beta + \gamma$ of two positive roots. The simple roots forms a basis of $\mathfrak{H}^*$ over F. For any root α, let h_α be such that $k(h_\alpha, h) = \alpha(h)$ and let $(\alpha, \beta) = k(h_\alpha, h_\beta)$ be the correponding bilinear form on $\mathfrak{H}^*$. The $(l \times l)$-matrix $A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ is the *Cartan Matrix* of $\mathfrak{L}$ (relative to $\mathfrak{H}$). To the matrix (A_{ij}) we associate the *Dynkin diagram* whose points are the simple roots $\{\alpha_1, \dots, \alpha_l\}$ and where α_i is connected to α_j by $A_{ij}A_{ji}$ lines. The Dynkin diagram of D_4 is:



Let ρ be an automorphism of $\mathfrak{L}$ which maps $\mathfrak{H}$ to itself. It follows from

$$\rho([h, x_\alpha]) = [\rho(h), \rho(x_\alpha)] = (\alpha \circ \rho^{-1})\big(\rho(h)\big)\rho(x_\alpha)$$

that $\alpha \circ \rho^{-1} = (\rho*)^{-1}(\alpha)$ is also a root of L. Thus $(\rho*)^{-1}$ permutes the roots, in fact it permutes the simple roots and induces an automorphism of the Dynkin diagram. Conversely, any automorphism of the Dynkin diagram comes from an automorphism of the Lie algebra (not uniquely, since inner automorphisms (i.e., given by conjugation with elements of GO^+_{2l}) induce the identity on the Dynkin diagram). For $\mathfrak{o}(8)$ of type D_4 the group of automorphisms of the Dynkin diagram is S_3 and there is an exact sequence:

$$1 \to \mathrm{Inn}_F\big(\mathfrak{o}(8)\big) \to \mathrm{Aut}_F\big(\mathfrak{o}(8)\big) \to S_3 \to 1 \tag{21}$$

The group $\mathrm{Inn}_F\big(\mathfrak{o}(8)\big)$ is the group of inner automorphisms of $\mathfrak{o}(8)$. The above computations with generic matrices can be used to show the surjectivity of the map to S_3:

Proposition 22. *The automorphisms d_ρ and d_π of $\mathfrak{o}(8)$ induce the full group of automorphisms of the Dynkin diagram.*

Proof. A Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{o}(8)$ is generated by the four diagonal blocks $\mathcal{E}_{12}$, $\mathcal{E}_{34}$, $\mathcal{E}_{56}$ and $\mathcal{E}_{78}$ (see Helgason, Differential Geometry, Lie groups, and Symmetric Spaces, p. 187). The action of d_ρ on $\mathfrak{H}$ with respect to the basis $h_1 = \mathcal{E}_{34}$, $h_2 = \mathcal{E}_{56}$, $h_3 = \mathcal{E}_{78}$ and $h_4 = \mathcal{E}_{12}$ is given by the orthogonal matrix

$$T = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & +1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

Since the matrix is orthogonal it is equal to its transpose inverse and we let it operate on the simple roots $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3 - e_4$ and $\alpha_4 = e_3 + e_4$. We get $T(\alpha_1) = \alpha_4$, $T(\alpha_2) = \alpha_2$, $T(\alpha_3) = \alpha_1$ and $T(\alpha_4) = \alpha_3$. This cyclic permutation of the roots $(\alpha_1, \alpha_4, \alpha_3)$ induces obviously an automorphism of order 3 of the Dynkin diagram. Finally conjugation with π on $\mathfrak{H}$ maps h_4 to $-h_4$ and leaves the other h_i fixed. Thus the action on roots maps e_4 to $-e_4$ and lets the other roots invariant. On the level of simple roots it permutes α_3 and α_4. This concludes the proof. □

Remark 23. In the considerations above we choose the octonion algebra with norm the identity form $< 1, 1, \ldots, 1 >$ to get simple formulas for the trialitarian action on generic matrices. There exist octonion algebras with norm

$$< 1, \alpha, \beta, \gamma, \alpha\beta, \beta\gamma, \alpha\gamma, \alpha\beta\gamma >$$

for any triple α, β, γ of nonzero elements of F (so-called 3-Pfister forms). In particular the form $q_s = \mathrm{diag}(1, 1, 1, 1, -1, -1, -1, -1)$ of maximal index can occur. The advantage of q_s is that the Cartan decomposition (20) of $\mathrm{Skew}\big(M_8(F), \sigma_{q_s}\big)$ holds over F. The drawback is that elements of $\mathrm{Skew}\big(M_8(F), \sigma_{q_s}\big)$ are more complicated than skew-symmetric matrices. To get a Cartan decomposition (20) for $\mathfrak{o}(8) = \mathrm{Skew}_8(F)$ one needs $\sqrt{-1} \in F$ (see the book of Helgason, p. 187, for explicit formulas).

Remark 24. The exact sequence (21), which holds a priori over an algebraic closure $\overline{F}$ of F, since it uses classification results valid over an algebraically closed field, holds in fact over F. The surjectivity of the map to S_3 follows from the construction of d_ρ and d_π, which are over F. We check exactness at $\mathrm{Aut}_F\big(\mathfrak{o}(8)\big)$. If $\alpha \in \mathrm{Aut}_F\big(\mathfrak{o}(8)\big)$ maps to the identity, then by exactness of (21) over $\overline{F}$, $\alpha \otimes 1_{\overline{F}}$ is inner, say $\alpha \otimes 1_{\overline{F}} = \mathrm{Int}(\bar{a})$, $\bar{a} \in \mathrm{GO}_8^+(\overline{F})$. Then $\mathrm{Int}(\bar{a})$ maps $\mathfrak{o}(8)$ to $\mathfrak{o}(8)$ and $M_8(\overline{F})$ to $M_8(\overline{F})$. Since the set $\mathfrak{o}(8) \subset M_8(F)$ generates $M_8(F)$ as an algebra, $\mathrm{Int}(\bar{a})$ is an automorphism of $M_8(F)$, hence of the form $\mathrm{Int}(b)$, $b \in \mathrm{GL}_8(F)$. It follows that $\mathrm{Int}(b) \otimes 1_{\overline{F}}) = \mathrm{Int}(\bar{a})$, hence $\bar{a} = \lambda b$, $\lambda \in \overline{F}^\times$. Replacing $\bar{a}$ by $\bar{a}\lambda^{-1}$, we may assume that $\bar{a} \in \mathrm{GL}_8(F) \cap \mathrm{PGO}_8^+(\overline{F}) = \mathrm{PGO}_8^+(F)$.

9. Similitudes and Triality

Any proper similitude $f \in \mathrm{GO}_8^+(F)$ induces an automorphism $C(f)$ of $\big(C_0(8), \tau_0\big)$ which leaves the center of $C_0(8)$ invariant (Proposition 9). Thus $\alpha_0 \circ C(f) \circ \alpha_0^{-1}$ is a pair of automorphisms of $\big(M_8(F), t\big)$, hence of the form $\big(\mathrm{Int}(f_1), \mathrm{Int}(f_2)\big)$ for similitudes f_1, f_2.

Proposition 25. *For any proper similitude $f \in \mathrm{GO}_8^+(F)$ there exist proper similitudes f_1, f_2 such that: 1) $\alpha_0 \circ C(f) \circ \alpha_0^{-1} = \big(\mathrm{Int}(f_1), \mathrm{Int}(f_2)\big)$.*
2)

$$m(f_1)^{-1} f_1(x \star y) = f(x) \star f_2(y), \tag{1}$$

$$m(f)^{-1}f(x \star y) = f_2(x) \star f_1(y) \tag{2}$$

and

$$m(f_2)^{-1}f_2(x \star y) = f_1(x) \star f(y). \tag{3}$$

The pair (f_1, f_2) *is determined by* t *up to a factor* (m, m^{-1})*,* $m \in F^\times$*, and we have*

$$m(f_1)m(f)m(f_2) = 1.$$

Furthermore, any of the formulas (1) to (3) implies the others.

Proof. Let f be a proper similitude with multiplier $m(f)$. The map $\mathbb{O} \to \mathrm{End}_F(\mathbb{O} \oplus \mathbb{O})$ given by

$$\varphi(f)\colon x \mapsto \begin{pmatrix} 0 & \ell_{f(x)} \\ m(f)^{-1}r_{f(x)} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & m(f)^{-1} \end{pmatrix} \alpha\big(f(x)\big)$$

is such that $(\varphi(f)(x))^2 = m(f)^{-1}n\big(f(x)\big) = n(x)$, so it induces a homomorphism

$$\widetilde{\varphi}(f)\colon C(8) \xrightarrow{\sim} \mathrm{End}_F(\mathbb{O} \oplus \mathbb{O}).$$

By dimension count $\widetilde{\varphi}(f)$ is an isomorphism. By the Skolem-Noether Theorem, the automorphism $\widetilde{\varphi}(f) \circ \alpha^{-1}$ of $\mathrm{End}_F(\mathbb{O} \oplus \mathbb{O})$ is inner. Let $\widetilde{\varphi}(f) \circ \alpha^{-1} = \mathrm{Int}\left(\begin{smallmatrix} s_0 & s_1 \\ s_3 & s_2 \end{smallmatrix}\right)$. Computing $\alpha^{-1} \circ \widetilde{\varphi}(f)$ on a product xy for $x, y \in \mathbb{O}$ shows that $\alpha^{-1} \circ \widetilde{\varphi}(f)|_{C_0} = C(f)$. Since f is proper, $C(f)$ is Z-linear. Again by Skolem-Noether we may write $\alpha \circ C(f) \circ \alpha^{-1} = \mathrm{Int}\left(\begin{smallmatrix} s_0' & 0 \\ 0 & s_2' \end{smallmatrix}\right)$. This implies $s_1 = s_3 = 0$ and we may choose $s_0' = s_0$, $s_2' = s_2$. We deduce from $\varphi(f)(x) = \mathrm{Int}\left(\begin{smallmatrix} s_0 & 0 \\ 0 & s_2 \end{smallmatrix}\right) \circ \big(\alpha_S(x)\big)$ that

$$\ell_{f(x)} = s_0\ell_x s_2^{-1} \text{ and } m(f)^{-1}r_{f(x)} = s_2 r_x s_0^{-1}$$

or

$$s_0(x \star y) = f(x) \star s_2(y) \text{ and } s_2(y \star x) = m(f)^{-1}s_0(y) \star f(x), \quad x,\, y \in \mathbb{O}.$$

The fact that $C(f)$ commutes with the involution τ of $C_0(8)$ implies that s_0, s_2 are similitudes and we have $m(s_0) = m(f)m(s_2)$. Putting $f_1 = m(s_0)^{-1}s_0$ and $f_2 = s_2$ we get (1) and (3). To obtain (2), we replace x by $y \star x$ in (1). We have

$$m(f_1)^{-1}n(y)f_1(x) = f(y \star x) \star f_2(y).$$

Multiplying with $f_2(y)$ on the left gives

$$m(f_1)^{-1}n(y)f_2(y) \star f_1(x) = f(y \star x)m(f_2)n(y).$$

By viewing y as “generic”, we may divide both sides by $n(y)$. This gives (2).

To show uniqueness of f_1, f_2 up to a unit, we first observe that f_1, f_2 are unique up to a pair (r_1, r_2) of scalars, since

$$\alpha C(f)\alpha^{-1} = \mathrm{Int}\left(\begin{smallmatrix} f_1 & 0 \\ 0 & f_2 \end{smallmatrix}\right).$$

Replacing (f_1, f_2) by $(r_1 f_1, r_2 f_2)$ gives

$$\mu(f_1)(r_1)^{-1} f_1(x \star y) = r_2 f_2(x) \star f(y) = \mu(f_1)^{-1} r_2 f_1(x \star y).$$

This implies ${r_1}^{-1} = r_2$. To show that f_1, (resp. f_2) is proper, we observe that

$$\alpha \circ C(f_1) \circ \alpha = \big(\mathrm{Int}(f), \mathrm{Int}(f_2)\big)$$

is the identity on the center. □

Examples 26. Right multiplication r_a is a similitude with multiplier $n(a)$ if $n(a) \neq 0$. The Moufang identity $(ax)(ya) = a(xy)a$, which holds in any alternative algebra, in particular in $\mathbb{O}$, implies the identity

$$(x \star a) \star (a \star y) = \overline{a} \star \big(a \star (x \star y)\big)$$

for the " $\star$ " multiplication. The corresponding similitudes are

$$(r_a)_1 = n(a)^{-1} \ell_a \text{ and } (r_a)_2 = n(a)^{-1} \ell_{\overline{a}} \circ \ell_a.$$

Another interesting case is given by reflections. If

$$\pi_a(x) = \frac{b_n(x,a)}{n(a)} a - x.$$

is the reflection with respect to some $a \in \mathbb{O}$ with $n(a) \neq 0$, then $C(\pi_a) = \mathrm{Int}(a)$ in $C(V,q)$, so that $\alpha \circ C(\pi_a) \circ \alpha^{-1} = \mathrm{Int}\big(\alpha(a)\big)$. This implies

$$\pi_a(x) \star (a \star y) = a \star (y \star x)$$

and

$$(\pi_a \circ \pi_b)(x) \star \big((b \star y) \star a)\big) = a \star \big((x \star y) \star b\big),$$

Thus

$$(\pi_a \circ \pi_b)_1 = \ell_a \circ r_b \text{ and } (\pi_a \circ \pi_b)_2 = \ell_b \circ r_a$$

Passing from GO_8^+ to $\mathrm{PGO}^+(8)$, we get well defined automorphisms of $\mathrm{PGO}^+(8)$, $\rho\colon [f] \mapsto [f_1]$, $\rho'\colon [f] \mapsto [f_2]$, and uniqueness in Proposition 25 implies that $\rho' = \rho^2$, $\rho^3 = 1$. Let π be the automorphism of $\mathrm{PGO}^+(8)$ induced by $\mathrm{Int}(\pi)$. It also follows from Proposition 25 and the identity $\pi(x \star y) = \pi(y) \star \pi(x)$ that $\pi \circ \rho = \rho^2 \circ \pi$. Thus :

Corollary 27 (Global triality)**.** *The set $\{\pi, \rho\}$ generate a subgroup of* $\mathrm{Aut}_F\big(\mathrm{PGO}^+(8)\big)$ *isomorphic to S_3.*

Proof. The fact that S_3 acts on $\mathrm{PGO}^+(8)$ follows from the relations given above. The fact that the action is faithful follows from the Examples 26. □

The action of S_3 on $\mathfrak{o}(8)$ given in Proposition 18 and the action on $\mathrm{PGO}^+(8)$ given in Corollary 27 are related:

Proposition 28. *Assume that* $[f] \in \mathrm{PGO}^+(8)$ *is represented by the matrix* $f \in \mathrm{GO}_8^+(F)$ *with* $ff^t = m(f)$ *and that* $\rho([f])$ *(resp.* $\rho^2([f])$ *is represented by* f_1 *(resp.* f_2*) as above. We then have for any skew symmetric matrix* $\mathcal{U}$

1) $\big(C(f) \circ ad^{-1}\big)(\mathcal{U}) = \mathrm{ad}^{-1}(f\mathcal{U}f^{-1})$,

2) $d_\rho(f\mathcal{U}f^{-1}) = f_1 d_\rho(\mathcal{U})f_1^{-1}$, $d_\rho(f_1\mathcal{U}f_1^{-1}) = f_2 d_\rho(\mathcal{U})f_2^{-1}$ *and* $d_\rho(f_2\mathcal{U}f_2^{-1}) = fd_\rho(\mathcal{U})f^{-1}$

3) $d_{\rho^2}(f\mathcal{U}f^{-1}) = f_2, d_\rho(\mathcal{U})f_2^{-1}$ $d_{\rho^2}(f_1\mathcal{U}f_1^{-1}) = fd_\rho(\mathcal{U})f^{-1}$ *and* $d_{\rho^2}(f_2\mathcal{U}f_2^{-1}) = f_1 d_\rho(\mathcal{U})f_1^{-1}$.

Furthermore, for the conjugation $\pi \in \mathrm{GO}(8)$, *we have*

4) $d_\rho(\pi\mathcal{U}\pi) = \pi d_{\rho^2}(\mathcal{U})\pi$ *and* $d_{\rho^2}(\pi\mathcal{U}\pi) = \pi d_\rho(\mathcal{U})\pi$.

Proof. The first formula is already in Proposition 9. We check the second. By definition we have $\alpha_0 \circ \mathrm{ad}^{-1}(\mathcal{U}) = \big(d_\rho(\mathcal{U}), d_{\rho^2}(\mathcal{U})\big)$. Thus

$$\begin{array}{rcl}\big(d_\rho(f\mathcal{U}f^{-1}), d_{\rho^2}(f\mathcal{U}f^{-1}\big) &=& \alpha \circ \mathrm{ad}^{-1}(f\mathcal{U}f^{-1}) \\ &=& \big(\alpha \circ C(f) \circ ad^{-1}\big)(\mathcal{U}) \\ &=& \big(\alpha \circ C(f)\alpha^{-1}\big) \circ \big(\alpha \circ ad^{-1}\big)(\mathcal{U}) \\ &=& \big(\alpha \circ C(f)\alpha^{-1}\big)\big(d_\rho(\mathcal{U}), d_{\rho^2}(\mathcal{U}\big) \\ &=& \big(f_1 d_\rho(\mathcal{U}f_1^{-1}), f_2 d_{\rho^2}(\mathcal{U})f_2^{-1}\big)\end{array}$$

The proofs of the other formulas are similar. □

For any algebraic group scheme G over F there is a Lie algebra $\mathrm{Lie}(G)$ over F defined as follows (see for example the book of Waterhouse, Introduction to Affine Group Schemes): Denote by $F[\varepsilon]$ the F-algebra of *dual numbers*, i.e., $F[\varepsilon] = F \cdot 1 \oplus F \cdot \varepsilon$ with multiplication given by $\varepsilon^2 = 0$. There is a unique F-algebra homomorphism $\kappa\colon F[\varepsilon] \to F$ with $\kappa(\varepsilon) = 0$. The kernel of $G(F[\varepsilon]) \xrightarrow{G(\kappa)} G(F)$ carries a natural F-vector space structure: addition is the multiplication in $G(F[\varepsilon])$ and scalar multiplication is defined by the formula $a \cdot g = G(\ell_a)(g)$ for $g \in G(F[\varepsilon])$, $a \in F$, where $\ell_a\colon F[\varepsilon] \to F[\varepsilon]$ is the F-algebra homomorphism defined by $\ell_a(\varepsilon) = a\varepsilon$. The kernel of $G(F[\varepsilon]) \xrightarrow{G(\kappa)} G(F)$ is the Lie algebra $\mathrm{Lie}(G)$. If $G \subset \mathrm{GL}_n(F)$

$$\mathrm{Lie}(G) = \{a \in M_n(F) \mid 1 + a\varepsilon \in G(F)\}.$$

The Lie algebra structure on $\mathrm{Lie}(G)$ can be recovered as follows (see Waterhouse, p. 94)). Consider the commutative F-algebra $R = F[\varepsilon, \varepsilon']$ with $\varepsilon^2 = 0 = \varepsilon'^2$. From $d, d' \in \mathrm{Lie}(G)$ we build two elements $g = 1 + d\varepsilon$ and $g' = 1 + d'\varepsilon'$ in $G(R)$. A computation of the commutator of g and g' in $G(R)$ yields $gg'g^{-1}g'^{-1} = 1 + d''\varepsilon\varepsilon'$ where $d'' = [d, d']$ in $\mathrm{Lie}(G)$. For example

$$\mathrm{Lie}(\mathrm{GO}_n^+) = \{a \in M_n(F) \mid a + a^t \in F\}$$

and

$$\mathrm{Lie}(\mathrm{PGO}_n^+) = \mathrm{Lie}(\mathrm{GO}_n^+)/F = \{a \in M_n(F) \mid a + a^t = 0\}$$

Any homomorphism of group schemes $f\colon G \to H$ induces a commutative diagram

$$\begin{array}{ccc} G(F[\varepsilon]) & \xrightarrow{f_{F[\varepsilon]}} & H(F[\varepsilon]) \\ {\scriptstyle G(\kappa)}\Big\downarrow & & \Big\downarrow{\scriptstyle H(\kappa)} \\ G(F) & \xrightarrow{f_F} & H(F) \end{array}$$

and hence defines an F-linear map $df\colon \operatorname{Lie}(G) \to \operatorname{Lie}(H)$, which is a Lie algebra homomorphism, called the *differential* of f.

Proposition 29. *The differential of the action of S_3 on PGO_8^+ in Corollary 27 is the action defined in Proposition 18.*

Proof. We only check that the differential of ρ in Corollary 27 is d_ρ as defined in Proposition 18. Let $[g] = [1 + a\varepsilon] \in \mathrm{PGO}_8^+\big(F[\varepsilon]\big)$, so that $[a] \in \operatorname{Lie}(\mathrm{PGO}_8^+) = \operatorname{Lie}(\mathrm{GO}_n^+)/F$. By definition of the differential we have $[\rho(g)] = [1 + d_\rho(a)\varepsilon]$ and by definition of triality

$$m(\rho(g)^{-1})\rho(g)(x \star y) = g(x) \star \rho^2(g)(y).$$

Thus

$$\big(1 + d_\rho(a)\varepsilon\big)(x \star y) = (1 + a\varepsilon)(x) \star \big(1 + d_{\rho^2}(a)\varepsilon\big)(y)$$

or

$$d_\rho(a)(x \star y) = a(x) \star y + x \star d_{\rho^2}(a)(y)$$

hence the claim by definition of triality on $\mathfrak{o}(8)$. $\square$

10. Triality and the group Spin(8)

Proposition 25 describes triality for similitudes: to any proper similitude f we associate two proper similitudes f_1 and f_2 such that

$$\alpha_0 \circ C(f) \circ \alpha_0^{-1} = \big(\operatorname{Int}(f_1), \operatorname{Int}(f_2)\big)$$

and $m(f_1)^{-1}f_1(x \star y) = f(x) \star f_2(y)$ holds. However the pair (f_1, f_2) is only defined up to a nonzero scalar. Let $\mathrm{O}^+(8)$ be the group of proper isometries (i.e., proper similitudes f with multiplier $m(f) = 1$) It is a natural question to ask if f is taken in $\mathrm{O}^+(8)$, can (f_1, f_2) be so normalized that they also belong to $\mathrm{O}^+(8)$? As we shall see this is not the case in general. We first go back to the case of quadratic space (V, q) of even dimension. Let $C_0(V, q)^\times$ be the group of units of the even Clifford algebra $C_0(V, q)$. The *even Clifford group* $\Gamma^+(V, q)$ is defined as

$$\Gamma^+(V, q) = \{c \in C_0(V, q)^\times \mid cVc^{-1} \subset V\}$$

For $c \in \Gamma^+(V, q)$ and all $v \in V$ we have

$$\tau_0(cvc^{-1}) = \tau_0(c^{-1})v\tau_0(c) = cvc^{-1}$$

Thus $v\tau_0(c)c = \tau_0(c)cv$ and $\mu(c) = \tau_0(c)c$ lies in the center F of C_0. Let

$$\mathrm{Spin}(V,q) = \{c \in \Gamma^+(V,q) \mid \tau(c)c = 1\}$$

The homomorphism

$$\chi\colon \mathrm{Spin}(V,q) \to \mathrm{O}^+(V,q) \text{ given by } \chi(c)(v) = cvc^{-1}$$

is called the *vector representation* of $\mathrm{Spin}(V,q)$. For any $f \in \mathrm{O}^+(V,q)$ $C(f)$ is an inner automorphism of $C_0(V,q)$, hence lifts to an element c of $\Gamma^+(V,q)$. The class of $\mu(c) = \tau_0(c)c$ in $F^\times/F^{\times 2}$ depends only on f. We note it $\mathrm{Sn}(f)$.

Proposition 30. *The vector representation χ fits into an exact sequence:*

$$1 \to \{\pm 1\} \longrightarrow \mathrm{Spin}(V,q) \xrightarrow{\chi} \mathrm{O}^+(V,q) \xrightarrow{\mathrm{Sn}} F^\times/F^{\times 2}.$$

Proof. The proof follows readily from the definition of the different maps and we leave it as an exercice. □

Going back to triality, we claim:

Lemma 31. *For $f \in \mathrm{O}^+(8)$ and $\rho(f) = f_1$, $\rho^2(f) = f_2$ we have $[m(f_1)] = [m(f_2)] = \mathrm{Sn}(f)$ in $F^\times/F^{\times 2}$. In particular if $f \in \mathrm{O}^+(8)$, then f_1, f_2 can be chosen in $\mathrm{O}^+(8)$ if and only $f \in \mathrm{O}^+(8)$ can be lifted to $\mathrm{Spin}(8)$.*

Proof. Let $f = \chi(c)$; we have $\alpha_0(c) = (f_1, f_2)$ and $\alpha_0\big(\tau_0(c)\big) = (f_1^t, f_2^t)$ since α_0 is an isomorphism of algebras with involution. Thus

$$\mathrm{Sn}(f) = c\tau_0(c) = (f_1 f_1^t, f_2 f_2^t) = \big(m(f_1), m(f_2)\big)$$

hence the claim. □

Lemma 31 can be used to give a nice description of $\mathrm{Spin}(8)$. For $c \in \mathrm{Spin}(8)$, let $\alpha_0(c) = (c^+, c^-) \in \mathrm{GO}^+(8) \times \mathrm{GO}^+(8)$; the two projections $\chi^+\colon c \to c^+$, $\chi^-\colon c \to c^-$are called the half-spin representations of $\mathrm{Spin}(8)$.

Proposition 32. *1) For any $c \in \mathrm{Spin}(8)$, c^+ and c^- are proper isometries, hence elements of $\mathrm{O}^+(8)$.*
2) There is an isomorphism

$$\mathrm{Spin}(8) \simeq \{(t, t^+, t^-) \mid t, t^+, t^- \in \mathrm{O}^+(8),\ t(x \star y) = t^-(x) \star t^+(y)\}$$

such that the vector representation $\chi\colon \mathrm{Spin}(8) \to \mathrm{O}^+(8)$ corresponds to the map $(t, t^+, t^-) \mapsto t$. The other projections $(t, t^+, t^-) \mapsto t^+$ and $(t, t^+, t^-) \mapsto t^-$ correspond to the half-spin representations χ^{pm} of $\mathrm{Spin}(S, n)$.

Proof. We have $\alpha_0\big(\tau_0(c) = (c^{+t}, c^{-t})$ since α_0 is an isomorphism of algebras with involution. Thus the condition $\tau_0(c)c = 1$ implies that c^+ and c^- are orthogonal matrices. We know already that they are proper. Let

$$\mathrm{O}'(8) = \{(t, t^+, t^-) \mid t^+, t, t^- \in \mathrm{O}^+(8),\ t(x \star y) = t^-(x) \star t^+(y)\},$$

Then $c \mapsto (\chi(c), t^+, t^-)$ defines an injective group homomorphism $\phi\colon \mathrm{Spin}(8) \to \mathrm{O}'(8$. It is also surjective, since, given $(t, t^+, t^-) \in \mathrm{O}'(8)$, we have $(t, t^+, t^-) = \phi(c)$ for $\alpha_0(c) = (t^+, t^-)$. □

Proposition 27 implies that if $(t, t^+, t^-) \in \mathrm{Spin}(8)$, then also (t^+, t^-, t) and $(t^-, t, t^+) \in \mathrm{Spin}(8)$. Let ρ be the automorphism of Spin(8) given by $(t, t^+, t^-) \mapsto (t^+, t^-, t)$. Let π be conjugation in $\mathbb{O}$. It follows from ... that if $(t, t^+, t^-) \in \mathrm{Spin}(8)$, then $(\pi t\pi, \pi t^-\pi, \pi t^+\pi) \in \mathrm{Spin}(8)$. So ρ and $\pi\colon (t, t^+, t^-) \mapsto (\pi t\pi, \pi t^-\pi, \pi t^+\pi)$ induce an action of S_3 on Spin(8) (triality for Spin(8) !)

Let $\mu_2 = \pm 1$ as a multiplicative group.

Lemma 33. *The center of* Spin(8) *can be identified with the group C defined by the exact sequence*

$$1 \to C \to \mu_2 \times \mu_2 \times \mu_2 \to \mu_2 \to 1$$

where the map $\mu_2 \times \mu_2 \times \mu_2 \to \mu_2$ is the multiplication map and the restriction of the action of S_3 on C is through permutations on $\mu_2 \times \mu_2 \times \mu_2$.

Proof. In fact the center consists of the triples

$$C = \{(1, 1, 1), \epsilon_0 = (1, -1, -1), \epsilon_1 = (-1, 1, -1), \epsilon_2 = \epsilon_0\epsilon_1 = (-1, -1, 1)\}$$

which readily implies the claim. □

Let $\chi' : \mathrm{Spin}(8) \to \mathrm{PGO}^+(8)$ be the vector representation χ composed with the projection $\mathrm{O}^+(8) \to \mathrm{PGO}^+(8)$.

Proposition 34. *We have an exact sequence*

$$1 \to C \to \mathrm{Spin}(8) \xrightarrow{\chi'} \mathrm{PGO}^+(8)$$

in which all the maps are equivariant under the action of S_3.

Proof. Exactness follows from the fact that the center of $\mathrm{O}^+ 8)$ is μ_2. The fact that the maps are equivariant follows from the definition of the actions of S_3. □

Remark 35. The fixed elements $\mathrm{Spin}(8)^{S_3}$ of Spin(8) under the action of S_3 (or even A_3) are isometries f such that $f(x \star y) = f(x) \star f(y)$ for all x, $y \in \mathbb{O}$. Such isometries are automorphisms of the octonion algebra $\mathbb{O}$. Assume that, instead of the octonions $\mathbb{O}$, we would have an 8-dimensional quadratic space (S, n) endoved with a bilinear multiplication

"$\star$" such that $x \star (y \star x) = (x \star y) \star x = n(x)y$. Such "algebras" are called *symmetric compositions* in the Book of Involutions. Then an action of A_3 can be defined on $\mathrm{PGO}^+(S, n)$ and $\mathrm{Spin}(S, n)$ (not a full S_3-action, because there is no identity element for the multiplication and no conjugation "π"). This is more than a formal generalization, since there exist such algebras which are very different from octonions. An example is given by the set $S = M_3(F)_0$ of 3×3-matrices of trace 0. Assume that F has characteristic different from 2 and 3 and contains a primitive cubic root of unity ω; set $\mu = \frac{1-\omega}{3}$ and define

$$x \star y = \mu xy + (1 - \mu) yx - \tfrac{1}{3} \operatorname{Tr}(yx) 1$$

Then $\star$ has the desired properties for the norm $n(x) = -\frac{1}{6}\big(\operatorname{Tr}(x)^2 - \operatorname{Tr}(x^2)\big)$. For the induced action of A_3 on $\mathrm{Spin}(S, n)$ we have

$$\mathrm{Spin}(S, n)^{A_3} = \mathrm{PGL}_3(F).$$

This and similar examples are discussed in the Book of Involution.

11. Central Simple Algebras with Involutions

Let A be a central simple F-algebra of even degree $n = 2l$, with an F-linear involution σ of orthogonal type. Let E/F be a finite Galois extension such that there exists

$$\beta \colon (A, \sigma) \otimes E \simeq \big(M_n(E), t\big).$$

(β is a *splitting*[3] of (A, σ).) For any $\gamma \in \Gamma = \mathrm{Gal}(E/F)$ let $\overline{\gamma}$ be the automorphism of $\big(M_n(E), t\big)$ defined by $\overline{\gamma} = \beta \circ (1_A \otimes \gamma) \circ \beta^{-1}$. Clearly $\overline{\gamma}$ is semilinear, i.e., $\overline{\gamma}(\lambda x) = \gamma(\lambda)\overline{\gamma}(x)$ for $\lambda \in E$ and $x \in M_n(E)$. We have $\overline{\gamma}_1 \overline{\gamma}_2 = \overline{\gamma_1 \gamma_2}$ for all γ_1, $\gamma_2 \in \Gamma$ and

$$A = \{x \in M_n(E) \mid \overline{\gamma}(x) = x \text{ for all } \gamma \in \Gamma\}. \tag{36}$$

The map

$$\beta_\gamma = \overline{\gamma} \circ \big(1_{M_n(F)} \otimes \gamma\big)^{-1}$$

is an E-linear automorphism of $\big(M_n(E), t\big)$ hence is of the form $\operatorname{Int}(f_\gamma)$ for $f_\gamma \in \mathrm{GO}_n(E)$. Thus there is a 1-1-correspondence between the the set of the $\overline{\gamma}$ and the set of the β_γ. The β_γ satisfy the relations

$$\beta_{\gamma_1\gamma_2} = \beta_{\gamma_1} \gamma_1(\beta_{\gamma_2}) \tag{37}$$

where for any $[f] \in \mathrm{PGO}_n(E)$, $\gamma([f]) = [\operatorname{Int}\big(1_{M_n(F)} \otimes \gamma\big)(f)]$. Conversely, given a set $\{\beta_\gamma,\ \gamma \in \Gamma\}$, satisfying (37), and putting $\overline{\gamma} = \beta_\gamma \circ \big(1_{M_n(E)} \otimes \gamma\big)$, then (36) defines a central simple algebra A over F of degree n with an orthogonal involution σ, split by E. The map

[3] this is not the usual definition; In the usual definition one considers the involution induced by the bilinear form of maximal index

$\Gamma \to \mathrm{PGO}_n(E)$, $\gamma \mapsto \beta_\gamma$, is a *cocycle* of Γ with values in $\mathrm{PGO}_n(E)$. Two cocycles β_γ and β'_γ are *equivalent* if there exists $[b] \in \mathrm{PGO}_n(E)$ satisfying

$$\beta'_\gamma = [b]\beta_\gamma \gamma([b])^{-1}.$$

Equivalent cocycles define isomorphic algebras with involution. The set of cocycles modulo equivalence is written $H^1\big(\Gamma, \mathrm{PGO}_n(E)\big)$. It is bijective with the isomorphism classes of central simple algebra A over F of degree n with an orthogonal involution σ, split by E.

As in Proposition 9 we define for each $\overline{\gamma}$, a semilinear automorphism $C(\overline{\gamma})$ of $\big(C_0(n), \tau_0\big)$ and obviously (!) $C(\overline{\gamma}_1) \circ C(\overline{\gamma}_2) = C(\overline{\gamma_1\gamma_2})$ for all γ_1, $\gamma_2 \in \Gamma$. The F-algebra

$$C(A, \sigma) = \{x \in C_0(n) \mid C(\overline{\gamma})(x) = x \text{ for all } \gamma \in \Gamma\}$$

is such that $C(A, \sigma) \otimes E \xrightarrow{\sim} C_0(n)$ and the involution τ_0 of $C_0(n)$ induces an involution $\underline{\sigma}$ of $C(A, \sigma)$. The algebra with involution $\big(C(A, \sigma), \underline{\sigma}\big)$ does not depend (up to canonical isomorphism) on the choice of the splitting field E/F and is the *Clifford algebra* of (A, σ). Equivalently, if (A, σ) is defined by the cocycle β_γ, then $C(A, \sigma)$ is defined by the cocycle $C(\beta_\gamma)$. The Clifford algebra of $(\mathrm{End}_F(V), \sigma_q)$ is canonically isomorphic to $C_0(V, q)$ for any quadratic space (V, q). The Clifford algebra $C(A, \sigma)$ was first defined by descent (as here) by Jacobson (1964) and later rationally (i.e., without descent) by Tits (1968). Another rational construction is in the Book of Involution. The construction is *functorial*, in the sense that any isomorphism $\beta\colon (A, \sigma) \xrightarrow{\sim} (A', \sigma')$ induces an isomorphism $C(\beta)\colon C(A, \sigma) \xrightarrow{\sim} C(A', \sigma')$. Assume for example that $\beta\colon (A, \sigma) \otimes E \xrightarrow{\sim} \big(M_n(E), t\big) = \big(M_n(F), t\big) \otimes E$ is a splitting of algebras with involution (one says that (A, σ) is a *twisted form* of $\big(M_n(F), t\big)$), then

$$C(\beta)\colon C(A, \sigma) \otimes E \xrightarrow{\sim} C\big(M_n(F), t\big) \otimes E = C_0(n).$$

The algebra $C(A, \sigma)$ has the same structure as $C_0(n)$. For example (still assuming n even) its center $Z = Z\big(C(A, \sigma)\big)$ is a separable quadratic extension of F and $C(A, \sigma)$ is central simple over Z if Z is a field. Let $Z = F[x]/(x^2 - d)$. We define the *discriminant* of σ as $\mathrm{disc}(\sigma) = [d] \in F^\times/F^\times 2$. If the discriminant is trivial (i.e., $\mathrm{disc}(\sigma) = 1$) then

$$C(A, \sigma) \simeq C(A, \sigma)^+ \times C(A, \sigma)^+ \qquad (38)$$

and $C(A, \sigma)^+$, $C(A, \sigma)^-$ are central simple algebras over F. Fix an isomorphism $\zeta : Z\big(C(A, \sigma)\big) \xrightarrow{\sim} F \times F$. The cocycle $C(\beta_\gamma)$ restricts to the identity of Z, hence lies in $\mathrm{PGO}_n^+(E)$. Conversely given any cocycle with values in $\mathrm{PGO}_n^+(E)$, $C(\beta_\gamma)$ will restrict to the identity on $Z\big(C_0(n)\big) \otimes E$, hence induce an isomorphism $\zeta : Z\big(C(A, \sigma)\big) \xrightarrow{\sim} F \times F$. Hence $H^1\big(\Gamma, \mathrm{PGO}_n^+(E)\big)$ classifies triples $\big(A, \sigma, \zeta : Z\big(C(A, \sigma)\big) \xrightarrow{\sim} F \times F\big)$ which are split over E. Observe that the image of $H^1\big(\Gamma, \mathrm{PGO}_n^+(E)\big)$ in $H^1\big(\Gamma, \mathrm{PGO}_n(E)\big)$ classifies algebras with involutions having trivial discriminant; however the map

$$H^1\big(\Gamma, \mathrm{PGO}_n^+(E)\big) \to H^1\big(\Gamma, \mathrm{PGO}_n(E)\big)$$

need not be injective.

12. Central Simple Algebras and Triality

Assume from now on that $n = 8$. Triality acts on $\mathrm{PGO}_8^+(E)$ and, by fonctoriality, on $H^1\big(\Gamma, \mathrm{PGO}_8^+(E)\big)$. Our next aim is to describe this action.

Let A be central simple of degree 8 with an orthogonal involution which is still supposed to have trivial discriminant. The algebras $C(A,\sigma)^+$ and $C(A,\sigma)^+$ in a decomposition (38) are also central simple of degree 8 and the transport of $\underline{\sigma}$ restricts to orthogonal involutions σ^+ of $C(A,\sigma)^+$, resp. σ^- of $C(A,\sigma)^-$. There a canonical choice of $C(A,\sigma)^+$ and $C(A,\sigma)^-$, related to the isomorphism $\alpha_0\colon C_0(8) \xrightarrow{\sim} M_8 \times M_8$ of Proposition 13:

Proposition 39. *Let $\beta_A\colon (A,\sigma_A)\otimes E \xrightarrow{\sim} \big(M_8(E), t\big)$ be a Galois splitting of (A,σ_A). There exist*
1) central simple algebras B, C, of degree 8 with orthogonal involutions σ_B, σ_C of trivial discriminant,
2) splittings $\beta_B\colon (B,\sigma_B)\otimes E \xrightarrow{\sim} \big(M_8(E), t\big)$, $\beta_C\colon (C,\sigma_C)\otimes E \xrightarrow{\sim} \big(M_8(E), t\big)$ and
3) an isomorphism $\alpha_A\colon C(A,\sigma_A) \xrightarrow{\sim} (B,\sigma_B)\times(C,\sigma_C)$
such that

$$\alpha_0 \circ C(\beta_A) = (\beta_B, \beta_C)\circ(\alpha_A \otimes 1_E).$$

Proof. If $\beta_\gamma \in \mathrm{PGO}_8^+(E)$ is a cocycle defining (A,σ_A), then the cocycles defining (B,σ_B) and (C,σ_C) are given by $\rho(\beta_\gamma)$ and $\rho^2(\beta_\gamma)$, where ρ acts as in Proposition 27. $\square$

Triality then implies:

Corollary 40. *The set $H^1\big(\Gamma, \mathrm{PGO}_8^+(E)\big)$ classifies triples (A,B,C) together with canonical isomorphisms of algebras with involution $\alpha_A\colon C(A,\sigma_A) \xrightarrow{\sim} B\times C$, $\alpha_B\colon C(B,\sigma_B) \xrightarrow{\sim} C\times A$, $\alpha_C\colon C(C,\sigma_C) \xrightarrow{\sim} A\times B$ and S_3 acts as permutations on (A,B,C).*

Remark 41. A characterization of the possible triples (A,B,C) in (40) is not known. A necessary condition is $[A][B][C] = 1 \in \mathrm{Br}(F)$, but the condition is not sufficient: one can show that the triple $\big(\mathrm{End}_F(V), A, A\big)$ occur if and only if A is a tensor product of three quaternions algebras. However there exist, over certain fields, central division algebras of degree 8, which admit orthogonal involutions with trivial discriminant, and which are not tensor products of three quaternions algebras. Examples of algebras of degree 8 with orthogonal involution, which are not tensor products of three quaternion algebras are due to Amitsur, Rowen, Tignol; the existence of an involution of trivial discriminant on such an algebra is due to Parimala, Sridharan, Suresh. The condition $[A][B][C] = 1 \in \mathrm{Br}(F)$ is however sufficient if $[A] = [(a,b)_F]$, $[B] = [(a,c)_F]$ and $[C] = [(a,bc)_F]$ for quaternion algebras $(p,q)_F$.

Remark 42. The isomorphism d_ρ of (18) restricts to an isomorphism of Lie algebras $\mathrm{Skew}(A,\sigma_A) \xrightarrow{\sim} \mathrm{Skew}(B,\sigma_B)$ even if $A \not\simeq B$. Similarly, let $\mathrm{PGO}^+(A,\sigma)$ be the connected component of $\mathrm{Aut}_F(A,\sigma)$. Then ρ induces an isomorphism $\mathrm{PGO}^+(A,\sigma_A) \xrightarrow{\sim} \mathrm{PGO}^+(B,\sigma_B)$.

The split exact sequence of algebraic groups

$$1 \to \mathrm{PGO}_8^+ \to \mathrm{PGO}_8^+ \rtimes S_3 \to S_3 \to 1$$

induces a sequence of pointed sets in Galois cohomology

$$\to H^1\big(\Gamma, \mathrm{PGO}_8^+(E)\big) \to H^1\big(\Gamma, \mathrm{PGO}_8^+(E) \rtimes S_3\big) \to H^1(\Gamma, S_3).$$

The set $H^1(\Gamma, S_3)$ classifies cubic étale F-algebras L which are split by E, i.e., such that $L \otimes E \simeq E \times E \times E$ and $H^1\big(\Gamma, \mathrm{PGO}_8^+\big)$ was described above. Following the Book of Involution, we introduce algebraic objects which are classified by $H^1\big(\Gamma, \mathrm{PGO}_8^+(E) \rtimes S_3\big)$.

We view the triple (A,B,C) as an algebra T over $F \times F \times F$ with an involution $\sigma_T = (\sigma_A, \sigma_B, \sigma_C)$. The triple $\alpha_T = (\alpha_A, \alpha_B, \alpha_C)$ then is an isomorphism of $C(T,\sigma_T)$ with the $F\times F\times F$-algebra $(B\times C)\times(C\times A)\times(A\times B)$, which in term can be viewed as $\rho\big(T\otimes(F\times F)\big)$ where

$$\rho\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} = \begin{pmatrix} b_1 & c_2 \\ c_1 & a_2 \\ a_1 & b_2 \end{pmatrix} \tag{43}$$

If $(A,\sigma) = \big(M_8(F), t\big)$ is split, then also $(B,\sigma) = (C,\sigma) = \big(M_8(F), t\big)$ and viewing $\mathfrak{o}(8) \times \mathfrak{o}(8) \times \mathfrak{o}(8)$ as a Lie subalgebra of $C_0(T)$ through ad^{-1}, we have

$$\alpha_T \circ \mathrm{ad}^{-1}(x,y,z) = \Big(\big(d_\rho(y), d_{\rho^2}(z)\big), \big(d_\rho(z), d_{\rho^2}(x)\big), \big(d_\rho(x), d_{\rho^2}(y)\big)\Big)$$

for $(x,y,z) \in \mathfrak{o}(8) \times \mathfrak{o}(8) \times \mathfrak{o}(8)$. We say that such a T is split.

Let L be a cubic étale F-algebra split by E (this is no restriction, since E can be taken as big as necessary) and let T be an L-algebra. If L is not a field we say that T is *central simple over* L if each component of T with respect to a field component of L is central simple. Equivalently $T \otimes \overline{F}$ is isomorphic to three copies of $M_8(\overline{F})$.

Let T be central simple over L and let σ_T be an orthogonal involution of T. To extend the definition of α_T given above for $L = F \times F \times F$ to arbitrary cubic L we need an isomorphism

$$\alpha_T \colon \big(C(T,\sigma_T), \underline{\sigma}\big) \xrightarrow{\sim} (T_2, \sigma_2)$$

where (T_2,σ_2) is a central simple algebra with involution over a quadratic extension of L, which is functorially associated with (T,σ_T). We take $L \otimes \Delta/L$ as a quadratic extension, where Δ/F is the discriminant of L, viewed as a quadratic F-algebra. Then $L \otimes \Delta/F$

is a Galois extension with group S_3 and $L \otimes \Delta/\Delta$ is a Galois extension with group the alternating group A_3, such that

$$\big(L \otimes \Delta\big) \otimes E \xrightarrow{\sim} (E \times E) \times (E \times E) \times (E \times E).$$

Let ρ be a generator of $\mathrm{Gal}(L \otimes \Delta/L)$; for any L-module V we denote ${}^{\rho}(V \otimes \Delta)$ the module $V \otimes \Delta$ with $L \otimes \Delta$-action twisted through ρ. We set

$$(T_2, \sigma_2) = {}^{\rho}\big((T, \sigma) \otimes_F \Delta\big)$$

and say that (T, σ_T) is a *trialitarian F-algebra* if there exists an isomorphism

$$\alpha_T \colon \big(C(T, \sigma_T), \underline{\sigma}\big) \xrightarrow{\sim} {}^{\rho}\big((T, \sigma) \otimes_F \Delta\big) \tag{44}$$

which over some Galois extension E/F reduces to the split α_T described above. A trialitarian algebra is given by a quadruple $(T, L, \sigma_T, \alpha_T)$ and an *isomorphism of trialitarian algebras*

$$\Psi \colon (T, L, \sigma_T, \alpha_T) \xrightarrow{\sim} (T', L', \sigma_{T'}, \alpha_{T'})$$

is a pair (ψ, ϕ) with $\phi \colon L \xrightarrow{\sim} L'$ and $\psi \colon (T, \sigma_T) \xrightarrow{\sim} (T', \sigma_{T'})$ such that

$$\alpha_{T'} \circ C(\psi) = {}^{\rho}\big(\psi \otimes \Delta(\phi)\big).$$

Trialitarian F-algebras which are split over the Galois extension E are classified by the pointed set $H^1\big(\Gamma, \mathrm{PGO}^+(8) \rtimes S_3\big)$. Let Z_T be the center of $C(T, \sigma_T)$. The isomorphism (44) restricts to an isomorphism $Z_T \xrightarrow{\sim} {}^{\rho}(L \otimes \Delta)$ of the centers, which we use to identify Z_T with $L \otimes \Delta$. In view of Lemma 8 we may identify $\mathrm{Skew}\big(C(T, \sigma_T), \underline{\sigma}\big)$ with $\mathrm{Skew}(T, \sigma_T) \otimes_L Z_T = \mathrm{Skew}(T, \sigma_T) \otimes_F \Delta$ so that α_T restricts to an isomorphism

$$\alpha_\rho \colon \mathrm{Skew}(T, \sigma_T) \otimes_F \Delta \xrightarrow{\sim} {}^{\rho}\big(\mathrm{Skew}(T, \sigma_T) \otimes_F \Delta\big)$$

which, in turn, can be viewed as a ρ-semilinear automorphism of $\mathrm{Skew}(T, \sigma_T) \otimes_F \Delta$ as a Lie algebra. In the split case $T = M_8 \times M_8 \times M_8$ we have by (43)

$$\alpha_\rho \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{pmatrix} = \begin{pmatrix} d_\rho(y_1) & d_{\rho^2}(z_2) \\ d_\rho(z_1) & d_{\rho^2}(x_2) \\ d_\rho(x_1) & d_{\rho^2}(y_2) \end{pmatrix}$$

Let $\alpha_\pi = 1_{\mathrm{Skew}(T, \sigma_T)} \otimes \iota$, where ι is conjugation on the quadratic algebra Δ. In the split case we have

$$\alpha_\pi \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{pmatrix} = \begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \\ z_2 & z_1 \end{pmatrix}.$$

Proposition 45. *The automorphisms α_ρ and α_π generate a subgroup of* $\mathrm{Aut}_F\big(\mathrm{Skew}(T,\sigma_T)\big)$ *of semilinear automorphisms isomorphic to* $S_3 = \mathrm{Gal}(L\otimes\Delta/F)$. *The fixed points of the action*

$$\mathfrak{o}(T) = \{x \in \mathrm{Skew}(T,\sigma_T) \mid \alpha_\gamma(x) = x \quad \textit{for all } \gamma \in S_3\}$$

is a Lie algebra of type D_4 associated with the triality T.

Proof. The first claim follows from the explicit description of α_ρ and α_π in the split case. The last follows by descent. □

If L/F is cubic cyclic the discriminant $\Delta(L)$ is split,

$$^{\rho}\big(L\otimes\Delta\big) = {}^{\rho}L \times {}^{\rho^2}L$$

and

$$\alpha_T\colon \big(C(T,\sigma_T),\underline{\sigma}\big) \xrightarrow{\sim} {}^{\rho}(T,\sigma) \times {}^{\rho^2}(T,\sigma)$$

We say in this case that T is *cyclic trialitarian.* Cyclic trialitarian algebras are classified by the pointed set $H^1\big(\Gamma, \mathrm{PGO}^+(8)\rtimes A_3\big)$. Denoting the restriction of the two components of α_T to $\mathrm{Skew}(T,\sigma_T)$ by $(\alpha_\rho,\alpha_{\rho^2})$, we have $\alpha_{\rho^2} = \alpha_\rho^2$ and get a Galois descent data for $\mathrm{Skew}(T,\sigma_T)$ from L to F.

Examples 46. 1) If L decomposes as $F\times Z$, Z a quadratic separable extension of F, we have a corresponding decomposition $T = A\times C$ for (C,σ_C) a central simple Z-algebra with orthogonal involution. then (Book of Involutions) $(C,\sigma_C)\simeq \big(C(A,\sigma_A),\underline{\sigma}\big)$.
2) Let L be a cubic cyclic field extension of F, with ρ a generator of $\mathrm{Gal}(E/F)$. We extend the multiplication $\star$ used in Section 6 on $\widetilde{\mathbb{O}} = L\otimes\mathbb{O}$ by $(l\otimes x)\star(l'\otimes x') = \rho(l)\rho^2(l')\otimes x\star x'$. Then α_0 extends to an isomorphism

$$C\big(\mathrm{End}_L(\widetilde{\mathbb{O}}),t\big) \xrightarrow{\sim} {}^{\rho}\mathrm{End}_L(\widetilde{\mathbb{O}}) \times {}^{\rho^2}\mathrm{End}_L(\widetilde{\mathbb{O}})$$

which defines a trialitarian structure on $\mathrm{End}_L(\widetilde{\mathbb{O}})$.
3) Using the generic formulas given in Section 7, it is possible to define a generic trialitarian algebra, which is a division algebra over its center. Details are in a forthcoming joint paper with Parimala and Sridharan.