On Cubic Étale Algebras

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Abstract. We describe explicitly a group structure on isomorphism classes of cubic étale algebras with prescribed discriminant. The case of trivial discriminant corresponds to the group of cubic cyclic algebras.

1 Introduction

Let F be a field and let F_{sep} be a separable closure of F. Cubic étale algebras are commutative associative F-algebras L of dimension 3 over F such that

$$L \otimes F_{\text{sep}} \simeq F_{\text{sep}} \times F_{\text{sep}} \times F_{\text{sep}}.$$

Aside from their intrinsic interest, cubic étale algebras are closely connected with central simple algebras of degree 3 ([2], [5]), exceptional Jordan algebras of dimension 27 ([5], [11], [12], [13]) and triality ([5]). They are of four possible types: ${}^1L \simeq F \times F \times F$, the *split algebra*, ${}^2L \simeq F \times K$, K a Galois quadratic field extension of F, ${}^3L$ a cyclic cubic field extension of F and ${}^6L$, a "general" separable field extension of degree 3 over F. Let C_3 be the cyclic group $\mathbb{Z}/3\,\mathbb{Z}$ with generator $\rho = 1 + 3\,\mathbb{Z}$. Types ${}^1L$ and ${}^3L$ are C_3-Galois F-algebras, i.e., $C_3 \subset \mathrm{Aut}_F(L)$, $|C_3| = [L : F]$ and $L^{C_3} = F$. Such pairs (L, ρ), with the obvious notion of isomorphism, are *cyclic cubic algebras*. There is a commutative group structure on isomorphism classes of cyclic cubic algebras which has been extensively studied in [1], [3], [4], [6], [7], [8], [9], [10], [14], [15]. In particular the product in the group has been described in terms of reduced equations for the algebra (see [3], [4], [7], [8]).

To any cubic étale algebra L there is a functorially associated *discriminant algebra* $\Delta(L)$, which is quadratic étale, and $\Delta(L)$ is split, i.e., $\Delta(L) \simeq F \times F$, if and only if L is cubic cyclic. Let K be a fixed quadratic étale F-algebra. The aim of this paper is to describe a group structure on isomorphism classes of pairs $(L, \phi : \Delta(L) \xrightarrow{\sim} K)$, "algebras with fixed discriminant" which, for $K = F \times F$, coincides with the group of cyclic cubic algebras. As far as we know such a group structure has not yet been explicitly described.

In the next two sections we use Kummer theory, hence we assume there that F is a field of characteristic not 3. We first describe the cohomological background referring mainly to [2] and [5]. Then, in an elementary way which is independent of (but of course motivated by) the cohomology part, we give a complete description of the group law in terms of reduced equations. The last section deals with the characteristic 3 case and relies on Artin-Schreier theory.

2 Galois cohomology

Let $\Gamma = \mathrm{Gal}(F_{\text{sep}}/F)$. For any continuous Γ-module M we denote by $H^i(F, M)$ the i-th cohomology group $H^i(\Gamma, M)$. We recall that $H^1(F, A)$ is also defined as a pointed set if A is a (not necessarily commutative) Γ-group. The group $H^1(F, C_n)$

(with trivial Γ-action on C_n) classifies cyclic F-algebras of degree n, the pointed set $H^1(F, S_n)$ (with trivial Γ-action on the group S_n of permutations of n elements) classifies étale F-algebras of dimension n, i.e., n-dimensional F-algebras L such that

$$L \otimes F_{\mathrm{sep}} \simeq F_{\mathrm{sep}} \times \ldots \times F_{\mathrm{sep}}.$$

Since the Γ-action on S_n is trivial, a 1-cocycle is a group homomorphism $\phi : \Gamma \to S_n$ and two cocycles are cohomologous if they are conjugate in S_n. Let X_n be a set with n elements. To a cocycle ϕ, we associate the F-algebra $L_\phi = \mathrm{Maps}(X_n, F_{\mathrm{sep}})^\Gamma$ where Γ acts on X_n via ϕ. Since $L_\phi \otimes_F F_{\mathrm{sep}} \simeq \mathrm{Maps}(X_n, F_{\mathrm{sep}}) = F_{\mathrm{sep}} \times \ldots F_{\mathrm{sep}}$, L_ϕ is étale. Moreover any étale F-algebra of dimension n is of this type since there is a canonical identification $L \xrightarrow{\sim} \mathrm{Maps}(X(L), F_{\mathrm{sep}})^\Gamma$ where $X(L)$ is the Γ-set $\mathrm{Alg}_F(L, F_{\mathrm{sep}})$ of F-algebra homomorphisms $L \to F_{\mathrm{sep}}$. Let $\pi : S_n \to S_2 = \mathbb{Z}/2\,\mathbb{Z}$ be the parity map. If $L = L_\phi$, we define the *discriminant algebra* of L as the quadratic étale F-algebra

$$\Delta(L) = L_{\pi\circ\phi} = \mathrm{Maps}(X_2, F_{\mathrm{sep}})^\Gamma,$$

where Γ acts on X_2 via $\pi \circ \phi$. If $L = K[T]/(T^3 - pT - q)$, then $\Delta(L)$ is related to the classical discriminant:

$$\Delta(L) \simeq \begin{cases} F[T]/(T^2 - d) & \text{with } \; d = 4p^3 - 27q^2 \;\text{ if }\; \mathrm{char}F \neq 2 \\ F[T]/(T^2 + T + d) & \text{with } \; d = 1 + p^3q^{-2} \;\text{ if }\; \mathrm{char}F = 2 \end{cases}$$

More precisely, if $\Delta(L)$ is not split, if $\Gamma_0 = \ker(\pi \circ \phi)$ and if $\{x_1, x_2, x_3\}$ is the set of roots of $T^3 - pT - q$ in F_{sep}, we identify

$$\Delta(L) \simeq (F_{\mathrm{sep}})^{\Gamma_0} = \begin{cases} F[(x_1 - x_2)((x_1 - x_3)(x_2 - x_3)] & \text{if } \mathrm{char}F \neq 2 \\ F[\frac{x_1}{x_1+x_2} + \frac{x_1}{x_1+x_3} + \frac{x_2}{x_2+x_3}] & \text{if } \mathrm{char}F = 2 \end{cases}$$

Since S_2 is commutative there is a group structure on the set of isomorphism classes of quadratic étale algebras, defined as:

$$[K_1] \cdot [K_2] = [K_1 \star K_2]$$

where

$$K_1 \star K_2 = \{x \in K_1 \otimes K_2 \mid \tau_1 \otimes \tau_2(x) = x\}$$

and τ_i is conjugation on K_i, $i = 1, 2$. If $\mathrm{char}F \neq 2$, the map

$$F^\times / F^{\times 2} \to H^1(F, S_2)$$

induced by $a \mapsto F[T]/(T^2 - a) = F(\sqrt{a})$ where $\sqrt{a} = T + (T^2 - a)$, is an isomorphism and

$$K(\sqrt{a_1}) \star K(\sqrt{a_2}) = K(\sqrt{a_1 a_2}).$$

In characteristic 2, the map $b \in F \mapsto F[T]/(T^2 + T + b)$ induces an isomorphism

$$F/\wp(F) \xrightarrow{\sim} H^1(F, S_2)$$

where $\wp(x) = x^2 + x$. Putting $K_i = F[T]/(T^2 + T + b_i)$, $b_i \in F$, $i = 1, 2$, we have

$$K_1 \star K_2 = F[T]/(T^2 + T + b_1 + b_2).$$

We may use continuous homomorphisms $\Phi : \Gamma \to S_2$ to twist Γ-actions. Let first $M = \mathbb{Z}$ with trivial action. We define $\mathbb{Z}_\Phi$ as $\mathbb{Z}$ with the action

$$\gamma \cdot z = \begin{cases} z & \text{if} \quad \Phi(\gamma) = e \\ -z & \text{if} \quad \Phi(\gamma) \neq e \end{cases}$$

Multiplication in $\mathbb{Z}$ induces a canonical isomorphism

$$\mathbb{Z}_{\Phi_1} \otimes \mathbb{Z}_{\Phi_2} = \mathbb{Z}_{\Phi_1 \cdot \Phi_2}.$$

For arbitrary Γ-modules M we define

$$M_\Phi = M \otimes_{\mathbb{Z}} \mathbb{Z}_\Phi.$$

Observe that $(M_{\phi_1})_{\Phi_2} = M_{\Phi_1 \cdot \Phi_2}$. Since continuous homomorphisms $\Phi : \Gamma \to S_2$ correspond bijectively to quadratic étale algebras K we use from now on the notation $M_{(K)}$ for M_Φ, so that

$$M_{1(K_1)} \otimes M_{2(K_2)} \simeq (M_1 \otimes M_2)_{(K_1 \star K_2)} \quad \text{and} \quad M_{(K_1)(K_2)} \simeq M_{(K_1 \star K_2)}.$$

The split exact sequence

$$(*) \qquad 1 \to C_3 \to S_3 \xrightarrow{\pi} S_2 \to e$$

induces a sequence in cohomology

$$\ldots \to H^1(F, C_3) \to H^1(F, S_3) \xrightarrow{\pi^1} H^1(F, S_2)$$

and the map π^1 associates to the class of L the class of its discriminant $\Delta(L)$. The group $H^1(F, C_3)$ classifies pairs $(L, \psi : \Delta(L) \xrightarrow{\sim} F \times F)$ or equivalently cyclic cubic algebras (L, ρ). If we twist the action of Γ on the groups in $(*)$ by the cocycle corresponding to a fixed quadratic algebra K, the group $H^1(F, (C_3)_{(K)})$ classifies isomorphism classes of pairs $(L, \psi : \Delta(L) \xrightarrow{\sim} K)$ where an isomorphism

$$\phi : (L, \psi : \Delta(L) \xrightarrow{\sim} K) \xrightarrow{\sim} (L', \psi' : \Delta(L') \xrightarrow{\sim} K)$$

is an isomorphism $\phi : L \xrightarrow{\sim} L'$ such that $\psi = \psi' \circ \Delta(\phi)$ (see [5]). Thus the set of isomorphism classes of pairs $(L, \psi : \Delta(L) \xrightarrow{\sim} K)$ admits a commutative group structure. We denote it by (L, K). Its neutral element is the class of $(F \times K, \mathrm{can})$, observing that there is a canonical isomorphism $\mathrm{can} : \Delta(F \times K) \xrightarrow{\sim} K$. The group operation is particularly simple to describe for Kummer algebras. We first describe it on the cohomological side. Putting

$$\mu_3(F) = \{x \in F \mid x^3 = 1\}$$

for any field F, $\mu_3 = \mu_3(F_{\text{sep}})$ is a Γ-module and we have, if $\text{char} F \neq 3$,

$$H^1(F, \mu_3) \simeq F^\times / F^{\times 3}.$$

From now on we assume $\text{char} F \neq 3$. Let

$$F(\omega) = F[X]/(X^2 + X + 1),\ \omega = X + (X^2 + X + 1).$$

We have

$$\mu_3 = (C_3)_{(F(\omega))}$$

as Γ-modules. Thus, for any quadratic étale F-algebra K,

$$(C_3)_{(K)} = (\mu_3)_{(K_1)}$$

where $K_1 = K \star F(\omega)$, and $H^1(F, (\mu_3)_{(K_1)}) \simeq (L, K)$. We compute $H^1(F, (\mu_3)_{(K)})$ for K quadratic étale arbitrary:

Proposition 1. *We have group isomorphisms*

$$\begin{aligned} H^1(F, (\mu_3)_{(K)}) &\simeq \{x \cdot K^{\times 3} \in K^\times / K^{\times 3} \mid x\tau(x) = 1\} \\ &\simeq K^\times / K^{\times 3} \cdot F^\times \\ &\simeq E^1 / (E^1)^3 \end{aligned}$$

where $E^1 = \{x \in K \mid x\tau(x) = 1\}$ *and* τ *is conjugation in* K.

Proof. (See [2] or [5], in fact corresponding claims hold for μ_n, n odd arbitrary). The canonical map

$$H^1(F, (\mu_3)_{(K)}) \to H^1(K, \mu_3)$$

is injective since $[K : F] = 2$ and $|\mu_3(F_{\text{sep}})| = 3$. Let

$$\text{Cor} : H^1(K, \mu_3) \to H^1(F, \mu_3)$$

be the map induced in cohomology by $x \mapsto N_{K/F}(x) = x\tau(x)$. In view of Proposition 21 of [2], we have

$$H^1(F, (\mu_3)_{(K)}) \simeq \text{Ker}\left(\text{Cor}\colon H^1(K, \mu_3) \to H^1(F, \mu_3)\right)$$

so that

$$H^1(F, (\mu_3)_{(K)}) \simeq \left\{x \cdot K^{\times 3} \in K^\times / K^{\times 3} \mid x\tau(x) \in F^{\times 3}\right\}$$

since $H^1(F, \mu_3) = F^\times / F^{\times 3}$. If $x \in K^\times$ is such that $x\tau(x) = \lambda^3$ with $\lambda \in F^\times$, then $x' = x(\lambda x^{-1})^3$ represents the same element and satisfies $x'\tau(x') = 1$. The second isomorphism is induced by $y \mapsto y\tau(y)^{-1}$ in view of Hilbert 90. For the third we have to show that if $x \in E^1$ is such that $x = y^3$, $y \in K^\times$, then there exists $u \in E^1$ such that $x = u^3$. As suggested by M. Rost we put $u = y(y\tau(y))^2$. $\square$

Corollary 2. *There is an isomorphism* $(L,K) \simeq E^1/(E^1)^3$ *where*

$$E^1 = \{x \in K_1 = K \star F(\omega) \mid x\tau(x) = 1\}$$

and τ *is conjugation in* K_1. $\square$

The isomorphism in Corollary 2 is through cohomology. We now describe it in terms of algebras. We first give an explicit isomorphism

$$\Psi : F^\times/F^{\times 3} \xrightarrow{\sim} (L, F(\omega)),$$

corresponding to the case $K = F(\omega)$. Let $F(\sqrt[3]{a}) = F[T]/(T^3 - a)$ for $a \in F^\times$, $\sqrt[3]{a} = T + (T^3 - a) \in F[T]/(T^3 - a)$. The algebra $F(\sqrt[3]{a})$ is étale and, in view of the above formulas,

$$\Delta(F(\sqrt[3]{a})) \simeq F[t] \text{ with } t^2 = \begin{cases} -27a^2 & \text{if } \operatorname{char} F \neq 2 \\ t+1 & \text{if } \operatorname{char} F = 2 \end{cases}$$

For $a \in F^\times$, let $L_a = F(\sqrt[3]{a})$. Assume that F does not contain a primitive cubic root of unity and that L_a is a field. Choosing an embedding $L_a \subset F_{\text{sep}}$, we write

$$\{z \in F_{\text{sep}} \mid z^3 = a\} = \{z_0, \omega_0 z_0, \omega_0^2 z_0\} = \{x_1, x_2, x_3\}$$

where ω_0 is a fixed primitive cubic root of unity in F_{sep} and we define

$$\psi_a : \Delta(L_a) \xrightarrow{\sim} F(\omega)$$

by

$$\psi_a\left((x_1 - x_2)(x_1 - x_3)(x_2 - x_3)\right) = 3a(2\omega + 1)$$

if $\operatorname{char} F \neq 2$ and

$$\psi_a\left(\tfrac{x_1}{x_1+x_2} + \tfrac{x_1}{x_1+x_3} + \tfrac{x_2}{x_2+x_3}\right) = 1 + \omega$$

if $\operatorname{char} F = 2$. Observe that $\psi_{-a} = \tau \circ \psi_a$ if $\operatorname{char} F \neq 2$ and $\psi_{1/a} = \tau \circ \psi_a$ if $\operatorname{char} F = 2$, denoting conjugation in $F(\omega)$ by τ. We let it as an exercise to define ψ_a if F contains a primitive cubic root of unity or if L_a is not a field, the case done being "generic".

Lemma 3. *The map* $a \mapsto (L_a, \psi_a)$ *induces an isomorphim of groups*

$$\Psi : F^\times/F^{\times 3} \xrightarrow{\sim} (L, F(\omega)).$$

Proof. The fact that Ψ is a well-defined homomorphism and is injective is straightforward. We check that it is surjective. Let L be such that $\psi : \Delta(L) \xrightarrow{\sim} F(\omega)$. If $F(\omega) \simeq F \times F$, i.e., F contains a primitive cubic root ω_0 of 1, L is cyclic, let ρ be a generator of $\operatorname{Gal}(L/F)$ and let α be a primitive element of L (the only case where a cubic étale algebra L does not have a primitive element is $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$ over $\mathbb{F}_2$; this case is easy to treat separately). The Lagrange resolvent

$$z = \alpha + \omega_0 \rho^2(\alpha) + \omega_0^2 \rho(\alpha)$$

is not zero (by Dedekind's lemma), satisfies $\rho(z) = \omega_0 z$, hence $z \notin F$ and $z^3 = N_{L/F}(z) = a \in F$. Furthermore the minimal equation of z over F has degree 3, since it has the three different roots z, $\rho(z)$, $\rho^2(z)$ and $L = F(z) = F(\sqrt[3]{a})$. If $(L,\psi) \simeq (F(\sqrt[3]{a}), \psi_a)$, we are done. If not, i.e., if $\psi = \tau \circ \psi_a$, we replace z by $-z$ if $\mathrm{char}F \neq 2$, resp. by $1/z$ if $\mathrm{char}F = 2$. Assume now that $F(\omega) \not\simeq F \times F$, so that L is not cyclic. Then $L \otimes F(\omega)$ is cyclic over $F(\omega)$, since its discriminant is trivial. Let $\rho \in \mathrm{Gal}\,(L \otimes F(\omega)/F(\omega))$ be a generator of the Galois group and let $\tau' = 1_L \otimes \tau$ where τ is conjugation in $F(\omega)$. Let α be a primitive element of L; as above

$$z = \alpha + \omega\rho^2(\alpha) + \omega^2\rho(\alpha)$$

is not in $F(\omega)$ and $z^3 = a \in F(\omega)$. Since L is not cyclic we have $\rho \circ \tau' = \tau' \circ \rho^2$ in $L(\omega)$ which implies that $\tau'(z) = z$ and $z \in L,\ z \notin F,\ z^3 \in F$. We conclude as in the case L cyclic. $\square$

We call a pair $(F(\sqrt[3]{a}), \psi_a)$ a *Kummer algebra.* If $(L, \psi : \Delta(L) \xrightarrow{\sim} K)$ is a cubic étale algebra with fixed discriminant, $(L,\psi) \otimes K_1$, where $K_1 = K \star F(\omega)$, is Kummer over K_1 in view of Lemma 3. This is used in the following proposition (see [5]), which gives an explicit description of the isomorphism described in Corollary 2.

Proposition 4. (1) *Let K be quadratic étale over F and let $x \in K$ be such that $x\tau(x) = 1$. Let $\tilde\tau$ be the automorphism of $K(\sqrt[3]{x})$ extending τ and such that $\tilde\tau(\sqrt[3]{x}) = \sqrt[3]{x}^{-1}$. Then*

$$L = \{u \in K(\sqrt[3]{x}) \mid \tilde\tau(u) = u\}$$

is cubic étale and $\psi_x|_L$ restricts to an isomorphism $\Delta(L) \xrightarrow{\sim} K \star F(\omega)$.
(2) *Conversely, if L is cubic étale and $\psi : \Delta(L) \xrightarrow{\sim} K$ is given then*

$$(L,\psi) \otimes K_1 \simeq (K_1(\sqrt[3]{x}), \psi_x)$$

where $K_1 = K \star F(\omega)$ and $x \in K_1$ is such that $x\tau(x) = 1$. Furthermore

$$L = \{u \in K_1(\sqrt[3]{x}) \mid \tilde\tau(u) = u\}$$

and $\psi = \psi_x|_L$.

Proof. (1) The case $K = F \times F$ follows from Lemma 3, so we may assume that K is a field. The multiplication in $K(\sqrt[3]{x})$ induces an isomorphism $L \otimes K \simeq K(\sqrt[3]{x})$, hence L is cubic étale. The fact that $\Delta(L) \simeq K \star F(\omega)$ follows from the computation of the discriminant in Lemma 3.
(2) Assume first that $L = F(\sqrt[3]{a})$, $a \in F^\times$; then $K_1 \simeq F \times F$ and we take $x = (a, a^{-1})$. If $L \not\simeq F(\sqrt[3]{a})$ for some $a \in F^\times$, K_1 is a field. We have

$$\begin{aligned} \Delta(L \otimes K_1) &= (\Delta(L) \star F(\omega)) \otimes \Delta(L) \\ &= \Delta(L) \otimes F(\omega) \\ &= (\Delta(L) \star F(\omega)) \otimes F(\omega) = K_1(\omega) \end{aligned}$$

so that $L \otimes K_1 \simeq K_1(\sqrt[3]{x})$ for some $x \in K_1^\times$. Let $z \in L \otimes K_1$ be such that $z^3 = x \in K_1^\times$. We have $z\tilde\tau(z) \in L$ and $(z\tilde\tau(z))^3 = x\tau(x) \in F^\times$. Since $L \not\simeq F(\sqrt[3]{a})$, $z\tilde\tau(z) = \lambda \in F^\times$. Then $x' = (\lambda x^{-1})^3 x$ defines the same algebra and $x'\tau(x') = 1$. The fact that we get L back follows from the injectivity of $H^1(F, (\mu_3)_{(K)}) \to H^1(K, \mu_3)$. $\square$

3 Reduced equations

We assume char$F \neq 3$ throughout this section.

Proposition 5. *Let L be a cubic étale algebra. If L is not isomorphic to one of the algebras $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$, $\mathbb{F}_2(\omega) \times \mathbb{F}_2(\omega) \times \mathbb{F}_2(\omega)$ or $\mathbb{F}_2 \times \mathbb{F}_2(\omega)$ then L is of the form $L = K[T]/(T^3 - 3T - c)$ for some $c \in F$.*

Proof. Since $L \not\xrightarrow{\sim} \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$, L has a primitive element and (since char$F \neq 3$) we may assume that $L = F[T]/(T^3 - pT - q)$. Let $\alpha = T + (T^3 - pT - q)$. If $p = 0$, $q \neq 0$, the element $\xi = \lambda\alpha + (q\lambda)^{-1}\alpha^2$, $\lambda \in F^\times$, satisfies

$$\xi^3 = 3\xi + q\lambda^3 + (q\lambda^3)^{-1}$$

and, by elementary linear algebra, $L = F(\xi)$ if and only if $q\lambda^3 \neq \pm 1$. If card$(F) \geq 5$ it is always possible to find $\lambda \in F^\times$ satisfying that condition. Since char$F \neq 3$ the problem restricts to $\mathbb{F}$ or $\mathbb{F}_2(\omega)$. Since L is not isomorphic to $\mathbb{F}_2(\omega) \times \mathbb{F}_2(\omega) \times \mathbb{F}_2(\omega)$ or $\mathbb{F}_2 \times \mathbb{F}_2(\omega)$ we have $q \neq 1$ and we may take $\lambda = 1$. Assume now that $p \neq 0$. The element

$$\xi = 2 - 3p^{-1}\alpha^2$$

satisfies

$$\xi^3 = 3\xi + c, \quad \text{with} \quad c = 2 - 27p^{-3}q^2$$

and $L = F(\xi)$. $\square$

In this section we describe explicitly the group structure on the set of classes $(L, K)_r$ of cubic étale algebras $F(\alpha)$ given by reduced equations

$$\alpha^3 = 3\alpha + c.$$

The cases char$F \neq 2$ and char$F = 2$ are treated separately.

I Char$F \neq 2$

Let $K = F[T]/(T^2 - \delta)$, $\delta \in F^\times$, be a fixed quadratic étale algebra; we have

$$\Delta(L) = F[T]/\left(T^2 - 27(4 - c^2)\right)$$

and the choice of an isomorphism $\psi :\ \Delta(L) \xrightarrow{\sim} K$ amounts to the choice of $s \in F^\times$ such that

$$27(4 - c^2) = \delta s^2.$$

Thus pairs $(L, \psi : \Delta(L) \xrightarrow{\sim} K)$ are parametrized (up to some relations) by rational points (c, s) on the torus E^1 with equation

$$(**) \qquad \frac{1}{4}X^2 + \frac{\delta}{27 \cdot 4}Y^2 = 1.$$

Let $K_1 = K \star F(\omega) = F(\sqrt{-3\delta})$ and let $\beta = \sqrt{-3\delta} \in K_1$. We have an isomorphism

$$E^1 \simeq \{z \in K_1 \mid z\tau(z) = 1\}$$

induced by

$$(c, s) \mapsto \frac{c}{2} + \frac{s}{18}\beta$$

which gives the natural group structure on E^1:

$$(c_1, s_1) \cdot (c_2, s_2) = \left(\tfrac{1}{2}(c_1c_2 - \tfrac{\delta}{27}s_1s_2), \tfrac{1}{2}(c_1s_2 + c_2s_1)\right)$$

with $e = (2, 0)$ as neutral element.

Proposition 6. *Let $K = F[T]/(T^2 - \delta)$, $L = F[T]/(T^3 - 3T - c)$ and $d = 27(4 - c^2)$. The map $(c, s) \mapsto (L, \psi)$, where $\psi : \Delta(L) = F[T]/(T^2 - d) \xrightarrow{\sim} K$ is given by $T + (T^2 - d) \mapsto sT + (T^2 - \delta)$, induces a group isomorphism $E^1/(E^1)^3 \xrightarrow{\sim} (L, K)_r$.*

For the proof of Proposition 6 we use the following lemma, clearly suggested by Proposition 4. We keep the same notations and we put

$$\beta = T + (T^2 + 3d) \in F[T]/(T^2 + 3d) = \Delta(L)_1 = \Delta(L) \star F(\omega)$$

and

$$\alpha = T + (T^3 - 3T - c) \in F[T]/(T^3 - 3T - c) = L.$$

Lemma 7. *The element*

$$z = -\frac{6\beta}{d} + \frac{d - 3c\beta}{2d}\alpha + \frac{3\beta}{d}\alpha^2 = \frac{\alpha}{2} + \frac{3\beta}{d}(\alpha^2 - \frac{c}{2}\alpha - 2) \in \Delta(L)_1 \otimes L$$

generates $\Delta(L)_1 \otimes L$ over $\Delta(L)_1$ and satisfies

$$z + \overline{z} = \alpha, \quad z\overline{z} = 1 \quad \textit{and} \quad z^3 = \frac{c}{2} + \frac{\beta}{18}$$

where $\overline{z} = (\tau \otimes 1_L)(z)$ and τ is conjugation in $\Delta(L)_1$. Furthermore z is uniquely determined by these relations.

Proof. The first two identities for z are straightforward, they imply that $z^2 = \alpha z - 1$ and determine z uniquely. Thus $z^3 = (\alpha^2 - 1)z - \alpha$ and direct computations show that the right hand side is equal to $\frac{c}{2} + \frac{\beta}{18}$. □

Proof of (6). Let $(c_i, s_i) \in E^1$, $s_i \neq 0$, $i = 1, 2$. Then $L_i = F[T]/(T^3 - 3T - c_i)$ is cubic étale and we have an isomorphism

$$\psi_i : \Delta(L_i) = F[T]/(T^2 - 27(4 - c_i^2)) \xrightarrow{\sim} K = F[T]/(T^2 - \delta)$$

given by $T_i + (T^2 - 27(4 - c_1^2)) \mapsto s_iT + (T^2 - \delta)$. Thus we get a surjective map $E^1 \setminus \{e\} \to (L, K)_r$ (observe that $e = (2, 0)$ gives a cubic algebra which is not étale). We show that it is multiplicative. Let

$$\beta_i = T + (T^2 + 3d_i) \in F[T]/(T^2 + 3d_i) = \Delta(L_i)_1 = \Delta(L_i) \star F(\omega),$$

where $d_i = 27(4 - c_i^2)$. Let further

$$\beta = T + (T^2 + 3\delta) \in F[T]/(T^2 + 3\delta) = K_1 = K \star F(\omega).$$

We have isomorphisms $\phi_i : \Delta(L_i)_1 \xrightarrow{\sim} K_1$ induced by $\beta_i \mapsto s_i\beta$. By Lemma 7 $L_i \otimes \Delta(L_i)_1$ is Kummer over $\Delta(L_i)_1$; thus $(L_i \otimes \Delta(L_i)_1) \otimes_{\phi_i} K_1$ is Kummer over K_1. The corresponding "Kummer" generator z_i satisfies

$$z_i^3 = \frac{c_i}{2} + \frac{s_i\beta}{18}.$$

It follows that

$$(z_1 z_2)^3 = \tfrac{1}{4}c_1c_2 - \tfrac{3\delta}{18^2}s_1s_2 + \tfrac{1}{36}(c_1s_2 + c_2s_1)\beta.$$

By Lemma 7, the cubic algebra determined by the generator z_1z_2 has parameters

$$(c, s) = \left(\tfrac{1}{2}(c_1c_2 - \tfrac{\delta}{27}s_1s_2), \tfrac{1}{2}(c_1s_2 + c_2s_1)\right)$$

and we get a surjective multiplicative map $E^1 \setminus \{e\} \to (L, K)_r$. The neutral element of (L, K) is the class of $F \times K$. This occurs if $T^3 - 3T - c$ has a root u in F, hence c has to be of the form $c = u^3 - 3u$ for $u \in F$. Let

$$E' = \{(c, s) \in E^1 \mid c = u^3 - 3u,\ u \in F\}$$

It is easy to check that $(E^1)^3 \subset E'$. We now show that $E' \subset (E^1)^3$. Let $(c, s) = (u^3 - 3u, s) \in E'$ and assume first that $u^2 \neq 1$; the element s satisfies

$$27\left(4 - (u^3 - 3u)^2\right) = s^2\delta$$

and the left hand side is equal to $27(4 - u^2)(u^2 - 1)^2$. We have

$$T^3 - 3T - u^3 + 3u = (T - u)(T^2 + uT + u^2 - 3) = (T - u)G$$

and

$$\Delta\left(F[T]/(T^3 - 3T - u^3 + 3u)\right) \simeq (F[T]/(G).$$

The discriminant of the polynomial $G = T^2 + uT + u^2 - 3$ is $3(4 - u^2)$. Thus there exists $t \in F^\times$ such that $\delta t^2 = 3(4 - u^2)$, s can be chosen as $s = 3(u^2 - 1)t$ and $(u, 3t)^3 = (u^3 - 3u, s)$. If $u^2 = 1$, $c = u^3 - 3u = \pm 2$, $s = 0$ and $(\pm 2, 0) = (\pm 2, 0)^3$. Thus $E' = (E^1)^3$ and, as claimed, $E^1/(E^1)^3 \xrightarrow{\sim} (L, K)_r$. □

Remark. Parametrizing rationally the curve $(**)$, the algebra $L = F[T]/(T^3 - 3T - c)$ has discriminant d if and only if there exists $t \in F$ such that

$$c = -2\frac{(3t^2 - d)}{(3t^2 + d)}$$

II Char$F = 2$

Let $K = F[T]/(T^2 + T + \delta)$ be a fixed quadratic étale F-algebra; we have $\Delta(L) = F[T]/(T^2+T+1+c^{-2})$ for $L = F[T]/(T^3+T+c)$ and the choice of an isomorphism $\psi : \Delta(L) \xrightarrow{\sim} K$ amounts to the choice of $s \in F$ such that $1 + c^{-2} = s^2 + s + \delta$. We may write $s = t/c$ since $c \in F^\times$, so that

$$t^2 + ct + c^2(1 + \delta) = 1$$

and (t, c) is a point on the torus E^1 with equation

$$X^2 + XY + Y^2(1 + \delta) = 1.$$

Let $K_1 = K \star F(\omega) = F(\beta)$ with $\beta^2 + \beta = 1 + \delta$. We have an isomorphism

$$E^1 \simeq \{z \in K_1 \mid z\tau(z) = 1\}$$

induced by

$$(t, c) \mapsto t + c\beta$$

which gives the natural group structure on E^1 with product

$$(t_1, c_1) \cdot (t_2, c_2) = (t_1t_2 + c_1c_2(1 + \delta), t_1c_2 + t_2c_1 + c_1c_2)$$

and neutral element $e = (1, 0)$.

Proposition 8. *Let $K = F[T]/(T^2+T+\delta)$, $L = F[T]/(T^3+T+c)$ and $d = 1 + c^{-2}$. The map $(t, c) \mapsto (L, \psi)$, where $\psi : \Delta(L) = F[T]/(T^2 + T + d) \xrightarrow{\sim} K$ is given by $T + (T^2 + T + d) \mapsto (t/c) + T + (T^2 + T + \delta)$, induces a group isomorphism $E^1/(E^1)^3 \xrightarrow{\sim} (L, K)_r$.*

As for (6), we need a lemma, whose proof is along the same lines as the proof of (6). Let

$$\beta = T + (T^2 + T + 1 + d) \in F[T]/(T^2 + T + 1 + d) = \Delta(L)_1 = \Delta(L) \star F(\omega)$$

and

$$\alpha = T + (T^3 + T + c) \in F[T]/(T^3 + T + c) = L.$$

Lemma 9. *The element $z = \beta\alpha + c^{-1}\alpha^2 \in \Delta(L)_1 \otimes L$ satisfies*

$$z\overline{z} = 1, \quad z + \overline{z} = \alpha \quad \textit{and} \quad z^3 = c\beta$$

where $\overline{z} = (\tau \otimes 1_L)(z)$ and τ is conjugation in $\Delta(L)_1$. Furthermore z is uniquely determined by these relations. □

Proof of (8)*:* Let $(t_i, c_i) \in E^1$, $c_i \neq 0$, $i = 1, 2$. Then $L_i = F[T]/(T^3 + T + c_i)$ is cubic étale and we have an isomorphism

$$\psi_i : \Delta(L_i) = F[T]/(T^2 + T + 1 + c_i^{-2}) \xrightarrow{\sim} K = F[T]/(T^2 + T + \delta)$$

given by $T + (T^2 + T + 1 + c_i^{-2})) \mapsto (t_i/c_i) + T + (T + (T^2 + T + \delta))$. Thus $(t_i, c_i) \mapsto (L_i, \psi_i)$ induces a surjective map $E^1 \setminus \{e\} \to (L, K)_r$. We claim that it is multiplicative: let

$$\beta_i = T + (T^2 + T + c_i^{-2}) \in F[T]/(T^2 + T + c_i^{-2}) = \Delta(L_i)_1 = \Delta(L_i) \star F(\omega)$$

and

$$\beta = T + (T^2 + T + 1 + \delta) \in F[T]/(T^2 + T + 1 + \delta) = K_1 = K \star F(\omega).$$

We have isomorphisms $\phi_i : \Delta(L_i)_1 \xrightarrow{\sim} K_1$ induced by $\beta_i \mapsto t_i/c_i + \beta$. By Lemma 9 $(L_i \otimes \Delta(L)_1) \otimes_{\phi_i} K_1$ is Kummer over K_1. The corresponding "Kummer" generator z_i satisfies

$$z_i^3 = c_i \beta_i.$$

It follows that

$$\begin{aligned}(z_1 z_2)^3 &= c_1 c_2 (\beta + \tfrac{t_1}{c_1})(\beta + \tfrac{t_2}{c_2}) \\ &= \beta(t_1 c_2 + t_2 c_1 + c_1 c_2) + t_1 t_2 + c_1 c_2 (1 + \delta),\end{aligned}$$

the cubic algebra corresponding to $z_1 z_2$ has parameters

$$(t, c) = (t_1 t_2 + c_1 c_2 (1 + \delta), (t_1 c_2 + t_2 c_1 + c_1 c_2)),$$

and we get a surjective multiplicative map $E^1 \setminus \{e\} \to (L, K)_r$. The neutral element of (L, K) is the class of $F \times K$. This corresponds to the case where $T^3 + T + c$ has a root in F, hence c has to be of the form $c = u^3 + u$ for $u \in F$. Let

$$E' = \{(t, c) \in E^1 \mid c = u^3 + u,\ u \in F\}$$

For $(t, u) \in E^1$ we have

$$(t, u)^3 = \big(t(u^2 + 1) + u, u^3 + u\big)$$

so $(E^1)^3 \subset E'$. Let now $(\ell, c) \in E'$, $c = u^3 + u$, $u \neq 0, 1$, so that

$$1 + \frac{1}{u^6 + u^2} = \left(\frac{\ell}{u^3 + u}\right)^2 + \frac{\ell}{u^3 + u} + \delta.$$

We have $T^3 + T + u^3 + u = (T + u)(T^2 + uT + u^2 + 1)$, hence, putting $L = F[T]/(T^3 + T + u^3 + u)$ and $\delta_L = 1 + \frac{1}{u^6 + u^2}$,

$$\Delta(L) = F[T]/(T^2 + T + \delta_L) \simeq F[T]/(T^2 + uT + u^2 + 1).$$

The (Arf) discriminant δ_G of $G = T^2 + uT + u^2 + 1$ is $\frac{u^2+1}{u^2}$, thus there exists $t \in F$ such that

$$\frac{u^2+1}{u^2} = (t/u)^2 + (t/u) + \delta$$

and this implies $(t, u) \in E^1$. On the other hand we have $\delta_L = \delta_G + \frac{u^2}{u^4+1}$ so that

$$(\frac{\ell}{u^3+u})^2 + \frac{\ell}{u^3+u} + \delta = \delta_L = \delta_G + \frac{u^2}{u^4+1} = (t/u)^2 + (t/u) + \delta + \frac{u^2}{u^4+1}$$

This can be reformulated as

$$\ell^2 + (u^3+u)\ell = (u^2t + t + u)^2 + (u^3+u)(u^2t + t + u)$$

hence either

$$\ell = u^2t + t + u \quad \text{or} \quad \ell = u^2t + t + u^3$$

In the first case we get $(\ell, u^3+u) = (t, u)^3$ and in the second $(\ell, u^3+u) = (t+u, u)^3$. Since (t, u) and $(t+u, u)$ belong to E^1, we have $(\ell, u^3+u) \in (E^1)^3$. Finally, if $u = 0, 1$, $\ell = 1$ and $(1, 0)^3 = (1, 0)$. Thus $E' = (E^1)^3$ and, as claimed, $E^1/(E^1)^3 \xrightarrow{\sim} (L, K)_r$. $\square$

4 Artin-Schreier Theory

Let F be a field of characteristic 3. The additive map

$$\wp : F \to F, \quad \wp(x) = x^3 - x$$

has kernel C_3 and it follows from the exact sequence

$$0 \to C_3 \to F_{\text{sep}} \to F_{\text{sep}} \to 0$$

that $H^1(F, C_3) \simeq F/\wp(F)$. More precisely we associate to $u \in F$ the cyclic algebra (L_u, ρ_u) with

$$L_u = F[T]/(T^3 - T - u) \quad \text{and} \quad \rho_u(T + (T^3 - T - u)) = T + 1 + (T^3 - T - u).$$

The product of cyclic cubic algebras, which in general is given by

$$(L_1, \rho_1) \star (L_2, \rho_2) = (L, \rho)$$

with $L = (L_1 \otimes L_2)^{\rho_1^{-1} \otimes \rho_2}$ and $\rho = \rho_1 \otimes 1$, reduces here to

$$(***) \qquad [u_1] + [u_2] \mapsto [L_{u_1+u_2}].$$

Observe that $T \mapsto -T$ induces an isomorphism $\phi : L_u \xrightarrow{\sim} L_{-u}$ such that

$$\phi \circ \rho_u \circ \phi^{-1} = \rho_u^2.$$

It is easy to check directly that any cubic cyclic algebra is of the form L_u: let $\xi \in L$ be primitive, then $\eta' = \xi - \rho(\xi)$ is also primitive and has trace zero. Hence

$$L \simeq F[T]/(T^3 - pT - q).$$

Since L has trivial discriminant we have $p^3 = t^2$ for some $t \in F^\times$ and $\eta = t^{-1}p\eta'$ satisfies $\eta^3 = \eta + u$.

Let $K = F[T]/(T^2 - d)$, $d \in F^\times$. Pairs $(L, \psi : \ \Delta(L) \xrightarrow{\sim} K)$ with L cubic étale are classified by $H^1(F, (C_3)_{(K)})$. Thus we have an induced group structure on (L, K). We first compute the cohomology part:

Proposition 10. *We have*

$$\begin{aligned} H^1(F,(C_3)_{(K)}) &\simeq \{x + \wp(K) \in K/\wp(K) \mid x + \tau(x) = 0\} \\ &\simeq F/\wp_d(F) \end{aligned}$$

where τ is conjugation in K and $\wp_d(x) = dx^3 - x$.

Proof. The map $H^1(F, (C_3)_{(K)}) \to H^1(K, C_3)$ is injective and by Proposition 21 of [2]

$$\begin{aligned} H^1(F,(C_3)_{(K)}) &\simeq \ker(\mathrm{Cor}: \ H^1(K, C_3) \to H^1(F, C_3)) \simeq \\ &\simeq \ker(t: \ K/\wp(K) \to F/\wp(F)) \end{aligned}$$

where t is induced by $x \mapsto x + \tau(x)$. If $x \in K$ is such that $x + \tau(x) = \lambda^3 - \lambda$ for $\lambda \in F$ then $x' = x + \lambda^3 - \lambda$ satisfies $x' + \tau(x') = 0$, hence the first claim. For the second, if $x + \tau(x) = 0$, we have $x = \beta v$ with $\beta = T + (T^2 - d) \in K$ and $x = \wp(z)$, with $z = a + b\beta$, implies $a^3 = a$, $v = b^3 d - b$. □

We now consider reduced equations for cubic étale algebras.

Proposition 11. *Let L be cubic étale over F with discriminant algebra $\Delta(L) = F[T]/(T^2 - d)$. There exists $a \in F$ such that $L \simeq F[T]/(T^3 - dT - a)$.*

Proof. Let $\alpha \in L$ be a primitive element. So there are $m, p, q \in F$ such that

$$\alpha^3 = m\alpha^2 + p\alpha + q$$

and

$$d = -4m^3q + m^2p^2 - 18mpq + 4p^3 - 27q^2 = -m^3q + m^2p^2 + p^3$$

since $\mathrm{char} F = 3$. The element

$$\xi = (2m^2 + p)\alpha + m\alpha^2$$

satisfies

$$\xi^3 = d\xi + a \quad \text{with} \quad a = (2m^4p + m^3q + p^3)q$$

and $L = F(\xi)$. □

Let $(K_z = K[T]/(T^3 - T - z), \rho_z)$ be cyclic, let $\psi_z : \Delta(K_z) \xrightarrow{\sim} F \times F$ be a fixed isomorphism. The analogue of (4) is

Proposition 12. (1) *Let $K = F[T]/(T^2-d)$ and let $z \in K$ be such that $z+\tau(z) = 0$. Let $\tilde{\tau}$ be the automorphism of $K_z = K[T]/(T^3-T-z)$ extending τ on K and such that $T+(T^3-t-z) \mapsto -T+(T^3-t-z)$. Then $L = \{x \in K_z \mid \tilde{\tau}(x) = x\}$ is isomorphic to $F[T]/(T^3-dT-u)$ for $u \in F$ and $\psi_z|_L : \Delta(L) \xrightarrow{\sim} K$. In particular L is cubic étale with discriminant algebra K.*
(2) *Conversely, given $(L, \psi : \Delta(L) \xrightarrow{\sim} K)$, there exists $z \in K$ such that $z+\tau(z) = 0$ and*

$$(L,\psi) \otimes K \simeq (K_z, \psi_z).$$

Furthermore

$$L = \{x \in K_z \mid \tilde{\tau}(x) = x\} \quad \text{and} \quad \psi = \psi_z|_L.$$

Proof. (1) Let $\xi = T+(T^3-T-z) \in K_z$ and $\beta = T+(T^2-d) \in K$. Then $\eta = \beta\xi$ is invariant under $\tilde{\tau}$ and satisfies $\eta^3 = d\eta - u$ with $u = dz\beta$.
(2) Conversely, let $L = F[T]/(T^3-dT-a)$ and let $\eta = T+(T^3-dT-a) \in L$. let $\beta = T+(T^2-d) \in K$. Then $\xi = \eta\beta^{-1}$ satisfies $\xi^3 = \xi + z$ with $z = ad^{-1}\beta^{-1}$. The last claim follows from the injectivity of $H^1(F,(C_3)_{(K)}) \to H^1(K,C_3)$. □

Let $K = F(\beta)$, $\beta^2 = \delta \in F^\times$. In view of (10) there exists an isomorphism $(L,K) \simeq F/\wp_\delta(F)$. This isomorphism can be explicitly described using reduced equations:

Proposition 13. *For $a \in F$ let $L_a = F[T]/(T^3-\delta T-a)$; the map $a \mapsto (L_a, 1_K)$ induces an isomorphism $F/\wp_\delta(F) \xrightarrow{\sim} (L,K)$.*

We prove (13) in two steps. We first give another description of $F/\wp_\delta(F)$. On $E = F^\times \times F$ we define a multiplication by

$$(t_1,a_1) \star (t_2,a_2) = (t_1t_2, a_1t_2^3 + a_2t_1^3)$$

This gives a group with neutral element $(1,0)$ and

$$\wp_\delta(E) = \{(t,a) \mid a = (\lambda^3\delta - \lambda)\delta^2t^3 \quad \text{for some} \quad \lambda \in F\}$$

is a subgroup of E.

Lemma 14. *The map $(t,a) \mapsto \frac{a}{\delta^2t^3}$ induces a group isomorphism*

$$E/\wp_\delta(E) \xrightarrow{\sim} F/\wp_\delta(F).$$

□

Let now $L_i = F(\alpha_i)$, $\alpha_i^3 = d_i\alpha_i + a_i$, $i = 1,2$, be cubic étale with discriminant $\Delta(L_i) = F(\beta_i)$, $\beta_i^2 = d_i$. An isomorphism $\psi_i : \Delta(L_i) \xrightarrow{\sim} K$ is given by $\beta_i \mapsto t_i\beta$, $t_i \in F^\times$. Thus we have a surjective map $\Psi : E \to (L,K)$, $(t_i,a_i) \mapsto (L_i,\psi_i)$.

Proposition 15. *The map Ψ induces an isomorphism of groups*

$$E/\wp_\delta(E) \xrightarrow{\sim} (L,K).$$

Proof. In view of (12) the product

$$(L, \psi) = (L_1, \psi_1) \star (L_2, \psi_2)$$

can be computed over K and there it is given by $(***)$. Putting

$$z_i = \alpha_i (t_i \beta)^{-1} \in L_i \otimes K,$$

we have $z_i^3 = z_i + c_i$ with $c_i = \frac{a_i}{t_i \delta^2} \beta$ and

$$(L_1 \otimes K) \star (L_2 \otimes K) = K(z)$$

with

$$z = z_1 + z_2, \quad z^3 = z + (c_1 + c_2), \quad c_1 + c_2 = \frac{a_1 t_2^3 + a_2 t_1^3}{(t_1 t_2)^3 \delta^2} \beta.$$

Thus $L = F(\alpha)$ with $\alpha = (t_1 t_2) \beta z = t_2 \alpha_1 + t_1 \alpha_2$ and

$$\psi : \ \Delta(L) = F[T]/(T^2 - (t_1 t_2)^2 \delta) \xrightarrow{\sim} K$$

is given by

$$T + (T^2 - (t_1 t_2)^2 \delta) \mapsto (t_1 t_2) T + (T^2 - \delta).$$

This shows that Ψ is multiplicative. The neutral element is $F \times K \simeq F[T]/(T^3 - \delta T)$ with can : $\Delta(F \times K) \xrightarrow{\sim} K$. Assume now that two pairs (L_i, ψ_i) are isomorphic. The corresponding cubic cyclic algebras $(L_i \otimes K, \rho_i)$ over K are isomorphic (and conversely). Hence there exists $\lambda = \lambda_0 + \lambda_1 \beta \in K$ such that $c_2 - c_1 = \lambda^3 - \lambda$ which implies

$$\lambda_0^3 - \lambda_0 \quad \text{and} \quad a_2 t_1^3 - a_1 t_2^3 = (\lambda_1^3 \delta - \lambda_1) \delta^2 (t_1 t_2)^3.$$

Thus we get, as claimed, an isomorphism $E/\wp_\delta(E) \xrightarrow{\sim} (L, K)$. □

Proof of (13). Combine (15) with (14). □

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