Journal of the Indian Math. Soc.
Vol. 57 (1991) pp. 143–151

INVOLUTIONS ON RANK 16 CENTRAL SIMPLE ALGEBRAS

M.-A. KNUS, R. PARIMALA, AND R. SRIDHARAN

(*In memory of Prof. Hansraj Gupta*)

(Received May 22, 1990)

Let D be a central division algebra of rank 16 over a field K which represents a 2-torsion element in the Brauer group of K. A classical result of Albert asserts that D splits as a tensor product of quaternion subalgebras. Further, D admits an involution which is trivial on K. We say that an involution σ on D *splits* if there exist quaternion subalgebras H_i of D with involutions σ_i such that $D = H_1 \otimes H_2$ and $\sigma = \sigma_1 \otimes \sigma_2$. An involution σ is split if and only if D has σ-invariant quaternion subalgebras. A natural question arises as to when an involution σ on D splits. Examples of rank 16 division algebras with involutions which admit no invariant quaternion subalgebras were constructed in ([2] and [8]). It was proved in ([7], Theorem B, p. 296) that if σ is of even symplectic type and char $K \neq 2$, then σ splits. One can modify easily the arguments in [7] to include the case char $K = 2$ also in the even symplectic case. In ([5], p. 196) an invariant called *pfaffian discriminant* was attached to any involution on a central simple algebra of even dimension with values in K^*/K^{*2}. It was shown in [6], using quadratic form theory and Clifford algebras that if char $K \neq 2$ and σ is of orthogonal type on a rank 16 algebra, σ splits if and only if the pfaffian discriminant of σ is trivial. The aim of this paper is to give a criterion for an involution σ on a rank 16 central simple algebra A over a field K to split, without restriction on the characteristic of K. We have included the proofs in the case char $K \neq 2$ to make the discussion self-contained.

The method of proof is to produce σ-invariant quadratic subalgebras of A using a certain pfaffian adjoint map defined in [5] on the set of alternating elements for σ in A.

§ 1. Generalities on involutions

Let A be a central simple algebra over a field K with an involution σ of the first kind. Let φ: $\bar{K}\otimes_K A \simeq M_n(\bar{K})$ be a splitting for A over the algebraic closure $\bar{K}$ of K. Let φ transport $1\otimes\sigma$ to the involution

$$X \longmapsto UX^tU^{-1},\ U\in GL_n(\bar{K}),\ U^t = \varepsilon U \text{ with } \varepsilon=\pm 1.$$

There exists $V\varepsilon GL_n(\bar{K})$ such that $VUV^t = I$ or E where I is the identity matrix and

$$E = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & \ddots & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$

If char $K \neq 2$, the first possibility arises if $\varepsilon=1$ (in which case, we call σ to be of *type* 1) and the second possibility arises if $\varepsilon=-1$. (In this case, we call σ to be *type*-1). If char $K=2$, $\varepsilon=1$, and both cases can arise. We say that an involution is of *orthogonal type* if it can be split over $\bar{K}$ as $X \longmapsto X^t$ and is of *even symplectic type* if it can be split over $\bar{K}$ as $X \longmapsto EX^tE^{-1}$.

§ 2. Pfaffian discriminant

Let A be a central simple algebra of rank 16 over a field K such that the class of A is 2-torsion in Br(K). Let σ be an involution on A of type ε. By Albert's theorem, we may write $A = H_1\otimes H_2$, H_i denoting quaternion subalgebras of A. Let τ_i denote the standard involution on H_i, viz., $\tau_i(x)=\text{Trd}\ x-x$, Trd denoting the reduced trace. Then $\sigma=\text{Int}\ u\circ(\tau_1\otimes\tau_2)$ for some unit u of A, Int u denoting the inner automorphism $x\longmapsto uxu^{-1}$. Since type $\tau_i = -1$, type $(\tau_1 \otimes \tau_2) = 1$ and type $\sigma = \varepsilon$ implies that $(\tau_1\otimes\tau_2)(u) = \varepsilon u$. Let $S_\sigma = \{x-\varepsilon\sigma(x),\ x \in A\}$ be the *set of alternating elements* for σ. Then dimension of S_σ is 6.

LEMMA 2.1. *There exists a K-linear isomorphism p_σ: $S_\sigma \to S_\sigma$ satisfying $x\cdot p_\sigma(x) = p_\sigma(x)\cdot x \in K$, $\forall x \in S_\sigma$. The map is unique upto scalars and $p_\sigma^2 \in K^*$*

Proof. Suppose first that $A = H_1\otimes H_2$, $\sigma=\tau=\tau_1\otimes\tau_2$, τ_i denoting the standard involutions on H_i. Then type $\tau = 1$ and it is easy to verify that $S_\tau = \{x\otimes 1 - 1\otimes y,\ x\in H_1,\ y\in H_2,\ \text{Trd}_{H_1}x = \text{Trd}_{H_2}y\}$. Trd denoting the reduced trace. The map $p_\tau = 1\otimes\tau_2$: $S_\tau \to S_\tau$ is K-linear. Further, for

$x\in H_1$, $y\in H_2$ with Trd x = Trd y, if $z = x\otimes 1 - 1\otimes y$, then,

$$z\cdot p_\tau(z) = p_\tau(z)\cdot z = -Nrd\ x + Nrd\ y \in K,$$
$$p_\tau^2 = 1_A$$

Let $\sigma = \text{Int}\ u\circ\tau$ with $\tau u = \varepsilon u$, ε being the type of σ. If $x \in S_\sigma$, then $u^{-1}x\in S_\tau$. We define p_σ: $S_\sigma \to S_\sigma$ to be the map $p_\sigma(x) = p_\tau(u^{-1}x)\cdot u^{-1}$. We have, for $x \in S_\sigma$,

$$\begin{aligned} xp_\sigma(x) &= x\cdot p_\tau(u^{-1}x)u^{-1} \\ &= u\{(u^{-1}x)\,p_\tau(u^{-1}x)\}\,u^{-1} \in K \end{aligned}$$

The uniqueness upto scalars of p_σ as well as the fact that $p_\sigma^2 \in K^*$ may be verified by going over to a splitting of A. Therefore we assume that σ: $M_4(K) \to M_4(K)$ is the involution given by $X \longmapsto UX^tU^{-1}$, $U^t = \varepsilon U$. Let p: $S_\sigma \to S_\sigma$ be a K-linear map satisfying $X\cdot p(X) \in K\ \forall X\in S_\sigma$. For $X \in S_\sigma$, $U^{-1}X \in \text{Alt}_4(K)$= the set of alternating matrices in $M_4(K)$. The map $\tilde{p}$: $\text{Alt}_4(K) \to \text{Alt}_4(K)$ defined by $\tilde{p}(Y) = p(UY)U$ satisfies $\tilde{p}(Y)Y \in K$, $\forall Y \in \text{Alt}_4(K)$. Then, upto a scalar, $\tilde{p}$ must coincide with the pfaffian adjoint map (cf [5])

$$\text{pfad}\begin{pmatrix} 0 & x & y & z \\ -x & 0 & u & v \\ -y & -u & x & t \\ -z & -v & -t & 0 \end{pmatrix} = \begin{pmatrix} 0 & -t & v & -u \\ t & 0 & -z & y \\ -v & z & 0 & -x \\ u & -y & x & 0 \end{pmatrix}$$

Further, $(\text{pfad})^2 = 1$. Thus $p(X) = \lambda\cdot\text{pfad}\,(U^{-1}X)\cdot U^{-1}$ for $\lambda \in K$. This proves the uniqueness of p_σ upto scalars. The formulae

$$\text{pfaffian}\,(UXU^t) = \det U\cdot\text{pfaffian}\,(X),$$
$$\text{pfad}\,(UXU^t) = (\det U)\,(U^t)^{-1}\,(\text{pfad}\,X)\,U^{-1},$$

for $X\in \text{Alt}_4(K)$ yield, for $X \in S_\sigma$,

$$\begin{aligned} p_\sigma^2(X) &= \text{pfad}\,(U^{-1}\,\text{pfad}\,(U^{-1}X)\,U^{-1})\,U^{-1} \\ &= (\det U)^{-1}\,X. \end{aligned}$$

This completes the proof of the lemma.

Definition. The scalar $p_\sigma^2 \in K^*$ modulo squares is defined to be the *pfaffiian discriminant of* σ, denoted by pf disc σ.

Remark 2.2. If $\sigma = \text{Int}\ u\circ\tau$ on $H_1\otimes H_2$, $\tau = \tau_1\otimes\tau_2$, τ_i denoting the standard involutions on H_i it is easily verified that pf disc σ = Nrd u, where Nrd denotes the reduced norm.

Remark 2.3. (cf [5], Prop. 3.4, p. 197) The pfaffian discriminant of an even symplectic involution is trivial.

Remark 2.4. Let $A = H_1 \otimes H_2$ and $\sigma = \sigma_1 \otimes \sigma_2$ σ_i denoting involutions on H_i. Then the pfaffian discriminant of σ is trivial. In fact, if $\sigma_i = \mathrm{Int}\ u_i \circ \tau_i$, u_i denoting units in H_i, τ_i denoting the standard involutions of H_i, then $\sigma = \mathrm{Int}\ (u_1 \otimes u_2) \circ (\tau_1 \otimes \tau_2)$ and by (2.2), pf disc σ is the class of Nrd $(u_1 \otimes u_2)$ in K^*/K^{*}_2. We have,

$$\mathrm{Nrd}\ (u_1 \otimes u_2) = (\mathrm{Nrd} u_1\ \mathrm{Nrd} u_2)^2 \in K^{*2}.$$

§ 3. Main Theorem

We prove the following

THEOREM 3.1. *Let A be a rank 16 central simple algebra over a field K with an involution σ of the first kind. Then (A, σ) splits as*

$$(H_1 \otimes H_2, \sigma_1 \otimes \sigma_2),$$

H_i denoting quaternion subalgebras of A, if and only if the pfaffian discriminant of σ is trivial.

We begin with the following special case of the theorem for even symplectic involutions, which is already contained in [7] for char $K \neq 2$. We include it here for the sake of completeness and to cover the case char $K = 2$ as well.

PROPOSITION 3.2. *Let σ be an even symplectic involution on a division algebra D of rank 16. Then σ has an invariant quaternion subalgebra.*

Proof. Every element $x \in S_\sigma$ satisfies a quadratic polynomial over K. In fact, in a splitting of σ as $X \to EX^tE^{-1}$, x looks like EY where Y is an alternating matrix and x satisfies the equation

$$\text{pfaffian } (E^{-1}T + Y) = 0$$

which is quadratic in T. Further, not every element of S_σ has its square a scalar. For, otherwise, the same would be true for $E \cdot \mathrm{Alt}_4(\bar{K})$. This however is not true since there are elements in $E \cdot \mathrm{Alt}_4(\bar{K})$ whose squares do not belong to $\bar{K}$. We choose $x \in S_\sigma$ which generates a separable quadratic algebra $K(x)$ over K. The involution σ restricts to identity on $K(x)$ since $\sigma(x) = x$. Let α be the nontrivial automorphism of $K(x)$ over

K. Let $u \in D^*$ be such that Int u extends α to D. The element $u^{-1}\sigma u$ commutes with x so that if $u + \sigma u \neq 0$, Int $(u + \sigma u) \mid K(x) = \text{Int } u \mid K(x)$. The element $y = u + \sigma u$ belongs to S_σ so that $K(y)/K$ is quadratic. Further, $yxy^{-1} = \alpha(x)$. Thus (x, y) generate a quaternion subalgebra of D invariant under σ. Suppose $u + \sigma u = 0$. Since u may be replaced by $u\lambda$ for any $\lambda \in D_1$ where D_1 is the commutant of $K(x)$ in D, it is enough to show that there exists $\lambda \in D_1$ such that $u\lambda + \sigma(u\lambda) \neq 0$. Suppose not. Then $u\lambda u^{-1} = \sigma(\lambda)\ \forall \lambda \in D_1$; i.e. $u(\lambda\mu)u^{-1} = \sigma(\lambda\mu)$, $\forall \lambda, \mu \in D_1$; i.e.,

$$\sigma(\lambda)\ \sigma(\mu) = \sigma(\mu)\ \sigma(\lambda)\ \ \forall \lambda, \mu \in D_1.$$

This implies that D_1 is commutative, contradicting $[D_1 : K] = 8$. Thus, one may choose $\lambda \in D_1$ such that $u\lambda + \sigma(u\lambda) \neq 0$. We set $y = u\lambda + \sigma(u\lambda)$. Then (x, y) generate a σ-invariant quaternion subalgebra of D.

PROPOSITION 3.3. *Let A be a central simple algebra of rank 16 over a field K with an even symplectic involution σ. Then A has σ-invariant quaternion subalgebras.*

Proof. We need only to consider the case $A = M_2(H)$, where H is a quaternion (not necessarily division) algebra over K. Let bar on H denote the standard involution on H and bar on $M_2(H)$ denote the involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}.$$

Then any even symplectic involution σ on $M_2(H)$ is of the form $\sigma(A) = U\bar{A}^tU^{-1}$, where $U = \begin{pmatrix} \lambda & \mu \\ \bar{\mu} & \delta \end{pmatrix}$, $\lambda, \delta \in K$, $\mu \in H$. There exists $V \in GL_2(H)$ such that

$$VU\bar{V}^t = \begin{pmatrix} \nu & 0 \\ 0 & \eta \end{pmatrix},$$

$\nu, \eta \in K$. Therefore, σ is conjugate to the involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} \nu & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \begin{pmatrix} \nu^{-1} & 0 \\ 0 & \eta^{-1} \end{pmatrix}$$

which clearly fixes both $M_2(K)$ and H.

LEMMA 3.4. (*cf* [4], *Lemma* 2.1) *Let* σ *be an involution on a rank* 16 *central simple algebra* A. *Let* $x \in A$ *be such that* $K(x)/K$ *is a separable quadratic subalgebra and* σ *restricts to the nontrivial automorphism of* $K(x)$ *over* K. *Then there is a quaternion subalgebra* H *of* A *such that* σ *restricts to the standard involution on* H.

Proof. Let D_1 be the commutant of $K(x)$ in A. Then σ restricts to an involution on the quaternion algebra D_1 which is nontrivial on its centre $K(x)$. Thus by [1, p. 161] which is valid even if $K(x) \xrightarrow{\sim} K \times K$, there exists a quaternion algebra $H_1 \subset D_1$ such that $D_1 = K(x) \cdot H_1 \simeq K(x) \otimes H_1$ and σ restricted to D_1 is simply $\alpha \otimes \tau$, α denoting the nontrivial automorphism of $K(x)$ over K and τ the standard involution on H_1. The quaternion subalgebra H_1 is the required one.

Let σ be an involution on a rank 16 central simple algebra A with pf disc σ trivial. Then one can chose $p_\sigma: S_\sigma \to S_\sigma$ such that $p_\sigma^2 = 1$. Let $W^+ = \{x + p_\sigma(x), x \in S_\sigma\}$, $W^- = \{x - p_\sigma(x), x \in S_\sigma\}$. If char $K = 2$, $W^+ = W^-$. Let $x \in W^+$, $x = y + p_\sigma(y)$, $y \in S_\sigma$. Then

$$\begin{aligned} x^2 &= (y + p_\sigma(y)) \cdot (y + p_\sigma(y)) \\ &= (y + p_\sigma(y))\,(p_\sigma^2(y) + p_\sigma(y)) \\ &= (y + p_\sigma(y))\, p_\sigma(y + p_\sigma(y)) \in K \end{aligned}$$

Thus W^+ is a K subspace of D such that every element of W^+ has its square in K.

PROPOSITION 3.5 *Let char* $K \neq 2$ *and* σ *an involution of orthogonal type on a rank* 16 *central-simple algebra* A *with pf disc* σ *trivial. Then there exist quaternion subalgebras* H_i *of* A *such that* $A = H_1 \otimes H_2$, $\sigma = \tau_1 \otimes \tau_2$, τ_i *denoting the standard involutions on* H_i.

Proof. Let $p_\sigma: S_\sigma \to S_\sigma$ be such that $p_\sigma^2 = 1$. Since σ is of orthogonal type, for $x \in S_\sigma$, $\sigma(x) = -x$. Thus, $W^+ \cap K = \{0\}$. In fact dim $W^+ = 3$ is seen from the fact that over the algebraic closure $\bar{K}$ of K, if σ is split as $X \to X^t$(σ being orthogonal), the space $\bar{K} \otimes W^+$ is simply the space

$$\left\{ \begin{pmatrix} 0 & \lambda & \mu & \nu \\ -\lambda & 0 & \nu & \mu \\ -\mu & -\nu & 0 & \lambda \\ -\nu & -\mu & -\lambda & 0 \end{pmatrix}, \lambda, \mu, \nu \in \bar{K} \right\}$$

Thus, we may choose $x \in W^+$, $x \neq 0$. Since $x^2 \in K$ and char $K \neq 2$, $K(x)/K$ is a separable quadratic algebra. Since $x \in W^+ \subset S_\sigma$, $\sigma(x) = -x$ so that σ restricts to the nontrivial automorphism of $K(x)$ over K. Hence by (3.2) there exists a quaternion subalgebra H_1 of A such that σ restricts to the standard involution τ_1 of H_1. Let H_2 be the commutant of H_1 in A and σ_2 the restriction of σ to H_2. Then $A = H_1 \otimes H_2$, $\sigma = \tau_1 \otimes \tau_2$. Since type $\sigma = 1$ and type $\tau_1 = -1$, it follows that type $\sigma_2 = -1$. The unique involution of symplectic type on a quaternion algebra is the standard involution so that $\sigma_2 = \tau_2$ and $\sigma = \tau_1 \otimes \tau_2$.

LEMMA 3.6. *Let* A *be a rank* 16 *algebra over* K *with char* $K = 2$. *Let* σ *be an involution on* A *with pf disc* σ *trivial. Let* W^+ *be as defined earlier. If* σ *is even symplectic*, $W^+ = K$. *If* σ *is orthogonal*, $K + W^+$ *is a maximal commutative subalgebra of* A *which is purely inseparable of index* 2 *over* K. *In particular it is a commutative Frobenius subalgebra of dimension* 4 *in* A.

Proof. To prove the lemma, we may assume that $A = M_4(K)$ and σ given by $X \longmapsto UX^tU^{-1}$ with det $U = 1$. We have $S_\sigma = U \cdot \mathrm{Alt}_4(K)$ and $p_\sigma: S_\sigma \to S_\sigma$ is given by $p_\sigma(Y) = \mathrm{pfad}\,(U^{-1}Y)U^{-1}$, pfad: $\mathrm{Alt}_4(K) \to \mathrm{Alt}_4(K)$ being the usual pfaffian adjoint map. One can explicitly compute W^+. Suppose σ is even symplectic and $U = E$, then, $W^+ = K$. If σ is given by $X \to X^t$, then,

$$W^+ = \left\{ \begin{pmatrix} 0 & \lambda & \mu & \nu \\ \lambda & 0 & \nu & \mu \\ \mu & \nu & 0 & \lambda \\ \nu & \mu & \lambda & 0 \end{pmatrix}, \lambda, \mu, \nu \in K \right\}$$

In this case, dim $W^+ = 3$, $W^+ \cap K = 0$ and one can verify directly that $K + W^+$ is a maximnl commutative subalgebra of $M_4(K)$.

PROPOSITION 3.7. *Let A be an algebra of rank 16 over a field K of characteristic 2. Let σ be an involution on A with pfdisc σ trivial. Then, A has a σ-invariant quaternion subalgebra.*

Proof. In view of (3.3), we may assume that σ is of orthogonal type. In this case, $L = K + W^+$ is a maximal commutative subalgebra of A of dimension 4 over K which is purely inseparable of index 2 over K. Let $x, y \in L$ be genarators of L over K with $x^2 = a$, $y^2 = b$, $a, b \in K$. Since L is a commutative Frobenius subalgebra of A of dimension 4, by ([3], Th. 3, p. 223), the K-derivation $d : L \to L$ defined by $dx = x$, $dy = 0$ extends to an inner derivation on A, i.e. $\exists \xi \in A$ such that $\xi x + x\xi = x$, $\xi y = y\xi$. The element $\xi^2 + \xi$ commutes with L and L being maximal commutative, $\xi^2 + \xi \in L$ and hence $(\xi^2 + \xi)^2 = c \in K$. Let $\eta = \xi^2$. Then $\eta^2 + \eta = c$ and $x\eta + \eta x = x$. Thus (x, η) generate a quaternion subalgebra of A. We show that this is invariant under σ. The element $\sigma\eta + \eta$ commutes with L and hence belongs to L so that $(\sigma\eta + \eta)^2 = d \in K$; i.e., $\sigma\eta = \eta + d$. This completes the proof of the proposition.

The propositions 3.3, and 3.5 and 3.7 lead to the theorem.

REFERENCES

1. ALBERT, A.A. Structure of Algebras, AMS Colloquium publications, Vol 24, 1939.
2. AMITSUR, A.S., L.H. ROWEN and J.P. TIGNOL, Division algebras of degrees 4 and 8 with involutions, Israel J. Math. Vol. 33, 133-148 (1979).
3. JACOBSON, N. Generation of separable and central simple algebras, J. Math. Pures Appl. Vol. 36, 217-227 (1957).
4. CHACRON, M., H. DHERTE, J.P. TIGNOL and A. WADSWORTH, Stable decomposition of division algebras with involutions, Preprint.
5. KNUS, M.A., R. PARIMALA and R. SRIDHARAN, A classification of rank 6 quadraric spaces via pfaffians, J. Reine u. Angew. Math. 398, 187-218 (1989).
6. KNUS, M.A., R. PARIMALA and R. SRIDHARAN, Pfaffians, Central simple algebras and similitudes, to appear in Math. Z.
7. ROWEN, L.H. Central Simple Algebras, Israel J. Math. Vol. 29, 285-301

8. SCHARLAU, W., Zur Existenz von Involutionen auf einfachen Algebren II, Math. Z. Vol. 176, 399–404 (1981).

Mathematik
ETH-Zentrum
Rämistrasse 74,
CH-8092 Zürich
Switzerland

School of Mathematics
Tata Institute of Fundamental Research
Bombay 400005
India