# Quadratic quaternion forms, involutions and triality

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Abstract. Quadratic quaternion forms, introduced by Seip-Hornix (1965), are special cases of generalized quadratic forms over algebras with involutions. We apply the formalism of these generalized quadratic forms to give a characteristic free version of different results related to hermitian forms over quaternions:
1) An exact sequence of Lewis
2) Involutions of central simple algebras of exponent 2.
3) Triality for 4-dimensional quadratic quaternion forms.

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## 1. Introduction

Let $F$ be a field of characteristic not 2 and let $D$ be a quaternion division algebra over $F$. It is known that a skew-hermitian form over $D$ determines a symmetric bilinear form over any separable quadratic subfield of $D$ and that the unitary group of the skew-hermitian form is the subgroup of the orthogonal group of the symmetric bilinear form consisting of elements which commute with a certain semilinear mapping (see for example Dieudonné [3]). Quadratic forms behave nicer than symmetric bilinear forms in characteristic 2 and Seip-Hornix developed in [9] a complete, characteristic-free theory of quadratic quaternion forms, their orthogonal groups and their classical invariants. Her theory was subsequently (and partly independently) generalized to forms over algebras (even rings) with involution (see [11], [10], [1], [8]).
Similitudes of hermitian (or skew-hermitian) forms induce involutions on the endomorphism algebra of the underlying space. To generalize the case where only similitudes of a quadratic form are considered, the notion of a quadratic pair was worked out in [6]. Relations between quadratic pairs and generalized quadratic forms were first discussed by Elomary [4].
The aim of this paper is to apply generalized quadratic forms to give a characteristic free presentation of some results on forms and involutions. After briefly recalling in Section 2 the notion of a generalized quadratic form (which, following the standard literature, we call a $(\varepsilon, \sigma)$-quadratic form) we give in Section

3 a characteristic-free version of an exact sequence of Lewis (see [7], [8, p. 389] and the appendix to [2]), which connects Witt groups of quadratic and quaternion algebras. The quadratic quaternion forms of Seip-Hornix are the main ingredient. Section 4 describes a canonical bijective correspondence between quadratic pairs and $(\varepsilon,\sigma)$-quadratic forms and Section 5 discusses the Clifford algebra. In particular we compare the definitions given in [10] and in [6]. In Section 6 we develop triality for 4-dimensional quadratic quaternion forms whose associated forms (over a separable quadratic subfield) are 3-Pfister forms. Any such quadratic quaternion form $\theta$ is an element in a triple $(\theta_1,\theta_2,\theta_3)$ of forms over 3 quaternions algebras $D_1$, $D_2$ and $D_3$ such that $[D_1][D_2][D_3]=1$ in the Brauer group of $F$. Triality acts as permutations on such triples.

## 2. Generalized quadratic forms

Let $D$ be a division algebra over a field $F$ with an involution $\sigma : x \mapsto \overline{x}$. Let $V$ be a finite dimensional right vector space over $D$. A $F$-bilinear form

$$k : V \times V \to D$$

is *sesquilinear* if $k(xa, yb) = \overline{a}k(x,y)b$ for all $x$, $y \in V$, $a$, $b \in D$. The additive group of such maps will be denoted by $\mathrm{Sesq}_\sigma(V,D)$. For any $k \in \mathrm{Sesq}_\sigma(V,D)$ we write

$$k^*(x,y) = \overline{k(y,x)}.$$

Let $\varepsilon \in F^\times$ be such that $\varepsilon\overline{\varepsilon} = 1$. A sesquilinear form $k$ such that $k = \varepsilon k^*$ is called *$\varepsilon$-hermitian* and the set of such forms on $V$ will be denoted by $\mathrm{Herm}^\varepsilon_\sigma(V,D)$. Elements of

$$\mathrm{Alt}^\varepsilon_\sigma(V,D) = \{g = f - \varepsilon f^* \mid f \in \mathrm{Sesq}_\sigma(V,D)\}.$$

are *$\varepsilon$-alternating forms*. We obviously have $\mathrm{Alt}^{-\varepsilon}_\sigma(V,D) \subset \mathrm{Herm}^\varepsilon_\sigma(V,D)$. We set

$$\mathrm{Q}^\varepsilon_\sigma(V,D) = \mathrm{Sesq}_\sigma(V,D)/\,\mathrm{Alt}^\varepsilon_\sigma(V,D)$$

and refer to elements of $\mathrm{Q}^\varepsilon_\sigma(V,D)$ as *$(\varepsilon,\sigma)$-quadratic forms*. We recall that $(\varepsilon,\sigma)$-quadratic forms were introduced by Tits [10], see also Wall [11], Bak [1] or Scharlau [8, Chapter 7]. For any algebra $A$ with involution $\tau$, let $\mathrm{Sym}^\varepsilon(A,\tau) = \{a \in A \mid a = \varepsilon\tau(a)\}$ and $\mathrm{Alt}^\varepsilon(A,\tau) = \{a \in A \mid a = c - \varepsilon\tau(c),\ c \in A\}$. To any class $\theta = [k] \in \mathrm{Q}^\varepsilon_\sigma(V,D)$, represented by $k \in \mathrm{Sesq}_\sigma(V,D)$, we associate a quadratic map

$$q_\theta : V \to D/\,\mathrm{Alt}^\varepsilon(D,\sigma), \quad q_\theta(x) = [k(x,x)]$$

where $[d]$ denotes the class of $d$ in $D/\,\mathrm{Alt}^\varepsilon_\sigma(D)$. The $\varepsilon$-hermitian form

$$b_\theta(x,y) = k(x,y) + \varepsilon k^*(x,y) = k(x,y) + \varepsilon\overline{k(y,x)}$$

depends only on the class $\theta$ of $k$ in $\mathrm{Q}^\varepsilon_\sigma(V,D)$. We say that $b_\theta$ is the *polarization* of $q_\theta$.

PROPOSITION 2.1. *The pair $(q_\theta, b_\theta)$ satisfies the following formal properties:*

$$\begin{array}{lcl} q_\theta(x+y) & = & q_\theta(x) + q_\theta(y) + [b_\theta(x,y)] \\ q_\theta(xd) & = & \overline{d} q_\theta(x) d \\ b_\theta(x,x) & = & q_\theta(x) + \varepsilon \overline{q_\theta(x)} \end{array} \tag{1}$$

*for all $x$, $y \in V$, $d \in D$. Conversely, given any pair $(q,b)$, $q : V \to D/\operatorname{Alt}^\varepsilon(D,\sigma)$, $b \in \operatorname{Herm}^\varepsilon_\sigma(V,D)$ satisfying* (1)*, there exist a unique $\theta \in \mathrm{Q}^\varepsilon_\sigma(V,D)$ such that $q = q_\theta$, $b = b_\theta$.*

*Proof.* The formal properties are straightforward to verify. For the converse see [11, Theorem 1]. □

EXAMPLE 2.2. Let $D = F$, $\sigma = Id_F$ and $\varepsilon = 1$. Then sesquilinear forms are $F$-bilinear forms, $\operatorname{Alt}^\varepsilon(D,\sigma) = 0$ and a $(\sigma,\varepsilon)$-quadratic form is a (classical) quadratic form. We denote the set of bilinear forms on $V$ by $\operatorname{Bil}(V,F)$. Accordingly we speak of $\varepsilon$-symmetric bilinear forms instead of $\varepsilon$-hermitian forms.

EXAMPLE 2.3. Let $D$ be a division algebra with involution $\sigma$ and let $D$ be a finite dimensional (right) vector space over $D$. We use a basis of $V$ to identify $V$ with $D^n$ and $\operatorname{End}_D(V)$ with the algebra $M_n(D)$ of $(n \times n)$-matrices with entries in $D$. For any $(n \times m)$-matrix $x = (x_{ij})$, let $x^* = \overline{x}^t$, where $t$ is transpose and $\overline{x} = (\overline{x}_{ij})$. In particular the map $a \mapsto a^*$ is an involution of $A = M_n(D)$. If we write elements of $D^n$ as column vectors $x = (x_1, \ldots, x_n)^t$ any sesquilinear form $k$ over $D^n$ can be expressed as $k(x,y) = x^* a y$, with $a \in M_n(D)$, and $k^*(x,y) = x^* a^* y$. We write $\operatorname{Alt}_n(D) = \{a = b - \varepsilon b^*\} \subset M_n(D)$, so that $\mathrm{Q}^\varepsilon_\sigma(V,D) = M_n(D)/\operatorname{Alt}_n(D)$.

EXAMPLE 2.4. Let $D$ be a quaternion division algebra, i.e. $D$ is a central division algebra of dimension 4 over $F$. Let $K$ be a maximal subfield of $D$ which is a quadratic Galois extension of $F$ and let $\sigma : x \mapsto \overline{x}$ be the nontrivial automorphism of $K$. Let $j \in K \setminus F$ be an element of trace 1, so that $K = F(j)$ with $j^2 = j + \lambda$, $\lambda \in F$. Let $\ell \in D$ be such that $\ell x \ell^{-1} = \overline{x}$ for $x \in K$, $\ell^2 = \mu \in F^\times$. The elements $\{1, j, \ell, \ell j\}$ form a basis of $D$ and $D = K \oplus \ell K$ is also denoted $[K, \mu)$. The $F$-linear map $\sigma : D \to D$, $\sigma(d) = \operatorname{Trd}_D(d) - d = \overline{d}$ is an involution of $D$ (the "conjugation") which extends the automorphism $\sigma$ of $K$. The element $\mathrm{N}(d) = d\sigma(d) = \sigma(d)d$ is the reduced norm of $d$. We have $\operatorname{Alt}^{-1}_\sigma(D) = F$ and $(\sigma,-1)$-quadratic forms correspond to the quadratic quaternion forms introduced by Seip-Hornix in [9]. Accordingly we call $(\sigma,-1)$-quadratic forms *quadratic quaternion forms.*

The restriction of the involution $\tau$ to the center $Z$ of $A$ is either the identity (*involutions of the first kind*) or an automorphism of order 2 (*involutions of the second kind*). If the characteristic of $F$ is different from 2 or if the involution is of second kind there exists an element $j \in Z$ such that $j + \sigma(j) = 1$. Under such conditions the theory of $(\sigma,\varepsilon)$-quadratic forms reduces to the theory of $\varepsilon$-hermitian forms:

PROPOSITION 2.5. *If the center of $D$ contains an element $j$ such that $j+\sigma(j) = 1$, then $\operatorname{Herm}^{-\varepsilon}_\sigma(V,D) = \operatorname{Alt}^{\varepsilon}_\sigma(V,D)$ and a $(\sigma,\varepsilon)$-quadratic form is uniquely determined by its polar form $b_\theta$.*

*Proof.* If $k = -\varepsilon k^* \in \operatorname{Herm}^{-\varepsilon}_\sigma(V,D)$, then $k = 1k = jk + \overline{j}k = jk - \overline{j}\varepsilon k^* \in \operatorname{Alt}^{\varepsilon}_\sigma(V,D)$. The last claim follows from the fact that polarization induces an isomorphism $\operatorname{Sesq}_\sigma(V,D)/\operatorname{Herm}^{-\varepsilon}_\sigma(V,D) \xrightarrow{\sim} \mathrm{Q}^{\varepsilon}_\sigma(V,D)$. □

For any left (right) $D$-space $V$ we denote by ${}^\sigma V$ the space $V$ viewed as right (left) $D$-space through the involution $\sigma$. If ${}^\sigma x$ is the element $x$ viewed as an element of ${}^\sigma V$, we have ${}^\sigma x d = {}^\sigma\big(\sigma(d)x\big)$. Let $V^*$ be the dual ${}^\sigma\operatorname{Hom}_D(V,D)$ as a right $D$-module, i.e., $({}^\sigma f d)(x) = {}^\sigma(\overline{d}f)(x)$, $x \in V$, $d \in D$. Any sesquilinear form $k \in \operatorname{Sesq}_\sigma(V,D)$ induces a $D$-module homomorphism $\widehat{k} : V \to V^*$, $x \mapsto k(x,-)$. Conversely any homomorphism $g : V \to V^*$ induces a sesquilinear form $k \in \operatorname{Sesq}_\sigma(V,D)$, $k(x,y) = g(x)(y)$ and the additive groups $\operatorname{Sesq}_\sigma(V,D)$ and $\operatorname{Hom}_D(V,V^*)$ can be identified through the map $h \mapsto \widehat{k}$. For any $f : V \to V'$, let $f^* : V'^* \to V^*$ be the transpose, viewed as a homomorphisms of right vector spaces. We identify $V$ with $V^{**}$ through the map $v \mapsto v^{**}$, $v^{**}(f) = \overline{f(v)}$. Then, for any $f \in \operatorname{Hom}_D(V,V^*)$, $f^*$ is again in $\operatorname{Hom}_D(V,V^*)$ and $\widehat{k}^* = \widehat{k^*}$. A $(\sigma,\varepsilon)$-quadratic form $q_\theta$ is called *nonsingular* if its polar form $b_\theta$ induces an isomorphism $\widehat{b_\theta}$. A pair $(V,q_\theta)$ with $q_\theta$ nonsingular is called a $(\sigma,\varepsilon)$-*quadratic space*. For any vector space $W$, the *hyperbolic space* $V = W \oplus W^*$ equipped with the quadratic form $q_\theta$, $\theta = [k]$ with

$$k\big((p,q),(p',q')\big) = q(p'),$$

is nonsingular. There is an obvious notion of orthogonal sum $V \perp V'$ and a quadratic space decomposes whenever its polarization does. Most of the classical theory of quadratic spaces extends to $(\sigma,\varepsilon)$-quadratic spaces. For example Witt cancellation holds and any $(\sigma,\varepsilon)$-quadratic space decomposes uniquely (up to isomorphism) as the orthogonal sum of its anisotropic part with a hyperbolic space. Moreover, if we exclude the case $\sigma = 1$ and $\varepsilon = -1$, any $(\sigma,\varepsilon)$-quadratic space has an orthogonal basis. A *similitude* of $(\sigma,\varepsilon)$-quadratic spaces $t : (V,q) \xrightarrow{\sim} (V',q')$ is a $D$-linear isomorphism $V \xrightarrow{\sim} V'$ such that $q'(tx) = \mu(t)q(x)$ for some $\mu(t) \in F^\times$. The element $\mu(t)$ is called the *multiplier* of the similitude. Similitudes with multipliers equal to 1 are *isometries*. As in the classical case there is a notion of Witt equivalence and corresponding Witt groups are denoted by $W^\varepsilon(D,\sigma)$.

## 3. AN EXACT SEQUENCE OF LEWIS

Let $D$ be a quaternion division algebra. We fix a representation $D = [K,\mu) = K \oplus \ell K$, with $\ell^2 = \mu$, as in (2.4). Let $V$ be a vector space over $D$. Any sesquilinear form $k : V \times V \to D$ can be decomposed as

$$k(x,y) = P(x,y) + \ell R(x,y)$$

with $P : V \times V \to K$ and $R : V \times V \to K$. The following properties of $P$ and $R$ are straightforward.

Lemma 3.1. *1)* $P \in \mathrm{Sesq}_\sigma(V,K)$, $R \in \mathrm{Sesq}_1(V,K) = \mathrm{Bil}(V,K)$.
*2)* $k^* = P^* - \ell R^t$, *where* $P^*(x,y) = \overline{P(y,x)}$ *and* $R^t(x,y) = R(y,x)$.

The sesquilinearity of $k$ implies the following identities:

$$\begin{array}{lrclrcl} & R(x\ell,y) & = & -P(x,y), & R(x,y\ell) & = & \overline{P(x,y)} \\ (2) & P(x\ell,y) & = & -\mu\overline{R(x,y)}, & P(x,y\ell) & = & \mu\overline{R(x,y)} \\ & P(x\ell,y\ell) & = & -\mu\overline{P(x,y)}, & R(x\ell,y\ell) & = & -\mu\overline{R(x,y)} \end{array}$$

Let $V^0$ be $V$ considered as a (right) vector space over $K$ (by restriction of scalars) and let $T : V^0 \to V^0, x \mapsto x\ell$. The map $T$ is a $K$-semilinear automorphism of $V^0$ such that $T^2 = \mu$. Conversely, given a vector spave $U$ over $K$, together with a semilinear automorphism $T$ such that $T^2 = \mu \in F^\times$, we define the structure of a right $D$-module on $U$, $D = [K,\mu)$, by putting $x\ell = T(x)$.

Lemma 3.2. *Let $V$ be a vector space over $D$. 1) Let $f_1 : V^0 \times V^0 \to K$ be a sesquilinear form over $K$. The form*

$$f(x,y) = f_1(x,y) - \ell\mu^{-1} f_1(Tx,y)$$

*is sesquilinear over $D$ if and only if $f_1(Tx,Ty) = -\mu\overline{f_1(x,y)}$.*
*2) Let $f_2 : V^0 \times V^0 \to K$ be a bilinear form over $K$. The form*

$$f(x,y) = -f_2(Tx,y) + \ell f_2(x,y)$$

*is sesquilinear over $D$ if and only if $f_2(Tx,Ty) = -\mu\overline{f_2(x,y)}$.*

*Proof.* The two claims follow from the identities (2). □

Let $f$ be a bilinear form on a space $U$ over $K$ and let $\lambda \in K^\times$. A semilinear automorphism $t$ of $U$ such that $f(tx,ty) = \lambda\overline{f(x,y)}$ for all $x \in U$ is a *semilinear similitude* of $(U,f)$, with *multiplier* $\lambda$. In particular $Tx = x\ell$ is a semilinear similitude of $R$ on $V^0$, such that $T^2 = \mu$ and with multiplier $-\mu$. The following nice observation of Seip-Hornix [9, p. 328] will be used later:

Proposition 3.3. *Let $R$ be a $K$-bilinear form over $U$ and let $T$ be a semilinear similitude of $U$ with multiplier $\lambda \in K^\times$ and such that $T^2 = \mu$. Then:*
*1)* $\mu \in F$,
*2) For any $\xi \in K$ and $x \in U$, let $\rho_\xi(x) = x\xi$. There exists $\nu \in K^\times$ such that $T' = \rho_\nu \circ T$ satisfies ${T'}^2 = \mu'$ and $R(T'x,T'y) = -\mu'\overline{R(x,y)}$.*

*Proof.* The first claim follows from $\mu = \lambda\overline{\lambda}$. For the second we may assume that $\lambda \neq \mu$ (if $\lambda = \mu$ replace $T$ by $T \circ \rho_k$ for an appropriate $k$). For $\nu = (1 - \mu\lambda^{-1})$ we have $\mu' = 2\mu - \lambda - \overline{\lambda}$. □

Assume that $k \in \mathrm{Sesq}_\sigma(V,D)$ defines a $(\sigma,\varepsilon)$-quadratic space $[k]$ on $V$ over $D$. It follows from (3.1) that $P$ defines a $(\sigma,\varepsilon)$-quadratic space $[P]$ on $V^0$ over $K$ and $R$ a $(Id,-\varepsilon)$-quadratic space $[R]$ on $V^0$ over $K$. Let $K = F(j)$ with $j^2 = j + \lambda$. Let $r(x,y) = R(x,y) - \varepsilon R(y,x)$ be the polar of $R$.

PROPOSITION 3.4. *1)* $q_{[P]}(x) = \overline{\varepsilon} j[r(x, Tx)]$
*2)* $q_{[k]}(x) = \overline{\varepsilon} j[r(x, Tx)] + \ell q_{[R]}(x)$
*3) The map $T$ is a semilinear similitude of $\big(q_{[R]}, V^0\big)$ with multiplier $-\mu$.*

*Proof.* It follows from the relations (2) that

$$\overline{P(x,x)} + \varepsilon P(x,x) = R(x,Tx) - \varepsilon R(Tx,x) = r(x,Tx) \tag{3}$$

and obviously this relation determines $P(x,x)$ up to a function with values in $\mathrm{Sym}^{-\varepsilon}(K,\sigma)$. Since $\mathrm{Sym}^{-\varepsilon}(K,\sigma) = \mathrm{Alt}^{+\varepsilon}(K,\sigma)$ by (2.5), $[P]$ is determined by (3). Since $\overline{r(x,Tx)} = \overline{\varepsilon} r(x,Tx)$ by (2), we have $\overline{\overline{\varepsilon} j r(x,Tx)} + \varepsilon\big(\overline{\varepsilon} j r(x,Tx)\big) = r(x,Tx)$ and 1) follows. The second claim follows from 1) and 3) is again a consequence of the identities (2). □

COROLLARY 3.5. *Any pair $\big([R], T\big)$ with $[R] \in \mathrm{Q}_1^{\varepsilon}(U,K)$ and $T$ a semilinear similitude with multiplier $-\mu \in F^{\times}$ and such that $T^2 = \mu$, determines the structure of a $(\sigma,\varepsilon)$-quadratic space on $U$ over $D = [K,\mu)$.*

PROPOSITION 3.6. *The assignments $h \mapsto P$ and $h \mapsto R$ induce homomorphisms of groups $\pi_1 : W^{\varepsilon}(D,-) \to W^{\varepsilon}(K,-)$ and $\pi_2 : W^{-\varepsilon}(D,-) \to W^{\varepsilon}(K,Id)$.*

*Proof.* The assignments are obviously compatible with orthogonal sums and Witt equivalence. □

We recall that $W^{\varepsilon}(K,-)$ can be identified with the corresponding Witt group of $\varepsilon$-hermitian forms (apply (2.5)). However, it is more convenient for the following computations to view $\varepsilon$-hermitian forms over $K$ as $(\sigma,\varepsilon)$- quadratic forms. Let $i \in K^{\times}$ be such that $\sigma(i) = -i$ (take $i = 1$ if $\mathrm{Char}\, F = 2$). The map $k \mapsto ik$ induces an isomorphism $s : W^{\varepsilon}(K,-) \xrightarrow{\sim} W^{-\varepsilon}(K,-)$ ("scaling"). For any space $U$ over $K$, let $U_D = U \otimes_K D$. We identify $U_D$ with $U \oplus U\ell$ through the map $u \otimes (x + \ell y) \mapsto (ux, u\overline{y}l)$ and get a natural $D$-module structure on $U_D = U \oplus U\ell$. Any $K$-sesquilinear form $k$ on $U$ extends to a $D$-sesquilinear form $k_D$ on $U_D$ through the formula

$$k_D(x \otimes a, y \otimes b) = \overline{a} k(x,y) b$$

for $x$, $y \in U$ and $a$, $b \in D$.

LEMMA 3.7. *The assignment $k \mapsto (ik)_D$ induces a homomorphism*

$$\beta : W^{\varepsilon}(K,-) \to W^{-\varepsilon}(D,-)$$

*Proof.* Let $\widetilde{k} = (ik)_D$. We have $(\widetilde{k})^* = -\widetilde{k^*}$. □

THEOREM 3.8 (Lewis). *With the notations above, the sequence*

$$W^{\varepsilon}(D,-) \xrightarrow{\pi_1} W^{\varepsilon}(K,-) \xrightarrow{\beta} W^{-\varepsilon}(D,-) \xrightarrow{\pi_2} W^{\varepsilon}(K,Id)$$

*is exact.*

*Proof.* This is essentially the proof given in Appendix 2 of [2] with some changes due to the use of generalized quadratic forms, instead of hermitian forms. We first check that the sequence is a complex. Let $[k] \in \mathrm{Q}_{\sigma}^{\varepsilon}(V,D)$ and let $V^0 = U$.

We write elements of $U_D = U \oplus U\ell$ as pairs $(x, y\ell)$ and decompose $k_D = P + \ell R$. By definition we have $\beta\pi_1([k]) = [\beta(P)]$ and

$$\begin{aligned}\beta(P)\big((x_1,y_1),(x_2,y_2)\big) &= i\big(P(x_1,x_2)+P(x_1,y_2)\ell+\ell P(y_1,x_2)\\ &\quad +\ell P(y_1,y_2)\ell\big).\end{aligned}$$

Let $(x\ell, x\ell) \in U \oplus U\ell$. We get $\beta(P)\big((x\ell,x\ell),(x\ell,x\ell)\big) = 0$ hence $W = \{(x\ell,x\ell)\} \subset U \oplus U\ell$ is totally isotropic. It is easy to see that $W \subset W^\perp$, so that $[\beta(P)]$ is hyperbolic and $\beta \circ \pi_1 = 0$. Let $[g] \in \mathrm{Q}^\varepsilon_\sigma(U,K)$. The subspace $W = \{(x,0) \in U \oplus U\ell\}$ is totally isotropic for $\pi_2\beta([g])$ and $W \subset W^\perp$. Hence $\pi_2\beta([g]) = 0$. We now prove exactness at $W^\varepsilon(K,-)$. Since the claim is known if Char $\neq 2$, we may assume that Char $= 2$ and $\varepsilon = 1$. Let $[g] \in \mathrm{Q}^\varepsilon_\sigma(U,K)$ be anisotropic such that $\beta([g]) = 0 \in W^{-\varepsilon}(D,-)$. In particular $\beta([g]) \in \mathrm{Q}^{-\varepsilon}_\sigma(U_D,D)$ is isotropic. Hence the exist elements $x_1,\ x_2 \in U$ such that $[\widetilde{g}]\big((x_1,x_2\ell),(x_1,x_2\ell)\big) = 0$. This implies (in Char 2) that

$$g(x_1,x_1)+\mu\overline{g(x_2,x_2)} \in F, \quad g(x_1,x_2)\ell+\ell g(x_2,x_1) = 0. \tag{4}$$

Let $V_1$ be the $K$-subspace of $V$ generated by $x_1$ and $x_2$. Since $[g]$ is anisotropic, $[g] = [g_1] \perp [g_2]$ with $g_1 = g|_{V_1}$. We make $V_1$ into a $D$-space by putting

$$(x_1a_1 + x_2a_2)\ell = \mu x_2\overline{a}_1 + x_1\overline{a}_2$$

To see that the action is well-defined, it suffices to show that $\dim_K V_1 = 2$. The elements $x_1$ and $x_2$ cannot be zero since $[g]$ is anisotropic, so assume $x_2 = x_1c,\ c \in K^\times$. Then (4) implies $g(x_1,x_1) + \mu c\overline{c}\overline{g(x_1,x_1)} \in F$, which contradicts the fact that $g$ is anisotropic. Let $g_1(x_1,x_1)+\mu\overline{g_1(x_2,x_2)} = z \in F$. Let $f \in \mathrm{Sesq}_\sigma(V_1,K)$. Replacing $g_1$ by $g_1 + f + f^*$ defines the same class in $\mathrm{Q}^\varepsilon_\sigma(V_1,K)$ (recall that $\mathrm{Char}\, F = 2$). Choosing $f$ as

$$f(x_1,x_1) = jz,\ \ f(x_2,x_2) = 0,\ \ f(x_1,x_2) = f(x_2,x_1) = 0,$$

we may assume that

$$g_1(x_1,x_1)+\mu\overline{g_1(x_2,x_2)} = 0, \quad g_1(x_1,x_2)\ell+\ell g_1(x_2,x_1) = 0. \tag{5}$$

By (3.2) we may extend $g_1$ to a sesquilinear form

$$g'(x,y) = g_1(x,y) + \ell\mu^{-1}g_1(x\ell,y)$$

over $D$ if $g_1$ satisfies

$$g_1(x\ell,y\ell) = -\mu\overline{g_1(x,y)}$$

This can easily be checked using (5) (and the definition of $x\ell$). Then $g_1$ is in the image of $\pi_1$. Exactness at $W^\varepsilon(K,-)$ now follows by induction on the dimension of $U$. We finally check exactness at $W^{-\varepsilon}(D,-)$. Let $[k]$ be anisotropic such that $\pi_2([k]) = 0$ in $W^{-\varepsilon}(K,Id)$. In particular $\pi_2([k])$ is isotropic; let $x \neq 0$ be such that $\pi_2k(x,x) = 0$ and let $W$ be the $D$-subspace of $V$ generated by $x$. Since $[k]$ is anisotropic, $[k'] = [k|_W]$ is nonsingular and $[k] = [k'] \perp [k'']$. The condition $\pi_2k(x,x) = 0$ implies $k(x,x) \in K$. Let $W_1$ be the $K$-subspace of $W$ generated by $x$. Define $g : W_1 \times W_1 \to K$ by $g(xa,xb) = k(xa,xb)i^{-1}$ for $a$,

$b \in K$. Then clearly $[g]$ defines an element of $W^\varepsilon(K,-)$ and $\beta(g) = k'$. Once again exactness follows by induction on the dimension of $V$. □

## 4. Involutions on central simple algebras

Let $D$ be a central division algebra over $F$, with involution $\sigma$ and let $b : V \times V \to D$ be a nonsingular $\varepsilon$-hermitian form on a finite dimensional space over $D$. Let $A = \mathrm{End}_D(V)$. The map $\sigma_b : A \to A$ such that $\sigma_b(\lambda) = \sigma(\lambda)$ for all $\lambda \in F$ and

$$b\big(\sigma_b(f)(x), y\big) = b\big(x, f(y)\big)$$

for all $x$, $y \in V$, is an involution of $A$, called the involution *adjoint to* $b$. We have $\sigma_b(f) = \widehat{b}^{-1} f^* \widehat{b}$, where $\widehat{b} : V \xrightarrow{\sim} V^*$ is the adjoint of $b$. Conversely, any involution of $A$ is adjoint to some nonsingular $\varepsilon$-hermitian form $b$ and $b$ is uniquely multiplicatively determined up to a $\sigma$-invariant element of $F^\times$.
Any automorphism $\phi$ of $A$ compatible with $\sigma_b$, i.e., $\sigma_b\big(\phi(a)\big) = \phi\big(\sigma_b(a)\big)$, is of the form $\phi(a) = uau^{-1}$ with $u : V \xrightarrow{\sim} V$ a similitude of $b$. We say that an involution $\tau$ of $A$ is a *q-involution* if $\tau$ is adjoint to the polar $b_\theta$ of a $(\sigma, \varepsilon)$-quadratic form $\theta$. We write $\tau = \sigma_\theta$. Two algebras with $q$-involutions are *isomorphic* if the isomorphism is induced by a similitude of the corresponding quadratic forms. Over fields $q$-involutions differ from involutions only in characteristic 2 and for symplectic involutions. In view of possible generalizations (for example rings in which $2 \neq 0$ is not invertible) we keep to the general setting of $(\sigma, \varepsilon)$-quadratic forms. Let $F_0$ be the subfield of $F$ of $\sigma$-invariant elements and let $T_{F/F_0}$ be the corresponding trace.

LEMMA 4.1. *The symmetric bilinear form on $A$ given by* $\mathrm{Tr}(x, y) = T_{F/F_0}\big(\mathrm{Trd}_A(xy)\big)$ *is nonsingular and* $\mathrm{Sym}(A, \tau)^\perp = \mathrm{Alt}(A, \tau)$.

*Proof.* If $\tau$ is of the first kind $F_0 = F$ and the claim is (2.3) of [6]. Assume that $\tau$ is of the second kind. Since the bilinear form $(x, y) \to \mathrm{Trd}_A(xy)$ is nonsingular, Tr is also nonsingular and it is straightforward that $\mathrm{Alt}(A, \tau) \subset \mathrm{Sym}(A, \tau)^\perp$. Equality follows from the fact that $\dim_{F_0} \mathrm{Alt}(A, \tau) = \dim_{F_0} \mathrm{Sym}(A, \tau) = \dim_F A$. □

PROPOSITION 4.2. *Let $(V, \theta)$, $\theta = [k]$ be a $(\sigma, \varepsilon)$-quadratic space over $D$ and let $h = \widehat{k} + \varepsilon\widehat{k}^* : V \xrightarrow{\sim} V^*$. The $F_0$-linear form*

$$f_\theta : \mathrm{Sym}(A, \sigma_\theta) \to F_0, \quad f_\theta(s) = \mathrm{Tr}(h^{-1}\widehat{k}s),\ s \in \mathrm{Sym}(A, \sigma_\theta)$$

*depends only on the class $\theta$ and satisfies* $f_\theta\big(x + \sigma_\theta(x)\big) = \mathrm{Tr}(x)$.

*Proof.* The first claim follows from (4.1) and the fact that if $k \in \mathrm{Alt}^\varepsilon_\sigma(V, D)$ then $h^{-1}\widehat{k} \in \mathrm{Alt}^1_{\sigma_\theta}(V, D)$. For the last claim we have:

$$\begin{aligned}
f_\theta\big(x + \sigma_\theta(x)\big) &= \mathrm{Tr}\big(h^{-1}\widehat{k}(x + \sigma_\theta(x)\big) \\
&= \mathrm{Tr}\big(h^{-1}\widehat{k}x\big) + \mathrm{Tr}\big(h^{-1}\widehat{k}h^{-1}x^*h\big) \\
&= \mathrm{Tr}\big(h^{-1}\widehat{k}x\big) + \mathrm{Tr}\big(\widehat{k}h^{-1}x^*\big) \\
&= \mathrm{Tr}\big(h^{-1}\widehat{k}x\big) + \mathrm{Tr}\big(x(h^{-1})^*\widehat{k}^*\big) \\
&= \mathrm{Tr}\big(h^{-1}\widehat{k}x\big) + \mathrm{Tr}\big(h^{-1}\varepsilon\widehat{k}^*x\big) = \mathrm{Tr}(x).
\end{aligned}$$

□

LEMMA 4.3. *Let $\tau$ be an involution of $A = \operatorname{End}_D(V)$ and let $f$ be A $F_0$-linear form on* $\operatorname{Sym}(A,\tau)$ *such that* $f\big(x+\tau(x)\big) = \operatorname{Tr}(x)$ *for all $x \in A$. There exists an element $u \in A$ such that $f(s) = \operatorname{Tr}(us)$ and $u + \tau(u) = 1$. The element $u$ is uniquely determined up to additivity by an element of* $\operatorname{Alt}(A,\tau)$. *We take $u = 1/2$ if* $\operatorname{Char} F \neq 2$.

*Proof.* The proof of (5.7) of [6] can easily be adapted. □

PROPOSITION 4.4. *Let $\tau$ be an involution of $A = \operatorname{End}_D(V)$ and let $f$ be A $F_0$-linear form on* $\operatorname{Sym}(A,\tau)$ *such that* $f\big(x+\tau(x)\big) = \operatorname{Tr}(x)$ *for all $x \in A$.*
*1) There exists a nonsingular $(\sigma,\varepsilon)$-quadratic form $\theta$ on $V$ such that $\tau = \sigma_\theta$ and $f = f_\theta$.*
*2) $(\sigma_\theta, f_\theta) = (\sigma_{\theta'}, f_{\theta'})$ if and only if $\theta' = \lambda\theta$ for $\lambda \in F_0$.*
*3) If $\tau = \sigma_\theta$ and $f = f_\theta$ with $f_\theta(s) = \operatorname{Tr}(us)$, the class of $u$ in* $A/\operatorname{Alt}(A,\sigma_\theta)$ *is uniquely determined by $\theta$.*

*Proof.* Here the proof of (5.8) of [6] can adapted. We prove 1) for completeness. Let $\tau(x) = h^{-1}x^*h$, $h = \varepsilon h^* : V \xrightarrow{\sim} V^*$. Let $f(s) = \operatorname{Tr}(us)$ with $u + \tau(u) = 1$ and let $k \in \operatorname{Sesq}_\sigma(V,D)$ be such that $\widehat{k} = hu : V \to V^*$. We set $\theta = [k]$. It is then straightforward to check that $h = k + \varepsilon k^*$. □

PROPOSITION 4.5. *Let* $\phi : \big(\operatorname{End}_D(V),\sigma_\theta\big) \xrightarrow{\sim} \big(\operatorname{End}_D(V'),\sigma_{\theta'}\big)$ *be an isomorphism of algebras with involution. Let $f_\theta(s) = \operatorname{Tr}(us)$ and $f_{\theta'}(s') = \operatorname{Tr}(u's')$. The following conditions are equivalent:*
*1) $\phi$ is an isomorphism of algebras with q-involutions.*
*2) $f_{\theta'}\big(\phi(s)\big) = f_\theta(s)$ for all $s \in \operatorname{Sym}\big(\operatorname{End}_D(V),\sigma_\theta\big)$.*
*3) $[\phi(u)] = [u'] \in \operatorname{End}_D(V')/\operatorname{Alt}\big(\operatorname{End}_D(V'),\sigma_{\theta'}\big)$.*

*Proof.* The implication 1) ⇒ 2) is clear. We check that 2) ⇒ 3). Let $\phi$ be induced by a similitude $t : (V,b_\theta) \xrightarrow{\sim} (V',b_{\theta'})$. Since $f_{\theta'}(\phi s) = f_\theta(s)$, we have $\operatorname{Tr}(t^{-1}u'ts) = \operatorname{Tr}(u'tst^{-1}) = \operatorname{Tr}(us)$ for all $s \in \operatorname{Sym}\big(\operatorname{End}_D(V),\sigma_\theta\big)$, hence $[\phi(u)] = [u']$. The implication 3) ⇒ 1) follows from the fact that $u$ can be chosen as $h^{-1}\widehat{k}$, $h = \widehat{k} + \varepsilon\widehat{k}^*$. □

REMARK 4.6. We call the pair $(\sigma_\theta, f_\theta)$ a $(\sigma,\varepsilon)$*-quadratic pair* or simply a quadratic pair. It determines $\theta$ up to the multiplication by a $\sigma$-invariant scalar $\lambda \in F^\times$. In fact $\sigma_\theta$ determines the polar $b_\theta$ up to $\lambda$ and $f_\theta$ determines $u$. We have $\theta = [\widehat{b}_\theta u]$.

EXAMPLE 4.7. Let $q : V \to F$ be a nonsingular quadratic form. The polar $b_q$ induces an isomorphism $\psi : V \otimes_F V \xrightarrow{\sim} \operatorname{End}_F(V)$ such that $\sigma_q\big(\psi(x\otimes y)\big) = \psi(y \otimes x)$. Thus $\psi(x\otimes x)$ is symmetric and $f_q\big(\psi(x\otimes x)\big) = q(x)$ (see [6, (5.11)]. More generally, if $V$ is a right vector space over $D$, we denote by ${}^*V$ the space $V$ viewed as a left $D$-space through the involution $\sigma$ of $D$. The adjoint $\widehat{b_\theta}$ of a $(\sigma,\varepsilon)$-quadratic space $(V,\theta)$ induces an isomorphism $\psi_\theta : V \otimes_D {}^\sigma V \xrightarrow{\sim} \operatorname{End}_D(V)$ and $\psi_\theta(xd \otimes x)$ is a symmetric element of $\big(\operatorname{End}_D(V),\sigma_\theta\big)$ for all $x \in V$ and all

$\varepsilon$-symmetric $d \in D$. One has $f_\theta\big(\psi(xd \otimes x)\big) = [dk(x,x)]$, where $\theta = [k]$ (see [4, Theorem 7]).

## 5. Clifford algebras

Let $\sigma$ be an involution of the first kind on $D$ and let $\theta$ be a nonsingular $(\sigma,\varepsilon)$-quadratic form on $V$. Let $\sigma_\theta$ be the corresponding $q$-involution on $A = \mathrm{End}_D(V)$. We assume in this section that over a splitting $A \otimes_F \tilde{F} \xrightarrow{\sim} \mathrm{End}_{\tilde{F}}(M)$ of $A$, $\theta_{\tilde{F}} = \theta \otimes 1_{\tilde{F}}$ is a $(Id,1)$-quadratic form $\tilde{q}$ over $\tilde{F}$, i.e. $\theta_{\tilde{F}}$ is a (classical) quadratic form. In the terminology of [6] this means that $\sigma_\theta$ is orthogonal if Char $\neq 2$ and symplectic if Char $= 2$. From now on we call such forms over $D$ *quadratic forms over* $D$, resp. *quadratic spaces* over $D$ if the forms are non-singular.

Classical invariants of quadratic spaces $(V,\theta)$ are the dimension $\dim_D V$ and the discriminant $\mathrm{disc}(\theta)$ and the Clifford invariant associated with the Clifford algebra. We refer to [6, §7] for the definition of the discriminant. We recall the definition of the Clifford algebra $\mathrm{Cl}(V,\theta)$, following [10, 4.1]. Given $(V,\theta)$ as above, let $\theta = [k]$, $k \in \mathrm{Sesq}_\sigma(V,D)$, $b_\theta = k + \varepsilon k^*$ and $h = \widehat{b_\theta} \in \mathrm{Hom}_D(V,V^*)$. Let $A = \mathrm{End}_D(V)$, $B = \mathrm{Sesq}_\sigma(V,D)$ and $B' = V \otimes_D {}^\sigma V$. We identify $A$ with $V \otimes_D {}^\sigma V^*$ through the canonical isomorphism $(x \otimes {}^\sigma f)(v) = xf(v)$ and $B$ with $V^* \otimes_D {}^\sigma V^*$ through $(f \otimes {}^\sigma g)(x,y) = \overline{g(x)}f(y)$. The isomorphism $h$ can be used to define further isomorphisms:

$$\varphi_\theta : B' = V \otimes_D {}^\sigma V \xrightarrow{\sim} A = \mathrm{End}_D(M),\ \varphi_\theta : x \otimes y \mapsto x \otimes h(y)$$

and the isomorphism $\psi_\theta$ already considered in (4.7):

$$\psi_\theta : A \xrightarrow{\sim} B,\ \psi_\theta : x \otimes {}^\sigma f \mapsto h(x) \otimes {}^\sigma f.$$

We use $\varphi_\theta$ and $\psi_\theta$ to define maps $B' \times B \to A$, $(b',b) \mapsto b'b$ and $A \times B' \to B'$, $(a,b') \mapsto ab'$:

$$(x \otimes {}^\sigma y)(h(u) \otimes g) = xb(y,u) \otimes {}^\sigma f \text{ and } (x \otimes, {}^\sigma f)(u \otimes, {}^\sigma v) = xf(u) \otimes {}^\sigma h(v)$$

Furthermore, let $\tau_\theta = \varphi_\theta^{-1}\sigma_\theta\varphi_\theta : B' \to B'$ be the transport of the involution $\sigma_\theta$ on $A$. We have $\tau_\theta(x \otimes {}^\sigma y) = \varepsilon y \otimes {}^\sigma x$. Let $S_1 = \{s_1 \in B' \mid \tau_\theta(s_1) = s_1\}$. We have $S_1 = \big(\mathrm{Alt}^\varepsilon(V,D)\big)^\perp$ for the pairing $B' \times B \to F$, $(b',b) \mapsto \mathrm{Trd}_A(b'b)$. Let Sand be the bilinear map $B' \otimes B' \times B \to B'$ defined by $\mathrm{Sand}(b'_1 \otimes b'_2, b) = b'_2 b b'_1$. The *Clifford algebra* $\mathrm{Cl}(V,\theta)$ of the quadratic space $(V,\theta)$ is the quotient of the tensor algebra of the $F$-module $B'$ by the ideal $I$ generated by the sets

$$\begin{aligned} I_1 &= \{s_1 - \mathrm{Trd}_A(s_1 k)1,\ s_1 \in S_1\} \\ I_2 &= \{c - \mathrm{Sand}(c,k) \mid \mathrm{Sand}\big(c, \mathrm{Alt}^\varepsilon(V,D)\big) = 0\}. \end{aligned}$$

The Clifford algebra $\mathrm{Cl}(V,\theta)$ has a canonical involution $\sigma_0$ induced by the map $\tau$. We have $\mathrm{Cl}(V,\theta) \otimes_F \widetilde{F} = \mathrm{Cl}(V \otimes_F \widetilde{F}, \theta \otimes 1_{\widetilde{F}})$ for any field extension $\widetilde{F}$ of $F$ and $\mathrm{Cl}(V,q)$ is the even Clifford algebra $C_0(V,q)$ of $(V,q)$ if $D = F$ ([10, Théorème 2]). The reduction is through Morita theory for hermitian spaces (see for example [5, Chapter I, §9] for a description of Morita theory). In [6, §8] the Clifford algebra $C(A,\sigma_\theta,f_\theta)$ of the triple $(A,\sigma_\theta,f_\theta)$ is defined as the quotient of the tensor algebra $T(A)$ of the $F$-space $A$ by the ideal generated by the sets

$$\begin{array}{rcl} J_1 & = & \{s - \mathrm{Trd}_A(us),\ s \in \mathrm{Sym}(A, \sigma_\theta)\} \\ J_2 & = & \{c - \mathrm{Sand}'(c, u),\ c \in A \text{ with } \mathrm{Sand}'\big(c, \mathrm{Alt}(A, \sigma_\theta)\big) = 0\} \end{array}$$

where $u = \widehat{b_\theta}^{-1} k$ and $\mathrm{Sand}' : (A \otimes A, A) \to A$ is defined as $\mathrm{Sand}'(a \otimes b, x) = axb$. The two definitions give in fact isomorphic algebras:

PROPOSITION 5.1. *The isomorphism* $\varphi_\theta : V \otimes_D {}^{\sigma}V \xrightarrow{\sim} \mathrm{End}_D(V)$ *induces an isomorphism* $\mathrm{Cl}(V, \theta) \xrightarrow{\sim} C(A, \sigma_\theta, f_\theta)$.

*Proof.* We only check that $\varphi_\theta$ maps $I_1$ to $J_1$. By definition of $\tau$ and $S_1$, $s = \varphi_\theta(s_1)$ is a symmetric element of $A$. On the other hand we have by definition of the pairing $B' \times B \to A$,

$$\begin{array}{rcl} \mathrm{Trd}_A(s_1 k) & = & \mathrm{Trd}_A\big(\varphi_\theta(s_1)\psi_\theta^{-1}(k)\big) \\ & = & \mathrm{Trd}_A\big(sh^{-1}\widehat{k}\big) = \mathrm{Trd}_A(su) = \mathrm{Trd}_A(us), \end{array}$$

hence the claim. □

In particular we have $C\big(\mathrm{End}_F(V), \sigma_q, f_q\big) = C_0(V, q)$ for a quadratic space $(V, q)$ over $F$. It is convenient to use both definitions of the Clifford algebra of a generalized quadratic space.

Let $D = [K, \mu) = K \oplus \ell K$ be a quaternion algebra with conjugation $\sigma$. Let $V$ be a $D$-module and let $V^0$ be $V$ as a right vector space over $K$ (through restriction of scalars). Let $T : V^0 \to V^0$, $Tx = x\ell$. We have $\mathrm{End}_D(V) \subset \mathrm{End}_K(V^0)$ and

$$\mathrm{End}_D(V) = \{f \in \mathrm{End}_K(V^0) \mid fT = Tf\}.$$

Let $\theta = [k]$ be a $(\sigma, -1)$-quadratic space and let $k(x, y) = P(x, y) + \ell R(x, y)$ as in Section 3. It follows from (3.1) that $R$ defines a quadratic space $[R]$ on $V^0$ over $K$.

PROPOSITION 5.2. *We have* $\sigma_{[R]}|_{\mathrm{End}_D(V)} = \sigma_\theta$ *and* $f_\theta = f_{[R]}|_{\mathrm{End}_D(V)}$.

*Proof.* We have an embedding $D \hookrightarrow M_2(K)$, $a + \ell b \mapsto \begin{pmatrix} a & \mu\overline{b} \\ b & \overline{a} \end{pmatrix}$ and conjugation given by $x \mapsto x^* = c^{-1}x^t c$, $c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The choice of a basis of $V$ over $D$ identifies $V$ with $D^n$, $V^0$ with $K^{2n}$, $\mathrm{End}_D(V)$ with $M_n(D)$ and $\mathrm{End}_K(V^0)$ with $M_{2n}(K)$, where $n = \dim_D V$. We further identify $V$ and $V^*$ through the choice of the dual basis. We embed any element $x = x_1 + \ell x_2 \in M_{k,l}(D)$, $x_i \in M_{k,l}(K)$ in $M_{2k,2l}(K)$ through the map $\iota : x \mapsto \xi = \begin{pmatrix} x_1 & \mu\overline{x_2} \\ x_2 & \overline{x_1} \end{pmatrix}$. In particular $D^n$ is identified with a subspace of the space of $(2n \times 2)$-matrices over $K$. Then $D \subset M_2(K)$ operates on the right through $(2 \times 2)$-matrices and $M_n(D) \subset M_{2n}(K)$ operates on the left through $(2n \times 2n)$-matrices. With the notations of Example (2.3) we have $\iota(x^*) = \mathrm{Int}(c^{-1})(x^t)$. Any $D$-sesquilinear form $k$ on $D^n$ can be written as $k(x,y) = x^*ay$, where $a \in M_n(D)$, as in (2.3). Let $a = a_1 + \ell a_2$, $a_i \in M_n(K)$ and let

$$\alpha = \iota(a) = \begin{pmatrix} a_1 & \mu\overline{a_2} \\ a_2 & \overline{a_1} \end{pmatrix}.$$

Let $\eta = \iota(y)$, $y = y_1 + \ell y_2$. We have

$$k(x,y) = x^*ay = \xi^*\alpha\eta = \begin{pmatrix} x_1 & \mu\overline{x_2} \\ x_2 & \overline{x_1} \end{pmatrix}^* \begin{pmatrix} a_1 & \mu\overline{a_2} \\ a_2 & \overline{a_1} \end{pmatrix} \begin{pmatrix} y_1 & \mu\overline{y_2} \\ y_2 & \overline{y_1} \end{pmatrix}.$$

On the other side it follows from $h = P + \ell R$ that $R(x,y) = \xi^t\rho\eta$ with

$$\rho = \begin{pmatrix} a_2 & \overline{a_1} \\ -a_1 & -\mu\overline{a_2} \end{pmatrix}.$$

Assume that $\theta = [k]$ , so that $\sigma_\theta$ corresponds to the involution $\mathrm{Int}(\gamma^{-1}) \circ *$, where $\gamma = \alpha - \alpha^*$. Similarly $\sigma_{[R]}$ corresponds to the involution $\mathrm{Int}(\widetilde{\rho}^{-1}) \circ t$ where $\widetilde{\rho} = \rho + \rho^t$. We obviously have $\rho = c\alpha$ with $c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so that $\rho^t = \alpha^t c^t = -\alpha^t c = -ca^*$ and $\rho + \rho^t = c(\alpha - \alpha^*)$ or $c\gamma = \widetilde{\rho}$. Now $^* = \mathrm{Int}(c^{-1}) \circ t$ implies $\sigma_{[R]}|_{M_n(D)} = \sigma_\theta$. We finally check that $f_\theta = f_{[R]}|_{\mathrm{Sym}(M_n(D),\sigma_\theta)}$. We have $f_\theta(s) = \mathrm{Trd}_{M_n(D)}(\gamma^{-1}\alpha s)$ and $f_{[R]}(s) = \mathrm{Trd}_{M_{2n}(K)}(\widetilde{\rho}^{-1}\rho s)$, hence the claim, since $\rho = c\alpha$ and $\widetilde{\rho} = c\gamma$ implies $\gamma^{-1}\alpha = \widetilde{\rho}^{-1}\rho$. □

COROLLARY 5.3. *The embedding* $\mathrm{End}_D(V) \hookrightarrow \mathrm{End}_K(V^0)$ *induces*
*1) an isomorphism* $\big(\mathrm{End}_D(V), \sigma_\theta, f_\theta\big) \otimes K \xrightarrow{\sim} \big(\mathrm{End}_K(V^0), \sigma_{[R]}, f_{[R]}\big)$,
*2) an isomorphism* $C\big(\mathrm{End}_D(V), \sigma_\theta, f_\theta\big) \otimes K \xrightarrow{\sim} C_0(V^0, [R])$.

In view of (2) the semilinear automorphism $T : V^0 \xrightarrow{\sim} V^0$, $Tx = x\ell$, is a semilinear similitude with multiplier $-\mu$ of the quadratic form $[R]$, such that $T^2 = \mu$.

LEMMA 5.4. *The map $T$ induces a semilinear automorphism $C_0(T)$ of $C_0(V^0, R)$ such that*

$$C_0(T)(xy) = (-\mu)^{-1}T(x)T(y) \text{ for } x,\, y \in V^0$$

*and $C_0(T)^2 = Id$.*

*Proof.* This follows (for example) as in [6, (13.1)] □

PROPOSITION 5.5.

$$C\big(\mathrm{End}_D(V), \sigma_\theta, f_\theta\big) = \{c \in C_0(V^0, R) \mid C_0(T)(c) = c\}.$$

*Proof.* The claim follows from the defining relations of $C\big(\mathrm{End}_D(V), \sigma_\theta, f_\theta\big)$ and the fact that

$$\mathrm{End}_D(V) = \{f \in \mathrm{End}_K(V^0) \mid T^{-1}fT = f\}.$$

□

We call $C\big(\mathrm{End}_D(V), \sigma_\theta, f_\theta\big)$ or equivalently $\mathrm{Cl}(V, \theta)$ the *Clifford algebra of the quadratic quaternion space* $(V, \theta)$.

Let $t$ be a semilinear similitude of a quadratic space $(U, q)$ of even dimension over $K$. Assume that $\mathrm{disc}(q)$ is trivial, so that $C_0(U, q)$ decomposes as product of two $K$-algebras $C^+(U, q)$ and $C^-(U, q)$. We say that $t$ is *proper* if $C_0(t)\big(C^\pm(U, q)\big) \subset C^\pm(U, q)$ and we say that $t$ is *improper* if $C_0(t)\big(C^\pm(U, q)\big) \subset C^\mp(U, q)$. In general we say that $t$ is *proper* if $t$ is proper over some field extension of $F$ which trivializes $\mathrm{disc}(q)$. For any semilinear similitude $t$, let $d(t) = 1$ is $t$ if proper and $d(t) = -1$ if $t$ is improper.

LEMMA 5.6. *Let $t_i$ be a semilinear similitude of $(U_i, q_i)$, $i = 1, 2$. We have $d(t_1 \perp t_2) = d(t_1)d(t_2)$.*

*Proof.* We assume that $\mathrm{disc}(q_i)$, $i = 1, 2$, is trivial. Let $e_i$ be an idempotent generating the center $Z_i$ of $C_0(q_i)$. We have $t_i(e_i) = e_i$ if $t_i$ is proper and $t_i(e_i) = 1 - e_i$ if $t_i$ is improper. The idempotent $e = e_1 + e_2 - 2e_1e_2 \in C_0(q_1 \perp q_2)$ generates the center of $C_0(q_1 \perp q_2)$ (see for example [5, (2.3), Chap. IV] ) and the claim follows by case checking. □

LEMMA 5.7. *Let $V$, $\theta$, $V^0$, $R$ and $T$ be as above. Let $\dim_K V^0 = 2m$. Then $T$ is proper if $m$ is even and is improper if $m$ is odd.*

*Proof.* The quadratic space $(V, \theta)$ is the orthogonal sum of 1-dimensional spaces and we get a corresponding orthogonal decomposition of $\big(V^0, [R]\big)$ into subspaces $(U_i, q_i)$ of dimension 2. In view of (5.6) it suffices to check the case $m = 1$. Let $\alpha = a = a_1 + \ell a_2 \in D$ and $\rho = \begin{pmatrix} a_2 & \overline{a_1} \\ -a_1 & -\mu\overline{a_2} \end{pmatrix}$. We choose $\mu = 1$, $a_1 = j$ ($j$ as in (2.4)), put $i = 1 - 2j$, so that $\overline{i} = -i$ and choose $a_2 = 0$. Let $x = x_1e_1 + x_2e_2 \in V^0$, so $[R](x_1, x_2) = ix_1x_2$ and $C([R])$ is generated by $e_1$, $e_2$ with the relations $e_1^2 = 0$, $e_2^2 = 0$, $e_1e_2 + e_2e_1 = i$. The element $e = i^{-1}e_1e_2$ is an idempotent generating the center. Since $T(x_1e_1 + x_2e_2) = \overline{x}_2e_1 + \overline{x}_1e_2$, we have $C_0(T)(e_1e_2) = -e_2e_1$ and $C_0(T)(e) = 1 - e$. Thus $T$ is not proper. □

Of special interest for the next section are quadratic quaternion forms $[k]$ such that the induced quadratic forms $\pi_2([k])$ are Pfister forms. For convenience we call such forms *Pfister quadratic quaternion forms*. Hyperbolic spaces of dimension $2^n$ are Pfister forms, hence spaces of the form $\beta([b])$, $b$ a hermitian form over $K$, are Pfister, in view of the exactness of the sequence of Lewis [7]. It is in fact easy to give explicit examples of Pfister forms using the following constructions:

EXAMPLE 5.8 ($\mathrm{Char}\, F \neq 2$). Let $q = <\lambda_1, \dots, \lambda_n>$ be a diagonal quadratic form on $F^n$, i.e., $q(x) = \sum \lambda_i x_i^2$. Let $[k]$ on $D^n$ be given by the diagonal form $\ell q$. Then the corresponding quadratic form $[R]$ on $K^{2n}$ is given by the diagonal form $< 1, -\mu > \otimes q$. In particular we get the 3-Pfister form $<< a, b, \mu >>$ choosing for $q$ the norm form of a quaternion algebra $(a, b)_F$.

EXAMPLE 5.9 ($\operatorname{Char} F = 2$). Let $b =< \lambda_1, \dots, \lambda_n >$ be a bilinear diagonal form on $F^n$, i.e., $b(x,y) = \sum \lambda_i x_i y_i$. Let $k = (j + \ell)b$ on $D^n$. Then the corresponding quadratic form $[R]$ over $K = R(j)$, $j^2 = j + \lambda$, is given by the form $[R] = b \otimes [1, \lambda]$ where $[\xi, \eta] = \xi x_1^2 + x_1 x_2 + \eta x_2{}^2$. In particular, for $b =< 1, a, c, ac >$, we get the 3-Pfister form $<< a, c, \lambda]]$ with the notations of [6], p. xxi.

## 6. TRIALITY FOR SEMILINEAR SIMILITUDES

Let $\mathfrak{C}$ be a Cayley algebra over $F$ with conjugation $\pi : x \mapsto \overline{x}$ and norm $\mathfrak{n}\colon x \mapsto x\overline{x}$. The new multiplication $x \star y = \overline{x}\,\overline{y}$ satisfies

$$x \star (y \star x) = (x \star y) \star x = \mathfrak{n}(x)y \tag{6}$$

for $x$, $y \in \mathfrak{C}$. Further, the polar form $b_{\mathfrak{n}}$ is *associative* with respect to $\star$, in the sense that

$$b_{\mathfrak{n}}(x \star y, z) = b_{\mathfrak{n}}(x, y \star z).$$

PROPOSITION 6.1. *For $x$, $y \in \mathfrak{C}$, let $r_x(y) = y \star x$ and $\ell_x(y) = x \star y$. The map $\mathfrak{C} \to \operatorname{End}_F(\mathfrak{C} \oplus \mathfrak{C})$ given by*

$$x \mapsto \begin{pmatrix} 0 & \ell_x \\ r_x & 0 \end{pmatrix}$$

*induces isomorphisms $\alpha\colon \big(C(\mathfrak{C}, \mathfrak{n}), \tau\big) \xrightarrow{\sim} \big(\operatorname{End}_F(\mathfrak{C} \oplus \mathfrak{C}), \sigma_{\mathfrak{n} \perp \mathfrak{n}}\big)$ and*

$$\alpha_0\colon \big(C_0(\mathfrak{C}, \mathfrak{n}), \tau_0\big) \xrightarrow{\sim} \big(\operatorname{End}_F(\mathfrak{C}), \sigma_{\mathfrak{n}}\big) \times \big(\operatorname{End}_F(\mathfrak{C}), \sigma_{\mathfrak{n}}\big), \tag{7}$$

*of algebras with involution.*

*Proof.* We have $r_x\big(\ell_x(y)\big) = \ell_x\big(r_x(y)\big) = \mathfrak{n}(x) \cdot y$ by (6). Thus the existence of the map $\alpha$ follows from the universal property of the Clifford algebra. The fact that $\alpha$ is compatible with involutions is equivalent to

$$b_{\mathfrak{n}}\big(x \star (z \star y), u\big) = b_{\mathfrak{n}}\big(z, y \star (u \star x)\big)$$

for all $x$, $y$, $z$, $u$ in $\mathfrak{C}$. This formula follows from the associativity of $b_{\mathfrak{n}}$. Since $C(\mathfrak{C}, \mathfrak{n})$ is central simple, the map $\alpha$ is an isomorphism by a dimension count. $\square$

Assume from now on that $\mathfrak{C}$ is defined over a field $K$ which is quadratic Galois over $F$. Any proper semilinear similitude $t$ of $\mathfrak{n}$ induces a semilinear automorphism $C(t)$ of the even Clifford algebra $\big(C_0(\mathfrak{C}, \mathfrak{n}), \tau_0\big)$, which does not permute the two components of the center of $C_0(\mathfrak{C}, \mathfrak{n})$. Thus $\alpha_0 \circ C_0(t) \circ \alpha_0^{-1}$ is a pair of semilinear automorphisms of $\big(\operatorname{End}_K(\mathfrak{C}), \sigma_{\mathfrak{n}}\big)$. It follows as in (4.5) that, for any quadratic space $(V, q)$, semilinear automorphisms of $(\operatorname{End}_K(V), \sigma_q, f_q)$ are of the form $\operatorname{Int}(f)$, where $f$ is a semilinear similitude of $q$. The following result is due to Wonenburger [12] in characteristic different from 2:

PROPOSITION 6.2. *For any proper semilinear similitude $t_1$ of $\mathfrak{n}$ with multiplier $\mu_1$, there exist proper semilinear similitudes $t_2$, $t_2$ such that*

$$\alpha_0 \circ C_0(t_1) \circ \alpha_0^{-1} = \big(\mathrm{Int}(t_2), \mathrm{Int}(t_3)\big)$$

*and*

$$\begin{array}{rcl} \mu_3^{-1} t_3(x \star y) & = & t_1(x) \star t_2(y), \\ \mu_1^{-1} t_1(x \star y) & = & t_2(x) \star t_3(y) \\ \mu_2^{-1} t_2(x \star y) & = & t_3(x) \star t_1(y). \end{array} \tag{8}$$

*Let $t_1$ be an improper similitude with multiplier $\mu_1$. There exist improper similitudes $t_2$, $t_3$ such that*

$$\begin{array}{ll} \mu_3^{-1} t_3(x \star y) & = t_1(y) \star t_2(x), \\ \mu_1^{-1} t_1(x \star y) & = t_2(y) \star t_3(x) \\ \mu_2^{-1} t_2(x \star y) & = t_3(y) \star t_1(x). \end{array}$$

*The pair $(t_2, t_3)$ is determined by $t_1$ up to a factor $(\lambda, \lambda^{-1})$, $\lambda \in K^\times$, and we have $\mu_1\mu_2\mu_3 = 1$.*
*Furthermore, any of the formulas in (8) implies the two others.*

*Proof.* The proof given in [6, (35.4)] for similitudes can also be used for semilinear similitudes. □

REMARK 6.3. The class of two of the $t_i$, $i = 1, 2, 3$, modulo $K^\times$ is uniquely determined by the class of the third $t_i$.

COROLLARY 6.4. *Let $T_1$ be a proper semilinear similitude of $(\mathfrak{C}, \mathfrak{n})$ such that $T_1^2 = \mu_1$, $\mu_1 \in K^\times$ and with multiplier $-\mu_1$. There exist elements $a_i \in K^\times$, $i = 1, 2, 3$, and proper semilinear similitudes $T_i$ of $(\mathfrak{C}, \mathfrak{n})$, with $T_i^2 = \mu_i$, $\mu_i \in K^\times$ and with multiplier $-\mu_i$, $i = 2, 3$, such that $a_i\overline{a_i}\mu_i = \mu_{i+1}\mu_{i+2}$ and*

$$\begin{array}{rcl} a_3 T_3(x \star y) & = & T_1(x) \star T_2(y) \\ a_1 T_1(x \star y) & = & T_2(x) \star T_3(y) \\ a_2 T_2(x \star y) & = & T_3(x) \star T_1(y \end{array}$$

*The class of any $T_i$ modulo $K^\times$ determines the two other classes and the $\mu_i$'s are determined up to norms from $K^\times$. Furthermore any of the three formulas determines the two others.*

*Proof.* Counting indices modulo 3, we have relations

$$T_i(x) \star T_{i+1}(y) = b_{i+2} T_{i+2}, \quad b_i \in K^\times$$

in view of (6.2). If we replace all $T_j$ by $T_j \circ \rho_{\nu_j}$, $\nu_j \in K^\times$, we get new constants $a_i$. The claim then follows from (3.3). □

## 7. Triality for quadratic quaternion forms

Let $D_1 = K \oplus \ell_1 K = [K, \mu_1)$ be a quaternion algebra over $F$ and let $(V_1, q_{\theta_1})$ be a quaternion quadratic space of dimension 4 over $D_1$. Let $\theta_1 = [h_1]$, $h_1(x, y) = P_1(x, y) + \ell R_1(x, y)$, so that $[R_1] = \pi_2(\theta_1)$ corresponds to a 8-dimensional (classical) quadratic form on $V_1^0$ over $K$. The map $T_1 : V_1^0 \to V_1^0$, $T_1(x) = x\ell_1$, is a semilinear similitude of $\big(V_1^0, [R_1]\big)$ with multiplier $-\mu_1$ and such that $T_1^2 = \mu_1$. We recall that by (3.5) it is equivalent to have a quadratic quaternion space $(V_1, q_{\theta_1})$ or a pair $\big(V_1^0, [T_1]\big)$. We assume from now on that the quadratic form $q_{[R_1]}$ is a 3-Pfister form, i.e.,the norm form $\mathfrak{n}$ of a Cayley algebra $\mathfrak{C}$ over $K$. In view of (6.4) $T_1$ induces two semilinear similitudes $T_2$, resp. $T_3$, with multipliers $\mu_2$, resp. $\mu_3$, which in turn define a quaternion quadratic space $(V_2, \theta_2)$ of dimension 4 over $D_2 = [K, \mu_2)$, resp. a quaternion quadratic space $(V_3, \theta_3)$ of dimension 4 over $D_3 = [K, \mu_3)$. Let $\mathrm{Br}(F)$ be the Brauer group of $F$.

Proposition 7.1. *1)* $[D_1][D_2][D_3] = 1 \in \mathrm{Br}(F)$,
*2) The restriction of* $\alpha : C_0\big(\mathfrak{C}, \mathfrak{n})\big) \xrightarrow{\sim} \mathrm{End}_K(\mathfrak{C}) \times \mathrm{End}_K(\mathfrak{C})$ *to* $C(V_i, D_i, \theta_i)$ *induces isomorphisms*

$$\alpha_i : \big(C(V_i, D_i, \theta_i), \tau\big) \xrightarrow{\sim} \big(\mathrm{End}_{D_{i+1}}(V_{i+1}), \sigma_{\theta_{i+1}}\big) \times \big(\mathrm{End}_{D_{i+2}}(V_{i+2}), \sigma_{\theta_{i+2}}\big)$$

*Proof.* The first claim follows from the fact that $\mu_1\mu_2 = \mu_3 \operatorname{Nrd}_{D_3}(a_3)$ and the second is a consequence of (5.5), (3.5) and the definition of $\alpha$. □

Example 7.2. Let $\mathfrak{C}_0$ be a Cayley algebra over $F$ and let $\mathfrak{C} = \mathfrak{C}_0 \otimes_F K$. For any $c \in \mathfrak{C}_0$ such that $c^2 = \mu_1 \in F^\times$, $T_1 : \mathfrak{C} \to \mathfrak{C}$ given by $T_1(k \otimes x) = \overline{k} \otimes xc$ is a semilinear similitude with multiplier $-\mu_1$ such that $T_1^2 = \mu_1$. The Moufang identity $(cx)(yc) = c(xy)c$ in $\mathfrak{C}$ implies that

$$(xc) \star (cy) = \overline{c}(x \star y)\overline{c}.$$

Thus $T_2(k \otimes y) = \overline{k} \otimes cy$ and $T_3(k \otimes z) = i\overline{k} \otimes \overline{c}z\overline{c}$ (where $i \in K^\times$ is such that $\overline{i} = -i$) satisfy (6.4). The corresponding triple of quaternion algebras is $\big([K, \mu_1), [K, \mu_1), [K, i\overline{i}\mu_1^2)\big)$, the third algebra being split.

Example 7.3. Let $D_i$, $i = 1, , 2, 3$, be quaternion algebras over $F$ such that $[D_1][D_2][D_3] = 1 \in \mathrm{Br}(F)$. We may assume that the $D_i$ contain a common separable quadratic field $K$ and that $D_i = [K, \mu_i)$, $\mu_i \in F^\times$ such that $\mu_1\mu_2\mu_3 \in {F^\times}^2$. In [6, (43.12)] similitudes $S_i$ with multiplier $\mu_i$, $i = 1, 2, 3$, of the split Cayley algebra $\mathfrak{C}_s$ over $F$ are given, such that 1) $\mu_3^{-1} S_3(x \star y) = S_1(x) \star S_2(y)$ and 2) $S_i^2 = \mu_i$. Let $\mathfrak{C} = K \otimes \mathfrak{C}_s$. Let $u \in K^\times$ be such that $\overline{u} = -u$. The semilinear similitudes $T_i(k \otimes x) = u\overline{k} \otimes S_i(x)$, $i = 1, 2, 3$, satisfy

$$a_3 T_3(x \star y) = T_1(x) \star T_2(y)$$

with $a_3 = u\mu_3^{-1}$ (we use the same notation $\star$ in $\mathfrak{C}_s$ and in $\mathfrak{C}$). Thus there exist a triple of quadratic quaternion forms $(\theta_1, \theta_2, \theta_3)$ corresponding to the three given quaternion algebras. We hope to describe the corresponding quadratic quaternion forms in a subsequent paper.

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